CHAPTER TWO

CONCERNING THE MOTION OF A POINT ON A GIVEN LINE IN A VACUUM.

[p. 125]

PROPOSITION 31.

Problem.

282. *With a uniform force present acting downwards, to find the curve AM* (Fig. 36), *upon which a body with a given initial speed moves thus, so that in equal times equal angles are completed about a fixed point C.*

Solution.

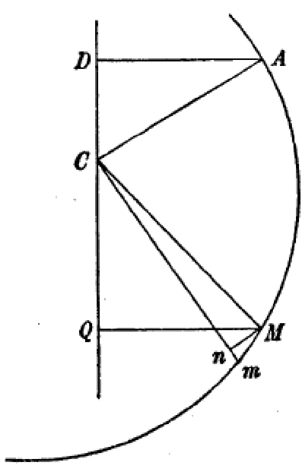

Fig. 36.

The beginning of the curve is taken at a certain place *A*, at which the line *CA* is normal to the curve itself, and let the speed at *A* corresponding to the height *b* and let *AC* = *a* ; the angular speed is as $\frac{\sqrt{b}}{a}$, [p. 126] to which quantity the angular speed expressed at individual points *M* must be equal. Let the speed at *M* correspond to the height *v* and *CM* = *x*, then we have *mn* = *dx*. The ratio is made so that

$$Mm : Mn = \sqrt{v} : \frac{Mn \cdot \sqrt{v}}{Mm},$$

[The component of the speed normal to the distance *CM* is taken.] since the quantity divided by *MC* gives the angular speed equal to $\frac{Mn.\sqrt{v}}{Mm.MC}$; which since this is equal to $\frac{\sqrt{b}}{a}$, we have this equation :

$$Mn \cdot a\sqrt{v} = Mm \cdot MC \cdot \sqrt{b} = Mm \cdot x\sqrt{b}.$$

Now let the line DCQ be drawn vertical and the sine of the angle *ACD* = *m*, the cosine of this is equal to $\sqrt{(1-m^2)}$ with the whole sine put equal to 1. Likewise the sine of the angle MCD is equal to *t*; and the cosine is equal to $\sqrt{(1-tt)}$. Therefore with these in place it follows that

$$CD = a\sqrt{(1-m^2)} \text{ and } CQ = -x\sqrt{(1-tt)}$$

and

$$\text{sine of angle } MCm = \frac{dt}{\sqrt{(1-tt)}} = \frac{Mn}{x},$$

thus the equation becomes

$$Mn = \frac{xdt}{\sqrt{(1-tt)}} \text{ and } Mm = \frac{\sqrt{(dx^2 - ttdx^2 + x^2dt^2)}}{\sqrt{(1-tt)}}.$$

But since the body has fallen from the height *DQ*, then

$$v = b + g.DQ = b + ga\sqrt{(1-m^2)} - gx\sqrt{(1-tt)}.$$

With which values substituted in the equation, this equation arises :

$$bdx^2(1-tt) = a^2bdt^2 + ga^3dt^2\sqrt{(1-m^2)} - ga^2xdt^2\sqrt{(1-tt)} - bx^2dt^2$$

or

$$dx\sqrt{b} = \frac{dt\sqrt{\left(a^2b + ga^3\sqrt{(1-m^2)} - ga^2x\sqrt{(1-tt)} - bx^2\right)}}{\sqrt{(1-tt)}}.$$

Which equation thus by integration, as $t = m$ makes $x = a$, expresses the nature of the curve sought. Q.E.I.

Corollary1.

283. If in place of the sines of the angles ACD and MCD the cosine of these are introduced, these become $\sqrt{(1-m^2)} = n$ and $\sqrt{(1-tt)} = q$, then we have

$$dx\sqrt{b} = \frac{-dq\sqrt{(a^2b + gna^3 - ga^2qx - bx^2)}}{\sqrt{(1-q^2)}}$$

or [p. 127]

$$\frac{-dq}{\sqrt{(1-qq)}} = \frac{dx\sqrt{b}}{\sqrt{(b(a^2-x^2) + ga^2(na-qx))}},$$

which thus has to be integrated, so that on putting $q = n$, we have $x = a$.

Corollary 2.

284. Where the curve is normal to the radius *CM*, with *dx* vanishing there, it becomes :

$$b(a^2 - x^2) = ga^2(qx - na).$$

Hence, whenever

$$q = \frac{a^2b + gna^3 - bx^2}{ga^2x},$$

the curve is normal to the radius *CM*. Moreover since *q* is contained between the limits + 1 and – 1, *x* cannot be a greater than the quantity given ; for with $x = \infty$, $q = \infty$, which is absurd.

Corollary 3.

285. If *CM* is normal to the curve, then the angular speed is equal to

$$\frac{\sqrt{v}}{MC} = \frac{\sqrt{(b+gna-gqx)}}{x},$$

which must be equal to $\frac{\sqrt{b}}{a}$. Therefore that is the maximum angular speed if $q = -1$. Moreover that angular speed is made less than that, when x is made greater. Now again the angular speed is less than that, when the angle of the curve to the radius *MC* is greater. Whereby the curve cannot descend below a certain distance *C*, as the distance giving x is found from this equation :

$$x\sqrt{b} = a\sqrt{(b+gna+gx)},$$

clearly,

$$x = \frac{ga^2}{2b} + a\sqrt{\left(\frac{g^2a^2}{4b^2} + \frac{gna}{b} + 1\right)}.$$

Therefore this is the maximum distance of the curve from the point *C*.

Corollary 4.

286. Therefore since the curve is not able to be at a greater distance from the fixed point C, [p. 127] this curve returns on itself. Clearly either after one revolution or after two or three etc., or even after an infinite number of revolutions it will return [to its starting conditions], as the letters *a, b, n* and *g* assumed.

Example.

287. If the force acting vanishes, then $g = 0$ and the body advances uniformly. Therefore this equation is then obtained to describe the curve :

$$\frac{dx}{\sqrt{(a^2-x^2)}} = \frac{dt}{\sqrt{(1-tt)}},$$

the integral by logarithms is :

$$\sqrt{-1}\, l\left(\frac{x\sqrt{-1}-\sqrt{(a^2-x^2)}}{c}\right) = \sqrt{-1}\, l(t\sqrt{-1} - \sqrt{(1-tt)})$$

or

$$x\sqrt{-1} - \sqrt{(a^2-x^2)} = ct\sqrt{-1} - c\sqrt{(1-tt)}.$$

Which on reduction gives :

$$(a^2-c^2)^2 = 4(a^2+c^2)ctx - 4c^2x^2 - 4a^2c^2t^2.$$

The line *AC* falls on the vertical *CD*; indeed likewise on account of the force g vanishing, this must become $x = a$ with $t = 0$, from which there arises $a^2 + c^2 = 0$ and

$$a^4 = a^2x^2 + a^4t^2 \text{ or } x^2 = a^2(1-tt).$$

Which is the equation of a circle of diameter *a* passing through the fixed point *C*. For when the motion is uniform on a circle, the motion is also uniform with respect to any point on the circumference.

Scholium.

288. Moreover it is evident in this case that the circumference of the circle is a satisfactory curve, the centre of which is at the fixed point *C*, which solution is Cleary the easiest and which can be produced spontaneously. On account of which it is a source of wonder that this case is not contained in the solution. [p. 129] Now the reason for this is clearly similar to that, as we deduced above (268), where we observed a similar paradox. Having designated *C* as the centre of the circle, it is necessary to set $x = a$ or $dx = 0$, which now, since *x* is considered as a variable quantity, is unable to be done, especially since in the same equation the solution is otherwise contained, in which *x* really is a variable quantity. Now from the first equation on putting $v = b$, which is $Mn.a = Mm.x$, it is understood that the circle can be a satisfactory solution ; for if $x = a$ everywhere, then also $Mn = Mm$. Moreover I consider the great help to be given, in producing the construction of the curves satisfying this problem, if a solution could be found by such a method, which at the same time should give the case of uniform circular motion about a centre *C*. For as the most simple case to be present in the solution has thus been removed, as it cannot be found, we conclude that it is often the case with other curves that simple curves are contained in the solution of some general curves, which are hard to solve.

PROPOSITION 32.

Problem.

282. *If the body is attracted by some force to the centre of force C* (Fig. 37), *to find the curve AM on which the body is carried with a speed equal to the uniform motion towards C.*

Solution. [p. 130]

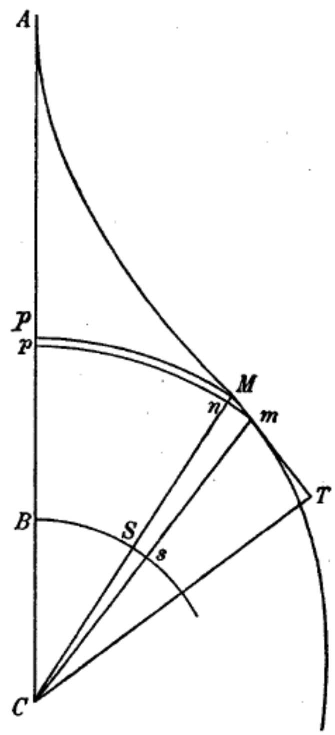

Fig. 37.

Let the minimum speed of the body at *A* correspond to the height *b*, the line *CA* is a tangent to the curve at *A*, since the body at *A* ought to move directly to *C*. Let $AC = a$ and $CM = x$, the speed at *M* corresponds to the height v and the centripetal force at *M* is equal to *P*; then we have $v = b - \int Pdx$, as the integral has to be taken in order that with $x = a$ it vanishes and makes $v = b$. Now the speed at *M* has to be of such a magnitude, since the element *Mm* is completed in the same increment of time as the element *Pp* with the speed $\sqrt{b}$. Hence it becomes $\sqrt{b} : \sqrt{v} = Pp : Mm = MT : MC$, hence this equation is produced [for the triangles *Mnm* and *MTC* are similar] :

$$b \cdot MC^2 = v \cdot MT^2 = b \cdot MT^2 - MT^2 \cdot \int Pdx.$$

The perpendicular *CT* to the tangent is equal to *p*; it becomes

$$bp^2 = -(x^2 - p^2)\int Pdx \text{ or } p^2 = \frac{-x^2\int Pdx}{b - \int Pdx}.$$

Or, if the sine of the angle *ACM* is put equal to *t*, it becomes [on differentiation]: $\frac{mn}{MC} = \frac{dt}{\sqrt{(1-tt)}}$. Hence the following equation is produced [as $mn = \frac{CT}{MT}.nM = \frac{p.-dx}{\sqrt{x^2-p^2}}$; and $x^2 - p^2 = \frac{bp^2}{\int Pdx}$, etc.] :

$$\frac{dt\sqrt{b}}{\sqrt{(1-tt)}} = \frac{-dx}{x}\sqrt{-\int Pdx}.$$

Of which for each case, if *P* is given in terms of *x*, a suitable curve can be constructed. Q.E.I.

Corollary 1.

290. If the centripetal force is proportional to some power of the distances, clearly

$$P = \frac{x^n}{f^n},$$

then we have :

$$\int Pdx = \frac{x^{n+1} - a^{n+1}}{(n+1)f^n},$$

With this substituted we have the following equation for the curve AM :

$$\frac{dt}{\sqrt{(1-tt)}} = \frac{-dx}{x}\sqrt{\frac{a^{n+1} - x^{n+1}}{(n+1)bf^n}}.$$

[p. 131] Which equation must thus be integrated, so that with $t = 0$ it becomes $x = a$.

Corollary 2.

291. If $\int \frac{dt}{\sqrt{(1-tt)}}$ is thus taken, so that it becomes equal to zero if $t = 0$, there is produced by integration from that equation:

$$\int \frac{dt}{\sqrt{(1-tt)}} = \frac{-2\sqrt{(a^{n+1} - x^{n+1})}}{(n+1)^{\frac{3}{2}}\sqrt{bf^n}} + \frac{a^{\frac{n+1}{2}}}{(n+1)^{\frac{3}{2}}\sqrt{bf^n}}\, l\, \frac{a^{\frac{n+1}{2}} + \sqrt{(a^{n+1} - x^{n+1})}}{a^{\frac{n+1}{2}} - \sqrt{(a^{n+1} - x^{n+1})}} = BS,$$

if with the centre C and with the radius $BC = 1$, the arc of the circle BSs is described. From which it is apparent that the curve AM has an infinite number of rotations before it arrives at C. For with $x = 0$, BS is made $\propto$.

Corollary 3.

292. Therefore the construction of this curve depends in part on the quadrature of the circle and in part on logarithms, if $n + 1$ is a positive number. But if $n + 1$ is a negative number, that term which was given by a logarithm, is also reduced to the quadrature of the circle.

Corollary 4.

293. This curve has a turning point [*flexus contrarii*]where $dp = 0$. [p. 132] Therefore in order that this point can be found, the equation is taken ;

$$pp = \frac{-x^2\int Pdx}{b - \int Pdx},$$

from which by differentiation, and on putting $dp = 0$, there is produced :

$$bPx = 2\left(\int Pdx\right)^2 - 2b\int Pdx.$$

Corollary 5.

294. Therefore in the case in which $P = \frac{x^n}{f^n}$, the turning point is at the place where :

$$(n+1)^2 bf^n x^{n+1} = 2(a^{n+1} - x^{n+1})^2 + 2(n+1)bf^n(a^{n+1} - x^{n+1}).$$

Thus, this equation arises:

$$\frac{a^{n+1} - x^{n+1}}{n+1} = -\frac{(n+3)bf^n}{4} + \frac{1}{4}\sqrt{((n+3)^2 b^2 f^{2n} + 8a^{n+1}bf^n)}.$$

Which on substitution into the integration gives the angle *ACM* at which the turning point lies.

Scholium 1.

295. But since from the nature of these kinds of curves it is difficult to produce turning points in general, it is considered that the principles are best effected by descending to the level of special cases.

Example 1.

296. If the centripetal force is proportional to the distances or $P = \frac{x}{f}$, in making $n = 1$.

Therefore with the arc *BS* put equal to *s*, the curve sought is expressed by this equation :

$$s = -\frac{\sqrt{(a^2 - x^2)}}{\sqrt{2bf}} + \frac{a}{2\sqrt{2bf}}\, l\, \frac{a + \sqrt{(a^2 - x^2)}}{a - \sqrt{(a^2 - x^2)}}.$$

[p. 133] From which equation with any distance of the point *M* from *C* the angle *BCS* is found, at which the body is without doubt present at that distance. Now between the distance $MC = x$ and the perpendicular $CT = p$, there is this equation :

$$pp = \frac{a^2x^2 - x^4}{2bf + a^2 - x^2}.$$

The turning point of this curve is where $dp = 0$, but this is where

$$x^4 = 2a^2x^2 + 4bfx^2 - a^4 - 2a^2bf$$

or

$$xx = a^2 + 2bf - \sqrt{(2a^2bf + 4b^2f^2)},$$

since x cannot be greater than a. Hence it becomes :

$$x = \sqrt{\left(a^2 + 2bf - \sqrt{(2a^2bf + 4b^2f^2)}\right)}$$

and

$$\frac{pp}{xx} = \frac{\sqrt{(2a^2+4bf)} - 2\sqrt{bf}}{\sqrt{(2a^2+4bf)}}.$$

Therefore the cosine of the angle that the curve makes at the turning point with the radius *CM* is equal to [Euler's original formula has here been corrected by P. S. in the *O. O.* to give the one presented here] :

$$\frac{\sqrt[4]{4bf}}{\sqrt[4]{(2a^2+4bf)}}.$$

Now the equation of the curve converted into a series on putting $\sqrt{(a^2 - x^2)} = y$ is this :

$$s\sqrt{2bf} = \frac{y^3}{3a^2} + \frac{y^5}{5a^4} + \frac{y^7}{7a^6} + \frac{y^9}{9a^8} + \text{ etc.}$$

Therefore in this series, the beginning of the curve, where *x* is not much less than *a*, or *y* is extremely small, will be

$$s\sqrt{2bf} = \frac{y^3}{3a^2}.$$

Then from this equation it is apparent on making $x = 0$ that s becomes $\propto$, whereby the curve goes round the centre *C* in an infinite number of spirals, and when the body is near to the centre , the equation becomes :

$$\frac{pp}{xx} = \frac{aa}{2bf + a^2}.$$

From which it follows that near the centre *C* the curve goes off in a logarithmic spiral.

Example 2.

296. Let $n = -1$ or $n + 1 = 0$, which case it to be extracted from the differential equation. For it becomes, on account of $P = \frac{f}{x}$

$$\int Pdx = fl\frac{x}{a},$$

thus this equation is obtained : [p. 134]

$$ds = \frac{dt}{\sqrt{(1-tt)}} = \frac{-dx}{x\sqrt{b}}\sqrt{fl\frac{a}{x}},$$

the integral of which is :

$$s = \frac{2f(la - lx)^{\frac{3}{2}}}{3\sqrt{bf}}.$$

The other equation between the perpendicular *p* and *x* is given by :

$$pp = \frac{fxx\, l\frac{a}{x}}{b + f\, l\frac{a}{x}}.$$

From which the turning point is found at that place, where

$$b = 2f\left(l\frac{a}{x}\right)^2 + 2b\, l\frac{a}{x}$$

or

$$l\frac{a}{x} = \frac{-b + \sqrt{(bb + 2bf)}}{2f}.$$

Therefore by taking this expression, there is obtained :

$$s = \frac{\left(-b + \sqrt{(bb + 2bf)}\right)^{\frac{3}{2}}}{3f\sqrt{2b}}.$$

Again it is evident, if x is made equal to 0, that $s = \infty$ or the curve makes an infinite number of turns around the centre C ; now in this case we have $\frac{pp}{xx} = 1$ or $p = x$. Therefore finally the curve goes round in an infinitely small circle.

Example 3.

298. Putting $n = -2$, in order that the centripetal force is inversely proportional to the square of the distances, the equation becomes

$$\frac{dt}{\sqrt{(1-tt)}} = \frac{-fdx}{x}\sqrt{\frac{a-x}{abx}} = ds.$$

Now $\int \frac{dy}{1+yy}$ expresses the arc, the tangent of which is y or $\sqrt{\frac{a-x}{x}}$; let this arc be equal to t, and the equation becomes :

$$t + \frac{s\sqrt{ab}}{2f} = y = \sqrt{\frac{a-x}{x}}.$$

Therefore everywhere with the distance given x the arc s taken and multiplied by $\frac{\sqrt{ab}}{2f}$ is equal to the difference between the tangent $\sqrt{\frac{a-x}{x}}$ and the corresponding arc with the radius put equal to 1. If x is put equal to 0, then this makes $s = \infty$, from which it follows that the curve falls towards the centre C in an infinite spiral. [p. 135] Besides, on account of

$$-\int Pdx = \frac{ff(a-x)}{ax}$$

it follows that

$$pp = \frac{ffxx(a-x)}{abx+ff(a-x)}.$$

From which it follows if *x* vanishes that $\frac{pp}{xx}=1$ or the curve finally also goes in an infinitely small circle.

If $ab = ff$, then

$$pp = \frac{xx(a-x)}{a} \text{ and } t+\frac{s}{2} = \sqrt{\frac{a-x}{x}}.$$

The turning point in this case lies at that place where $2ax = 3xx$, where either $x = 0$ or $x = \frac{2a}{3}$. But if it is not the case that $ab = 4ff$, then

$$t+s = \sqrt{\frac{a-x}{x}}$$

and the turning point is found by taking [either $x = 0$ or] $x = a\sqrt{\frac{1}{3}}$.

Scholium 2.

299. Moreover from which it is apparent how the infinite spirals can be compared, if the centripetal force is proportional to some power of the distance, or $P = \frac{x^n}{f^n}$, and the equation between *p* and *x* is considered, which is

$$\frac{pp}{xx} = \frac{a^{n+1}-x^{n+1}}{(n+1)bf^n+a^{n+1}-x^{n+1}}.$$

Here two cases are to be distinguished, the one in which $n + 1$ is a positive number, and the other in which it is negative.

If $n + 1$ is a positive number, on making $x = 0$ the equation becomes

$$\frac{pp}{xx} = \frac{a^{n+1}}{(n+1)bf^n+a^{n+1}}.$$

Hence in this case the curve *AM* goes around the centre *C* in a logarithmic spiral.

But if $n + 1$ is a negative number, on making $x = 0$ the equation becomes $\frac{pp}{xx}=1$. Therefore in these cases the curve at *C* becomes an infinitely small circle. The speed of approaching C in these cases becomes infinitely large, and on account of this, unless the curve becomes a circle [p. 136] the body approaches *C* with an infinite speed, which is contrary to the condition of the problem. Therefore with the curves determined, upon which the body uniformly approaches the centre of forces, we will investigate these curves in which the body is carries around the centre of force with a uniform motion.

PROPOSITION 33.

Problem.

300. *If a body is always attracted to the centre of force C* (Fig. 38), *to determine the curve AM, upon which the body is moving with a uniform angular motion around the centre C.*

Solution.

Let *A* be the highest point of the curve, where the curve is normal to the radius *AC*, and let the speed of the body at *A* correspond to *b* and *AC* = *a*; the angular speed at A = $\frac{\sqrt{b}}{a}$, to which quantity the angular speed at some particular point *M* must be equal. Putting *CM* = *x*, to which *CP* is taken equal, and the centripetal force at *M* is equal to *P*; then the speed at *M* corresponds to the height $b-\int Pdx$, with the integration of $\int Pdx$ thus taken, so that it vanishes on putting *x* = *a*. With the tangent *MT* drawn, the perpendicular sent from *C* to that is called *CT* = *p*; then we have $x : p = Mm : mn$. [p. 137; again, the infinitesimal triangle *Mnm* and the finite triangle *MTC* are similar] On account of this, the speed passing along *mn* is equal to

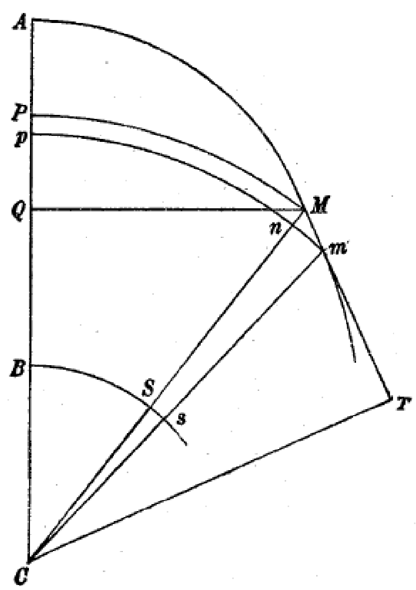

Fig. 38.

$$\frac{p\sqrt{(b-\int Pdx)}}{x}$$

and the angular speed is equal to

$$\frac{p\sqrt{(b-\int Pdx)}}{x^2},$$

which must be equal to $\frac{\sqrt{b}}{a}$. Hence the following equation is produced :

$$bx^4 = a^2bp^2 - a^2p^2\int Pdx$$

or

$$p = \frac{x^2\sqrt{b}}{a\sqrt{(b-\int Pdx)}}.$$

With centre *C* and with radius *BC* = 1 the arc of the circle *BS* is described, which is called equal to *s*, then $1 : ds = x : mn$, and hence $mn = xds$ and $Mm = \sqrt{(dx^2 + x^2ds^2)}$. Since now we have $x : p = \sqrt{(dx^2 + x^2ds^2)} : xds$, the equation becomes

$$p = \frac{xxds}{\sqrt{(dx^2 + x^2ds^2)}}.$$

With which value substituted in the equation found gives

$$bdx^2 + bx^2ds^2 = a^2bds^2 - a^2ds^2\int Pdx$$

and hence

$$ds = \frac{-dx\sqrt{b}}{\sqrt{(a^2b - bx^2 - a^2\int Pdx)}}.$$

From which equation the curve sought can be constructed. Q.E.I.

Corollary 1.

301. When *x* is made smaller, then the larger $b - \int Pdx$ becomes, whereby when *x* is made smaller, with that also $\frac{p}{x}$ becomes less, or the sine of the angle CMT. For $\frac{p}{x} = \frac{x\sqrt{b}}{a\sqrt{(b - \int Pdx)}}$.

Corollary 2.

302. Again, as by hypothesis since in this equation *x* cannot become greater than *a* ; for it would make $p > x$. On account of which none of the radii *CM* can be normal to the curve, unless it is at a maximum, clearly equal to *AC*.

Scholium 1.

303. Indeed it is evident through these that for whatever hypothesis of centripetal force, it is satisfactory for a circle with centre C to be described [p. 138] ; for the body must be moving uniformly on a given circle. Moreover even if the general equation does not seem to include the circle, yet no less it must be contained, as we have now intimated above.

[Part of Euler's problem was the imperfect state of affairs at the time regarding dynamics : we would now make some reference to angular momentum, which he was later to clarify, in the analysis. This of course makes these special curves all the more fascinating for us to look back on; thus, Euler's use of a potential energy or a work related function was introduced as a means of simplifying problems – only to be discovered later that this was how the world really worked, although Euler understood that he had to base his dynamics on known physical facts, such as Huygens' pendulum as a means of finding the acceleration of gravity, and Galileo's inclined plane : it was the latter that gave rise to the potential energy function that relates height fallen from rest to the square of the final speed.]

Scholium 2.

304. Moreover it is evident that no other curve going around the centre except the circle is able to satisfy the condition sought. For in curves of this kind it is not possible that all the lines drawn from the centre and normal to the curve are equal to each other. Therefore which curves besides the circle which solve the problem, these must pass through the centre *C* itself, so that not more than one radius *MC* is normal to the curve. These curves are of this kind that we look at in the following example.

Example.

305. Let the centripetal force be directly proportional to the distances from the centre or $P=\frac{x}{f}$; the equation becomes :

$$-\int Pdx=\frac{a^2-x^2}{2f}.$$

With which substituted, the following equation is produced for the curve :

$$ds=\frac{-dx\sqrt{b}}{\sqrt{\left(b+\frac{a^2}{2f}\right)(a^2-x^2)}}.$$

[p. 139] Now the arc is $\int\frac{dx}{\sqrt{(a^2-x^2)}}$, the sine of which is $\frac{x}{a}$ with the total sine arising equal to 1. This arc is denoted by $A.\frac{x}{a}$. Let the sine of the arc $BS=t$, then $s=A.t$; hence the equation becomes :

$$A.t=\sqrt{\frac{2bf}{a^2+2bf}}\cdot\left(A.1-A.\frac{x}{a}\right).$$

Or the arc, of which the cosine is $\frac{x}{a}$, is equal to :

$$A.t\sqrt{\frac{a^2+2bf}{2bf}}.$$

Hence the construction of the curve easily follows on, and it is an algebraic curve, whenever $\sqrt{\frac{a^2+2bf}{2bf}}$ is a rational number. Let

$$\sqrt{\frac{a^2+2bf}{2bf}}=m \text{ or } 2bf=\frac{a^2}{m^2-1},$$

giving

$$\frac{mdt}{\sqrt{(1-tt)}}=\frac{-dx}{\sqrt{(a^2-x^2)}},$$

the integral of this, by means of imaginary logarithms, is given by :

$$ml(t\sqrt{-1}+\sqrt{(1-tt)})=l\left(\frac{x+\sqrt{(x^2-a^2)}}{a}\right)$$

or

$$(t\sqrt{-1}+\sqrt{(1-tt)})^m=\frac{x+\sqrt{(x^2-a^2)}}{a}.$$

The perpendicular $MQ = y$ is sent from M to AC and on putting $CQ = u$ there is $1 : t = x : y$ and so $t=\frac{y}{x}$. Hence,

$$\left(\frac{y\sqrt{-1}+\sqrt{(x^2-y^2)}}{x}\right)^m=\frac{x+\sqrt{(x^2-a^2)}}{a}.$$

Or, if $m = 2$ or $bf=\frac{a^2}{6}$, then this equation is obtained :

$$\left(\frac{y\sqrt{-1}+u}{x}\right)^2=\frac{x+\sqrt{(x^2-a^2)}}{a}.$$

which reduced gives :

$$x^3=au^2-ay^2=ax^2-2ay^2$$

or

$$y=x\sqrt{\frac{a-x}{2a}}\quad \text{et}\quad u=x\sqrt{\frac{a+x}{2a}}.$$

But if an equation is desired between the orthogonal coordinates u and y, then it is this equation of the sixth order :

$$(y^2+u^2)^3=a^2(u^2-y^2)^2.$$

In this curve the applied line is at a maximum if $x=\frac{2b}{3}$ or if we take

$$CQ=\frac{2}{3}a\sqrt{\frac{5}{6}}=a\sqrt{\frac{10}{27}};$$

for then it becomes:

$$QM=\frac{2}{3}a\sqrt{\frac{2}{27}}.$$

Now in other examples of this, with the values m , the maximum applied line is where

$$my\sqrt{(a^2-x^2)}=ux.$$

PROPOSITION34.

Problem.

306. *Let the uniform force acting be g pulling downwards everywhere and the given curve AT* (Fig. 39) [p. 140] *; to find the curve AM, upon which a body thus descends, so that the time to pass through any arc AM is proportional to the square root of the corresponding applied line PT of the given curve AT.*

Solution.

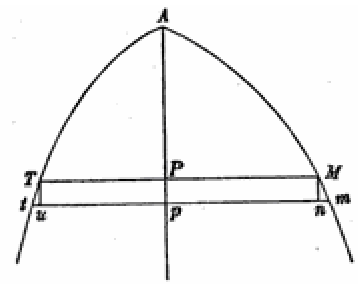

Fig. 39.

The common abscissa $AP = x$, the applied line of the curve AT is $PT = t$; hence the equation between x and t is given, since the curve AT is given, which must be such that with $x = 0$ makes $t = 0$ also, since the initial motion is put at A and the times reckoned from the point A. Again let the curve sought be AM with the applied line $PM = y$ and the arc $AM = s$. Let the initial speed at A correspond to the height b. Hence the speed at M corresponds to the height $b + gx$ and the time in which the arc is completed is equal to

$$\int \frac{ds}{\sqrt{(b+gx)}},$$

which must be equal to $\sqrt{t}$. Hence this equation is obtained :

$$\int \frac{ds}{\sqrt{(b+gx)}} = \sqrt{t} \text{ or } \frac{ds}{\sqrt{(b+gx)}} = \frac{dt}{2\sqrt{t}}.$$

Hence

$$dt^2(b+gx) = 4tds^2 = 4tdx^2 + 4tdy^2$$

and

$$dy = \frac{\sqrt{(bdt^2 + gxdt^2 - 4tdx^2)}}{2\sqrt{t}}.$$

From which equation, since t is given through x, the curve sought AM can be constructed. Moreover this has to be constructed so that $x = 0$ also gives $y = 0$, as the start of the curve AM is at A. Q.E.I.

Corollary 1.

307. Therefore in order that the curve is real, it is necessary that $bdt^2 + gxdt^2$ shall be greater than $4tdx^2$, or [p. 141]

$$\frac{dt}{2\sqrt{t}} > \frac{dx}{\sqrt{(b+gx)}} \quad \text{or on integrating} \quad \sqrt{t} > \frac{2\sqrt{(b+gx)} - 2\sqrt{b}}{g}.$$

For if the relation becomes

$$\sqrt{t} = \frac{2\sqrt{(b+gx)} - 2\sqrt{b}}{g},$$

then the curve *AM* becomes a vertical straight line, upon which the quickest descent is made.

Corollary 2.

308. Therefore if the curve *AT* somewhere makes $\frac{dt}{2\sqrt{t}}$ equal to $\frac{dx}{\sqrt{b+gx}}$, then the tangent there corresponding to the curve *AM* is vertical. And if beyond this point it satisfies $\frac{dt}{2\sqrt{t}} < \frac{dx}{\sqrt{b+gx}}$ then the curve *AM* does not descent to that point, but has a turning point at that point where the tangent is vertical.

Corollary 3.

309. If the angle that the curve *AT* makes to the vertical *AP* at *A* is acute, the tangent of which is *m*, then at the beginning *A* :

$$t = mx \text{ and } \frac{dt}{2\sqrt{t}} = \frac{mdx}{2\sqrt{mx}} > \frac{dx}{\sqrt{(b+gx)}};$$

hence $m(b+gx)$ must be greater than 4*x*, which always happens if *b* is not equal to 0. Moreover it then becomes :

$$dy = \frac{dx\sqrt{(bm^2 + gm^2x - 4mx)}}{2\sqrt{mx}}.$$

Therefore with $x = 0$, $\frac{dy}{dx} = \infty$, or in these cases the tangent to the curve *AM* at *A* is horizontal, unless $b = 0$. But if $b = 0$, then it becomes

$$dy = \frac{dx\sqrt{(gm-4)}}{2}.$$

Therefore least the curve *AM* becomes imaginary, *gm* must be greater than 4 and then the curve *AM* makes an acute angle with *AP* at *A*, the tangent of which is $\frac{\sqrt{(gm-4)}}{2}$.

Corollary 4. [p. 142]

310. Now if the angle that the curve *AT* makes with the vertical *AP* at *A* is right, then $m = \infty$. Therefore in this case the tangent to the curve *AM* at *A* is always horizontal, then either *b* is made equal to zero, or otherwise.

Corollary 5.

311. If the speed at *A* is equal to 0 and at the start *A* the curve *AT* is combined with the curve, the equation of which is $t = \alpha x^n$ with the number *n* taken as positive, so that as *x* increases, so too does *t*, then we have :

$$dt = \alpha n x^{n-1} dx \quad \text{and} \quad dy = \frac{dx\sqrt{(\alpha^2 g n^2 x^{2n-1} - 4\alpha x^n)}}{2\sqrt{\alpha x^n}}.$$

Now least *dy* becomes imaginary on making *x* = 0, it must be true that $n > 2n-1$ or $n < 1$, in which case clearly the curve *AT* is normal to *AP* at *A*. For now at the point *A*,

$$dy = \frac{n dx \sqrt{\alpha g}}{2x^{\frac{1-n}{2}}} \text{ and } y = \frac{n x^{\frac{n+1}{2}} \sqrt{\alpha g}}{n+1}$$

and the radius of osculation of the curve *AM* at *A* is equal to $\frac{n^2 \alpha g x^n}{2(n-1)}$. From which it follows for the curve *AM*, the tangent of which is horizontal at *A*, that the radius of osculation at *A* must be infinitely small, if the body is able to start from rest on that curve. For unless the radius of osculation is infinitely small, the body remains at rest at *A* for ever.

Corollary 6. [p. 143]

312. Therefore if the body placed at *A* descends from rest, so that the curve *AM* is made real, then $\frac{dt}{2\sqrt{t}}$ must be greater than $\frac{dx}{\sqrt{gx}}$, even at the start of the curve *AT*. Whereby if we put

$$\frac{dt}{2\sqrt{t}} = \frac{dx}{\sqrt{gx}} + p dx,$$

where *p* is a positive quantity, even if *x* is made exceedingly large, then we have :

$$\sqrt{t} = \frac{2\sqrt{gx}}{g} + \int p dx,$$

where $\int pdx$ must thus be taken, so that it vanishes on putting *x* = 0. Moreover with this value in place, on substituting for $\frac{dt}{2\sqrt{t}}$, there is produced

$$\frac{ds}{\sqrt{gx}} = \frac{dx}{\sqrt{gx}} + pdx \text{ or } s = x + \int pdx\sqrt{gx}$$

for the curve *AM* sought. Or this equation is obtained between *x* and *y* :

$$y = \int dx\sqrt{(2p\sqrt{gx} + gppx)}.$$

[This follows directly by substituting $\frac{1}{2\sqrt{t}}\frac{dt}{dx} = \frac{1}{\sqrt{gx}} + p$ into the above expression for *dy* written in the form : $\frac{dy}{dx} = \sqrt{gx\left(\frac{1}{2\sqrt{t}}\frac{dt}{dx}\right)^2 - 1}$ and integrating.]

Now it is to be noted that *p* cannot be such a quantity that $\int pdx$ can be made infinitely large when integrated in the prescribed manner.

Corollary 7.

313. From what has been said it is understood that as long as the value *p* is kept positive, the body descends the curve *AM* ; if we male $p = 0$ and then negative, then the curve has a cusp at that place and returns up again. If $p = \infty$ with $\int pdx$ yet remaining finite, then the curve *AM* has a horizontal tangent there.

Corollary 8.

314. If *b* is not put equal to 0, from the same curve *AT* innumerable curves AM can be found; since the initial speed can indeed be taken as greater or less, with another curve AM produced.

Scholium. [p. 144]

315. The greatest use of this problem is in the solutions of the following indeterminate problems, in which all the curves are required, upon which a body in the same time arrives at either a given straight or curved line. On account of this we will investigate the innate character of the quantities *t* and *p* with more care, so that we are allowed to use these in the following propositions.

PROPOSITION 35.

Problem.

316. *With a uniform force acting in the downwards direction, to find all the curves AMC* (Fig. 40)*; upon which a body beginning to descend from rest at A, arrives at the horizontal line BC in a given time.*

Solution.

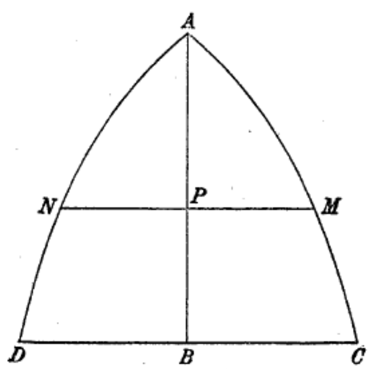

Fig. 40.

Putting $AP = x$, $PM = y$ and $AB = a$. In the curve *AND*, *PN* expresses the above assumed quantity $\int pdx$, and a property of this curve must be that it meets the axis *AB* at *A*, and that the applied lines increase on being continued even as far as *D*, so that clearly *pdx* is positive. Now by taking (312)

$$y = \int dx\sqrt{(2p\sqrt{gx} + gppx)}$$

the time to traverse AMC = $\frac{2\sqrt{ga}}{g} + BD$. [From $\frac{dt}{2\sqrt{dt}} = \frac{dx}{\sqrt{gx}} + pdx$ on integrating. Do realise that the curve in Fig. 40 is no longer the given curve AT in Fig. 39, which is still part of the calculation, but the extra arbitrary component for which $NP = \int pdx$.] On account of which, since an infinity of curves of this kind can be substituted in place of the curve *AND*, from these an infinite number of curves *AMC* can arise, upon all of which a body reaches the horizontal line *BC* in the same time from A [p. 145]. Therefore in order that this may be obtained, such a quantity must be taken for $\int pdx$, which vanishes when x = 0 and which becomes equal to *BD* on putting $x = a$, with p retaining a positive value everywhere along *AND*. Q.E.I.

Corollary 1.

317. If on making $x = a$, $p = 0$, or if the curve *AND* at *D* stands perpendicular to the horizontal *CD*, then the curve *AMC* also remains perpendicular to *DC*.

Corollary 2.

318. And if on putting $x = 0$ also $p = 0$, the tangent to the curve *AMC* is vertical at *A* ; now likewise also it comes about if $p\sqrt{x}$ becomes equal to zero on putting x = 0. But if $p\sqrt{x}$ becomes infinite on putting $x = 0$, then the curve *AMC* has a horizontal tangent at *A*.

Scholium 1.

319. Therefore the problem is understood to be indeterminate to a large extent, since in an infinite number of ways an infinite number of curves *AMC* can be found. On account of which in the following examples we indicate however many series of the infinite curves sought satisfying the question that we have been pleased to find.

Example 1. [p. 146]

320. Putting $PN = \int pdx = z$ and $BD = \sqrt{b}$, thus in order that the descent time must be equal to $\frac{2\sqrt{ga}}{g} + \sqrt{b}$. For the curve *AND*, this equation is taken: $z = \alpha x^2 + \beta x$, which now has this property, as $\int pdx$ or z vanishes on putting $x = 0$. Now since on making $x = a$ we must have $z = \sqrt{b}$, there is obtained $\sqrt{b} = \alpha a^2 + \beta a$ and hence $\beta = \frac{\sqrt{b}}{a} - \alpha a$ and thus $z = \alpha x^2 + \frac{x\sqrt{b}}{a} - \alpha ax.$

Then since p or $\frac{dz}{dx}$ must always have a positive value, if $x < a$, it is necessary that $2\alpha x + \frac{\sqrt{b}}{a} - \alpha a$ is positive. Whereby it is required that $\sqrt{b} > \alpha a^2$; therefore on putting $\sqrt{b} = \alpha a^2 + \alpha af$, then $\alpha = \frac{\sqrt{b}}{a^2 + af}$. With which substituted, there is obtained $z = \frac{x^2\sqrt{b} + fx\sqrt{b}}{a^2 + af}$, which equation, with innumerable positive values substituted in place of f, gives the curves *AND*. Moreover there becomes :

$$p = \frac{dz}{dx} = \frac{2x\sqrt{b} + f\sqrt{b}}{a^2 + af} \text{ and } p\sqrt{gx} = \frac{2x\sqrt{gbx} + f\sqrt{gbx}}{a^2 + af},$$

from which it is apparent that all the curves *AMC* hence arising are tangents to the line *AB* at *A*. Now the equation for the curves *AMC* is this [from (312)] :

$$y = \int \frac{dx}{a^2 + af}\sqrt{\left(2a(a+f)(2x+f)\sqrt{gbx} + gbx(2x+f)^2\right)}.$$

Which contains an infinitely of curves satisfying the problem, upon which the descent time for all to the horizontal line is equal to $\frac{2\sqrt{ga}}{g} + \sqrt{b}$.

Corollary 3.

321. Now all these line are rectifiable. For since [p. 147]

$$\frac{ds}{\sqrt{gx}} = \frac{dx}{\sqrt{gx}} + pdx,$$

then we have

$$s = x + \int pdx\sqrt{gx}.$$

Now

$$\int pdx\sqrt{gx} = \frac{\frac{4}{5}x^2\sqrt{gbx} + \frac{2}{3}fx\sqrt{gbx}}{a^2 + af}.$$

Thus the total area under the curve AMC is equal to :

$$a + \frac{\left(\frac{4}{5}a + \frac{2}{3}f\right)\sqrt{gab}}{a+f}.$$

Corollary 4.

322. Therefore among these curves *AMC* the longest is produced if $f = 0$; for then $AMC = a + \frac{4}{5}\sqrt{gab}.$ And for this, that equation becomes :

$$y = \int \frac{2\,dx}{a^2}\sqrt{\left(a^2x\sqrt{gbx} + gbx^3\right)}.$$

Now the shortest is obtained by making $f = \infty$; for then it becomes $AMC = a + \frac{2}{3}\sqrt{gab}.$ And the equation for this curve is :

$$y = \int \frac{dx}{a}\sqrt{\left(2a\sqrt{gbx} + gbx\right)}.$$

Scholium 2.

323. All the curves *AND* contained under the equation

$$z = \frac{x^2\sqrt{b} + fx\sqrt{b}}{a^2 + af}$$

are parabolas, thus so that by parabolas alone innumerable curves are found satisfying the problem. Now neither are all the parabolas contained by this equation, as in place of this equation, if this other equation is used :

$$z^2 + z\sqrt{f} = \frac{x(b+\sqrt{bf})}{a},$$

which also contains an infinite number of parabolas, again an infinite number of curves *AMC* are found, upon which a body completes the descent in the given time. From which it is to be understood that if only conic sections are substituted in place of the curve *AND*, then such an infinite number of curves *AMC* can be found. [p. 148] For taking this equation for the curve *AND* : $z^2 + \alpha z = \beta x^2 + \gamma x + \delta xz$, which contains all the conic sections passing through the point A, then it must become [for BD]:

$$b + \alpha\sqrt{b} = \beta a^2 + \gamma a + \delta a\sqrt{b}$$

and both $\frac{\gamma}{\alpha}$ and $\frac{\beta a+\gamma+\delta\sqrt{b}}{\alpha+2\sqrt{b}-\delta a}$ must be positive quantities; which is easily seen, since it can be done in an infinite number of ways. If then all the algebraic curves are considered and afterwards also the transcendental curves likewise, the greatest wealth of all the curves described in the same way con be conceived.

Example 2.

324. This general equation $z = \frac{x^n\sqrt{b}}{a^n}$ is taken for the curve *AND* with *n* denoting some positive number; *z* vanishes on putting *x* = 0 and the equation becomes $z = \sqrt{b}$ on putting *x* = *a*, as it is requires; now besides also the quantity *p* or $\frac{dz}{dx} = \frac{nx^{n-1}\sqrt{b}}{a^n}$ is positive.

Therefore since [the appropriate quantity] becomes $p\sqrt{gx} = \frac{nx^{n-\frac{1}{2}}\sqrt{gb}}{a^n}$, then the equation for *y* is :

$$y = \int \frac{dx}{a^n}\sqrt{\left(2na^n x^{n-\frac{1}{2}}\sqrt{gb} + n^2 gbx^{2n-1}\right)}.$$

Which equation includes an infinite number of curves AMC, which are all rectifiable. For the arc becomes :

$$AM = x + \frac{2nx^{n+\frac{1}{2}}\sqrt{gb}}{(2n+1)a^n}$$

and thus

$$AMC = a + \frac{2n\sqrt{gab}}{2n+1}.$$

Corollary 5. [p. 149]

325. If $n = \frac{1}{2}$, then the equation becomes

$$y = \int \frac{dx\sqrt{(gb+4\sqrt{gab})}}{2\sqrt{a}} \text{ and } y = \frac{x\sqrt{(gb+4\sqrt{gab})}}{2\sqrt{a}}$$

and

$$AM = x\left(1 + \frac{\sqrt{gab}}{2a}\right).$$

Whereby the curve becomes an inclined straight line, upon which the descent is in a time equal to $\frac{2\sqrt{ga}}{g} + \sqrt{b}$. Therefore it is evident the shorter lines [i.e. curves] are given by this inclined straight line, upon which a body from A arrives at the horizontal *BC* in a given time; indeed on making $n < \frac{1}{2}$ the line AMC becomes shorter.

Scholium 3.

326. Otherwise if a single curve *AND* is given, and it is required to provide a curve *AMC*, from that curve itself innumerable others can be found. For with one equation given between *z* and *x* taking

$$PN = \frac{(ma - (m-1)x)z}{a},$$

hence with diverse values of *m*, innumerable curves are found. In a similar manner also we can put :

$$PN = \frac{(max^n - (m-1)x^{n+1})z}{a^{n+1}};$$

for it becomes $PN = z = \sqrt{b}$, on putting $x = a$. And in general, if *P* is any function of *x* and *z*, *A* is now the same function that is produced on making $x = a$ and $z = \sqrt{b}$, and it can be taken that $PN = \frac{Pz}{A}$. Moreover *P* must be such a function that *Pz* vanishes on making $x = 0$ and $z = 0$, and the differential of *PN* divided by *dx* must be a positive quantity, for as long as $x < a$.

Scholium 4. [p. 150]

327. In a similar manner the most general problem can be solved, if on designating *P* as some function of x vanishing if $x = 0$, and *A* is that quantity which P becomes if $x = a$, on taking $z = \frac{P\sqrt{b}}{A}$, for the most general equation for the curve *AND*. Then we have

$dP = Qdx$, *Q* must be a positive quantity, as long as *x* is not greater than *a* ; then $p = \frac{Q\sqrt{b}}{A}$ and hence

$$y = \int \frac{dx}{A} \sqrt{(2AQ\sqrt{gbx} + gbQQx)},$$

which is the most general equation for the curves AMC, which are all completed by a body descending in a proposed time. It is apparent in this way that transcendental curves can also be substituted in place of the curves *AND*, in which cases the time, in which some part of the curve *AMC* is completed, cannot be defined algebraically. If $Q\sqrt{gbx}$ is put equal to *R*, then we have :

$$y = \int \frac{dx}{A} \sqrt{(2AR + RR)}.$$

Therefore with some function of x taken in place of R in order to find A, $\frac{Rdx}{\sqrt{gbx}}$ must be integrated, so that it thus vanishes on putting $x = 0$; then it is required to put $x = a$, and as that comes about, it is equal to A. Now here only this has advise has to be given, that a positive quantity is taken for R, for as long as x does not exceed a, and it must be warned that $\int \frac{Rdx}{\sqrt{gbx}}$ should not becomes infinite, if the integral is taken in the prescribed manner.

CAPUT SECUNDUM

DE MOTU PUNCTI SUPER DATA LINEA IN VACUO.

[p. 125]

PROPOSITIO 31.

Problema.

282. *Potentia sollicitante existente uniformi et deorsum tendente invenire curvam AM* (Fig. 36)*, super qua corpus data cum celeritate initiali ita moveatur, ut aequalibus temporibus aequales angulos circa punctum fixum C absolvat.*

Solutio.

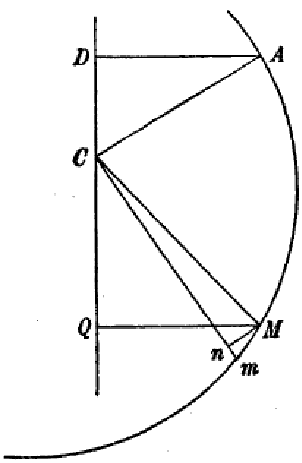
Fig. 36.

Sumatur initium curvae in loco quodam *A*, in quo recta *CA* in ipsam curvam est normalis, sitque celeritas in *A* debita altitudini *b* et *AC* = *a* ; erit celeritas angularis ut $\frac{\sqrt{b}}{a}$, [p. 126] cui quantitati celeritas angularis in singularis punctis *M* expressa debet esse aequalis. Sit celeritas in *M* debita altutudini *v* et *CM* = *x*, erit *mn* = *dx*. Fiet ut

$$Mm : Mn = \sqrt{v} : \frac{Mn \cdot \sqrt{v}}{Mm},$$

quae quantitas per *MC* divisa dat celeritatem angularem = $\frac{Mn.\sqrt{v}}{Mm.MC}$; quae cum aequalis esse debeat ipsi $\frac{\sqrt{b}}{a}$, habitur haec aequatio

$$Mn \cdot a\sqrt{v} = Mm \cdot MC \cdot \sqrt{b} = Mm \cdot x\sqrt{b}.$$

Sit iam ducta verticali *DCQ* sinus ang. *ACD* = *m*, erit cosinus eius = $\sqrt{(1-m^2)}$ posito sinu toto = 1. Item sinus ang. MCD sit = *t*; erit cosinus = $\sqrt{(1-tt)}$. His igitur positis erit

$$CD = a\sqrt{(1-m^2)} \text{ et } CQ = -x\sqrt{(1-tt)}$$

atque

$$\text{sinus ang. } MCm = \frac{dt}{\sqrt{(1-tt)}} = \frac{Mn}{x},$$

unde fit

$$Mn = \frac{xdt}{\sqrt{(1-tt)}} \quad \text{et} \quad Mm = \frac{\sqrt{(dx^2 - ttdx^2 + x^2dt^2)}}{\sqrt{(1-tt)}}.$$

At quia corpus ex altitudine *DQ* est delapsum, erit

$$v = b + g.DQ = b + ga\sqrt{(1-m^2)} - gx\sqrt{(1-tt)}.$$

Quibus valoribus in aequatione inventa substitutis orietur haec aequatio

$$bdx^2(1-tt) = a^2bdt^2 + ga^3dt^2\sqrt{(1-m^2)} - ga^2xdt^2\sqrt{(1-tt)} - bx^2dt^2$$

seu

$$dx\sqrt{b} = \frac{dt\sqrt{(a^2b + ga^3\sqrt{(1-m^2)} - ga^2x\sqrt{(1-tt)} - bx^2)}}{\sqrt{(1-tt)}}.$$

Quae aequatio ita integrata, ut posito $t = m$ fiat $x = a$, exprimit naturam curvae quaesitae. Q.E.I.

Corollarium 1.

283. Si loco sinuum angulorum ACD, MCD eorum cosinus introducantur fiatque $\sqrt{(1-m^2)} = n$ et $\sqrt{(1-tt)} = q$, erit

$$dx\sqrt{b} = \frac{-dq\sqrt{(a^2b + gna^3 - ga^2qx - bx^2)}}{\sqrt{(1-q^2)}}$$

seu [p. 127]

$$\frac{-dq}{\sqrt{(1-qq)}} = \frac{dx\sqrt{b}}{\sqrt{(b(a^2-x^2) + ga^2(na-qx))}},$$

quae ita est integranda, ut posito $q = n$ fiat $x = a$.

Corollarium 2.

284. Ubi curva ad radium *CM* est normalis, ibi ob evanescens *dx* erit

$$b(a^2 - x^2) = ga^2(qx - na).$$

Quoties ergo est

$$q = \frac{a^2b + gna^3 - bx^2}{ga^2x},$$

erit curva in radium *CM* normalis. Quia autem *q* intra limites + 1 et – 1 continetur, *x* non potest esse maior data quantitate; nam posito $x = \infty$, $q = \infty$, quod esset absurdum.

Corollary 3.

285. Si CM est normalis in curvam, erit celeritas angularis =

$$\frac{\sqrt{v}}{MC} = \frac{\sqrt{(b + gna - gqx)}}{x},$$

quae aequalis esse debet ipsi $\frac{\sqrt{b}}{a}$. Maxima ergo est illa celeritas angularis, si q = – 1. Ille autem motus angularis eo fit minor, quo maior est *x*. Eo vero minor porro erit motus angularis, quo magis obliqua est curva ad radium *MC*. Quare curva non ultra datam distantiam infra *C* descendere poterit, quam distantiam dabit *x* ex hac aequatione

$$x\sqrt{b} = a\sqrt{(b + gna + gx)},$$

nempe

$$x = \frac{ga^2}{2b} + a\sqrt{\left(\frac{g^2a^2}{4b^2} + \frac{gna}{b} + 1\right)}.$$

Haec ergo est maxima curvae a puncto *C* distantia.

Corollarium 4.

286. Cum igitur curva non ultra datam distantiam a centro fixo C distare queat, [p. 127] curva haec erit in se rediens. Scilicet vel post unam revolutionem vel post duas vel post tres etc. vel etiam post infinitas revolutiones in se redibit, prout litterae *a, b, n* et *g* fuerint assumtae.

Exemplum.

287. Sit potentia sollicitans evanescit, fit $g = 0$ et corpus aequabiliter provovebitur. Tum igitur pro curva descripta haec habebitur aequatio

$$\frac{dx}{\sqrt{(a^2 - x^2)}} = \frac{dt}{\sqrt{(1 - tt)}},$$

cuius integralis per logarithmos est

$$\sqrt{-1}\, l\left(\frac{x\sqrt{-1} - \sqrt{(a^2 - x^2)}}{c}\right) = \sqrt{-1}\, l(t\sqrt{-1} - \sqrt{(1 - tt)})$$

seu

$$x\sqrt{-1} - \sqrt{(a^2 - x^2)} = ct\sqrt{-1} - c\sqrt{(1 - tt)}.$$

Quae reducta dat

$$(a^2 - c^2)^2 = 4(a^2 + c^2)ctx - 4c^2x^2 - 4a^2c^2t^2.$$

Incidat recta *AC* in verticalem *CD*; hoc enim perinde est ob evanescentem potentiam *g*; debebit ergo fieri $x = a$ posito $t = 0$, ex quo fit $a^2 + c^2 = 0$ atque

$$a^4 = a^2x^2 + a^4t^2 \text{ seu } x^2 = a^2(1-tt).$$

Quae aeqatio est pro circulo diametri a per punctum fixum C transeunte. Quando enim motus in circulo est aequalis, motus quoque respectu cuiusque puncti in peripheria erit aequabilis.

Scholion.

288. Perspicuum autem est hoc casu peripheriam circuli quoque satisfacere, cuius centrum est in puncto fixo C, quippe quae solutio est facillima et sua sponte se prodit. Quamobrem maxime mirandam est hunc casum in solutione non contineri. [p. 129] Ratio vero huius similis prorsus est eius, quam supra (268) dedimus, ubi simile paradoxum observavimus. Ad circulum centrum in C habentem designandum prodire debuisset $x = a$ seu $dx = 0$, quod vero, quia x ut quantitas variabilis consideratur, non fieri potuit, praesertim cum in eadem aequatione solutio alia sit contenta, in qua x est quantitas revera variabilis. Ex prima vero aequatione positio $v = b$, quae est $Mn.a = Mm.x,$ intelligi potest circulum satisfacere; nam si ubique est $x = a$, erit quoque $Mn = Mm$. Magnum autem arbitror subsidium ad construendas curvas huic problemati satisfacientes proditurum, si talis methodo solutio inveniri posset, quae sponte pro casu motus aequabilis circulum centrum in C habentem esset datura. Cum enim casus simplicissimus ita sit involutus est abditus, ut elici vix queat, coniicere licet alias saepe curvas simplices in generali quapiam solutione contineri, quae sint erutu difficillimae.

PROPOSITIO 32.

Problema.

282. *Si corpus attrahantur vi quacunque ad centrum virium C* (Fig. 37), *invenire curvam AM, super qua corpus data cum celeritate descendens motu aequabili versus C feratur.*

Solutio. [p. 130]

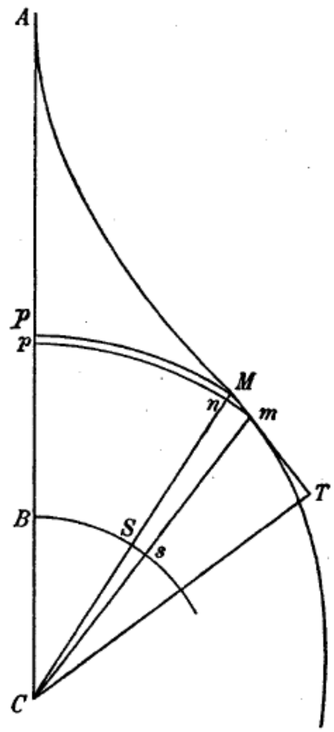

Fig. 37.

Sit corporis in *A* celeritas minima debita altitudini *b*, erit recta *CA* tangens curvae in *A*, quia corpus in *A* directe ad *C* moveri debet. Sit *AC* = *a* et *CM* = *x*, celeritas in *M* debita altitudini *v* et vis centripeta in *M* = *P*; erit $v = b - \int Pdx$, quod integrale ita est accipiendum, ut facto *x* = *a* evanescat fiatque *v* = *b*. Celeritas vero in *M* tanta esse debet, qua elementum *Mm* eodem tempusculo absolvatur, quo elementum *Pp* celeritate $\sqrt{b}$. Erit ergo $\sqrt{b} : \sqrt{v} = Pp : Mm = MT : MC$, unde prodibit ista aequatio

$$b \cdot MC^2 = v \cdot MT^2 = b \cdot MT^2 - MT^2 \cdot \int Pdx.$$

Dicatur perpendiculum *CT* in tangentem = *p*; erit

$$bp^2 = -(x^2 - p^2)\int Pdx \quad \text{seu} \quad p^2 = \frac{-x^2\int Pdx}{b - \int Pdx}.$$

Vel si sinus ang. *ACM* ponatur = *t*, erit

$$\frac{mn}{MC} = \frac{dt}{\sqrt{(1-tt)}}.$$

Unde sequens emergit aequatio

$$\frac{dt\sqrt{b}}{\sqrt{(1-tt)}} = \frac{-dx}{x}\sqrt{-\int Pdx}.$$

Quarum utraque, si quidem *P* per *x* datur, ad curvam construendam est apta. Q.E.I.

Corollarium 1.

290. Si vis centripeta potestati cuicunque distantiarum fuerit proportionalis, nempe

$$P = \frac{x^n}{f^n},$$

erit

$$\int Pdx = \frac{x^{n+1} - a^{n+1}}{(n+1)f^n},$$

Hoc substituto habebitur pro curva AM sequens aequatio

$$\frac{dt}{\sqrt{(1-tt)}} = \frac{-dx}{x}\sqrt{\frac{a^{n+1} - x^{n+1}}{(n+1)bf^n}}.$$

[p. 131] Quae aequatio ita debet integrari, ut facto $t = 0$ fiat $x = a$.

Corollarium 2.

291. Si $\int \frac{dt}{\sqrt{(1-tt)}}$ ita accipiatur, ut fiat = 0, si $t = 0$. prodibit ex illa aequatione integrata haec

$$\int \frac{dt}{\sqrt{(1-tt)}} = \frac{-2\sqrt{(a^{n+1} - x^{n+1})}}{(n+1)^{\frac{3}{2}}\sqrt{bf^n}} + \frac{a^{\frac{n+1}{2}}}{(n+1)^{\frac{3}{2}}\sqrt{bf^n}}\, l\, \frac{a^{\frac{n+1}{2}} + \sqrt{(a^{n+1} - x^{n+1})}}{a^{\frac{n+1}{2}} - \sqrt{(a^{n+1} - x^{n+1})}} = BS,$$

si centro C radio $BC = 1$ descriptus fuerit arcus circuli BSs. Ex quo patet curvam AM infinitos habere gyros, antequam corpus in C perveniat. Nam posito $x = 0$ fit $BS = \infty$.

Corollarium 3.

292. Pendet igitur constructio huius curvae partim a quadratura circuli, partim a logarithmis, si $n + 1$ est numerus affirmativus. At si $n + 1$ est numerus negativus, is terminus, qui per logarithmos erat datus, ad quadraturam circuli quoque reducitur.

Corollarium 4.

293. Curva haec punctum habebit flexus contrarii, ubi est $dp = 0$. [p. 132] Ad hoc igitur inveniendum sumatur aequatio

$$pp = \frac{-x^2\int Pdx}{b - \int Pdx},$$

ex qua differentiata positoque $dp = 0$ prodibit

$$bPx = 2\left(\int Pdx\right)^2 - 2b\int Pdx.$$

Corollarium 5.

294. In casu igitur, quo $P=\frac{x^n}{f^n}$, punctum flexus contrarii ibi erit, ubi est

$$(n+1)^2bf^nx^{n+1}=2(a^{n+1}-x^{n+1})^2+2(n+1)bf^n(a^{n+1}-x^{n+1}).$$

Unde haec oritur aequatio

$$\frac{a^{n+1}-x^{n+1}}{n+1}=-\frac{(n+3)bf^n}{4}+\frac{1}{4}\sqrt{((n+3)^2b^2f^{2n}+8a^{n+1}bf^n)}.$$

Quae in integrali substituta dabit angulum *ACM*, in quo est punctum flexus contrarii.

Scholion 1.

295. Cum autem de natura huiusmodi curvarum difficile sit in genere quicquam producere, ad casus speciales descendendum erit principales, id quod in sequentibus exemplis efficere visum est.

Exemplum 1.

296. Sit vis centripeta ipsis distantiis proportionalis seu $P=\frac{x}{f}$, fiet $n=1$. Posito ergo arcu $BS=s$ curva quaesita exprimetur ista aequatione

$$s=-\frac{\sqrt{(a^2-x^2)}}{\sqrt{2bf}}+\frac{a}{2\sqrt{2bf}}\,l\,\frac{a+\sqrt{(a^2-x^2)}}{a-\sqrt{(a^2-x^2)}}.$$

[p. 133] Ex qua aequatione data quavis puncti *M* a *C* distantia reperitur angulus *BCS*, quo absoluto corpus in ea distantia existit. Inter distantia $MC=x$ vero et perpendiculum $CT=p$ aequatio haec erit

$$pp=\frac{a^2x^2-x^4}{2bf+a^2-x^2}.$$

Huius curvae punctum flexus contrarii erit, ubi est $dp=0$, hoc autem, ubi est

$$x^4=2a^2x^2+4bfx^2-a^4-2a^2bf$$

seu

$$xx=a^2+2bf-\sqrt{(2a^2bf+4b^2f^2)},$$

quia x non maior esse potest quam a. Hinc fit

$$x=\sqrt{\left(a^2+2bf-\sqrt{(2a^2bf+4b^2f^2)}\right)}$$

atque

$$\frac{pp}{xx} = \frac{\sqrt{(2a^2+4bf)} - 2\sqrt{bf}}{\sqrt{(2a^2+4bf)}}.$$

Anguli ergo, quem curva in puncto flexus contrarii constituit cum radio *CM*, consinus erit =

$$\frac{\sqrt[4]{4bf}}{\sqrt[4]{(2a^2+4bf)}}.$$

Aequatio vero curvae in seriem conversa erit posito $\sqrt{(a^2-x^2)} = y$ haec

$$s\sqrt{2bf} = \frac{y^3}{3a^2} + \frac{y^5}{5a^4} + \frac{y^7}{7a^6} + \frac{y^9}{9a^8} + \text{etc.}$$

In ipso ergo, curvae principio, ubi *x* non multo minor est quam *a* seu *y* valde parvum, erit

$$s\sqrt{2bf} = \frac{y^3}{3a^2}.$$

Deinde ex ipsa aequatione apparet facto $x = 0$ fore $s = \infty$, quare curva infinitis spiris ambit centrum *C*, eritque, quando corpus centro iam proximum est,

$$\frac{pp}{xx} = \frac{aa}{2bf + a^2}.$$

Ex quo sequitur proxime circa centrum *C* curvam abire in logarithmicam spiralem

Exemplum 2.

296. Sit $n = -1$ seu $n + 1 = 0$, qui casus ex ipsa aequatione differentiali est eruendus. Fit enim ob $P = \frac{f}{x}$

$$\int Pdx = fl\frac{x}{a},$$

unde habebitur ista aequatio [p. 134]

$$ds = \frac{dt}{\sqrt{(1-tt)}} = \frac{-dx}{x\sqrt{b}}\sqrt{fl\frac{a}{x}},$$

cuius integralis est

$$s = \frac{2f(la-lx)^{\frac{3}{2}}}{3\sqrt{bf}}.$$

Altera aequatio inter perpendiculum *p* et *x* erit haec

$$pp = \frac{fxxl\frac{a}{x}}{b + fl\frac{a}{x}}.$$

Ex qua invenitur punctum flexus contrarii in eo loco, in quo est

$$b = 2f\left(l\frac{a}{x}\right)^2 + 2b\,l\frac{a}{x}$$

seu

$$l\frac{a}{x} = \frac{-b + \sqrt{(bb + 2bf)}}{2f}.$$

Hoc ergo habebitur sumendo

$$s = \frac{\left(-b + \sqrt{(bb + 2bf)}\right)^{\frac{3}{2}}}{3f\sqrt{2b}}.$$

Perspicitur porro, si fiat $x = 0$, fore $s = \infty$ seu curvam infinitis spiris centrum C circumdare; hoc vero casu erit $\frac{pp}{xx} = 1$ seu $p = x$. Ultimo ergo curva in circulum infinite parvum abit.

Exemplum 3.

298. Ponatur $n = -2$, ut vis centripeta sit quadratis distantiarum reciproce proportionalis; erit

$$\frac{dt}{\sqrt{(1-tt)}} = \frac{-f\,dx}{x}\sqrt{\frac{a-x}{abx}} = ds.$$

Ponatur vero $\int \frac{dy}{1+yy}$ arcum, cuius tangens est y seu $\sqrt{\frac{a-x}{x}}$; sit hic arcus $= t$, erit

$$t + \frac{s\sqrt{ab}}{2f} = y = \sqrt{\frac{a-x}{x}}.$$

Ubique ergo data distantia x capiendus est arcus s in $\frac{\sqrt{ab}}{2f}$ ductus aequalis differentiae inter tangentem $\sqrt{\frac{a-x}{x}}$ et arcum respondentem posito radio = 1. Si x ponatur = 0, fiet $s = \infty$, ex quo sequitur curvam per infinitas spiras ad centrum C descendere. [p. 135] Praeteria ob

$$-\int Pdx = \frac{ff(a-x)}{ax}$$

erit

$$pp = \frac{ffx(a-x)}{abx + ff(a-x)}.$$

Ex quo sequitur, si x evanescat, fore $\frac{pp}{xx} = 1$ seu curvam ultimo quoque in circulum infinite parvum abire.

Si fuerit $ab = ff$, erit

$$pp = \frac{xx(a-x)}{a} \quad \text{et} \quad t + \frac{s}{2} = \sqrt{\frac{a-x}{x}}.$$

Punctum flexus contrarii hoc ergo casu incidet in eum locum, ubi est $2ax = 3xx$ seu vel $x = 0$ vel $x = \frac{2a}{3}$. Sin autem fuerit $ab = 4ff$, erit

$$t + s = \sqrt{\frac{a-x}{x}}$$

et punctum flexus contrarii habebitur capiendo [vel $x = 0$ vel] $x = a\sqrt{\frac{1}{3}}$.

Scholion 2.

299. Quo autem appareat, quomodo spirae infinitae sint comparatae, si vis centripeta fuerit potestati cuicunque distantiarum proportionalis seu $P = \frac{x^n}{f^n}$ consideretur aequatio inter p et x, quae erit

$$\frac{pp}{xx} = \frac{a^{n+1} - x^{n+1}}{(n+1)bf^n + a^{n+1} - x^{n+1}}.$$

Ubi duo distinguendi sunt casus, alter, quo $n + 1$ est numerus affirmativus, alter, quo est negativus.

Si $n + 1$ est numerus affirmativus, facto $x = 0$ fit

$$\frac{pp}{xx} = \frac{a^{n+1}}{(n+1)bf^n + a^{n+1}}.$$

Hoc ergo casu curva *AM* circa centrum *C* abit in logarithmicam spiralem.

At si $n + 1$ est numerus negativus, facto $x = 0$ fit $\frac{pp}{xx} = 1$.

His ergo in casibus curva in *C* abit in circulum infinite parvum. Fit his casibus corporis ad C accedentis celeritas infinite magna [p. 136] et hanc ob rem, nisi curva in circulum abiret, corpus celeritate infinite magna ad *C* accederet, quod esset contra conditionem problematis. Determinatis igitur curvis, super quibus corpus aequalibiter ad centrum virium accedit, investigabimus eas curvas, super quibus motu aequabili circa centrum virium circumfertur.

PROPOSITIO 33.

Problema.

300. *Si corpus attrahantur perpetuo ad centrum virium C* (Fig. 38), *determinare curvam AM, super qua corpus motu angulari circa centrum C aequalibiter movetur.*

Solutio.

Sit *A* curvae punctum supremum, ubi curva normalis erit in radium *AC*, sitque celeritatas corporis in *A* debita altitudini *b* et *AC* = *a*; erit motus angularis in A = $\frac{\sqrt{b}}{a}$, cui quantitati motus angularis in singulis punctis *M* debet esse aequalis. Ponatur *CM* = *x*, cui aequalis capiatur *CP*, et sit vis centripeta in *M* = *P*; erit celeritas in *M* debita altitudini $b-\int Pdx$ integrali $\int Pdx$ ita accepto, ut evanescat posito *x* = *a*. Ducta tangente *MT* vocetur perpendiculum ex *C* in eam demissum *CT* = *p*; erit $x : p = Mm : mn.$ [p. 137] Hanc ob rem celeritas per *mn* =

$$\frac{p\sqrt{(b-\int Pdx)}}{x}$$

et celeritas angularis =

$$\frac{p\sqrt{(b-\int Pdx)}}{x^2},$$

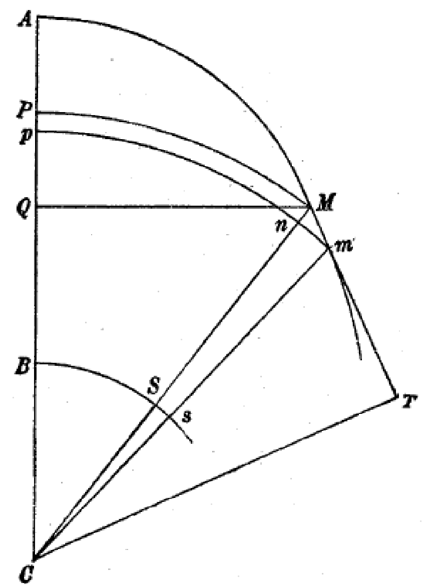

Fig. 38.

quae aequalis esse debet ipsi $\frac{\sqrt{b}}{a}$. Hinc prodit sequens aequatio

$$bx^4 = a^2bp^2 - a^2p^2\int Pdx$$

seu

$$p = \frac{x^2\sqrt{b}}{a\sqrt{(b-\int Pdx)}}.$$

Centro *C* radio *BC* = 1 describatur arcus circuli *BS*, qui dicatur = *s*, erit $1 : ds = x : mn$, unde erit *mn* = *xds* et $Mm = \sqrt{(dx^2 + x^2ds^2)}$. Cum nunc sit $x : p = \sqrt{(dx^2 + x^2ds^2)} : xds,$ fiet

$$p = \frac{xxds}{\sqrt{(dx^2 + x^2ds^2)}}.$$

Quo valore in aequatione inventa substituto habebitur

$$bdx^2 + bx^2ds^2 = a^2bds^2 - a^2ds^2\int Pdx$$

hincque

$$ds = \frac{-dx\sqrt{b}}{\sqrt{(a^2b - bx^2 - a^2\int Pdx)}}.$$

Ex qua aequatione curva quaesita poterit construi. Q.E.I.

Corollarium 1.

301. Quo minor fit *x*, eo maior fiet $b - \int Pdx$, quare, quo minor fit *x*, eo minor quoque fiet $\frac{p}{x}$ seu sinus anguli CMT. Est enim $\frac{p}{x} = \frac{x\sqrt{b}}{a\sqrt{(b - \int Pdx)}}$.

Corollarium 2.

302. Porro tam ex hypothesi quam hac aequatione *x* non potest fieri maior quam *a* ; fieret enim $p > x$. Quamobrem radiorum *CM* nullus potest esse normalis in curvam, nisi qui est maximus, nempe = *AC*.

Scholion 1.

303. Per se quidem manifestum est in quaqcunque vis centripetae hypothesi circulum centro C descriptum satisfacere [p. 138] ; corpus enim super circulo uniformite moveri debebit. Etiamsi autem aequatio generalis circulum non comprehendere videatur, nihilo tamen minus in ea contentus esse debet, ut iam supra innuimus.

Scholion 2.

304. Perspicuum autem est nullam aliam curvam centrum *C* cingentem praeter circulum quaesito satisfacere posse. Nam in huiusmodi curvis fieri non potest, ut omnes rectae ex *C* eductae et in curvam normales sint inter se aequales. Quae igitur curvae praeter circulum problema solvunt, eae per ipsum centrum *C* transire debent, ut plus uno radio *MC* non sit in curvam normali. Cuiusmodi ergo sint hae curvae, in sequente exemplo videamus.

Exemplum.

305. Sit vis centripeta distantiis a centro directe proportionalis seu $P = \frac{x}{f}$; erit

$$-\int Pdx = \frac{a^2 - x^2}{2f}.$$

Quo substituto pro curva sequens prodit aequatio

$$ds = \frac{-dx\sqrt{b}}{\sqrt{\left(b + \frac{a^2}{2f}\right)(a^2 - x^2)}}.$$

[p. 139] Est vero $\int \frac{dx}{\sqrt{(a^2-x^2)}}$ arcus, cuius sinus est $\frac{x}{a}$ existente toto sinu = 1. Notetur hic arcus per $A.\frac{x}{a}$. Sit sinus arcus $BS = t$, erit $s = A.t$; unde fiet

$$A.\,t = \sqrt{\frac{2bf}{a^2+2bf}}\cdot\left(A.\,1 - A.\frac{x}{a}\right).$$

Seu arcus, cuius cosinus est $\frac{x}{a}$, erit =

$$A.\,t\sqrt{\frac{a^2+2bf}{2bf}}.$$

Unde constructio curvae facilis fluit eritque curva algebraica, quoties $\sqrt{\frac{a^2+2bf}{2bf}}$ est numerus rationalis. Sit

$$\sqrt{\frac{a^2+2bf}{2bf}} = m \quad \text{seu} \quad 2bf = \frac{a^2}{m^2-1},$$

erit

$$\frac{mdt}{\sqrt{(1-tt)}} = \frac{-dx}{\sqrt{(a^2-x^2)}},$$

cuius integralis per logarithmos imaginarios est

$$ml(t\sqrt{-1}+\sqrt{(1-tt)}) = l\left(\frac{x+\sqrt{(x^2-a^2)}}{a}\right)$$

seu

$$(t\sqrt{-1}+\sqrt{(1-tt)})^m = \frac{x+\sqrt{(x^2-a^2)}}{a}.$$

Demittatur ex M in AC perpendiculum $MQ = y$ et posito $CQ = u$ erit $1 : t = x : y$ atque $t = \frac{y}{x}$. Propterea prodibit

$$\left(\frac{y\sqrt{-1}+\sqrt{(x^2-y^2)}}{x}\right)^m = \frac{x+\sqrt{(x^2-a^2)}}{a}.$$

Ut sit $m = 2$ seu $bf = \frac{a^2}{6}$, habibitur ista aequatio

$$\left(\frac{y\sqrt{-1}+u}{x}\right)^2 = \frac{x+\sqrt{(x^2-a^2)}}{a}.$$

Quae reducta dat hunc

$$x^3 = au^2 - ay^2 = ax^2 - 2ay^2$$

seu

$$y = x\sqrt{\frac{a-x}{2a}} \quad \text{et} \quad u = x\sqrt{\frac{a+x}{2a}}.$$

At sin inter coordinatas orthogonales u et y aequatio desideretur, ea erit ordinis sexti haec

$$(y^2+u^2)^3 = a^2(u^2-y^2)^2.$$

In hac curva applicata erit maxima, si $x=\frac{2b}{3}$ seu si sumatur

$$CQ=\frac{2}{3}a\sqrt{\frac{5}{6}}=a\sqrt{\frac{10}{27}};$$

tum enim erit

$$QM=\frac{2}{3}a\sqrt{\frac{2}{27}}.$$

In aliis vero ipsius *m* valoribus maxima applicata erit, ubi est

$$my\sqrt{(a^2-x^2)}=ux.$$

PROPOSITIO 34.

Problema.

306. *Sit potentia sollicitans uniformis g et ubique deorsum tendat deturque curva AT* (Fig. 39) [p. 140] *; invenire curvam AM, super qua corpus ita descendat, ut tempus per arcum quemcunque AM proportionale sit radici quadratae ex applicata respondente PT curvae datae AT.*

Solutio.

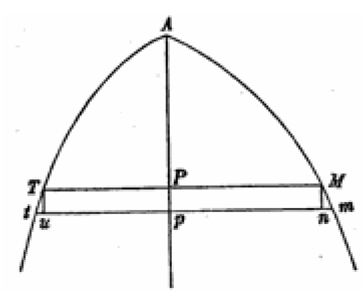

Fig. 39.

Ponatur abscissa communis $AP = x$, curvae *AT* applicata $PT = t$; dabitur ergo, quia curva *AT* datur, aequatio inter *x* et *t*, quae talis esse debebit, ut evanescente *x* fiat quoque $t = 0$, quia motus initium in *A* ponitur et tempora a puncto *A* computantur. Sit porro curvae quaesitae *AM* applicata $PM = y$ et arcus $AM = s$. Debita sit celeritas initialis in *A* altitudini *b*. Erit ergo celeritas in *M* debita altitudini $b + gx$ et tempus quo arcus *AM* absolvitur, =

$$\int\frac{ds}{\sqrt{(b+gx)}},$$

quod aequale esse debet ipsi $\sqrt{t}$. Habebitur ergo haec aequatio

$$\int\frac{ds}{\sqrt{(b+gx)}}=\sqrt{t} \quad \text{seu} \quad \frac{ds}{\sqrt{(b+gx)}}=\frac{dt}{2\sqrt{t}}.$$

Unde

$$dt^2(b+gx)=4tds^2=4tdx^2+4tdy^2$$

atque

$$dy = \frac{\sqrt{(bdt^2 + gxdt^2 - 4tdx^2)}}{2\sqrt{t}}.$$

Ex qua aequatione, cum *t* per *x* detur, curva quaesita *AM* constui poterit. Ita autem est construenda, ut posito $x = 0$ fiat quoque $y = 0$, quo curvae *AM* initium sit in *A*. Q.E.I.

Corollarium 1.

307. Quo igitur curva sit realis, oportet, ut $bdt^2 + gxdt^2$ sit maius quam $4tdx^2$ seu [p. 141]

$$\frac{dt}{2\sqrt{t}} > \frac{dx}{\sqrt{(b+gx)}} \quad \text{sive integrando} \quad \sqrt{t} > \frac{2\sqrt{(b+gx)} - 2\sqrt{b}}{g}.$$

Si enim fuerit

$$\sqrt{t} = \frac{2\sqrt{(b+gx)} - 2\sqrt{b}}{g},$$

curva *AM* fit recta verticalis, super qua descensus fit celerrimus.

Corollarium 2.

308. Si igitur in curva *AT* alicubi fiat $\frac{dt}{2\sqrt{t}}$ aequale ipsi $\frac{dx}{\sqrt{b+gx}}$, ibi tangens curvae *AM* respondens erit verticalis. Atque si infra hunc locum sit $\frac{dt}{2\sqrt{t}} < \frac{dx}{\sqrt{b+gx}}$ curva *AM* non eousque descendet, sed habebit punctum reversionis in eo loco, ubi tangens est verticalis.

Corollarium 3.

309. Si angulus, quem curva *AT* in *A* cum verticali *AP* constituit, fuerit acutus, cuius tangens = *m*, erit in initio *A*

$$t = mx \quad \text{et} \quad \frac{dt}{2\sqrt{t}} = \frac{mdx}{2\sqrt{mx}} > \frac{dx}{\sqrt{(b+gx)}},$$

unde $m(b+gx)$ maius esse debet quam 4*x*, id quod semper accidit, si *b* non fuerit = 0. Tum autem erit

$$dy = \frac{dx\sqrt{(bm^2 + gm^2x - 4mx)}}{2\sqrt{mx}}.$$

Posito igitur $x = 0$ fiet $\frac{dy}{dx} = \infty$, seu his casibus curvae *AM* tangens in *A* erit horizontalis, nisi sit $b = 0$. At si $b = 0$, erit

$$dy = \frac{dx\sqrt{(gm-4)}}{2}.$$

Ne igitur curva *AM* fiat imaginaria, debet *gm* maius esse quam 4 atque tum curva *AM* cum *AP* in *A* angulum acutum constituet, cuius tangens erit $\frac{\sqrt{(gm-4)}}{2}$.

Corollarium 4. [p. 142]

310. Sin vero angulus, quem curva *AT* in *A* cum verticali *AP* facit, sit rectus, fit $m = \infty$. Hoc ergo casu curvae *AM* tangens in *A* semper erit horizontalis, sive *b* fit = 0 sive secus.

Corollarium 5.

311. Si celeritatas in *A* est = 0 et in principio *A* curva *AT* confundatur cum curva, cuius aequatio est $t = \alpha x^n$ existente *n* numero affirmativo, quo crescente *x* quoque *t* crescat, erit

$$dt = \alpha n x^{n-1} dx \quad \text{et} \quad dy = \frac{dx\sqrt{(\alpha^2 g n^2 x^{2n-1} - 4\alpha x^n)}}{2\sqrt{\alpha x^n}}.$$

Nunc ne *dy* fiat imaginarium facto *x* = 0, debebit esse $n > 2n-1$ seu $n < 1$, quibus casibus scilicet curva *AT* in *A* est normalis ad *AP*. Tum vero erit in puncto *A*

$$dy = \frac{n dx\sqrt{\alpha g}}{2x^{\frac{1-n}{2}}} \quad \text{et} \quad y = \frac{n x^{\frac{n+1}{2}}\sqrt{\alpha g}}{n+1}$$

et radius osculi curvae *AM* in *A* = $\frac{n^2\alpha g x^n}{2(n-1)}$. Ex quo sequitur curvae *AM*, cuius tangens in *A* est horizontalis, radium osculi in *A* debere esse infinite parvum, si corpus ex quiete super ea descendere posse debeat. Nisi enim radius osculi fuerit infinite parvus, corpus perpetuo in *A* quiescens permanebit.

Corollarium 6. [p. 143]

312. Si igitur corpus ex quiete descendere ponatur in *A*, quo curva *AM* fiat realis, debebit $\frac{dt}{2\sqrt{t}}$ maius esse quam $\frac{dx}{\sqrt{gx}}$, saltem in initio curvae *AT*. Quare si ponatur

$$\frac{dt}{2\sqrt{t}} = \frac{dx}{\sqrt{gx}} + p dx,$$

ubi *p* est quantitas affirmativa, saltem nisi *x* ponatur nimis magnum, erit

$$\sqrt{t} = \frac{2\sqrt{gx}}{g} + \int p dx,$$

ubi $\int pdx$ ita accipi debet, ut evanescat facto *x* = 0. Hoc autem valore loco $\frac{dt}{2\sqrt{t}}$ substituto prodibit

$$\frac{ds}{\sqrt{gx}} = \frac{dx}{\sqrt{gx}} + p dx \quad \text{seu} \quad s = x + \int p dx\sqrt{gx}$$

pro curva quaesita *AM*. Vel inter *x* et *y* habebitur aequatio

$$y = \int dx\sqrt{(2p\sqrt{gx} + gppx)}.$$

Notandum vero est *p* non talem esse posse quantitatem, ex qua $\int pdx$ praescripto modo acceptum fiat infinite magnum.

Corollarium 7.

313. Ex dictis intelligitur, quamdiu p valorem affirmativum retineat, tamdiu curvam *AM* descendere; si fit $p = 0$ et deinceps negativum, curva *AM* in illo loco habebit cuspidem et revertitur sursum. Si $p = \infty$ manente tamen $\int pdx$ finito, curva *AM* ibi habebit tangentem horizontalem.

Corollarium 8.

314. Si *b* non ponatur = 0, ex eadem curva *AT* innumerabiles inveniri poterunt curvae *AM*; prout enim celeritas initialis maior minorve accipiatur, alia prodit curva AM.

Scholion. [p. 144]

315. Problematis huius maximus erit usus in solutionibus sequentium problematum indeterminatorum, in quibus omnes curvae requiruntur, super quibus corpus eodem tempore vel ad datam rectam vel curvam lineam perveniat. Hanc ob rem indolem quantitatum *t* et *p* diligentius investigavimus, quo iis in sequentibus uti liceat.

PROPOSITIO 35.

Problema.

316. *Posita potentia sollicitante uniformi g et deorsum directa invenire omnes curvas AMC* (Fig. 40)*; super quibus corpus in A ex quiete descensum incipiens dato tempore ad rectam horizontalem BC perveniat.*

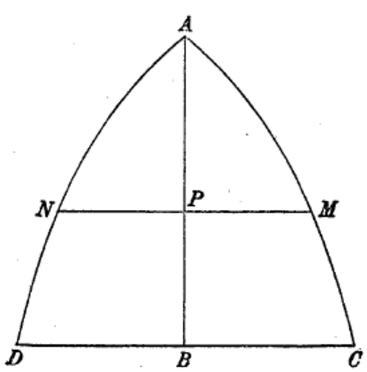

Fig. 40.

Solutio.

Ponatur $AP = x$, $PM = y$ et $AB = a$. In curva *AND* exprimat *PN* supra sumtam quantitatem $\int pdx$, cuius curvae haec debet esse proprietas, ut in *A* cum axe *AB* concurrat eiusque applicatae continuo usque ad *D* saltem crescant, quo scilicet *pdx* sit affirmativum. Nunc sumto

$$y = \int dx\sqrt{(2p\sqrt{gx} + gppx)}$$

erit tempus per AMC = $\frac{2\sqrt{ga}}{g} + BD$ (312). Quamobrem cum infinitae curvae huius indolis in locum curvae *AND* substitui queant, ex iis infinitae orientur curvae *AMC*, super quibus omnibus corpus [p. 145]eodem tempore ex A ad lineam horizontalem BC pertingit. Ad hoc ergo obtinendum pro $\int pdx$ talis quantitas accipi debet, quae evansecat posito $x = 0$ et fiat = *BD* posito $x = a$, retinente *p* ubique per *AND* affirmitavum valorem. Q.E.I.

Corollarium 1.

317. Si facto $x = a$ fiat $p = 0$, seu si curva *AND* in *D* perpendiculariter insistat horizontali *CD*, curva *AMC* quoque horizontali *DC* perpendicularer insistet.

Corollarium 2.

318. Atque si posito $x = 0$ fiat quoque $p = 0$, tangens curvae *AMC* in *A* erit verticalis; idem vero quoque accidit, si $p\sqrt{x}$ fiat = 0 posito x = 0. At si $p\sqrt{x}$ fiat infinitum posito $x = 0$, curva *AMC* in *A* habebit tangentem horizontalem.

Scholion 1.

319. Intelligitur ergo problema hoc maxime esse indeterminatum, cum infinitis modis infinitae curvae AMC possint inveniri. Quamobrem in sequentibus examplis modum indicabimus quotcunque libuerit series infinitarum curvarum quaesito satisfacientium inveniendi.

Exemplum 1. [p. 146]

320. Ponatur $PN = \int pdx = z$ et $BD = \sqrt{b}$, ita ut tempus descensus esse debeat = $\frac{2\sqrt{ga}}{g} + \sqrt{b}$. Sumatur pro curva *AND* haec aequatio $z = \alpha x^2 + \beta x$, quae hanc iam habet proprietatem, ut $\int pdx$ seu *z* evanescat posito $x = 0$. Nunc quia facto $x = a$ fieri debet $z = \sqrt{b}$, habebitur $\sqrt{b} = \alpha a^2 + \beta a$ hincque $\beta = \frac{\sqrt{b}}{a} - \alpha a$ ideoque $z = \alpha x^2 + \frac{x\sqrt{b}}{a} - \alpha ax$. Deinde quia *p* seu $\frac{dz}{dx}$ affirmativum semper habere debet valorem, si $x < a$, debebit esse $2\alpha x + \frac{\sqrt{b}}{a} - \alpha a$ affirmativum. Quare oportet esse $\sqrt{b} > \alpha a^2$; ponatur ideo $\sqrt{b} = \alpha a^2 + \alpha af$, erit $\alpha = \frac{\sqrt{b}}{a^2 + af}$. Quo substituto habebitur $z = \frac{x^2\sqrt{b} + fx\sqrt{b}}{a^2 + af}$, quae aequatio substituendis loco *f* innumerabilibus valoribus affirmativis infinitas dat curvas *AND*. Fiet autem

$$p = \frac{dz}{dx} = \frac{2x\sqrt{b} + f\sqrt{b}}{a^2 + af} \quad \text{et} \quad p\sqrt{gx} = \frac{2x\sqrt{gbx} + f\sqrt{gbx}}{a^2 + af},$$

ex qua patet omnes hinc orientes curvas *AMC* tangere rectam *AB* in *A*. Aequatio vero pro curvis *AMC* erit haec

$$y = \int \frac{dx}{a^2 + af} \sqrt{\left(2a(a+f)(2x+f)\sqrt{gbx} + gbx(2x+f)^2\right)}.$$

Quae infinitas continet curvas problemati satisfacientes, super quibus omnibus tempus descensus ad lineam horizontalem est = $\frac{2\sqrt{ga}}{g} + \sqrt{b}$.

Corollarium 3.

321. Hae autem lineae omnes sunt rectificabiles. Nam cum sit [p. 147]

$$\frac{ds}{\sqrt{gx}} = \frac{dx}{\sqrt{gx}} + pdx,$$

erit

$$s = x + \int pdx\sqrt{gx}.$$

Est vero

$$\int pdx\sqrt{gx} = \frac{\frac{4}{5}x^2\sqrt{gbx} + \frac{2}{3}fx\sqrt{gbx}}{a^2 + af}.$$

Unde tota curva AMC erit =

$$a + \frac{\left(\frac{4}{5}a + \frac{2}{3}f\right)\sqrt{gab}}{a+f}.$$

Corollarium 4.

322. Inter has igitur curvas AMC longissima prodit, si f = 0; erit enim tum $AMC = a + \frac{4}{5}\sqrt{gab}.$ Et pro hac erit aequatio ista

$$y = \int \frac{2\,dx}{a^2} \sqrt{\left(a^2x\sqrt{gbx} + gbx^3\right)}.$$

Brevissima vero habetur facto $f = \infty$; tum enim erit $AMC = a + \frac{2}{3}\sqrt{gab}.$ Et aequatio pro hac curva erit

$$y = \int \frac{dx}{a} \sqrt{\left(2a\sqrt{gbx} + gbx\right)}.$$

Scholion 2.

323. Omnes curvae *AND* sub aequatione

$$z = \frac{x^2\sqrt{b} + fx\sqrt{b}}{a^2 + af}$$

contentae sunt parabolae, adeo ut per solas parabolas innumerabiles inventae sint curvae problemati satisfacientes. Neque vero omnes parabolae in hac aequatione continentur. sed loco illius aequationis si adhibeatur haec

$$z^2 + z\sqrt{f} = \frac{x(b + \sqrt{bf})}{a},$$

quae etiam infinitas parabolas continet, iterum infinitae curvae *AMC* invenientur, super quibus corpus dato tempore descensum absolvit. Ex quo intelligi potest, quoties infinitae invenire queant curvae *AMC*, si tantum sectiones conicae in locum curvae *AND* substituantur. [p. 148] Sumta enim pro curvae *AND* hac aequatione $z^2 + \alpha z = \beta x^2 + \gamma x + \delta xz$, quae omnes continet sectiones conicas per punctum A transeuntes, fieri debet

$$b + \alpha\sqrt{b} = \beta a^2 + \gamma a + \delta a\sqrt{b}$$

atque $\frac{\gamma}{\alpha}$ et $\frac{\beta a + \gamma + \delta\sqrt{b}}{\alpha + 2\sqrt{b} - \delta a}$ debent esse quantitaties positivae; quod quam infinitis modis fieri possit, facile perspicitur. Si deinde omnes curvae algebraicae considerentur atque postmodum quoque curvae transdendentes simul, maxima copia curvarum simul descriptarum concipi poterit.

Exemplum 2.

324. Sumatur pro cura *AND* haec aequatio generalis $z = \frac{x^n\sqrt{b}}{a^n}$ denotante *n* numerum affirmativum quemcunque; evanescet *z* posito *x* = 0 fietque $z = \sqrt{b}$ posito $x = a$, ut requiritur; praeterea vero quoque erit *p* seu $\frac{dz}{dx} = \frac{nx^{n-1}\sqrt{b}}{a^n}$ quantitas affirmativa. Cum igitur sit $p\sqrt{gx} = \frac{nx^{n-\frac{1}{2}}\sqrt{gb}}{a^n}$, erit

$$y = \int \frac{dx}{a^n}\sqrt{\left(2na^n x^{n-\frac{1}{2}}\sqrt{gb} + n^2 gbx^{2n-1}\right)}.$$

Quae aequatio infinitas curvas AMC complectitur, quae omnes erunt rectificabiles. Erit enim

$$AM = x + \frac{2nx^{n+\frac{1}{2}}\sqrt{gb}}{(2n+1)a^n}$$

ideoque

$$AMC = a + \frac{2n\sqrt{gab}}{2n+1}.$$

Corollarium 5. [p. 149]

325. Si fuerit $n = \frac{1}{2}$, erit

$$y = \int \frac{dx\sqrt{(gb + 4\sqrt{gab})}}{2\sqrt{a}} \quad \text{atque} \quad y = \frac{x\sqrt{(gb+4\sqrt{gab})}}{2\sqrt{a}}$$

et

$$AM = x\left(1 + \frac{\sqrt{gab}}{2a}\right).$$

Quare curva abit in lineam rectam inclinatam, super qua descensus fit tempore = $\frac{2\sqrt{ga}}{g} + \sqrt{b}$. Perspicitur ergo dari lineas breviores recta hac inclinata, super quibus corpus dato tempore ex *A* ad horizontalem *BC* pervenit; facto enim $n < \frac{1}{2}$ linea AMC fit brevior.

Scholion 3.

326. Ceterum si detur unica curva *AND* desideratam curvam *AMC* praebens, ex ea ipsa innumerabiles aliae poterunt inveniri. Data enim unica aequatione inter *z* et *x* capiatur

$$PN = \frac{(ma - (m-1)x)z}{a},$$

unde pro diverso ipsius m valore innumerabiles curvae orientur. Simili modo poni etiam potest

$$PN = \frac{(max^n - (m-1)x^{n+1})z}{a^{n+1}};$$

fit enim $PN = z = \sqrt{b}$, si ponatur $x = a$. Atque generaliter, si fuerit *P* functio quaecunque ipsarum *x* et *z*, *A* vero eadem functio, quae prodit facto $x = a$ et $z = \sqrt{b}$, accipi poterit $PN = \frac{Pz}{A}$. Debebit autem *P* talis esse functio, ut *Pz* evanescat facto $x = 0$ et $z = 0$, et diff. *PN* divisum per *dx* debet esse quantitas affirmativa, saltem quamdiu est $x < a$.

Scholion 4. [p. 150]

327. Simili modo problema generalissime solvetur, si designante *P* quamcunque functionem ipsius *x* evanescentem, si est $x = 0$, et *A* eam quantitatem, in quam abit *P*, si fit $x = a$, sumatur $z = \frac{P\sqrt{b}}{A}$ pro generalissima aequatione curvae *AND*. Sit deinde $dP = Qdx$, debebit *Q* esse quantitas affirmativa, quamdiu *x* non superat *a*; erit $p = \frac{Q\sqrt{b}}{A}$ atque hinc

$$y = \int \frac{dx}{A}\sqrt{\left(2AQ\sqrt{gbx} + gbQQx\right)},$$

quae est generalissima aequatio pro curvis AMC, quae omnes a corpore descendente proposito tempore absolventur. Apparet hoc modo curvas transcendentes quoque in locum curvarum *AND* substituti posse, quibus casibus tempus, quo quaevis curvae *AMC* portio absolvitur, algebraice non potest definiri. Si $Q\sqrt{gbx}$ ponatur = *R*, erit

$$y = \int \frac{dx}{A} \sqrt{(2AR + RR)}.$$

Sumta ergo loco *R* quacunque functione ipsius *x* ad inveniendam *A* integrari debet $\frac{Rdx}{\sqrt{gbx}}$, ita ut evanescat posito $x = 0$; deinde poni oportet $x = a$, et quod provenit, erit = *A*. Hic vero hoc tantum est monendum, ut pro *R* sumatur quantitas affirmativa, quamdiu *x* non excedit *a*, et caveri debet, ne $\int \frac{Rdx}{\sqrt{gbx}}$ fiat infinitum, si praescripto modo accipiatur.