CHAPTER TWO

CONCERNING THE MOTION OF A POINT ON A GIVEN LINE IN A VACUUM.

[p. 151]

PROPOSITION 36.

Problem.

328. *With a uniform force g acting downwards everywhere, to find all the curves AMC* (Fig. 41)*, upon which a body descends from rest at A in a given time to a straight line BC inclined at any angle to the horizontal. .*

Solution.

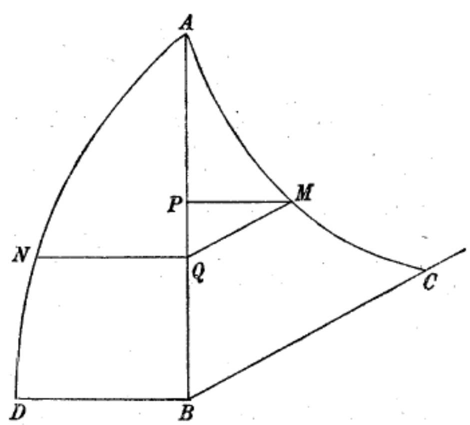

Fig. 41.

The applied line *BD* of the curve *AND* expresses the time, in which the body from *A* reaches the line straight *BC*, and from any point M the straight line *MQ* is drawn parallel to the given line *BC*, on cutting the vertical *AB* at *Q*, and *QN* expresses the time in which the body has traversed the part *AM* . Whereby, if an infinitude of curves *AND* are considered, which all have the same applied line *BD* at *B*, all these generate curves *AMC*, upon which the body in the given time arrives at *BC* from *A*. Moreover the curves *AND*, as advised above, must concur at *A* with the vertical *AB* and as far as *D* must diverge from *AB*. Now putting the tangent of the angle *ABC* = *k*, and putting $AP = x$, $PM = y$, $AQ = u$, $QN = t$ and $AB = a$; then $PQ = \frac{y}{k}$ and thus $x + \frac{y}{k} = u$.

Moreover since the speed at *M* must correspond to the height *gx*, the time to traverse *AM* $= \int \frac{\sqrt{(dx^2+dy^2)}}{\sqrt{gx}}$, which has to be equal to $QN = t$; therefore we have exactly

$dt = \frac{\sqrt{(dx^2+dy^2)}}{\sqrt{gx}}$ and $gxdt^2 = dx^2 + dy^2$. But on account of the given curve *AND*, *t* is given in terms of *u*, and since *u* is given by $u = x + \frac{y}{k}$, *t* is given by *x* and *y* ; on account of which the equation is obtained between x and y for the curve sought AMC. Or since we have $y = ku - kx$, then the equation becomes :

$$gxdt^2 = (k^2 + 1)dx^2 - 2k^2 dudx + k^2 du^2$$

and [p. 152] it is allowed to find x in terms of u . In order to do this, let $dt = pdu$, [i.e. p is the gradient of the *AND* curve with the abscissa u vertical] then

$$gxp^2 du^2 = (k^2 + 1)dx^2 - 2k^2 dudx + k^2 du^2$$

and

$$dx = \frac{k^2 du \pm \sqrt{((g(k^2+1)p^2 x - k^2)du^2)}}{k^2+1}.$$

Therefore the curve *AND* must be taken everywhere such that [the gradient] p is greater than $\frac{k}{\sqrt{gx(k^2+1)}}$. Moreover that equation must be integrated thus, so that on making u = 0 it becomes x = 0. With which done, the equation between x and y for the curve sought can be obtained. Q.E.I.

Corollary 1.

329. The curve *AMC* is a tangent to the straight line *AB* at *A*, if $dy = 0$ on putting $x = 0$; now it must then be the case that $du = dx$ and $1 = \sqrt{(g(k^2 + 1)p^2 x - k^2)}$. Whereby this comes about, if on making $ppx = \frac{1}{g}$ on making $x = 0$. Moreover since in this case y is infinitely smaller than x, in the beginning $x = u$; from which it follows that the curve *AMC* is a tangent to the vertical *AB* at *A*, if $ppu = \frac{1}{g}$ with $u = 0$. [Note that this degenerates to the equation $dx = gdt^2$ in the neighbourhood of A, on setting $du = dx$, recognisable as an equation of motion; the customery $\frac{1}{2}$ is missing as usual, as all such expressions involving free fall are compared against each other by Euler, rather than dealt with in an absolute manner, which could not be done at the time, due to the lack of an adequate set of units. The reader has no doubt observed that two curves are always used in these demonstrations : AND is the vertical displacement x versus time t, while AMD is the vertical displacement x versus horizontal displacement y.]

Corollary 2.

330. Then the normal to the curve *AMC* is along *QM*, if $PQ = \frac{y}{k} = \frac{ydy}{dx}$ or $dx = kdy = k^2 du - k^2 dx$ or $dx = \frac{k^2 du}{k^2+1}$. Now this eventuates [from above] when $p^2 x = \frac{kk}{g(k^2+1)}$.

Corollary 3.

331. At the point *A*, the expression ppx either has a finite value and that is greater than the value $\frac{kk}{g(k^2+1)}$ it has on making $x = 0$ [one presumes that Euler has in mind by this statement, taking the limit as x tends towards zero], or it is infinitely great. In the latter case [$dx = \frac{k^2 du}{k^2+1}$] becomes $dx = \pm \propto du$, and since $du = dx + \frac{dy}{k}$, then $dy = -kdx$, [as $\frac{du}{dx} = 0$; p. 153] In which cases, the tangent to the curve AMC at A is parallel to the line BC.

Example.

332. Let the curve *AND* be any parabola, so that thus

$$t = \frac{2\alpha\sqrt{u}}{\sqrt{g}};$$

then

$$dt = \frac{\alpha du}{\sqrt{gu}} \text{ and } p = \frac{\alpha}{\sqrt{gu}},$$

hence this equation is obtained [from $dx = \frac{k^2 du \pm \sqrt{((g(k^2+1)p^2 x - k^2)du^2)}}{k^2+1}$]:

$$(k^2+1)dx - k^2 du = \pm \frac{du\sqrt{(\alpha^2(k^2+1)x - k^2 u)}}{\sqrt{u}}.$$

The integral of this equation is :

$$C = \left(\pm\sqrt{(\alpha^2(k^2+1)x - k^2 u)} + A\sqrt{u}\right)^{\pi}\left(\pm\sqrt{(\alpha^2(k^2+1)x - k^2 u)} + B\sqrt{u}\right)^{\varrho}$$

with

$$A = \frac{\alpha^2 + \sqrt{(\alpha^4 - 4(1-\alpha^2)k^2)}}{2} \text{ and } B = \frac{\alpha^2 - \sqrt{(\alpha^4 - 4(1-\alpha^2)k^2)}}{2}$$

with

$$\pi = \frac{-2A}{A-B} \text{ and } \varrho = \frac{2B}{A-B}.$$

[Clearly a convenient way to demonstate this integral is to assume the final form, and to differentiate the logarithms of this product, at which point an attempt is made to generate the given differential equation by equating coefficients and making use of partial fractions. Note that the l.h.s. of the differential equation is the derivative of part of the numerator of the r.h.s., while the rest is $\frac{du}{\sqrt{u}}$. Euler concerns himself with this aspect below.]

[If $\alpha < 1$, then *B* and hence also the number ρ is a positive number.] Therefore since the number π is a negative number, then

$$C\left(\pm\sqrt{(\alpha^2(k^2+1)x - k^2 u)} + A\sqrt{u}\right)^{-\pi} = \left(\pm\sqrt{(\alpha^2(k^2+1)x - k^2 u)} + B\sqrt{u}\right)^{\varrho},$$

where *C* denotes a constant, which is effected so that on putting *x* = 0, *u* = 0 also. Moreover it is clear, whatever the constant should be, that always *u* = 0 on making ρ or π to disappear. But it is not possible for π to vanish, and in this case ρ now vanishes, since $\alpha = 1$; therefore in this case it must be that $C = \propto$ and the equation becomes

$$\pm \sqrt{(\alpha^2(k^2+1)x - k^2u)} + \alpha^2\sqrt{u} = 0$$

or $u = x$ and $y = 0$; whereby in this case the condition for the vertical line *AB* is satisfied. For the remaining cases, since *C* is now an arbitrary quantity, from one curve *AND* innumerable curves *AMC* can be found.

Again the single case has to be treated on its own, if $A = B$ or

$$k^2 = \frac{\alpha^4}{4(1-\alpha^2)};$$

for then it becomes :

$$lu = C - 2l\left(r \mp \frac{\alpha^2}{2}\right) \pm \frac{\alpha^2}{r \mp \frac{\alpha^2}{2}}$$

with arising

$$r = \frac{\sqrt{(\alpha^2(k^2+1)x - k^2u)}}{\sqrt{u}}.$$

Consequently for this case this equation is obtained : [p. 154]

$$C = 2l\left(\sqrt{(\alpha^2(k^2+1)x - k^2u)} \mp \frac{\alpha^2\sqrt{u}}{2}\right) \mp \frac{\alpha^2\sqrt{u}}{\sqrt{(\alpha^2(k^2+1)x - k^2u)} \mp \frac{\alpha^2\sqrt{u}}{2}},$$

where C too does not have to be found.

If $\alpha > 1$, then B and hence ρ too becomes a negative number; then it follows that $C = \infty$. Hence in this case either

$$A\sqrt{u} = \sqrt{(\alpha^2(k^2+1)x - k^2u)}$$

or

$$B\sqrt{u} = \sqrt{(\alpha^2(k^2+1)x - k^2u)}.$$

Which two equations are for straight lines inclined in a certain manner and passing through *A* and these also are satisfactory in general. So if we have $\alpha = 1$, then $A = 1$ and $B = 0$ and hence these equations

$$u = x \text{ or } y = 0 \text{ and } u = \frac{(k^2+1)x}{kk} = x + \frac{y}{k} \text{ or } x = ky,$$

which is a straight line perpendicular to *BC*; for this is traversed by a body in the same time as the vertical *AB*.

Corollary 4.

332a. [Numbering error] Therefore other than $\alpha > 1$ or $\alpha = 1$, innumerable curved lines can be found that satisfy the problem; hence all are completed in a smaller time than the perpendicular AB.

Corollary 5.

333. Therefore since an infinitude of curves *AMC* are able to arise from the one curve *AND*, it is readily understood that infinitely more curves satisfying this question than in the preceeding.

Corollary 6. [p. 155]

334. If $\alpha < 1$ then the determination of the constant *C* can be brought about, so that the curve sought passes through a given point on the line *BC*. The for other assumed curves *AND* in a similar manner an infinitude of curves can be found, upon which the body ont only reaches the given line *BC* in a given time, but reaches some given point *C* on that line.

Scholium 1.

335. In this example two cases occur in which $\alpha = 1$; indeed in the first in turn only the vertical line satisfying the problem has been found, and in the other besides this line the other inclined straight line has been found, yet we have used both in the same way in the general equation. Moreover now more often cases of this kind are touched upon, in which the differential equations contain amongst themselves the integral equations, which nevertheless cannot be expressed by integration. As in the case $\alpha = 1$, this has the differential equation

$$\frac{(k^2+1)dx - k^2 du}{\sqrt{((k^2+1)x - k^2 u)}} = \frac{\pm du}{\sqrt{u}},$$

which on integration gives $x = u$. Yet meanwhile it is evident that this equation $k^2 u = (k^2+1)x$ is contained in that too, even if it cannot be produced by integration. And on this account the equation is equally satisfied by the latter integral as the former $x = u$. Hence it is permitted generally to gather together the differential equation into the form

$$\frac{dt}{T} = Vdu,$$

in which *T* is such a function of *t*, which vanishes on putting *t* = 0, and *V* is some function of *u*, equally understood that this integration vanishes when *t* = 0 and this [p. 156]

$$\int \frac{dt}{T} = \int Vdu,$$

is is elicited by integration. Indeed generally the case *t* = 0 can be disregarded if *t* a simple quantity; but if *t* is a composit quantity as in our case, it is wrongly omitted. We had a similar case above (300) in the equation

$$ds = \frac{-dx\sqrt{b}}{\sqrt{(a^2 b - bx^2 - a^2\int Pdx)}},$$

where we observed the equation *x* = *a* to be contained in that, even if it could not be integrated by integration. For on putting $a - x = t$ then $-dx = dt$ and

$$\sqrt{(a^2 b - bx^2 - a^2\int Pdx)}$$

is a function of *t*, which becomes equal to zero, if we make *t* = 0 or *x* = *a*; in as much as $\int Pdx$ is thus prescribed to be accepted, so that it vanishes on putting *x* = *a*. Hence on putting this function of *t* equal to *T* the equation becomes $ds = \frac{dt\sqrt{b}}{T}$, hence in which equation it is allowed to conclude without risk that the equation *t* = 0 or *x* = *a* is satisfied, and thus the circle to satisfy that problem, as we agreed upon there (303). More generally now in this equation $Vdu = \frac{dt}{T}$, if T does not vanish on putting $t = 0$, it is understood that the equation $T = 0$, from which t is equal to some constant quantity and thus $dt = 0$. Hence it is understood that T = 0 is contained in the proposed equation $Vdu = \frac{dt}{T}$. And hence, if t should be a composite quantity for argument's sake from u and x, then the equation of the integral is not likely to be had at once by integration.

Scholium 2.

336. Here the case remains, which requires a particular resolution, [p. 157] when the straight line *BC* meets the line *BA* above the point *A*, and when it is parallel. But we consider only the case in the following problem, in which *BC* is made parallel to the vertical line *AB* and put at a given distance ; from which in a similar manner the case of the line inclined at some angle can be deduced.

PROPOSITION 37.

Problem.

337. *Let the body be always acted on by a constant uniform downwards force g ; to find the innumerable curves upon which the body, starting to move from rest at A* (Fig. 42), *arrives at the vertical lineEC in a given time.*

Solution.

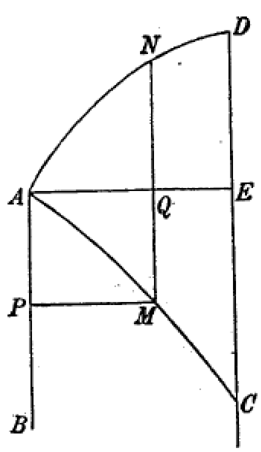

Fig. 42.

Let *AMC* be any of the curves sought and the vertical line *AB* is taken for the axis; calling $AP = x$, $PM = AQ = y$; then the speed at *M* corresponds to the height *gx* and the time for *AM* is equal to

$$\int \frac{\sqrt{(dx^2+dy^2)}}{\sqrt{gx}}$$

Again let *AND* be a curve, of which any applied line *QN* expressed the time for *AM*, and let $QN = t$ be a function of y; from the given curve *AND* it is possible to find the curve *AMC*. Whereby if an infinity of curves *AND* are considered, which all have the common point of apllication *DE* at *E*, all produce the curves *AMC*, upon which in a given time expressed by *DE* arrive at *CE* from *A*. And thus the equation becomes ;

$$t = \int \frac{\sqrt{(dx^2+dy^2)}}{\sqrt{gx}}$$

and on putting $dt = pdy$ it becomes :

$$gp^2xdy^2 = dx^2 + dy^2 \text{ and } dx = dy\sqrt{(gp^2x-1)}$$

[p. 158] Which thus on integration, since putting $x = 0$ makes $y = 0$, gives the curves sought *AMC*.

Let R be some function of y and $\int Rdy$ thus taken, so that it vanishes on putting $y = 0$.

Then it becomes $\int Rdy = A$ with $y = AE = a$ and with $DE = \sqrt{b}$ arising, taking

$$t = \frac{\sqrt{b}\int Rdy}{A};$$

there becomes :

$$p = \frac{R\sqrt{b}}{A} \text{ and } dx = \frac{dy}{A}\sqrt{(gbR^2x - A^2)}.$$

Which equation, whatever is substituted for R, gives innumerable curves satisfying the question. Q.E.I.

Corollary 1.

338. Therefore in the simplest case, if we put $R = 1$; then indeed a separable equation is produced. Now it becomes $t = \frac{y\sqrt{b}}{a}$ as $A = a$. hence on this account the equation becomes

$$\frac{adx}{\sqrt{(gbx-a^2)}} = dy \text{ and } \frac{2a}{gb}\sqrt{(gbx-a^2)} = y.$$

But which is of no use as the value of y is imaginary.

Corollary 2.

339. Moreover since $\sqrt{(gbR^2x - A^2)}$ cannot be an imaginary quantity, it is required that $R^2x > \frac{A^2}{gb}$; even if $x = 0$. Whereby R neither can be a constant quantity nor a function of y, which vanishes on making $y = 0$. Hence on this account R must be such a function of y, which becomes infinite, if $y = 0$. Yet besides it must be a function of the kind, such that $\int Rdy$ is not made infinite, which comes about if we should make $R = \frac{1}{y}$ or $\frac{1}{y^2}$ etc.

Example. [p. 159]

340. Therefore we put $R = \frac{1}{\sqrt{y}}$; and the integral becomes

$$\int Rdy = 2\sqrt{y} \text{ and } A = 2\sqrt{a}.$$

Hence this equation is obtained:

$$2dx\sqrt{ay} = dy\sqrt{(gbx - 4ay)}.$$

Which equation, since it is homogeneous, on putting $x = qy$; there is found

$$2qdy\sqrt{a} + 2ydq\sqrt{a} = dy\sqrt{(gbq - 4a)}$$

or

$$\frac{dq}{\sqrt{(gbq-4a)}-2q\sqrt{a}}=\frac{dy}{2y\sqrt{a}}.$$

On putting $\frac{gb}{4a}=n$ or $\sqrt{(nq-1)}=r$ then we have

$$\frac{dy}{y}=\frac{-2rdr}{r^2-nr+1}.$$

Which latter formula depends on the quadrature of the circle, if $n<2$.

Now in the other case $n>2$ the integral is :

$$ly=lC+\frac{n-\sqrt{(n^2-4)}}{\sqrt{(n^2-4)}}l\left(2r-n+\sqrt{(n^2-4)}\right)+\frac{-n-\sqrt{(n^2-4)}}{\sqrt{(n^2-4)}}l\left(2r-n-\sqrt{(n^2-4)}\right).$$

Which since

$$r=\frac{\sqrt{(nx-y)}}{\sqrt{y}}$$

becomes equal to this :

$$C\left(2\sqrt{(nx-y)}-(n-\sqrt{(n^2-4)})\sqrt{y}\right)^{\frac{n-\sqrt{(n^2-4)}}{\sqrt{(n^2-4)}}}$$
$$=\left(2\sqrt{(nx-y)}-(n+\sqrt{(n^2-4)})\sqrt{y}\right)^{\frac{n+\sqrt{(n^2-4)}}{\sqrt{(n^2-4)}}}.$$

Where any constant is allowed to be accepted for C, since the equation thus is comparable, as on putting $x = 0$ makes $y = 0$. Moreover by the method treated above (335) this integral is at once had from the differential equation :

$$2q\sqrt{a}=\sqrt{(gbq-4a)} \text{ or } 2x\sqrt{a}=\sqrt{(gbxy-4ayy)},$$

hence there arises :

$$\frac{y}{x}=\frac{gb\pm\sqrt{(g^2b^2-64a^2)}}{8a},$$

which gives two straight lines, unless $gb<8a$, in which case the equation is imaginary.

In the case in which $n = 2$, it becomes

$$\frac{dy}{y}=\frac{-2rdr}{(r-1)^2} \text{ or } ly=-2l(r-1)+\frac{2}{r-1},$$

hence it becomes :

$$ly=-2l\left(\sqrt{(2x-y)}-\sqrt{y}\right)+2l\sqrt{y}+\frac{2\sqrt{y}}{\sqrt{(2x-y)}-\sqrt{y}}+lC$$

or

$$l\left(\sqrt{(2x-y)}-\sqrt{y}\right)-lC=\frac{\sqrt{y}}{\sqrt{(2x-y)}-\sqrt{y}}.$$

Also there any quantity can be taken for C.

If $n < 2$, then the construction of the integal depends in part on logarithms, and in part on the quadrature of the circle[p. 160] ; indeed on account of the imaginary

$\sqrt{(n^2-4)}$ imaginary logarithms are found. Therefore in this case it is expedient for the construction to be carried out analytically.

PROPOSITION 38.

Problem.

341. *Let the body always be acted on by a uniform downwards force g and any curve is given BSC* (Fig. 43) ; *to find all the curves AMC, upon which the body by descending from A in a given time arrives at the given curve BSC.*

Solution.

Let any of the given curves sought be *AMC,* through some point *M* of this, the curve *MQ* similar to the [known]curve *BSC* with respect to the fixed point *A* is drawn, and the applied line NQ expresses the time for the arc AM of the curve *AND* to be completed; hence the applied line *BD* sets out the time for the whole curve *AMC* to be completed. With which done, the curve *AMC* can be found in turn from the given curve *AND*. Whereby if an infinity of curves *AND* is considered, which all have the common line of application *BD* at *B*, these generate the infinity of curves *AMC*, upon all of which the body arrives at the curve BSC by descending from *A* in the given time. If now

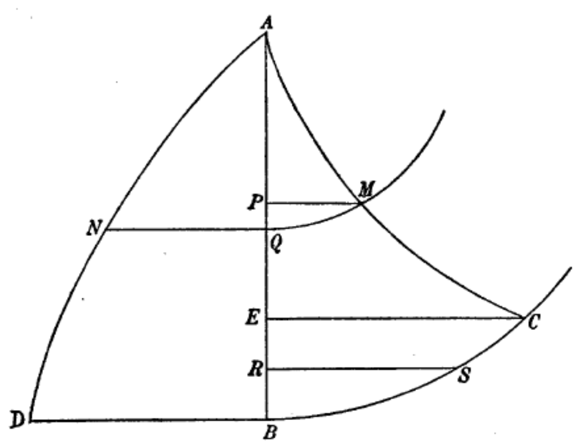

Fig. 43.

$AB = a,\ BD = \sqrt{b}$ and for the arc *QM* a similar arc BS is cut off from the given curve BSC; then we have :

$$AQ : AB = PM : RS = AP : AR = PQ : BR. \text{ [p. 161]}$$

Again putting $AP = x, PM = y, AQ = u, QN = t, AR = r, RS = s;$ on account of the given curve *BSC* there is an equation given between *r* and *s* and on account of the given curve *AND* there is an equation given between *t* and *u*. But on account of the similitude there is the ratio $u : a = y : s = x : r,$ hence this becomes : $y = \frac{us}{a}$ and $x = \frac{ur}{a}$.

Then the speed of the body at *M* corresponds to the height *gx*, from which the time to pass along *AM* is equal to :

$$\int \frac{\sqrt{(dx^2+dy^2)}}{\sqrt{gx}},$$

that must be put equal to *t*; hence this equation arises $gxdt^2 = dx^2 + dy^2$.

Moreover since *t* is given through *u*, let $dt = pdu$ and *p* is a function of *u*; and because

$$dx = \frac{udr+rdu}{a} \text{ and } dy = \frac{uds+sdu}{a}$$

this equation goes over into this :

$$garup^2du^2 = (udr + rdu)^2 + (uds + sdu)^2$$

But since the curve *BSC* is given, then *s* is a function of *r* and it becomes $ds = qdr$ with *q* present some function of *r*. With these put in place, an equation is obtained between *u* and *r*:

$$garup^2du^2 = (udr + rdu)^2 + (uqdr + sdu)^2$$

Which with the square root extracted gives :

$$\frac{dr}{du} = \frac{-r - sq \pm \sqrt{(2rsq - s^2 - r^2q^2 + garup^2(1+qq))}}{u(1+qq)}.$$

From which if the equation between *r* and *u* can be found, then likewise the equation between *x* and *y* for the curve sought is obtained.

Moreover since it is restricted to the curve *AND*, let *P* be some function of *u* and thus $\int Pdu$ integrated, so that it vanishes on making *u* = 0 and it becomes equal to *A* on puttng *u* = *a*; then we can take $t = \frac{\sqrt{b}\int Pdu}{A}$ for the equation of the curve *AND*. Hence $p = \frac{P\sqrt{b}}{A}$, where any allowed function of *u* can be put in place for the function *P*. Q.E.I.

Corollary 1. [p. 162]

342. If *u* is put equal to 0, for that both *x* and y vanish, unless perhaps *r* or *s* become infinite. Therefore in that case in the integration of differential equation some constant found can be added, since there is no need that *r* should have a given value, if *u* is made equal to 0.

Corollary 2.

343. Therefore then on account of the arbitrary constant to be added, from one given curve *AND*, innumerable curves *AMC* can be found satisfying what is sought.

Corollary 3.

344. If the curve *BSC* has thus been prepared, so that neither *r* nor *s* is able to become infinitely great, an infinite number of curves sought *AMC* are given always from a single curve *AND*. Not only do they have this property so that bodies descending on these arrive at the same time at the given line *BSC*, but also likewise they can reach any given similar curve *QM* .

Corollary 4.

345. Therefore since in the integration of the equation found it is allowed to add some constant[p. 163], thus that can always be assumed, so that the curve *AMC* is directed towards a given point *C* of the given *BSC*. And in this way an infinite number of curves *AMC* can be found, which all come together at a given point *C*.

Scholium 1.

346. We have put the curves *QM* similar to the curve *BSC*, so that curve itself raised to *A* becomes infinitely small and all the points of the curve *BSC* come together at *A* and both *x* and *y* vanish on putting $u = 0$. Moreover in the same manner we have put the curves QM either to be congruent with *BSC*, or disagreeing according to some condition. As *Q* is some function of *u* vanishing on setting $u = 0$ and that goes into *B* on making $u = a$, the curve *QM* thus is able to depend on the curve *BSC*, so that it becomes

$$x = \frac{Qr}{B} \text{ et } y = \frac{Qs}{B};$$

in as much as on making $u = a$ the curve *QM* is changed into *BSC* itself and at *A* the curve is changed into the point, unless the curve *BSC* should become infinite. But also in this case a funtion can be accepted for *Q*, in order that, even if on making $r = \infty$, yet *Qr* and *Qs* become equal to 0, if $u = 0$. Moreover on putting $dQ = Vdu$ the following general equation is obtained :

$$Qdr(1+qq) + Vdu(r+sq) = du\sqrt{\left(\frac{gbBP^2Qr(1+qq)}{A^2} - (Vs - Vrq)^2\right)}.$$

Which equation appears to be the most wide, and from a single curve *AND* an infinite number of curves *AMC* can be furnished, and it can also be supposed that an infinite number of curves pass through the point C.

Scholium 2. [p. 164]

347. Moreover, however general this equation is, yet the curve *QM* is similar to the curve BSC, since it satisfies $x : y = r : s$. Whereby thus the more general solution can be shown, in which curves *QM* dissimilar in some manner can be put in place of the curve *BSC*, yet of the same kind, such that *QM* becomes *BSC* on making $u = a$. Now this solution is obtained, if some function *R* of *u* is taken vanishing on making $u = 0$ and *R* is evaluated at *D* on putting $u = a$, and it is given by $dR = Wdu$. Indeed taking

$$x = \frac{Qr}{B} \text{ and } y = \frac{Rs}{D};$$

it changes to *x* at *r* and *y* at *s*, on putting *u* = *a*, and with *u* vanishing as long as *x* and *y* vanish, for whatever *r* may be. Moreover this most following general equation arises :

$$dr(D^2Q^2 + B^2R^2q^2) + du(D^2QVr + B^2RWqs)$$
$$= \pm du \sqrt{\left(\frac{gBD^2bP^2Qr(D^2Q^2 + B^2R^2q^2)}{A^2} - B^2D^2(RVqr - QWs)^2\right)}.$$

In this equation in place of *dr*, *ds* can be introduced by putting $\frac{ds}{q}$ in place of *dr*, or even in place of r, *x* can be introduced by putting the value of this $\frac{Bx}{Q}$ in place of *r*, and then the equation is obtained between *u* and *x*. But it should be noted, since *Q* vanishes on making *u* = 0, then *P* must be such a function of *u* that P^2Q, on putting *u* = 0, either becomes a finite quantity or it becomes infinitely great; but yet one is required to beware, lest $\int Pdu$ on being assumed in this manner, becomes infinitely great.

Corollary 5.

348. The equation found in the solution is separable if $P = \frac{1}{\sqrt{u}}$; it becomes $A = 2\sqrt{a}$ and $p = \frac{\sqrt{b}}{2\sqrt{au}}$. [p. 165] For there is obtained :

$$\frac{du}{u} = \frac{dr(1+qq)}{-r - sq \pm \sqrt{\left(\frac{gbr}{4}(1+qq) - (s-rq)^2\right)}};$$

for *s* and *q* given in terms of *r*.

Corollary 6.

349. In a similar manner the equation of scholium 1 can be separated, if

$$P^2Q = V^2 \text{ or } P = \frac{V}{\sqrt{Q}}.$$

For there is obtained :

$$\frac{dr(1+qq)}{-r - sq \pm \sqrt{\left(\frac{gBbr(1+qq)}{AA} - (s-rq)^2\right)}} = \frac{Vdu}{Q} = \frac{dQ}{Q},$$

in which the indeterminates *u* and *r* are between themselves in turn separated.

Example 1.

350. On retaining $P^2Q = V^2$ or $\int Pdu = 2\sqrt{Q}$ on account of $Vdu = dQ$, on putting $n = a, \quad A = 2\sqrt{B}.$ Hence the equation becomes :

$$\frac{dQ}{Q} = \frac{dr(1+qq)}{-r-sq \pm \sqrt{\left(\frac{gbr}{4}(1+qq)-(s-rq)^2\right)}}.$$

Therefore $\int Pdu$ has the required property, so that it vanishes on making $u = 0$; for Q vanishes. Now let the curve BSC be a circle described with the diameter AB ; it becomes

$$s = \sqrt{(ar - r^2)} \text{ and } q = \frac{a-2r}{2\sqrt{(ar-r^2)}}$$

and

$$1 + qq = \frac{a^2}{4(ar-r^2)};$$

with these values put in place of s and q this equation is produced :

$$\frac{dQ}{Q} = \frac{adr}{\left(-2\sqrt{(ar-rr)} \pm \sqrt{(gbr-4r^2)}\right)\sqrt{(ar-r^2)}}.$$

[p. 166] Which equation not only has the indeterminates separated from each other, but also in general it can be integrated by logarithms ; for indeed in the equation :

$$\frac{dQ}{Q} = \frac{adr}{-2ar+2r^2 \pm r\sqrt{(gb-4r)(a-r)}}$$

with the irrational part to be effected with a rational one. Moreover, this integral is produced :

$$lQ = \frac{4a}{4a-gb} l \frac{2\sqrt{(a-r)} \mp \sqrt{(gb-4r)}}{\sqrt{r}} \pm \frac{\sqrt{gab}}{4a-gb} l \frac{\sqrt{a(gb-4r)}+\sqrt{gb(a-r)}}{\sqrt{a(gb-4r)}-\sqrt{gb(a-r)}} + lC.$$

Here it is to be noted the case is in agreement, in which $gb = 4a$ or $\sqrt{b} = \frac{2\sqrt{a}}{\sqrt{g}}$, in which the time for some curve AMC is made equal to the time of descent along the vertical line AB; for then it becomes :

$$\frac{dQ}{Q} = \frac{adr}{-2(ar-r^2) \pm 2(ar-r^2)}.$$

Therefore if the + sign prevales, then $dr = 0$ and r = const. = c, hence the equation becomes :

$$s = \sqrt{(ac - c^2)}$$

and

$$x : y = \sqrt{c} : \sqrt{(a - c)} \text{ or } y\sqrt{c} = x\sqrt{(a - c)},$$

which equation gives gives all the chords in the semicircle drawn from A, as we have now thus shown(102) the times for individual chords are equal to each other. Should the – sign prevale, then the equation becomes

$$\frac{4dQ}{Q} = \frac{-adr}{ar - rr}$$

and hence

$$Q^{-4} = \frac{C^{-4}r}{a - r} \text{ or } \frac{r}{a - r} = \frac{C^4}{Q^4}.$$

Hence the equation becomes

$$Q = C\sqrt{\sqrt{\frac{a - r}{r}}}$$

and since $s = \sqrt{(ac - c^2)}$ there is obtained

$$x = \frac{Cr}{B}\sqrt{\sqrt{\frac{a - r}{r}}} \text{ and } y = \frac{C}{B}\sqrt{\sqrt{r(a - r)^3}};$$

therefore with r eliminated, this equaton is produced (on puting $\frac{C}{B} = m$) :

$$y^2 + x^2 = ma\sqrt{xy}.$$

Therefore these curves have this property, so that the arc of these cut by a semicircle are completed by descending in the same time, clearly in that time in which the chords of the semicircle are traversed.

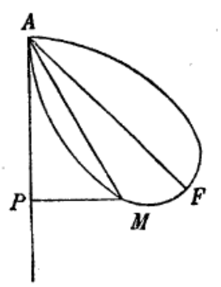

Fig. 44.

Corollary 7. [p. 167]

351. The figure of this is curve is *AMFA* (Fig. 44), of which the equation is $y^2 + x^2 = ma\sqrt{xy}$; clearly the curve has the diameter *AF* making a 45 degree angle to the vertical *AP*, and having a node at *A*. But now all these curves are similar to each other and all can be applied to all circles.

Corollary 8.

352. Therefore if some point *M* is taken on this curve and through this point and *A* a circle is considered crossing the curve, and having the centre on the vertical *AP*, the body traverses the arc *AM* in the same time in which the diameter of the circle or in which the chord *AM* is traversed. Whereby this curve has the property that any arc *AM* is completed by a body descending from *A* in the same time as the subtangent chord *AM*.

Corollary 9.

353. Therefore in this case, in which $P^2Q = V^2$ (349), likewise, either $Q = u$ or otherwise; for the same equation is produced between *x* and *y*, as this is understood either from the example or from the equation.

Example 2.

354. With keeping $P^2Q = V^2$, so that the equation becomes : [p. 168]

$$\frac{dQ}{Q} = \frac{dr(1+qq)}{-r-sq \pm \sqrt{\left(\frac{gbr}{4}(1+qq)-(s-rq)^2\right)}},$$

let the curve *BSC* be a circle described with centre *A* and with radius $AB = a$; then

$$s = \sqrt{(a^2 - r^2)}$$

and

$$q = \frac{-r}{\sqrt{(a^2-r^2)}} \text{ and } 1+qq = \frac{a^2}{a^2-r^2}.$$

With which put in place the following equation is produced :

$$\frac{dQ}{Q} = \frac{\pm a\,dr}{\sqrt{\left(\frac{gbr}{4}-a^2\right)(a^2-r^2)}},$$

where *gb* must be greater than 4*a*, since *r* cannot exceed *a*. Hence at once that radius becomes known which satisfies the question, on putting

$$\frac{gbr}{4} = a^2 \text{ or } r = \frac{4a^2}{gb},$$

in which case the equation becomes

$$s = \frac{a\sqrt{(g^2b^2-16a^2)}}{gb}.$$

Hence

$$x : y = 4a : \sqrt{(g^2b^2-16a^2)} \text{ and } \frac{y}{x} = \frac{\sqrt{(g^2b^2-16a^2)}}{4a},$$

which is the tangent of the angle of that radius, upon which the body in the given time $\sqrt{b}$ arrives at the periphery with the vertical *AB*. In addition, algebraic curves are not given, since the differential formula cannot be made rational.

Example 3.

355. With the most general equation taken from (347) the straight line *BSC* is put horizontal ; then $r = a$ and $dr = 0$. On account of this in place of dr the value of $\frac{ds}{q}$ is introduced, where q is infinitely great. Therefore with the terms deleted, which vanish before q, this equation arises :

$$ABRds + ABWsdu = \pm Ddu\sqrt{(gBabP^2Q - A^2a^2V^2)}.$$

Which equation, since $Wdu = dR$, and P, Q and V are given in terms of u, can be integrated. [p. 169] Now it becomes

$$\frac{ABRs}{D} = C \pm \int du\sqrt{(gBabP^2Q - A^2a^2V^2)}.$$

From which equation s can therefore be found. Then, on setting $y = \frac{Rs}{D}$, but y met vanish on making $u = 0$, it must be that $C = 0$, if indeed the integral

$$\int du\sqrt{(gBabP^2Q - A^2a^2V^2)}$$

is thus taken, in order that it disappears on putting $u = 0$. Therefore it then becomes :

$$x = \frac{Qa}{B} \text{ and } y = \frac{\pm 1}{AB}\int du\sqrt{(gBabP^2Q - A^2a^2V^2)}.$$

Which is the general equation for all the curves, upon which the body descents from *A* to the given horizontal.

Example 4.

356. The most general equation found above (347) is considere, and the straight line *BSC* is put parallel to the vertical *AB* and at a distance f from that; then $s = f$ and $q = 0$. Whereby this equatin is obtained :

$$ADQdr + ADrdQ = \pm du\sqrt{(gBD^2bP^2Qr - A^2B^2f^2W^2)}.$$

Moreover when we set $Qr = Bx$, with this substituted there is produced :

$$ADdx = \pm du\sqrt{(gD^2bP^2x - A^2f^2W^2)},$$

thus on finding x, $y = \frac{fR}{D}$. But since in that equation the indeterminates x and u cannot be separated from each other in turn, it is not possible to derive much from that equation.

Scholium 3.

357. From the general solution of this problem, when a single curve *AND* gives an infinity of curves *AMC*, it is allowed to collect the solution of this problem, when an infinity of curves are required[p. 170] on all of which the body arrives at a given point from *A*. Indeed some curve *AND* gives a single curve crossing through a given point of the curve *BSC* and in this way innumerable curves of this kind are found. But when the solution in this manner should be exceedingly difficult and laborious, it is agreed that another more fundamental method should be produced. But the manner in which we use this is thus prepared, so that it is necessary to know one curve, from which we show how to deduce innumerable other curves. Therefore this curve, which must be known, is elicited from one method of treatment, as from (350), where the curve can be found crossing through some point of the semicircle in a time to be described.

PROPOSITION 39.

Problem.

358. *Let the body be acted on always by a uniform force g downwards and let the curve AMC* (Fig. 45) *be given, upon which the body released from A arrives at some given point C; to find all the curves ANC upon which the body in the same time descends in the same time to the point C from A.*

Solution.

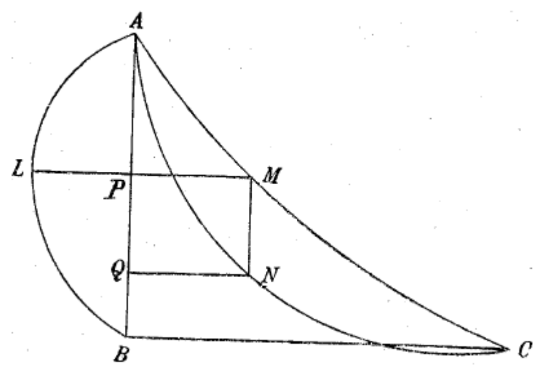

Fig. 45.

With the vertical line *AB* taken for the axis of all the curves, let the abscissa of the given curve *AMC* be $AP = t$, with the applied line $PM = u$ and let the curve *ANC* be one of the sought curves ; in that the arc *AN* is taken, [p. 171] which is completed in the same time as the arc *AM*; the points *M* and *N* are joined by the straight line *MN* and the curve *ALB* is constructed such that the applied line *PL* is equal to the line *MN*. Hence this curve *ALB* crosses the axis *AB* at the points *A* and *B*; for with the point *M* falling on *A* the point *N* also falls on *A* and on putting *M* at *C* the point *N* is also at *C*, since the arcs *AMC* and *ANC* are put to be transversed in the same time. Moreover, it is understood from the curve *ALB* that it is possible to find the curve *ANC*; whereby if an infinitude of curves of this kind *ALB* are considered at *A* and *B* coinciding with *AB*, a certain one of these gives the curve *ANC* and in this manner an innumerable number of curves *ANC* satisfying the question are produced. Now let $PL = r$; then r is some function of $AP = t$; moreover putting the abscissa of the curve *ANC*, $AQ = x$ and $QN = y$. With these in place we have :

$MN = \sqrt{((x-t)^2 + (u-y)^2)} = r$ and thus $y = u \pm \sqrt{(r^2 - (x-t)^2)}$.
Again, since the time for *AM* is put equal to the time for *AN*, there arises

$$\int \frac{\sqrt{(dt^2 + du^2)}}{\sqrt{gt}} = \int \frac{\sqrt{(dx^2 + dy^2)}}{\sqrt{gx}}$$

or

$$xdt^2 + xdu^2 = tdx^2 + tdy^2.$$

But

$$dy = du \pm \frac{(rdr - xdx + tdx + xdt - tdt)}{\sqrt{(r^2 - (x-t)^2)}}.$$

Let $du = pdt$ and $dr = qdt;$ where *r, p* and *q* are functions of *t*; again for brevity put $x - t = z$ or $x = t + z;$ then the equation arises :

$$dy = pdt \pm \frac{qrdt - zdz}{\sqrt{(r^2 - z^2)}}$$

and

$$dy^2 + dx^2$$
$$= p^2dt^2 + dt^2 + 2dtdz + dz^2 \pm \frac{2pqrdt^2 - 2pzdtdz}{\sqrt{(r^2 - z^2)}} + \frac{q^2r^2dt^2 - 2qrzdtdz + z^2dz^2}{r^2 - z^2}.$$

Hence this equation is obtained :

$$\frac{tr^2dz^2}{r^2 - z^2} = \frac{2tqrzdtdz - tq^2r^2dt^2}{r^2 - z^2} \pm \frac{2pztdtdz - 2pqtrdt^2}{\sqrt{(r^2 - z^2)}} - 2tdtdz + zdt^2 + zp^2dt^2,$$

from which *z* and *t* can be determined ; an equation can be obtained between *x* and *y*. [p. 172] Moreover where it is apparent, a function of *t* of this kind ought to be accepted in place of *r*, so that *r* vanishes on putting *t* = 0 since *t* = *AB* = *a*, let *P* be some function of *t* vanishing on putting *t* = 0 and *Q* is also such a function vanishing on putting *t* = 0; Now *Q* can be changed at *A*, if on making *t* = *a*; it is hence able to put $r = P(A - Q)$. And with this value whatever substituted in place of P and Q, the equation for the curve sought is obtained. Q.E.I.

Scholium 1.

359. Indeed from this equation it can be concluded that it is a little complex to be resolved, even if it is considered to be fundamental. Moreover often the equation found must be *reductio ad absurdum*, if the curve given AMC were the line of shortest descent, in which case it is not possible to give other curves, upon which the descent can be made in the same time. Therefore it is seen to be in agreement with our intention that the lines of quickest descent are to be treated, and this problem resolved, in which between all the curves either of the same length or having some other common property that is sought the line which is completed in the minimum time; and also as between all the lines upon which the descent is made in the same time, that line is to be found, which has some given property provided. For even if it is most difficult, all the lines having the same time of descent can be shown, yet from these [p. 173] any can be found with the property it possesses before all the rest. Moreover in order to carry out this program, it is required to deal with the method of isoperimetrics, that as it is set out everywhere, we will not explain here.

Scholium 2.

360. Moreover the solution of this problem is obtained in the following manner by(348). There it was found that :

$$\frac{du}{u} = \frac{dr(1+qq)}{-r-sq \pm \sqrt{\left(\frac{gbr}{4}(1+qq)-(s-rq)^2\right)}}.$$

Let *C* be the point(Fig. 43), at which it is agreed that all the curves must meet, and put $AE = f$ and $EC = h$; if therefore on putting $r = f$, then it is also the case that $s = h$. According to this, let *S* be some function of *r*, which goes over into *F* on putting $r = f$, with which done, put $s = \frac{Sh}{F}$. This value is substituted into the above equation and that is thus integrated, so that on putting $u = a$, it gives $r = f$. Then from that equation there is produced an equation between the coordinates of the curve *AMC* sought, clearly $AP = x$ and $PM = y$, from which, that is

$$x = \frac{ur}{a} \text{ and } y = \frac{us}{a}.$$

And the value of the arbitrary *S* gives an infinity of curves *AMC* joining the points *A* and *C* an on which the body descending in a given time equal to $\sqrt{b}$ arrives at *C* from *A*. Moreover let
$dS = Tdr$; then $q = \frac{hT}{F}$ and [p. 174]

$$\frac{du}{u} = \frac{dr(F^2+h^2T^2)}{-F^2r-h^2ST \pm \sqrt{\left(\frac{gF^2br}{4}(F^2+h^2T^2)-(FhS-FhTr)^2\right)}}$$

or

$$\frac{du}{u} = \frac{dr(F^2+h^2T^2)}{-F^2r-h^2ST \pm F\sqrt{\left(\frac{gbr}{4}(F^2+h^2T^2)-(hS-hTr)^2\right)}}.$$

Which equation can thus be integrated, as on putting $u = a$ makes $r = f$; with which done

$$x = \frac{ur}{a} \text{ and } y = \frac{hSu}{Fa}$$

and an equation is obtained between *x* and *y* for an infinite number of curves *AMC* satisfying the equation.

PROPOSITION 40.

Problem.

361. *To find the general rule, following which a curve ought to be disposed, in order that a body descending on that curve arrives at any point on the curve in the shortest possible time.*

Solution.

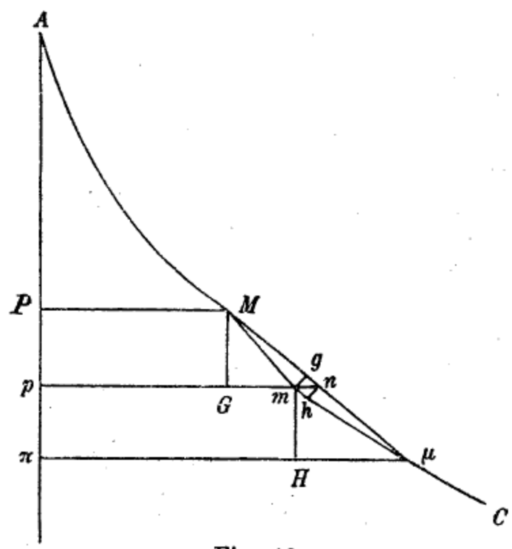

Fig. 46.

Let *AMC* (Fig. 45) be a curve of this kind, upon which a body arrives at C from A in a shorter time than on any other curve crossing from A to C. Therefore for any two point M and μ between these, the curve intercepted must be compared, in order that its motion along the arc AMC is completed in a shorter time than any other, if it should be intercepted. Now the nearby points M and μ are joined by the two elements $Mm, m\mu$, and the time to pass through $Mm\mu$ is a minimum, or by the rules of the method of maxima and minima, equal to the time for the nearby elements $Mn, n\mu$. [p. 175] The applied lines *MP, mp*, and $\mu\pi$ are drawn to the axis AP and with the elements *Pp*, $p\pi$ equal to each, or also *MG* = *mH* and *pm*, if there is a need, is produced to *n* and *mn* in made infinitely small with respect to the elements *Mm* and $n\mu$. Therefore it must be :

$$t.Mm + t.m\mu = t.Mn + t.n\mu.$$

[The time to traverse the elements Mm and $m\mu$ is equal to the time to traverse the infinitesimally close elements Mn and $n\mu$, the condition for the time variable to be a stationary point w.r.t. the independent variable *x*, for which $Pp = p\pi = dx$. Meanwhile the curve is described at these points $M, m,$ and μ by $y, y+dy,$ and $y+dy+ddy$]

Let the speed that the body has at *M* correspond to the height *v*, with which therefore both the elements *Mm* and *Mn* are traversed. Moreover the speed that the body has at *m*, corresponds to the height $v + du$ and the speed that it has at *n*, corresponds to the height $v + du + ddw$ [note that *du* includes contributions from the vertical force, as well as from a general horizontal force and some tangential force acting along the curve; the final speed includes second order contributions from the horizontal and tangential forces, as we are shown eventually in the corollaries. Recall that in Euler's analysis, an increment is always added at the start of an element, and holds a constant value within the increment : thus, the speed at *M* corresponds to *v* throughout *Mm*, the speed at *m* corresponds to *v* + *du*, and is held constant throughout $m\mu$, etc]; moreover with that [former] speed the element $m\mu$ is traversed, as now with this [latter] speed so the element $n\mu$ is traversed. Hence therefore this equation is obtained :

$$\frac{Mm}{\sqrt{v}}+\frac{m\mu}{\sqrt{(v+du)}}=\frac{Mn}{\sqrt{v}}+\frac{n\mu}{\sqrt{(v+du+ddw)}};$$

now

$$\frac{1}{\sqrt{(v+du+ddw)}}=\frac{1}{\sqrt{(v+du)}}-\frac{ddw}{2(v+du)\sqrt{(v+du)}},$$

hence with the centres M and μ and with the arclets *mg* and *nh* drawn, there arises :

$$\frac{ng}{\sqrt{v}}=\frac{mh}{\sqrt{(v+du)}}+\frac{n\mu\cdot ddw}{2(v+du)\sqrt{(v+du)}}.$$

Again,

$$\frac{1}{\sqrt{(v+du)}}=\frac{1}{\sqrt{v}}-\frac{du}{2v\sqrt{v}} \text{ and } \frac{1}{(v+du)\sqrt{(v+du)}}=\frac{1}{v\sqrt{v}}-\frac{3du}{2v^2\sqrt{v}}.$$

From which, with the negligible parts ignored, on substituting there arises

$$2v(mh-ng)=mh\cdot du-n\mu\cdot ddw=mh\cdot du-Mm\cdot ddw.$$

Now on account of the [four] similar triangles *nmg, mMG* and *nmh*, μmH, it follow that,

$$ng:mn=mG:mM \text{ or } ng=\frac{mG\cdot mn}{Mm}$$

and

$$mh:mn=\mu H:m\mu \text{ or } mh=\frac{\mu H\cdot mn}{m\mu}.$$

On account of which the above equation is given by :

$$2v\left(\frac{\mu H}{m\mu}-\frac{mG}{Mm}\right)=\frac{mG\cdot du}{Mm}-\frac{Mm\cdot ddw}{mn}=2v\text{ diff. }\frac{mG}{Mm}.$$

Which equation is homogeneous and determines the nature of the curve *AMC*, the so-called *brachistochrones*, [p. 176] upon which the body in the shortest time arrives at C from *A*. Q.E.I.

Corollary 1.

362. Therefore, if the lengths are called :

$$MG = mH = dx, \quad mG = dy \text{ and } Mm = \sqrt{(dx^2 + dy^2)} = ds,$$

then

$$H\mu = dy + ddy \text{ and } m\mu = ds + dds.$$

With these substituted, there is obtained :

$$2vd.\frac{dy}{ds} = \frac{dydu}{ds} - \frac{dsddw}{mn}.$$

In which, if from the forces acting *v*, *du,* and *ddw* can be determined, and the equation for the brachystochrone curve can be obtained. But always *ddw* thus involves *mn*, as from the calculation it exceeds *mn*.

Corollary 2.

363. Let the radius of osculation of the curve $Mm\mu = r$; and this lies in the region directed away from the axis *AP*, given by :

$$r = \frac{ds^3}{dxddy};$$

but

$$d.\frac{dy}{ds} = \frac{dsddy - dydds}{ds^2} = \frac{dx^2 ddy}{ds^3}.$$

[as $dx^2 + dy^2 = ds^2$, $ddx = 0$, and $dsdds = dyddy$, giving $\frac{dx}{r}$ as the differential of $\frac{dy}{ds}$;] hence this equation is produced :

$$\frac{2vdx}{r} = \frac{dydu}{ds} - \frac{dsddw}{mn}.$$

Where it is to be noted that $\frac{2v}{r}$ is the centrifugal force, which is sent along the normal to the curve at M .

Corollary 3.

364. If, from the forces acting, it follows that [we now interpret this equation as the sum of the works done per unit mass for increments *dx* with the force *P*, *dy* with force *Q*, and force *R* along *ds*] :

$$dv = Pdx + Qdy + Rds,$$

then

$$du = Pdx + Qdy + Rds \text{ and } ddw = Q\cdot mn + R\cdot ng,$$

since with the point *m* translated to *n* the small distance *dy* is increased by *mn* and *ds* by the small amount *ng*. [p. 177] Moreover since $ng = \frac{dy.mn}{ds}$, then

$$\frac{ddw}{mn} = -Q - \frac{Rdy}{ds},$$

with which substituted this equation is obtained :

$$\frac{2v}{r} = \frac{Pdy - Qdx}{ds}.$$

Scholium 1.

365. From the solution, it is understood that the formula found extends the widest and can be extended to any forces acting, and also including resistance. For whatever forces should be acting, so *du* as well as *ddw* can be determined, and which values put in place gives the equation for the brachystochrone sought. But yet these are only in place, if the directions of the forces are in the same plane; for the curve found is situated in the same plane. Yet no less, if the forces should not be acting in the same plane, a brachystochrone curve can be found in a given plane with the help of this formula. Indeed in any given particular plane it pleases, a brachystochrone curve is given, whatever the forces acting should be. Now the other question is, if a brachystochrone curve is sought from among all the curves joining two given points, also not situated in one given plane. Now as often as the directions of the forces acting in the same plane are put in place, there is no doubt that the brachystochrone curve is in the same plane. For if the curves are not in the same plane, oblique forces are acting and therefore they do not give the body as great an acceleration as is possible. [p. 178] Therefore from this solution, so the brachystochrone curve is found completely, if the directions of the forces acting are in the same plane, and also a brachystochrone is present in a given plane, whatever should be the forces acting. [However, we understand that the time will be larger for such curves where the forces are not coplanar.]

Scholium 2.

366. This question about brachystochrone curves or lines of quickest descent was first set out by the most celebrated Joh. Bernoulli [actually Leibniz and Jas. Bernoulli] and many solutions of this are extant, as in the *Act. Lips*., so also in the English *Transactions*, and in the *Comm. Acad. Paris*, and in others, where the solution for this problem, under the hypothesis of forces acting directed downwards and for centripetal forces, have been given. But no one has solved the fundamental problem that we have given here, set out so widely, so that it can also be extended to include forces of any kind, as well as resistance. For all have supposed that $ddw = 0$, because that is always wrong, unless the direction of the force is *MG* or *mH*. And the most celebrated Hermann stumbled over this, while he used such a proposition in finding the brachystochrones in a resistive medium in the

Comm. Acad. Petrop. A 1727; and I have corrected that very problem, in the same *Comm.* A. 1734.
[Ioh. Bernoulli, *Curvatura radii in diaphanis non uniformis solutionque problematis a se* in Actis 1696, p. 269, *propositi de invenienda linea brachystochrona*, Acta erud. 1697, p. 206; *Opera Omnia*, Tom. I, p. 187; Lettre de Mr. Jean Bernoulli a Mr. Basnage, *sur le problème de isopérimètres*, Histoire des Ouvrages des Sçavans, Paris 1697, p. 452; Opera Omnia, Tom. I, p. 194; *Remarques sur ce qu'on a donné jusqu'ici de solutions des problèmes isopérimètres*, Mém. de l'acad.d. sc. de Paris 1718, p. 100; *Opera omnia*, Tom. II, p. 225.
Iac. Hermann, *Theoria generalis motuum, qui nascuntur a potentiis quibusvis in corpora indesinenter agentibus*, Comment. acad. sc. Petrop. 2 (1727), 1729, p. 139.
L. Euleri Commentatio (E042) : *De linea celerrimi descensus in medio quocunque resistente*. Comment. acad. sc. Petrop. 2 (1734/5), 1740, p. 135. L. Euleri *Opera Omnia*, series I, vol. 25. Commentatio (E056) : *Curvarum maximi minimive proprietate gaudentium inventio nova et facilis*, Comment. acad. sc. Petrop. 2 (1736), 1741, p. 172. L. Euleri *Opera Omnia*, series I, vol. 25. Notes by P.St.]

CAPUT SECUNDUM

DE MOTU PUNCTI SUPER DATA LINEA IN VACUO.

[p. 151]

PROPOSITIO 36.

Problema.

328. *Posita potentia sollicitante uniformi g et ubique deorsum directa invenire omnes curvas AMC* (Fig. 41)*, super quibus corpus ex A dato tempore ad rectam BC ad horizontalem utcunque inclinationam descendat.*

Solutio.

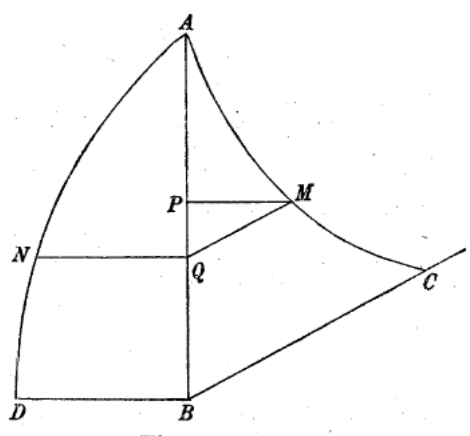

Fig. 41.

Exprimat curvae *AND* applicata *BD* tempus, quo corpus ex *A* ad rectam *BC* pertingit, et ducta per quodvis punctum *M* recta *MQ* parallela rectae datae *BC* secante verticalem *AB* in *Q* exprimat applicata *QN* tempus, quo corpus partem *AM* percurrit. Quare si infinitae curvae *AND* concipiantur, quae omnes in *B* eandem habeant applicatam *BD*, hae omnes generabunt curvas *AMC*, super quibus corpus dato tempore ab *A* ad rectam *BC* pervenit. Curvae autem *AND*, ut supra monitum, concurrere debent in *A* cum verticali *AB* et usque ad *D* divergere debent ab *AB*. Ponatur nunc tangens anguli *ABC* = *k* sitque *AP* = *x*, *PM* = *y*, *AQ* = *u*, *QN* = *t* et *AB* = *a*; erit $PQ = \frac{y}{k}$ ideoque $x + \frac{y}{k} = u$.

Quia autem celeritas in *M* debita est altutudini *gx*, erit tempus per $AM = \int \frac{\sqrt{(dx^2+dy^2)}}{\sqrt{gx}}$,

quod aequale est debet ipsi $QN = t;$ erit adeo $dt = \frac{\sqrt{(dx^2+dy^2)}}{\sqrt{gx}}$ et $gxdt^2 = dx^2 + dy^2$. At

ob curvam *AND* datam dabitur *t* in *u*, et cum sit $u = x + \frac{y}{k}$, dabitur *t* per *x* et *y* ;

quamobrem habebitur aequatio inter x et y pro curva quaesita AMC. Vel cum sit $y = ku - kx$, erit

$$gxdt^2du^2 = (k^2+1)dx^2 - 2k^2dudx + k^2du^2,$$

ex qua aequatione x per u invenire licebit. Sit ad hoc $dt = pdu$, erit

$$gxp^2du^2 = (k^2+1)dx^2 - 2k^2dudx + k^2du^2$$

atque [p. 152]

$$dx = \frac{k^2du \pm \sqrt{((g(k^2+1)p^2x - k^2)du^2)}}{k^2+1}.$$

Curva igitur *AND* talis accipi debet, ut ubique p maius sit quam $\frac{k}{\sqrt{gx(k^2+1)}}$.

Aequatio illa autem ita debet integrari, ut facto $u = 0$ fiat $x = 0$. Quo facto quoque eruetur aequatio inter x et y pro curva quaesita. Q.E.I.

Corollarium 1.

329. Curva *AMC* tanget in *A* rectam *AB*, si fit $dy = 0$ posito $x = 0$; tum vero debebit esse $du = dx$ atque $1 = \sqrt{(g(k^2+1)p^2x - k^2)}$. Quare hoc eveniet, si fit $ppu = \frac{1}{g}$ facto $x = 0$. Quia autem hoc casu est y infinities minor quam x, erit in ipso initio $x = u$; ex quo sequitur curvam *AMC* in *A* tangere verticalem *AB*, si fuerit $ppx = \frac{1}{g}$ positu $u = 0$.

Corollarium 2.

330. Deinde curva *AMC* normalis erit in *QM*, si fuerit $PQ = \frac{y}{k} = \frac{ydy}{dx}$ seu $dx = kdy = k^2du - k^2dx$ sive $dx = \frac{k^2du}{k^2+1}$. Hoc vero eveniet, ubi erit $p^2x = \frac{kk}{g(k^2+1)}$.

Corollary 3.

331. In ipso puncto *A* expressio *ppx* vel finitum valorem eumque maiorem quam $\frac{kk}{g(k^2+1)}$ habebit facto $x = 0$ vel infinite magnum. In posteriore casu erit $dx = \pm \infty du$, et cum sit $du = dx + \frac{dy}{k}$, erit $dy = -kdx$. [p. 153] Quibus casibus tangens curvae AMC in A parallela erit rectae BC.

Exemplum.

332. Sit curva *AND* parabola quaecunque, ita ut sit

$$t = \frac{2\alpha\sqrt{u}}{\sqrt{g}};$$

erit

$$dt = \frac{\alpha du}{\sqrt{gu}} \text{ et } p = \frac{\alpha}{\sqrt{gu}},$$

unde habebitur ista aequatio

$$(k^2+1)dx - k^2 du = \pm \frac{du\sqrt{(\alpha^2(k^2+1)x - k^2u)}}{\sqrt{u}}.$$

Huius aequationis integralis est

$$C = \left(\pm\sqrt{(\alpha^2(k^2+1)x - k^2u)} + A\sqrt{u}\right)^{\pi}\left(\pm\sqrt{(\alpha^2(k^2+1)x - k^2u)} + B\sqrt{u}\right)^{\varrho}$$

existente

$$A = \frac{\alpha^2 + \sqrt{(\alpha^4 - 4(1-\alpha^2)k^2)}}{2} \quad \text{et} \quad B = \frac{\alpha^2 - \sqrt{(\alpha^4 - 4(1-\alpha^2)k^2)}}{2}$$

et

$$\pi = \frac{-2A}{A-B} \quad \text{atque} \quad \varrho = \frac{2B}{A-B}.$$

[Si est $\alpha < 1$, fit *B* et hinc quoque ρ numerus positivus.] Cum igitur sit π numerus negativus, erit

$$C\left(\pm\sqrt{(\alpha^2(k^2+1)x - k^2u)} + A\sqrt{u}\right)^{-\pi} = \left(\pm\sqrt{(\alpha^2(k^2+1)x - k^2u)} + B\sqrt{u}\right)^{\varrho},$$

ubi *C* denotat constantem, quae efficiat, ut posito $x = 0$ fiat $u = 0$. Manifestum autem est, quaecunque fuerit constans, semper fieri $u = 0$ posito ρ vel π evanescit. At π evanescere non potest, ρ vero evanescit casu, quo $\alpha = 1$; hoc igitur casu debet esse $C = \infty$ fietque

$$\pm\sqrt{(\alpha^2(k^2+1)x - k^2u)} + \alpha^2\sqrt{u} = 0$$

seu $u = x$ et $y = 0$; quare satisfacit hoc casu recta verticalis *AB*. Reliquis casibus vero ob *C* arbitrariam quantitatem ex unica curva *AND* innumerabiles curvae *AMC* inveniuntur.

Unicus porro casu est seorsim tractandus, si $A = B$ seu

$$k^2 = \frac{\alpha^4}{4(1-\alpha^2)};$$

tum enim erit

$$lu = C - 2l\left(r \mp \frac{\alpha^2}{2}\right) \pm \frac{\alpha^2}{r \mp \frac{\alpha^2}{2}}$$

existente

$$r = \frac{\sqrt{(\alpha^2(k^2+1)x - k^2u)}}{\sqrt{u}}.$$

Consequenter pro hoc casu habebitur haec aequatio [p. 154]

$$C = 2l\left(\sqrt{(\alpha^2(k^2+1)x - k^2u)} \mp \frac{\alpha^2\sqrt{u}}{2}\right) \mp \frac{\alpha^2\sqrt{u}}{\sqrt{(\alpha^2(k^2+1)x - k^2u)} \mp \frac{\alpha^2\sqrt{u}}{2}},$$

ubi C quoque determinatione non opus habet.

Si est $\alpha > 1$, fit B et hinc quoque ρ numerus negativus; tum ergo debet esse $C = \infty$. Hoc ergo casu erit vel

$$A\sqrt{u} = \sqrt{(\alpha^2(k^2+1)x - k^2u)}$$

vel

$$B\sqrt{u} = \sqrt{(\alpha^2(k^2+1)x - k^2u)}.$$

Quae duae aequationes sunt pro lineis rectis certo modo inclinitis et per *A* transeuntibus haeque etiam generaliter satisfaciunt. Ut si fuerit $\alpha = 1$, erit *A* = 1 et *B* = 0 hincque hae aequationes

$$u = x \quad \text{seu} \quad y = 0 \quad \text{et} \quad u = \frac{(k^2+1)x}{kk} = x + \frac{y}{k} \quad \text{seu} \quad x = ky,$$

quae est linea recta perpendicularis in *BC*; haec enim a corpore eodem tempore percurritur quo verticalis *AB*.

Corollarium 4.

332a. Nisi igitur sit $\alpha > 1$ vel $\alpha = 1$, innumerabiles lineae curvae inveniuntur problemati satisfacientes; quae ergo omnes minore tempore absolventur quam perpendicularis AB.

Corollarium 5.

333. Cum ergo ex unica curva *AND* infinitae oriri queant curvae *AMC*, facile intelligitur infinities plures curvas huic quaestioni satisfacere quam praecedenti.

Corollarium 6. [p. 155]

334. Si $\alpha < 1$, effici potest determinanda constante *C*, ut curva quaesita per datum punctum rectae *BC* transeat. Deinde aliis assumendis curvis *AND* simili modo infinitae curvae poterunt inveniri, super quibus corpus non solum dato tempore ad rectam *BC* perveniat, sed ad quodvis in ea punctum datum *C*.

Scholion 1.

335. In hoc exemplo casus, quo $\alpha = 1$, bis occurrit; prima enim vice linea recta verticalis tantum satisfaciens est inventa, altera vice praeter hanc rectam alia inclinata, utroque tamen modo eadem aequatione generali sumus usi. Saepius autem iam huiusmodi casus obtigerunt, in quibus aequationes differentiales continent in se aequationes integrales, quae nihilhominus per integrationes non eruuntur. Ut in casu $\alpha = 1$ haec habetur aequatio differentialis

$$\frac{(k^2+1)dx - k^2du}{\sqrt{((k^2+1)x - k^2u)}} = \frac{\pm du}{\sqrt{u}},$$

quae integrata dat *x* = *u*. Interim tamen perspicuum est hanc aequationem $k^2u = (k^2+1)x$ in illla quoque contineri, etiamsi per integrationem non prodeat. Et hanc ob rem posterior aequatio integralis aeque satisfacit ac prior *x* = *u*. Hinc generaliter colligere licet aequationem differentialem

$$\frac{dt}{T} = Vdu,$$

in qua *T* est talis functio ipsius *t*, quae evanescat posito *t* = 0, et *V* functio quaecunque ipsius *u*, aeque comprehendere hanc integralem *t* = 0 ac hanc [p. 156]

$$\int \frac{dt}{T} = \int V du,$$

quae per integrationem elicitur. Plerumque quidem casus $t = 0$, si t est simplex quantitas, negligi potest; at si t est quantitas composit ut in nostro casu, perperam omittitur. Similem casum supra habuimus (300) in aequatione

$$ds = \frac{-dx\sqrt{b}}{\sqrt{(a^2b - bx^2 - a^2\int Pdx)}},$$

ubi observavimus aequationem $x = a$ in ea contineri, etiamsi integratio nequidem possit perfici. Nam posito $a - x = t$ erit $-dx = dt$ et

$$\sqrt{(a^2b - bx^2 - a^2\int Pdx)}$$

erit functio ipsius t, quae fit = 0, si fit $t = 0$ seu $x = a$; namque $\int Pdx$ ita accipi iubebatur, ut evanescat posito $x = a$. Posita ergo hac ipsius t functione = T erit $ds = \frac{dt\sqrt{b}}{T}$, ex qua aequatione ergo tuto concludi licet satisfacere aequationem $t = 0$ seu $x = a$ ideoque problemati illi satisfacere circulum, ut ibi innuimus (303). Magis universaliter vero in hac aequatione $Vdu = \frac{dt}{T}$, si T non evanescit posito $t = 0$, comprehendetur ista aequatio $T = 0$, ex qua erit t = constanti quantitati ideoque dt = 0. Unde intelligitur $T = 0$ contineri in aequatione proposita $Vdu = \frac{dt}{T}$. Atque hinc, si t fuerit quantitas composita v.g. ex u et x, statim habetur aequatio integralis per integrationem vix eruenda.

Scholion 2.

336. Casus hic superest, qui peculiarem resolutionem requirit, [p. 157] quando recta *BC* supra punctum *A* cum *BA* concurrit et quando est parallela. Considerabimus autem sequente problemate tantum casum, quo fit *BC* parallela verticali *AB* et in data distantia posita; ex quo simili modo casum rectae *BC* utcunque inclinatae deducere licebit.

PROPOSITIO 37.

Problema.

337. *Sollicitetur corpus perpetuo deorsum vi uniformi g ; invenire curvas innumerabiles, super quibus corpus ex A* (Fig. 42) *motum a quiete incipiendo dato tempore ad rectam verticalem EC perveniat.*

Solutio.

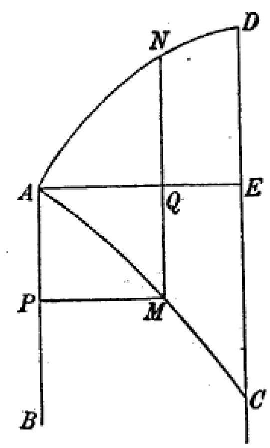

Fig. 42.

Sit *AMC* curva quaecunque quaesitarum et pro axe sumatur recta verticalis *AB*; dicatur *AP* = *x*, *PM* = *AQ* = *y*; erit celeritas in *M* debita altitudini *gx* et tempus per *AM* =

$$\int \frac{\sqrt{(dx^2+dy^2)}}{\sqrt{gx}}$$

Sit porro *AND* curva, cuius quaevis applicat *QN* exprimat tempus per *AM*, et sit *QN* = *t* functioni ipsius *y*; poterit ex curva *AND* data curva *AMC* inveniri. Quare si infinitae curvae *AND* concipiantur, quae omnes in *E* communem habeant applicatam *DE*, omnes producent curvas *AMC*, super quibus corpus dato tempore per *DE* expresso ex *A* ad *CE* perveniet. Erit itaque

$$t = \int \frac{\sqrt{(dx^2+dy^2)}}{\sqrt{gx}}$$

et posito $dt = pdy$ erit

$$gp^2xdy^2 = dx^2 + dy^2 \text{ atque } dx = dy\sqrt{(gp^2x-1)}$$

[p. 158] Quae ita integrata, ut posito $x = 0$ fiat $y = 0$, dabit curvas *AMC* quaesitas.
Sit *R* functio quaecunque ipsius *y* et $\int Rdy$ ita capiatur, ut evanescat posito $y = 0$. Tum fiat $\int Rdy = A$ posito $y = AE = a$ et existente $DE = \sqrt{b}$ sumatur

$$t = \frac{\sqrt{b}\int Rdy}{A};$$

erit

$$p = \frac{R\sqrt{b}}{A} \quad \text{atque} \quad dx = \frac{dy}{A}\sqrt{(gbR^2x - A^2)}.$$

Quae aequatio, quicquid pro R substituatur, dabit innumeras curvas quaesito satisfacientes. Q.E.I.

Corollarium 1.

338. Casus ergo habetur simplicissimus, si fuerit $R = 1$; tum enim aequatio separabilis prodit. Erit vero $t = \frac{y\sqrt{b}}{a}$ ob $A = a$. Hanc ob rem fit

$$\frac{adx}{\sqrt{(gbx - a^2)}} = dy \quad \text{atque} \quad \frac{2a}{gb}\sqrt{(gbx - a^2)} = y.$$

Qui autem nullius est utilitatis ob valorem ipsius y imaginarium.

Corollarium 2.

339. Quia autem $\sqrt{(gbR^2x - A^2)}$ non potest esse quantitas imaginaria, oportet sit $R^2x > \frac{A^2}{gb}$; etiamsi $x = 0$. Quare R neque quantitas constans esse potest neque functio ipsius y, quae evanescat facto $y = 0$. Hanc ob rem R talis esse debet functio ipsius y, quae fiat $= \infty$, si ponatur $y = 0$. Praeterea tamen eiusmodi esse debet, ut $\int Rdy$ non fiat infinitum, quod evenerit, si esset $R = \frac{1}{y}$ vel $\frac{1}{y^2}$ etc.

Exemplum. [p. 159]

340. Ponamus ergo esse $R = \frac{1}{\sqrt{y}}$; erit

$$\int Rdy = 2\sqrt{y} \text{ et } A = 2\sqrt{a}.$$

Hinc habebitur ista aequatio

$$2dx\sqrt{ay} = dy\sqrt{(gbx - 4ay)}.$$

Quae aequatio quia est homogenea, ponatur $x = qy$; erit

$$2qdy\sqrt{a} + 2ydq\sqrt{a} = dy\sqrt{(gbq - 4a)}$$

seu

$$\frac{dq}{\sqrt{(gbq - 4a)} - 2q\sqrt{a}} = \frac{dy}{2y\sqrt{a}}.$$

Posito $\frac{gb}{4a} = n$ et $\sqrt{(nq - 1)} = r$ erit

$$\frac{dy}{y} = \frac{-2rdr}{r^2 - nr + 1}.$$

Quae posterior formula a quadratura circuli pendebit, si $n < 2$.

Altereo vero casu $n > 2$ erit integrale

$$ly = lC + \frac{n - \sqrt{(n^2 - 4)}}{\sqrt{(n^2 - 4)}}\, l\left(2r - n + \sqrt{(n^2 - 4)}\right) + \frac{-n - \sqrt{(n^2 - 4)}}{\sqrt{(n^2 - 4)}}\, l\left(2r - n - \sqrt{(n^2 - 4)}\right).$$

Quae ob

$$r = \frac{\sqrt{(nx - y)}}{\sqrt{y}}$$

abit in hanc

$$C\left(2\sqrt{(nx-y)}-(n-\sqrt{(n^2-4)})\sqrt{y}\right)^{\frac{n-\sqrt{(n^2-4)}}{\sqrt{(n^2-4)}}}$$
$$=\left(2\sqrt{(nx-y)}-(n+\sqrt{(n^2-4)})\sqrt{y}\right)^{\frac{n+\sqrt{(n^2-4)}}{\sqrt{(n^2-4)}}}.$$

Ubi pro *C* constantem quamcunque accipere licet, quia ipsa aequatio ita est comparata, ut posito $x = 0$ fiat $y = 0$. Per methodum autem supra traditam (335) ex aequatione differentilai statim habetur haec integralis

$$2q\sqrt{a}=\sqrt{(gbq-4a)} \quad \text{seu} \quad 2x\sqrt{a}=\sqrt{(gbxy-4ayy)},$$

unde oritur

$$\frac{y}{x}=\frac{gb\pm\sqrt{(g^2b^2-64a^2)}}{8a},$$

quae dat duas lineas rectas, nisi sit $gb<8a$, quo casu aequatio est imaginaria.

In casu, quo $n = 2$, erit

$$\frac{dy}{y}=\frac{-2rdr}{(r-1)^2} \quad \text{seu} \quad ly=-2l(r-1)+\frac{2}{r-1},$$

unde fit

$$ly=-2l\left(\sqrt{(2x-y)}-\sqrt{y}\right)+2l\sqrt{y}+\frac{2\sqrt{y}}{\sqrt{(2x-y)}-\sqrt{y}}+lC$$

seu

$$l\left(\sqrt{(2x-y)}-\sqrt{y}\right)-lC=\frac{\sqrt{y}}{\sqrt{(2x-y)}-\sqrt{y}}.$$

Ubi etiam pro C quantitatem quamcunque accipere licet.

Si $n < 2$, tum constructio curvae partim a logarithmis, [p. 160] partim a quadratura circuli pendet; fiunt enim ob $\sqrt{(n^2-4)}$ imaginarium logarithmi inventi imaginarii. Hoc igitur casu expedit constructionem perficere prae expressione analytica.

PROPOSITIO 38.

Problema.

341. *Sollicitetur corpus perpetuo deorsum vi uniformi g dataque sit curva quaecunque BSC* (Fig. 43) ; *invenire omnes curvas AMC, super quibus corpus descendo ex A dato tempore ad curvam BSC perveniat.*

Solutio.

Sit curvarum quaesitarum quaecunque *AMC*, per cuius quodvis punctum *M* ducatur curva *MQ* similis curvae *BSC* respectu puncti fixi *A*, et exprimat curvae *AND* applicatae *NQ* tempus per arcum *AM*; exponet ergo applicata *BD* tempus per totam curvam *AMC*. Quo facto poterit vicissim ex data curva *AND* curva *AMC* inveniri. Quare si infinitae curvae *AND* concipiantur, quae omnes in *B* habeant applicatam *BD* communem, eae generabunt infinitas curvas *AMC*, super quibus omnibus corpus ex *A* descendendo dato tempore per *BD* expresso ad curvam *BSC* perveniat. Si nunc

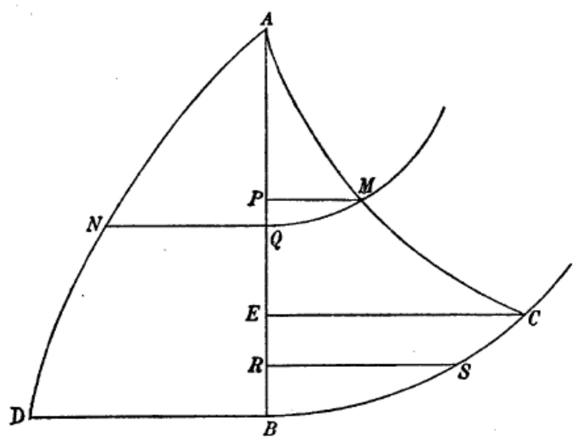

Fig. 43.

$AB = a$, $BD = \sqrt{b}$ et arcui QM abscindatur arcus similis BS ex curva data BSC; erit

$$AQ : AB = PM : RS = AP : AR = PQ : BR.$$ [p. 161]

Ponatur porro $AP = x, PM = y, AQ = u, QN = t, AR = r, RS = s;$ dabitur ob curvam *BSC* datam aequatio inter *r* et *s* atque ob curvam *AND* datam dabitur aequatio inter *t* et *u*. At ob similitudinem erit $u : a = y : s = x : r$, unde erit $y = \frac{us}{a}$ et $x = \frac{ur}{a}$.

Celeritas deinde corporis in *M* debita est altitudini *gx*, ex qua tempus per *AM* erit =

$$\int \frac{\sqrt{(dx^2 + dy^2)}}{\sqrt{gx}},$$

quod aequae poni debet ipsi *t*; unde oritur ista aequatio $gxdt^2 = dx^2 + dy^2$.

Quia autem *t* per *u* datur, sit $dt = pdu$ et *p* sit functio ipsius *u*; atque ob

$$dx = \frac{udr+rdu}{a} \text{ et } dy = \frac{uds+sdu}{a}$$

transibit illa aequatio in hanc

$$garup^2du^2 = (udr + rdu)^2 + (uds + sdu)^2$$

At quia curva *BSC* datur, erit *s* functio ipsius *r* sitque $ds = qdr$ existente *q* functione ipsius *r* quacunque. His substitutis habebitur aequatio inter *u* et *r* ista

$$garup^2du^2 = (udr + rdu)^2 + (uqdr + sdu)^2$$

Quae radice quadrata extracta dat

$$\frac{dr}{du} = \frac{-r - sq \pm \sqrt{(2rsq - s^2 - r^2q^2 + garup^2(1+qq))}}{u(1+qq)}.$$

Ex qua si aequatio inter *r* et *u* inveniatur, inde habebitur simul aequatio inter *x* et *y* pro curva quaesita.

Quod autem ad curvam *AND* attinet, sit *P* functio quaecunque ipsius *n* et $\int Pdu$ ita integratum, ut evanescat facto $u = 0$ et fiat = *A* posito $u = a$; tum sumatur $t = \frac{\sqrt{b}\int Pdu}{A}$ pro aequatio curvae *AND*. Erit ergo $p = \frac{P\sqrt{b}}{A}$, ubi pro *P* functionem quamvis ipsius *u* ponere licet. Q.E.I.

Corollarium 1. [p. 162]

342. Si *u* ponatur = 0, eo ipso quoque *x* et y evanescunt, nisi forte fiat *r* vel *s* infinitum. Illo igitur casu in integratione aequationis differentialis inventae constantem quamcunque addere licet, quia non opus est, ut *r* datum habeat valorem, si *u* fit = 0.

Corollarium 2.

343. Tum igitur ob constantem arbitrariam addendam ex unica curva *AND* data innumerabiles inveniuntur curvae *AMC* quaesito satisfacientes.

Corollarium 3.

344. Si curva *BSC* ita est comparata, ut nusquam neque *r* neque s fieri queat infinite magnum, semper unica curva *AND* infinitas dabit curvas quaesitas *AMC*. Quae non solum hanc habebunt proprietatem, ut corpora super iis descendentia simul ad datam *BSC* perveniant, sed quoque simul ad quamvis datae similem curvam *QM* pertingent.

Corollarium 4.

345. Cum igitur in integratione aequationis inventae constantem quamcunque addere liceat [p. 163], ea ita poterit assumi, ut curva *AMC* ad datum punctum *C* curvae datae *BSC* dirigatur. Hocque modo infinitae curvae *AMC* poterunt inveniri, quae omnes in dato puncto *C* conveniant.

Scholion 1.

346. Posuimus curvas *QM* similes curvae *BSC*, ut curva ipsa in *A* erecta fiat infinite parva et omnia puncta curvae *BSC* in *A* conveniant et *x* et *y* evanescant posito $u = 0$. Potuissemus autem eodem modo curvas *QM* vel cum *BSC* congruentes ponere vel discrepantes lege quacunque. Ut sit *Q* functio ipsius *u* quaecunque evanescens posito $u = 0$ abeatque ea in *B* facto $u = a$, curva *QM* ita pendere poterit a curva *BSC*, ut sit

$$x = \frac{Qr}{B} \text{ et } y = \frac{Qs}{B};$$

namque facto $u = a$ curva *QM* transibit in ipsam *BSC* et in *A* curva in punctum transibit, nisi curva *BSC* in infinitum progrediatur. At etiam hoc casu pro *Q* talis accipi poterit functio, ut, etiamsi fiat $r = \infty$, tamen *Qr* et *Qs* fiat = 0, si $u = 0$. Posito autem $dQ = Vdu$ habebitur aequatio generalis sequens

$$Qdr(1+qq) + Vdu(r+sq) = du\sqrt{\left(\frac{gbBP^2Qr(1+qq)}{A^2} - (Vs - Vrq)^2\right)}.$$

Quae aequatio latissime patet et ex unica curva *AND* infinite infinitas curvas *AMC* suggeret, quae etiam infinitas suppeditabit, quae per datum punctum *C* transeunt.

Scholion 2. [p. 164]

347. Quantumvis generalis autem est haec aequatio, tamen curva *QM* est similis curvae BSC, quia est $x : y = r : s.$ Quare adhuc generalior solutio poterit exhiberi, in qua curvae *QM* utcunque dissimiles ponuntur curvae *BSC*, eiusmodi tamen, ut *QM* in *BSC* abeant facto $u = a$. Obtinebitur vero haec solutio, si *R* sumatur functio quaecunque ipsius *u* evanescens facto $u = 0$ abeatque *R* in *D* posito $u = a$ sitque $dR = Wdu$. Sumatur enim

$$x = \frac{Qr}{B} \text{ et } y = \frac{Rs}{D};$$

abibit *x* in *r* et *y* in *s*, si fiat $u = a$, atque evanescente *u* tam *x* quam *y* evanescent, quicquid sit *r*. Hinc autem sequens orietur aequatio generalissima

$$dr(D^2Q^2 + B^2R^2q^2) + du(D^2QVr + B^2RWqs)$$
$$= \pm du\sqrt{\left(\frac{gBD^2bP^2Qr(D^2Q^2 + B^2R^2q^2)}{A^2} - B^2D^2(RVqr - QWs)^2\right)}.$$

In hac aequatione loco *dr* introduci potest *ds* ponendo $\frac{ds}{q}$ loco *dr*, vel etiam loco *r* poterit *x* introduci ponendo loco *r* eius valorem $\frac{Bx}{Q}$ et tum habebitur aequatio inter *u* et *x*. Notandum autem est, quia *Q* evanescit facto $u = 0$, *P* talem esse debere functionem ipsius *u*, ut P^2Q posito $u = 0$ vel fiat quantitas finita vel infinita magna; at tamen cavendum, ne $\int Pdu$ debito modo sumtum fiat infinite magnum.

Corollarium 5.

348. Aequatio in solutione inventa fit separabilis, si fuerit $P = \frac{1}{\sqrt{u}}$; erit $A = 2\sqrt{a}$ et $p = \frac{\sqrt{b}}{2\sqrt{au}}$. [p. 165] Habebitur enim

$$\frac{du}{u} = \frac{dr(1+qq)}{-r-sq \pm \sqrt{\left(\frac{gbr}{4}(1+qq)-(s-rq)^2\right)}};$$

dantur enim *s* et *q* per *r*.

Corollarium 6.

349. Simili modo aequatio scholii 1 separationem admittet, si fuerit

$$P^2Q = V^2 \text{ seu } P = \frac{V}{\sqrt{Q}}.$$

Habebitur enim

$$\frac{dr(1+qq)}{-r-sq \pm \sqrt{\left(\frac{gBbr(1+qq)}{AA}-(s-rq)^2\right)}} = \frac{Vdu}{Q} = \frac{dQ}{Q},$$

in qua indeterminatae *u* er *r* sunt a se invicem separatae.

Exemplum 1.

350. Manente $P^2Q = V^2$ seu $\int Pdu = 2\sqrt{Q}$ ob $Vdu = dQ$, erit facto $n = a \quad A = 2\sqrt{B}.$ Unde fit

$$\frac{dQ}{Q} = \frac{dr(1+qq)}{-r-sq \pm \sqrt{\left(\frac{gbr}{4}(1+qq)-(s-rq)^2\right)}}.$$

Habebit ergo $\int Pdu$ requisitam proprietatem, ut evanescat facto $u = 0$; evanescit enim Q. Sit nunc curva *BSC* circulus super diametro *AB* descriptus; erit

$$s = \sqrt{(ar-r^2)} \text{ et } q = \frac{a-2r}{2\sqrt{(ar-r^2)}}$$

atque

$$1+qq = \frac{a^2}{4(ar-r^2)};$$

his valoribus loco *s* et *q* substitutis prodibit ista aequatio

$$\frac{dQ}{Q} = \frac{adr}{\left(-2\sqrt{(ar-rr)} \pm \sqrt{(gbr-4r^2)}\right)\sqrt{(ar-r^2)}}.$$

[p. 166] Quae aequatio non solum indeterminatas a se invicem habet separatas, sed etiam generaliter per logarithmos integrari potest; potest enim in aequatione

$$\frac{dQ}{Q} = \frac{adr}{-2ar + 2r^2 \pm r\sqrt{(gb-4r)(a-r)}}$$

membrum irrationale rationale effici. Prodibit autem integralis haec

$$lQ = \frac{4a}{4a-gb} l \frac{2\sqrt{(a-r)} \mp \sqrt{(gb-4r)}}{\sqrt{r}} \pm \frac{\sqrt{gab}}{4a-gb} l \frac{\sqrt{a(gb-4r)} + \sqrt{gb(a-r)}}{\sqrt{a(gb-4r)} - \sqrt{gb(a-r)}} + lC.$$

Notari hic convenit casum, quo $gb = 4a$ seu $\sqrt{b} = \frac{2\sqrt{a}}{\sqrt{g}}$, quo tempus per quamvis curvam *AMC* aequale ponitur tempori descensus per rectam verticalem *AB*; tum enim erit

$$\frac{dQ}{Q} = \frac{adr}{-2(ar-r^2) \pm 2(ar-r^2)}.$$

Si igitur signum + valeat, erit $dr = 0$ et r = const. = c, unde fit

$$s = \sqrt{(ac - c^2)}$$

et

$$x : y = \sqrt{c} : \sqrt{(a-c)} \quad \text{seu} \quad y\sqrt{c} = x\sqrt{(a-c)},$$

quae, aequatio omnes dat chordas in hoc semicirculo ex A ductas, quemadmodum iam demonstrvimus (102) tempora per singulas chordas esse inter se aequalia. Valeat signum – , erit

$$\frac{4dQ}{Q} = \frac{-adr}{ar - rr}$$

atque hinc

$$Q^{-4} = \frac{C^{-4}r}{a-r} \quad \text{seu} \quad \frac{r}{a-r} = \frac{C^4}{Q^4}.$$

Erit ergo

$$Q = C\sqrt{\sqrt{\frac{a-r}{r}}}$$

atque ob $s = \sqrt{(ac - c^2)}$ habebitur

$$x = \frac{Cr}{B}\sqrt{\sqrt{\frac{a-r}{r}}} \quad \text{et} \quad y = \frac{C}{B}\sqrt{\sqrt{r(a-r)^3}};$$

eliminata ergo r prodibit ista aequatio (posito $\frac{C}{B} = m$)

$$y^2 + x^2 = ma\sqrt{xy}.$$

Hae ergo curvae hanc habent proprietatem, ut arcus earum a semicirculo abscissi absolvantur descendendo eodem tempore, eo scilicet tempore, quo singulae semicirculi chordae percurruntur.

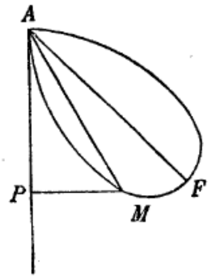

Fig. 44.

Corollarium 7. [p. 167]

351. Huius autem curvae, cuius aequatio est $y^2 + x^2 = ma\sqrt{xy}$, Figura est *AMFA* (Fig. 44); habet nimirum diametrum *AF* cum verticali *AP* angulum semirectum constituentem et in *A* nodum. At vero omnes hae curvae sunt inter se similes et omnes ad omnes circulos accomodari possunt.

Corollarium 8.

352. Si ergo in hac curva sumatur quodcunque punctum *M* et per hoc et *A* circulus transiens concipiatur centrum habens in verticali *AP*, corpus arcum *AM* eodem tempore percurret, quo diametrum circuli seu quo chordam *AM*. Quare haec curva hanc habet proprietatem, ut quivis arcus *AM* a corpore ex *A* descendente absolvatur eadem tempore, quo subtensa *AM*.

Corollarium 9.

353. Hoc ergo casu, quo $P^2Q = V^2$ (349), perinde est, sive sit $Q = u$ sive secus; eadem enim prodit aequatio inter *x* et *y*, uti tam ex exemplo hoc quam ex aequatione intelligitur.

Exemplum 2.

354. Manente $P^2Q = V^2$, ut sit [p. 168]

$$\frac{dQ}{Q} = \frac{dr(1+qq)}{-r - sq \pm \sqrt{\left(\frac{gbr}{4}(1+qq) - (s-rq)^2\right)}},$$

sit curva *BSC* circulus centro *A* radio *AB* = *a* descriptus; erit $s = \sqrt{(a^2 - r^2)}$ et

$$q = \frac{-r}{\sqrt{(a^2 - r^2)}} \quad \text{atque} \quad 1 + qq = \frac{a^2}{a^2 - r^2}.$$

Quibus substitutis prodibit sequens aequatio

$$\frac{dQ}{Q} = \frac{\pm adr}{\sqrt{\left(\frac{gbr}{4} - a^2\right)(a^2 - r^2)}},$$

ubi *gb* maius esse debet quam 4*a*, quia *r* non excedere potest *a*. Hinc statim ille radius innotescet, qui quaesito satisfacit, ponendo

$$\frac{gbr}{4} = a^2 \quad \text{seu} \quad r = \frac{4a^2}{gb},$$

quo casu erit

$$s = \frac{a\sqrt{(g^2b^2 - 16a^2)}}{gb}.$$

Unde erit

$$x : y = 4a : \sqrt{(g^2b^2 - 16a^2)} \quad \text{et} \quad \frac{y}{x} = \frac{\sqrt{(g^2b^2 - 16a^2)}}{4a},$$

quae est tangens anguli illius radii, super quo corpus dato tempore $\sqrt{b}$ ad peripheriam pervenit, cum verticali *AB*. Curvae praeterea algebraicae non dantur, quia formula differentialis non effici potest rationalis.

Exemplum 3.

355. Sumta aequatione generalissima ex (347) ponatur linea *BSC* recta horizontalis; fiet $r = a$ et $dr = 0$. Hanc ob rem loco *dr* introducatur eius valor $\frac{ds}{q}$, ubi *q* erit infinite magnum. Deletis ergo terminis, qui prae *q* evanescunt, proveniet ista aequatio

$$ABRds + ABWsdu = \pm Ddu\sqrt{(gBabP^2Q - A^2a^2V^2)}.$$

Quae aequatio ob $Wdu = dR$, et *P*, *Q* et *V* data per *u* integrationem admittit. [p. 169] Erit nempe

$$\frac{ABRs}{D} = C \pm \int du\sqrt{(gBabP^2Q - A^2a^2V^2)}.$$

Ex qua aequatione ergo invenitur *s*. Deinde, cum sit $y = \frac{Rs}{D}$, at *y* evanescere debeat facto $u = 0$, debebit esse $C = 0$, si quidem integrale

$$\int du\sqrt{(gBabP^2Q - A^2a^2V^2)}$$

ita sumatur, ut evanescat posito $u = 0$. Tum ergo erit

$$x = \frac{Qa}{B} \quad \text{et} \quad y = \frac{\pm 1}{AB}\int du\sqrt{(gBabP^2Q - A^2a^2V^2)}.$$

Quae est aequatio generalis pro omnibus curvis, super quibus corpus ex *A* ad horizontalem datam descendit.

Exemplum 4.

356. Teneatur aequatio generalissima supra inventa (347) et ponatur linea *BSC* recta verticalis parallela ipsi *AB* et ad distantiam *f* ab ea posita; erit $s = f$ et $q = 0$. Quare habebitur ista aequatio

$$ADQdr + ADrdQ = \pm du\sqrt{(gBD^2bP^2Qr - A^2B^2f^2W^2)}.$$

Cum autem sit $Qr = Bx$, hoc substituto prodibit

$$ADdx = \pm du\sqrt{(gD^2bP^2x - A^2f^2W^2)},$$

unde invento x erit $y = \frac{fR}{D}$. At quia in illa aequatione indeterminatae x et u non sunt a se invicem separatae, non multum ex ea derivare licet.

Scholion 3.

357. Ex generali huius problematis solutione, quando unica curva *AND* infinitas dat curvas *AMC*, colligere licet solutionem huius problematis, quo infinitae requiruntur curvae, [p. 170] super quibus omnibus corpus ex *A* ad datum punctum pervenit. Quaelibet enim curva *AND* unam dabit curvam per datum punctum curvae *BSC* transeuntem hocque modo innumerabiles huiusmodi curvae obtinebuntur. Sed cum hoc modo solutio nimis esset difficilis et operosa, aliam genuinam magis afferre convenit. Modus autem, quo utemur, ita est comparatus, ut unam curvam iam nosse oporteat, ex qua innumerabiles deducere docebimus. Haec ergo curva, quae nota esse debet, ex alterutra traditarum methodo eliciatur, ut ex (350), ubi curva per quodvis punctum semicirculi transiens inveniri potest dato tempore describenda.

PROPOSITIO 39.

Problema.

358. *Sollicitetur corpus perpetuo deorsum vi uniformi g dataque sit curva AMC* (Fig. 45); *super qua corpus ex A ad punctum datum C pervenit; invenire omnes curvas AnC, super quibus corpus eodem tempore ad punctum C ex A descendit.*

Solutio.

Sumta verticali *AB* pro axe omnium curvarum sit curvae datae *AMC* abscissa *AP* = *t*, applicata *PM* = *u* sitque curva *ANC* una quaesitarum; capiatur in ea arcus *AN*, [p. 171] qui eodem tempore absolvatur quo arcus *AM*; iungantur puncta *M* et *N* recta *MN* et construatur curva *ALB* talis, ut applicata *PL* aequalis sit rectae *MN*. Haec ergo curva *ALB* occurret axi *AB* in punctis *A* et *B*; nam incidente puncto *M* in *A* punctum *N* quoque in *A* incidet et posito *M* in *C* punctum *N* quoque erit in *C*, quia arcus *AMC* et *ANC* eodem tempore percurri ponuntur. Intelligitur autem ex curva *ALB* inveniri posse curvam *ANC*; quare si infinitae huiusmodi curvae *ALB* concipiantur in *A* et *B* incidentes in *AB*, earum quaeque dabit curvam *ANC* hocque modo innumerabiles prodibunt curvae *ANC* quaesito satisfacientes. Sit nunc *PL* =

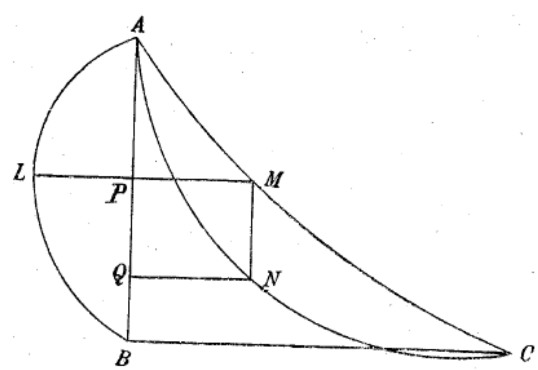

Fig. 45.

r; erit *r* functio quaedam ipsius *AP* = *t*; curvae autem *ANC* ponatur abscissa *AQ* = *x* et *QN* = *y*. His positis erit

$MN = \sqrt{((x-t)^2 + (u-y)^2)} = r$ ideoque $y = u \pm \sqrt{(r^2 - (x-t)^2)}$.

Porro, quia tempus per *AM* aequale ponitur tempori per *AN*, erit

$$\int \frac{\sqrt{(dt^2 + du^2)}}{\sqrt{gt}} = \int \frac{\sqrt{(dx^2 + dy^2)}}{\sqrt{gx}}$$

seu

$$xdt^2 + xdu^2 = tdx^2 + tdy^2 .$$

Est autem

$$dy = du \pm \frac{(rdr - xdx + tdx + xdt - tdt)}{\sqrt{(r^2 - (x-t)^2)}} .$$

Sit $du = pdt$ et $dr = qdt;$ erunt *r, p* et *q* functiones ipsius *t*; ponatur porro brevitatis gratia $x - t = z$ seu $x = t + z;$ erit

$$dy = pdt \pm \frac{qrdt - zdz}{\sqrt{(r^2 - z^2)}}$$

et

$$dy^2 + dx^2$$
$$= p^2dt^2 + dt^2 + 2dtdz + dz^2 \pm \frac{2pqrdt^2 - 2pzdtdz}{\sqrt{(r^2 - z^2)}} + \frac{q^2r^2dt^2 - 2qrzdtdz + z^2dz^2}{r^2 - z^2} .$$

Hinc obtinetur ista aequatio

$$\frac{tr^2dz^2}{r^2 - z^2} = \frac{2tqrzdtdz - tq^2r^2dt^2}{r^2 - z^2} \pm \frac{2pztdtdz - 2pqtrdt^2}{\sqrt{(r^2 - z^2)}} - 2tdtdz + zdt^2 + zp^2dt^2,$$

ex qua *z* et *t* determinetur; habebitur aequatio inter *x* et *y*. [p. 172] Quo autem appareat, cuiusmodi functio ipsius *t* loco *r* debeat accipi, ut *r* evanescat tam posito *t* = 0 quam *t* = *AB* = *a*, sit *P* functio quaecunque ipsius *t* evanescens posito *t* = 0 et *Q* sit etiam talis functio evanescens posito *t* = 0; abeat vero *Q* in *A*, si fiat *t* = *a*; poterit ergo poni $r = P(A - Q).$ Hocque valore substituto quicquid loco P et Q substituatur, habebitur aequatio pro curvis quaesitis. Q.E.I.

Scholion 1.

359. Ex hac quidem aequatione maxime intricata parum concludi potest ad propositum, etiamsi haec methodus genuina esse videatur. Saepe autem aequatio inventa ad absurdum deducere debet, ut si curva data AMC fuerit linea brevissimi descensus, quo casu non dari potest alia curva, super qua descensus fiat eodem tempore. Ad nostrum ergo institutum conveniens videtur de lineis celerrimi descensus tractare eaque problemata resolvere, in quibus inter omnes curvas vel eiusdem longitudinis vel aliam proprietatem communem habentes ea quaeritur, quae minimo tempore absolvatur, atque etiam quemadmodum inter omnes lineas, super quibus descensus fit eodem tempore, ea sit invenienda, quae data quapiam proprietate sit praedita. Etiamsi enim difficillimum sit omnes lineas idem [p. 173] descensus tempus habentes exhibere, tamen ex iis quaelibet potest inveniri ex proprietate, quam prae reliquis omnibus possidet. Requiritur autem ad hanc rem pertractatandam methodus isoperimetricorum, quam ut passim expositam hic non explicabimus.

Scholion 2.

360. Huius autem problematis solutio per (348) sequenti modo habetur. Erat ibi

$$\frac{du}{u}=\frac{dr(1+qq)}{-r-sq\pm\sqrt{\left(\frac{gbr}{4}(1+qq)-(s-rq)^2\right)}}.$$

Sit punctum C (Fig. 43), ad quod omnes curvae convenire debent, ponaturque $AE = f$ et $EC = h$; si ergo fit $r = f$, fieri debet $s = h$. Ad hoc sit S functio quaecunque ipsius r, quae abeat in F posito $r = f$, quo facto ponatur $s=\frac{Sh}{F}$. Substituatur hic valor in superiore aequatione eaque ita integretur, ut posito $u = a$ fiat $r = f$. Deinde ex ea aequatione prodibit aequatio inter coordinas curvae quaesitae AMC, nempe $AP = x$ et $PM = y$, ex quo, quod est

$$x=\frac{ur}{a} \text{ et } y=\frac{us}{a}.$$

Atque arbitrarius valor ipsius S dabit infinitas curvas AMC puncta A et C iungentes et super quibus corpus descendens tempore dato = $\sqrt{b}$ perveniet ex A ad C. Sit autem $dS = Tdr$; erit $q=\frac{hT}{F}$ atque [p. 174]

$$\frac{du}{u}=\frac{dr(F^2+h^2T^2)}{-F^2r-h^2ST\pm\sqrt{\left(\frac{gF^2br}{4}(F^2+h^2T^2)-(FhS-FhTr)^2\right)}}$$

seu

$$\frac{du}{u}=\frac{dr(F^2+h^2T^2)}{-F^2r-h^2ST\pm F\sqrt{\left(\frac{gbr}{4}(F^2+h^2T^2)-(hS-hTr)^2\right)}}.$$

Quae aequatio ita integretur, ut posito $u = a$ fiat $r = f$; quo facto ponatur

$$x=\frac{ur}{a} \text{ et } y=\frac{hSu}{Fa}$$

atque habebitur aequatio inter x et y pro infinitis curvis AMC quaesito satisfacientibus.

PROPOSITIO 40.

Problema.

361. *Invenire legem generalem, secundum quam curva disposita esse debet, ut corpus super ea descendens citissime perveniat ad quodvis curvae punctum.*

Solutio.

Sit *AMC* (Fig. 45) curva huiusmodi, super qua corpus ex *A* ad *C* tempore breviore perveniat quam supe quavis alia curva per puncta *A* et *C* transeunte. Sumtis ergo in ea duobus quibusque punctis *M* et μ curva inter ea intercepta ita debet esse comparata, ut corpus in motu suo per *AMC* arcum breviore tempore absolvat quam quemvis alium, si esset interceptus. Sint nunc puncta *M* et μ proxima iuncta duobus elementis *Mm,* $m\mu$, et debebit tempus per $Mm\mu$ esse minimum seu per regulas methodi maximorum et minimorum aequale tempori per elementa proxima *Mn,* $n\mu$. [p. 175]Ducantur ad axem AP applicatae *MP, mp,* $\mu\pi$ sumtisque elementis *Pp,* $p\pi$ inter se aequalibus seu quoque *MG* = *mH* et *pm*, si opus est, ad *n* producta erit mn infinite parvum respectu elementorum *Mm* et $m\mu$. Debebit ergo esse

$$t.Mm + t.m\mu = t.Mn + t.n\mu.$$

Sit celeritas, quam corpus in *M* habet, debita altitudini *v*, qua ergo tam elementum *Mm* quam *Mn* percurret. Celeritas autem, quam in *m* habebit, debita sit altitudini $v + du$ et celeritas, quam in *n* habebit, debita sit altitudini $v + du + ddw$; illa autem celeritate percurret elementum $m\mu$, hac vero elementum $n\mu$. Hinc ergo habebitur ista aequatio

$$\frac{Mm}{\sqrt{v}} + \frac{m\mu}{\sqrt{(v+du)}} = \frac{Mn}{\sqrt{v}} + \frac{n\mu}{\sqrt{(v+du+ddw)}};$$

est vero

$$\frac{1}{\sqrt{(v+du+ddw)}} = \frac{1}{\sqrt{(v+du)}} - \frac{ddw}{2(v+du)\sqrt{(v+du)}},$$

unde ductis centris M et μ arculis *mg* et *nh* erit

$$\frac{ng}{\sqrt{v}} = \frac{mh}{\sqrt{(v+du)}} + \frac{n\mu \cdot ddw}{2(v+du)\sqrt{(v+du)}}.$$

Porro est

$$\frac{1}{\sqrt{(v+du)}} = \frac{1}{\sqrt{v}} - \frac{du}{2v\sqrt{v}} \quad \text{et} \quad \frac{1}{(v+du)\sqrt{(v+du)}} = \frac{1}{v\sqrt{v}} - \frac{3du}{2v^2\sqrt{v}}.$$

Quibus, neglectis negligendis, substitutis oritur

$$2v(mh - ng) = mh \cdot du - n\mu \cdot ddw = mh \cdot du - Mm \cdot ddw.$$

Est vero propter triangula simila *nmg*, *mMG* et *nmh*, μ *mH*, ut sequitur,

$$ng : mn = mG : mM \quad \text{seu} \quad ng = \frac{mG \cdot mn}{Mm}$$

et

$$mh : mn = \mu H : m\mu \quad \text{seu} \quad mh = \frac{\mu H \cdot mn}{m\mu}.$$

Quamobrem erit

$$2v\left(\frac{\mu H}{m\mu} - \frac{mG}{Mm}\right) = \frac{mG \cdot du}{Mm} - \frac{Mm \cdot ddw}{mn} = 2v \text{ diff. } \frac{mG}{Mm}.$$

Quae aequatio est homogenae et determinat naturam curvae *AMC brachystochronae* vocatae, [p. 176] super qua corpus tempore brevissimo ex *A* ad *C* pervenit. Q.E.I.

Corollarium 1.

362. Si ergo dicatur

$$MG = mH = dx, \quad mG = dy \quad \text{et} \quad Mm = \sqrt{(dx^2 + dy^2)} = ds,$$

erit

$$H\mu = dy + ddy \quad \text{et} \quad m\mu = ds + dds.$$

His substitutis habebitur

$$2vd.\frac{dy}{ds} = \frac{dydu}{ds} - \frac{dsddw}{mn}.$$

In qua si ex potentiis sollicitantibus determinatur *v*, *du* et *ddw*, habebitur aequatio pro curva brachystochrona. At semper *ddw* ita involvet *mn*, ut *mn* ex calculo excedat.

Corollarium 2.

363. Sit radius osculi curvae $Mm\mu = r$; isque in plagam aversam ab axe *AP* directus erit

$$r = \frac{ds^3}{dxddy};$$

at est

$$d \cdot \frac{dy}{ds} = \frac{dsddy - dydds}{ds^2} = \frac{dx^2 ddy}{ds^3}.$$

hinc prodibit ista aequatio

$$\frac{2vdx}{r} = \frac{dydu}{ds} - \frac{dsddw}{mn}.$$

Ubi notandum est esse $\frac{2v}{r}$ vim centrifugam, qua curva in *M* secundum normalem ad curvam premitur.

Corollarium 3.

364. Si ex potantiis sollicitanibus fluat

$$dv = Pdx + Qdy + Rds,$$

erit

$$du = Pdx + Qdy + Rds \quad \text{et} \quad ddw = Q \cdot mn + R \cdot ng,$$

quia puncto *m* in *n* translato crescit *dy* particula *mn* et *ds* particula *ng*. [p. 177] Quia autem est $ng = \frac{dy.mn}{ds}$, erit

$$\frac{ddw}{mn} = -Q - \frac{Rdy}{ds},$$

quibus substitutis habebitur ista aequatio

$$\frac{2v}{r} = \frac{Pdy - Qdx}{ds}.$$

Scholion 1.

365. Ex solutione intelligitur formulam inventam latissime patere atque ad potentias sollicitantes quascunque extendi, etiam resistentia non excepta. Quaecunque enim fuerint potentiae sollicitantes, determinari potest tam *du* quam *ddw*, qui valores substituti dabunt aequationem pro brachystochrona quaesita. Attamen haec tantum locum habent, si potentiarum directiones sint in eodem plano; curva enim inventa est in eodem plano sita. Nihilo tamen minus, si potentiae non fuerint in eodem plano, curva brachystochrona in dato plano ope formulae huius poterit inveniri. In quolibet enim plano dato peculiaris erit curva brachystochrona, quaecunque fuerint potentiae sollicitantes. Alia vero quaestio est, si quaeritur linea brachystochrona inter omnes omnino lineas data duo puncta iungentes, etiam non in uno plano sitas. Quoties vero potentiarum sollicitatium directiones in eodem plano sunt positae, dubium non est lineam brachystochronam in eodem positam esse plano. Nam si curvae non essent in eodem plano, potentiae oblique agerent et propterea corpus non tantum, quantum fieri potest, accelerarent. [p. 178] Ex hac igitur solutione tam linea absolute brachystochrona, si potentiarum sollicitantium directiones sunt in eodem plano, invenitur quam linea, quae in dato plano est brachystochrona, quaecunque fuerint potentiae sollicitantes.

Scholion 2.

366. Questionem hanc de linea brachystochrona seu celerrimi descensus primus produxit Cel. Ioh. Bernoulli atque plures eius solutiones extant tam in Act. Lips. quam Transactionibus Angl. et Comm. Acad. Paris. et alibi, ubi hoc problema tam in hypothesi potentiae sollicitantis deorsum directae quam pro viribus centripetis solution dederunt. Nemo autem problema fundamentale, quae hic dedimus, tam late patens praemisit, ut ad potentias quascunque et resistantiam etiam extendi posset. Sumserunt enim omnes $ddw = 0$, quod semper perperam fit, nisi directio potentiae sit *MG* vel *mH*. Et hanc ob rem Cel. Hermannus cespitavit, dum tali propositione ad brachystochronas in medio resistente inveniandas est usus in Comm. Acad. Petrop. A 1727; quasque correctas dedi in iisdem Comm. A. 1734 ex hoc ipso problemate.

[Ioh. Bernoulli, *Curvatura radii in diaphanis non uniformis solutionque problematis a se* in Actis 1696, p. 269, *propositi de invenienda linea brachystochrona*, Acta erud. 1697, p. 206; *Opera Omnia*, Tom. I, p. 187; Lettre de Mr. Jean Bernoulli a Mr. Basnage, *sur le problème de isopérimètres*, Histoire des Ouvrages des Sçavans, Paris 1697, p. 452; Opera Omnia, Tom. I, p. 194; *Remarques sur ce qu'on a donné jusqu'ici de solutions des problèmes isopérimètres*, Mém. de l'acad.d. sc. de Paris 1718, p. 100; *Opera omnia*, Tom. II, p. 225.
Iac. Hermann, *Theoria generalis motuum, qui nascuntur a potentiis quibusvis in corpora indesinenter agentibus*, Comment. acad. sc. Petrop. 2 (1727), 1729, p. 139.
L. Euleri Commentatio (E042) : *De linea celerrimi descensus in medio quocunque resistente*. Comment. acad. sc. Petrop. 2 (1734/5), 1740, p. 135. L. Euleri *Opera Omnia*, series I, vol. 25. Commentatio (E056) : *Curvarum maximi minimive proprietate gaudentium inventio nova et facilis*, Comment. acad. sc. Petrop. 2 (1736), 1741, p. 172. L. Euleri *Opera Omnia*, series I, vol. 25.
Relati per P.St.]