CHAPTER THREE

CONCERNING THE MOTION OF A POINT ON A GIVEN LINE IN A MEDIUM WITH RESISTANCE.

[p. 431]

PROPOSITIO 86.

Problem.

782. *With everything put in place as before, to find the case in which the two curves MA and AN* (Fig. 84) *constitute a single continuous curve, upon which the descent and the following ascent are completed in equal times.*

Solution.

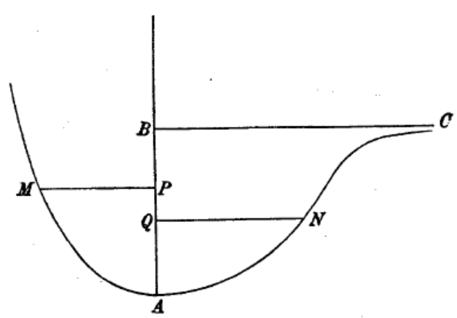

Fig. 84.

With the same denominations in place, as we have used in the previous proposition, clearly *AP* = *x*, *AM* = *s*, *AQ* = *t* and *AN* = *r*, besides the two equations found there :

$$2 = e^{\frac{r}{2k}} + e^{\frac{-s}{2k}} \text{ and } e^{\frac{-s}{k}}dx = e^{\frac{r}{k}}dt,$$

it has to be put into effect that both the equation between *s* and *x*, and between *r* and *t* are understood in terms of the same equation. According to this, we take a new variable *z*, from which the point *M* on the curve *AM* can be determined, thus in order that, if *z* becomes negative, then likewise the point *N* is obtained on the other curve. Hence on this account it is required that *s* be a function of *z* of this kind, in order that likewise, if – *z* is put in place of *z* , the arc *AN* is given, which on account of being made negative, is – *r*, thus so that *s* becomes – *r* on putting – *z* in place of *z*. [Thus, in modern notation, $s(z) = s$ and $s(-z) = -r$.] [So that this can be done,] there is put in place

$$e^{\frac{-s}{2k}} = 1 + Q;$$

then

$$e^{\frac{r}{2k}} = 1 - Q$$

since $2 = e^{\frac{r}{2k}} + e^{\frac{-s}{2k}}$, and thus *Q* is an odd function of *z*, since *Q* becomes negative on making *z* negative. [p. 432]

Hence the equations become :

$$e^{\frac{-s}{k}} = (1+Q)^2 \text{ and } e^{\frac{r}{k}} = (1-Q)^2 .$$

Again moreover it must be the case that :

$$dx(1+Q)^2 = dt(1-Q)^2$$

and x must be such a function of z, which changes into t on putting z negative.
[Thus $x(z) = x$ and $x(-z) = t$, while $Q(-z) = -Q$.]

Put $dx = Mdz$ and M changes to N on making z negative; hence $dt = -Ndz$. On account of which [note that $M = \frac{dx}{dz}$ etc., and is not related to the points M and N above on the diagram] :

$$M(1+Q)^2 = -N(1-Q)^2 .$$

Hence :

$$M = P(1-Q)^2$$

with P also made equal to an odd function of z; and then likewise

$$N = -P(1+Q)^2$$

and thus there is the same equality between $M(1+Q)^2$ and $-N(1-Q)^2$, as is required. Therefore for argument's sake on accepting odd functions of z in place of P and Q then

$$dx = Pdz(1-Q)^2 \text{ or } x = \int Pdz(1-Q)^2$$

and

$$s = 2kl\frac{1}{1+Q} .$$

Hence innumerable curves MA arise, of which the continued parts of the ascent AN produce isochrones with respect to the descents made on MA. Moreover two functions P and Q occur, in order that if $Q = -z$ then the other is determined; for then

$$s = 2kl\frac{1}{1-z} \text{ and } x = \int Pdz(1+z)^2 .$$

In the second equation of which the value of z from the first is substituted, which is $1-e^{\frac{-s}{2k}}$, and the equation between x and s for the curve sought is obtained. Q.E.I.

Corollary 1.

783. If $z = 0$, then also s becomes equal to 0. Whereby the integral of $Pdz(1+z)^2$ thus must be taken, so that it vanishes on putting $z = 0$. For with the arc s vanishing, the abscissa x also has to vanish.

Corollary 2. [p. 433]

784. Since it is the case that $s = 2kl\frac{1}{1-z}$, then

$$ds = \frac{2kdz}{1-z}$$

and since $dx = Pdz(1+z)^2$, then

$$\frac{ds}{dx} = \frac{2k}{P(1-z^2)(1+z)};$$

therefore unless $P(1-z^2)(1+z)$ is greater than 2*k*, the curve is real.

Corollary 3.

785. At the lowest point *A*, since *z* vanishes, then

$$\frac{ds}{dx} = \frac{2k}{P}.$$

Where *P* must be such an odd function of *z*, in order that, if *z* = 0, it must be less than 2*k*; moreover this cannot come about, unless *P* is such a function of *z* that vanishes on putting *z* = 0, and in this case the tangent at *A* is horizontal.

Example 1.

786. Since *P* must be an odd function of *z*, there is put in place $P = \frac{az}{(1-z^2)^2}$.

On putting this in place, we then have :

$$x = \int \frac{azdz}{(1-z)^2} \text{ or } dx = \frac{azdz}{(1-z)^2}.$$

Now

$$z = 1 - e^{\frac{-s}{2k}},\ dz = \frac{1}{2k} e^{\frac{-s}{2k}} ds, \text{ and } 1 - z = e^{\frac{-s}{2k}}.$$

With which substituted, there is obtained :

$$dx = \frac{ae^{\frac{-s}{2k}} ds\left(1 - e^{\frac{-s}{2k}}\right)}{2ke^{\frac{-s}{k}}} = \frac{ads}{2k}\left(e^{\frac{s}{2k}} - 1\right).$$

Which is the equation for the tautochrone curve found above, upon a part of this curve *MA* all the descents are completed in the same time, and moreover all the ascents are completed in the same time on the other part *AN*.

Example 2. [p. 434]

787. Let

$$P = \frac{6az - 2az^3}{(1-zz)^2};$$

then

$$x = \int \frac{6azdz - 2az^3dz}{(1-z)^2} = \frac{3az^2 + az^3}{1-z}.$$

Moreover since we have

$$z = 1 - e^{\frac{-s}{2k}} \text{ and } 1 - z = e^{\frac{-s}{2k}},$$

then

$$x = \frac{a\left(1 - e^{\frac{-s}{2k}}\right)^2\left(4 - e^{\frac{-s}{2k}}\right)}{e^{\frac{-s}{2k}}} = a\left(1 - e^{\frac{-s}{2k}}\right)^2\left(4e^{\frac{s}{2k}} - 1\right) = a\left(4e^{\frac{s}{2k}} - \left(3 - e^{\frac{-s}{2k}}\right)^2\right).$$

Which equation converted into a series gives :

$$\frac{4k^2x}{3a} = ss + \frac{s^3}{6k} - \frac{s^4}{48k^2} + \frac{s^5}{96k^3} - \frac{s^6}{640k^4} + \text{etc.} = bx$$

with the constant a changed into $\frac{4k^2}{3b}$.

PROPOSITION 87.

Problem.

788. *According to the hypothesis of uniform gravity acting downwards and for a uniform medium with resistance in the ratio of the square of the speeds, if some curve is given MA* (Fig. 87), *upon which the descending body completes the descent, to find a suitable curve AN joined to that for the ascent, such that all the semi-oscillations which are made on the curve MAN are completed in equal times.*

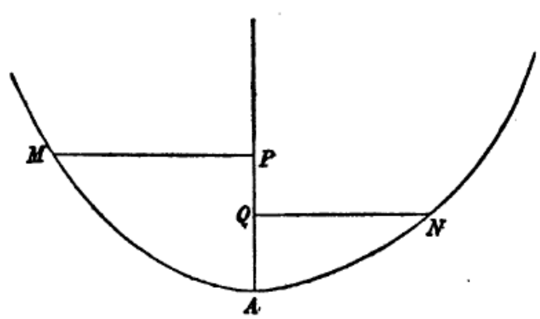

Fig. 87.

Solution.

With the force acting g put in place as up to this point, and with the exponent of the resistance k, let the abscissa of the given curve MA be $AP = x$, the arc $AM = s$, the abscissa of the sought curve $AQ = t$, and the arc $AN = r$. [p. 435] Now the descent begins at some point A of the curve MA and let the speed acquired at A correspond to the height b, with which [initial] speed the body completes the following ascent on the curve AN. With these in place the height of the descending body corresponding to the speed at M is equal to :

$$e^{\frac{s}{k}}\left(b - g\int e^{\frac{-s}{k}}dx\right)$$

and the height of the body corresponding to the speed at N is equal to :

$$e^{\frac{-r}{k}}\left(b-g\int e^{\frac{r}{k}}dt\right).$$

From these the time, in which the arcs *MA* and *AN* of the semi–oscillation are traversed, is equal to :

$$\int\frac{ds}{e^{\frac{s}{2k}}\sqrt{\left(b-g\int e^{\frac{-s}{2k}}dx\right)}}+\int\frac{dr}{e^{\frac{-r}{2k}}\sqrt{\left(b-g\int e^{\frac{r}{2k}}dt\right)}},$$

and this whole expression gives the time of the semi–oscillation, if after integration there is put in place :

$$g\int e^{\frac{-s}{k}}dx=b \text{ and } g\int e^{\frac{r}{k}}dt=b.$$

Therefore since this time must always have the same constant value, which does not depend on the letter *b*, from this condition the equation can be determined between *t* and *r* with the help of the given equation between *x* and *s*. For the sake of brevity we put :

$$g\int e^{\frac{-s}{k}}dx=X \text{ and } g\int e^{\frac{r}{k}}dt=T$$

and

$$\frac{ds}{e^{\frac{s}{2k}}}=dS \text{ and } \frac{dr}{e^{\frac{-r}{2k}}}=dR.$$

With these substituted,

$$\int\frac{dS}{\sqrt{(b-X)}}+\int\frac{dR}{\sqrt{(b-T)}}$$

must have a constant value, if on integrating there is put in place $X = b$ and $T = b$. Therefore generally $T = X$ is put in place, since *T* does not depend on *X* ; we have this expression for the time :

$$\int\frac{dS+dR}{\sqrt{(b-X)}},$$

which thus must be compared, so that after the integration on making $X = b$ the letter *b* clearly vanishes from the calculation. But this is done if [p. 436]

$$dS + dR = \frac{\alpha dX}{\sqrt{X}};$$

for the time of the semi–oscillation is equal to :

$$\int \frac{\alpha dX}{\sqrt{(bX - XX)}} = \pi\alpha$$

with π denoting the periphery of the circle, of which the diameter is equal to 1. Let

$$\alpha = \frac{\sqrt{2f}}{\sqrt{g}};$$

f denotes the length of the pendulum completing *in vacuo* and with gravity equal to *g*, the smallest semi–oscillations in the same time, in which these semi–oscillations are performed on the curves *MA* and *AN* (167). Therefore since

$$dS + dR = \frac{dX\sqrt{2f}}{\sqrt{gX}},$$

then

$$S + R = 2\sqrt{\frac{2fX}{g}} = 2\sqrt{2f}\int e^{\frac{-s}{k}}dx.$$

Now

$$S = 2k\left(1 - e^{\frac{-s}{2k}}\right) \text{ and } R = 2k\left(e^{\frac{r}{2k}} - 1\right),$$

hence

$$ke^{\frac{r}{2k}} - ke^{\frac{-s}{2k}} = \sqrt{2f}\int e^{\frac{-s}{k}}dx$$

or

$$e^{\frac{r}{2k}} = e^{\frac{-s}{2k}} + \frac{1}{k}\sqrt{2f}\int e^{\frac{-s}{k}}dx$$

and thus there arises :

$$r = 2kl\left(e^{\frac{-s}{2k}} + \frac{1}{k}\sqrt{2f}\int e^{\frac{-s}{k}}dx\right).$$

Now for this value of *r* there corresponds the value of *t* determined from this equation *T* = *X* or

$$e^{\frac{r}{k}}dt = e^{\frac{-s}{k}}dx.$$

From which there is found :

$$t = \int \frac{dx}{\left(1 + \frac{1}{k}e^{\frac{s}{2k}}\sqrt{2f}\int e^{\frac{-s}{k}}dx\right)^2},$$

from which the construction of the curve becomes known. But the equation for the curve sought *AN* is more conveniently found from the given equation between *x* and *s*, if in place of *s* there is substituted :

$$-2kl\left(e^{\frac{r}{2k}}-\frac{1}{k}\sqrt{2f}\int e^{\frac{r}{k}}dt\right)$$

and this value in place of x :

$$\int\frac{dt}{\left(1-\frac{1}{k}e^{\frac{-r}{2k}}\sqrt{2f}\int e^{\frac{r}{k}}dt\right)^2}.$$

For with these substituted the equation arises between r and t, which is sought for the curve *AN*. Q.E.I. [p. 437]

Corollary 1.

789. Since the smallest oscillations agree with the oscillations *in vacuo*, if the tangent to the curve *MA* at *A* is horizontal or the radius of osculation at *A* becomes infinitely small, the radius of osculation of the curve sought *AN* is 4*f* at *A*. [See the notes to E001 in this series of translations for an explanation of the appearance of 4*f* here.] For in this case the time of the shortest descent is equal to 0 and the time of the ascent is equal to $\frac{\pi\sqrt{2f}}{\sqrt{g}}$.

Corollary 2.

790. But if the radius of osculation of the curve *MA* at *A* radius is of finite magnitude, such as *h*, then the time of minimum descent is equal to $\frac{\pi\sqrt{2h}}{\sqrt{g}}$ (166). Therefore since the time of the semi–oscillation is equal to $\frac{\pi\sqrt{2f}}{\sqrt{g}}$, the radius of oscillation of the curve *AN* at *A* is equal to $(2\sqrt{f}-\sqrt{h})^2$; but this must $h < 4f$ or $f > \frac{1}{4}h$, lest the curve *AN* becomes imaginary.

Corollary 3.

791. Therefore the curves *MA* and *AN* have both a common horizontal tangent and radius of osculation at *A*, if $f = h$. For in this case the radius of osculation of the curve *AN* at *A* also becomes equal to *h*.

Scholium 1.

792. As here from the given curve of the descents we have determined the curve of the ascents, thus in a similar manner it is evident that it is possible to find the curve of the descents from the given curve of the ascents ; if indeed the equation is given between t and r, since it is [p. 438]

$$s=-2kl\left(e^{\frac{r}{2k}}-\frac{1}{k}\sqrt{2f}\int e^{\frac{r}{k}}dt\right)$$

and

$$x=\int\frac{dt}{\left(1-\frac{1}{k}e^{\frac{-r}{2k}}\sqrt{2f\int e^{\frac{r}{k}}dt}\right)^2},$$

with these values being substituted, the equation for the curve of the descents between *s* and *x* is obtained.

Corollary 4.

793. Since *f* is able to have innumerable values, as long as $f>\frac{1}{4}h$, to some given curve either of the descents or of the ascents innumerable curves of this kind can be adjoined, so that the semi–oscillations made upon these are all isochronous, generally possible to be used *in vacuo*.

Corollary 5.

794. Since in the solution we have put $T = X$, this relation is contained in an equation between some arc of the descent and the arc of the corresponding ascent. Thus, if the arc of the descent is *s*, then the arc of the ascent

$$r=2kl\left(e^{\frac{-s}{2k}}+\frac{1}{k}\sqrt{2f\int e^{\frac{-s}{k}}dx}\right).$$

Example 1.

795. Let the line of the descents be the given vertical right line *PA*, for which $s = x$. It is also the case that $ds = dx$ and

$$\int e^{\frac{-s}{k}}dx=k\left(1-e^{\frac{-s}{k}}\right).$$

Therefore [we have the ascending arc]

$$r=2kl\left(e^{\frac{-s}{2k}}+\sqrt{\frac{2f}{k}\left(1-e^{\frac{-s}{k}}\right)}\right)$$

or

$$e^{\frac{r}{2k}}=e^{\frac{-s}{2k}}+\sqrt{\frac{2f}{k}\left(1-e^{\frac{-s}{k}}\right)}.$$

Moreover again, [the abscissa] [p. 439]

$$t=\int\frac{ds}{\left(1+e^{\frac{s}{2k}}\sqrt{\frac{2f}{k}\left(1-e^{\frac{-s}{k}}\right)}\right)^2}\quad\text{or}\quad e^{\frac{r+s}{k}}dt=ds.$$

Therefore with *s* eliminated, this equation for the curve of the ascents is produced:

$$(2f+k)^2dt=k(2f-k)dr-\frac{2kf(2f+k)dr-4fk^2e^{\frac{r}{k}}dr}{e^{\frac{r}{2k}}\sqrt{\left(4ff+2fk-2fke^{\frac{r}{k}}\right)}},$$

in which the variables are separable from each other in turn, whereby that suffices for the construction of the curve. Now the integration of this equation depends on the quadrature of the circle. The equation for the vacuum is elicited from this equation on making $k = \infty$:

$$dt + dr = \frac{dr\sqrt{2f}}{\sqrt{(2f-r)}} \text{ or } t = 4f - r - 2\sqrt{2f(2f-r)},$$

which equation can be reduced to that, that we found in the preceding chapter[466].

Example 2.

796. Let the line of the descents be the given tautochrone of the descents found above [719], and the equation of this is

$$adx = kds\left(e^{\frac{s}{2k}} - 1\right).$$

Hence

$$\int e^{\frac{-s}{k}} dx = \frac{k}{a}\int ds\left(e^{\frac{-s}{2k}} - e^{\frac{-s}{k}}\right) = \frac{2k^2}{a}\left(\frac{1}{2} - e^{\frac{-s}{2k}} + \frac{1}{2}e^{\frac{-s}{k}}\right) = \frac{k^2}{a}\left(1 - e^{\frac{-s}{2k}}\right)^2$$

and

$$\sqrt{\int e^{\frac{-s}{k}} dx} = \frac{k\left(1 - e^{\frac{-s}{2k}}\right)}{\sqrt{a}}.$$

On account of which the equation becomes :

$$e^{\frac{r}{2k}} = e^{\frac{-s}{2k}} + \frac{\sqrt{2f} - e^{\frac{-s}{2k}}\sqrt{2f}}{\sqrt{a}} = \frac{\sqrt{2f} + e^{\frac{-s}{2k}}(\sqrt{a} - \sqrt{2f})}{\sqrt{a}}$$

and[p. 440]

$$e^{\frac{-s}{2k}} = \frac{e^{\frac{r}{2k}}\sqrt{a} - \sqrt{2f}}{\sqrt{a} - \sqrt{2f}} \text{ and } ds = \frac{e^{\frac{r}{2k}} dr\sqrt{a}}{\sqrt{2f} - e^{\frac{r}{2k}}\sqrt{a}}$$

and

$$adx = \frac{ake^{\frac{r}{2k}}dr\left(e^{\frac{r}{2k}} - 1\right)}{\left(e^{\frac{r}{2k}}\sqrt{a} - \sqrt{2f}\right)^2}.$$

Moreover again, this becomes

$$e^{\frac{-s}{k}} dx = e^{\frac{r}{k}} dt,$$

then

$$ae^{\frac{r}{k}}dt = \frac{ake^{\frac{r}{2k}}dr\left(e^{\frac{r}{2k}}-1\right)}{(-\sqrt{a}+\sqrt{2f})^2} \text{ or } adt = \frac{akdr\left(1-e^{\frac{-r}{2k}}\right)}{(-\sqrt{a}+\sqrt{2f})^2}$$

or

$$(-\sqrt{a}+\sqrt{2f})^2dt = kdr\left(1-e^{\frac{-r}{2k}}\right),$$

where $\sqrt{2f}$ must be greater than $\sqrt{a}$. But this equation found includes all the tautochrones of the ascents ; which indeed joined with any of the tautochrones of the descents, all the semi–oscillations on the curve composed from these must be isochrones. If we take $f = 2a$, then this equation arises :

$$adt = kdr\left(1-e^{\frac{-r}{2k}}\right),$$

which is for the tautochrone of the ascents, upon which all the ascents are completed in the same time as the descent on the given tautochrone of the descents, and that is by a continuation of the tautochrone of the descents.

Example 3.

797. Let the given line of the descents *MA* be the tautochrone of the ascents and there is sought, such curves when joined with produce isochronous semi–oscillations. Now the equation for this curve *MA* is

$$adx = kds\left(1-e^{\frac{-s}{2k}}\right).$$

Hence there is produced : [p. 441]

$$\int e^{\frac{-s}{k}}dx = \frac{k}{a}\int ds\left(e^{\frac{-s}{k}}-e^{\frac{-3s}{2k}}\right) = \frac{k^2}{a}\left(\frac{1}{3}-e^{\frac{-s}{k}}+\frac{2}{3}e^{\frac{-3s}{2k}}\right) = \frac{k^2}{3a}\left(1-3e^{\frac{-s}{k}}+2e^{\frac{-3s}{2k}}\right)$$
$$= \frac{k^2}{3a}\left(1+2e^{\frac{-s}{2k}}\right)\left(1-e^{\frac{-s}{2k}}\right)^2.$$

From these there arises :

$$e^{\frac{r}{2k}} = e^{\frac{-s}{2k}} + \left(1-e^{\frac{-s}{2k}}\right)\sqrt{\frac{2f}{3a}\left(1+2e^{\frac{-s}{2k}}\right)}$$

and

$$t = \frac{k}{a}\int \frac{ds\left(1-e^{\frac{-s}{2k}}\right)}{\left(1+\left(e^{\frac{s}{2k}}-1\right)\sqrt{\frac{2f}{3a}\left(1+2e^{\frac{-s}{2k}}\right)}\right)^2},$$

from which the construction of the curve follows.

Scholium 2.

798. Thus we have produced this example, in order that it is apparent, where it is required that any tautochrone of the ascents be joined with the curve, by which all the semi–oscillations are completed in equal times. Moreover from the formulas found it is apparent that the curve sought is not an isochrone of the descents; for the equation

$$cdt = kdr\left(e^{\frac{r}{2k}} - 1\right)$$

is not present in these formulas, which risk in the making is at once apparent. On account of which if *MA* is the tautochrone of the descents and *AN* of the ascents, even if all the departures along *MAN* are completed in equal times, yet the return or the following semi–oscillations along *NAM* are not isochrones. Therefore a pendulum, that is made to oscillate along the curves *MA* and *AN*, does not make isochronous oscillations, even if the other semi–oscillations in which the descent starts on the curve [p. 442] MA , are performed in equal times. Consequently this composite curve *MAN* is not suitable in being equally effective in the motion of pendulums in a resistive medium. Now the best remedy to this inconvenience is brought forwards, if the case can be determined, in which a curve *AN* similar and equal to the curve *MA* can be produced.
[This revelation is at once apparent from the nature of the resistance, which changes direction while the body returns to the starting point. Thus, carefully crafted isochrones in one direction along the curve do not work in the other direction.]

PROPOSITION 88.

Problem.

799. *If the curves MA and AN* (Fig. 87) *have that property, that all the semi–oscillations which begin on the curve MA, are between themselves isochrones in a medium that resists in the square ratio of the speeds, to determine the case in which these two curves MA and AN joined together constitute one continuous curve.*

Solution.

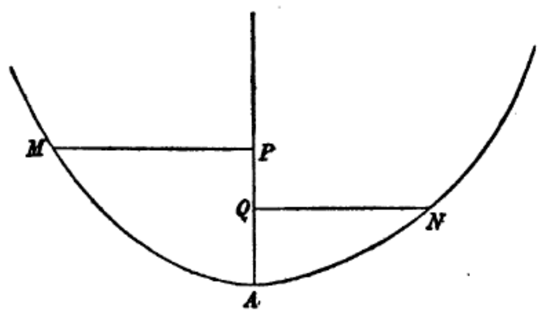

Fig. 87.

With the same denominations put in place as in the previous proposition, clearly $AP = x$, $AM = s$, $AQ = t$, $AN = r$, and f is equal to the length of the isochronous pendulum *in vacuo* and with gravity equal to g, there we found these two equations :

$$ke^{\frac{r}{2k}} - ke^{\frac{-s}{2k}} = \sqrt{2f}\int e^{\frac{-s}{k}}dx \text{ and } e^{\frac{r}{k}}dt = e^{\frac{-s}{k}}dx,$$

in which the relation between each curve is contained. Now since the curves *MA* and *NA* must be the roots of a continuous curve, the equation between x and s thus must be compared so that, if x changes to t, then s becomes equal to $-r$ on account of the negative in place. According to this we accept the new variable z, and both s and x are such functions of this, that on making z negative, x changes to t and s to $-r$. Let [p. 443]

$$\int e^{\frac{-s}{k}}dx = z^2;$$

then [above] we have

$$ke^{\frac{r}{2k}} - ke^{\frac{-s}{2k}} = z\sqrt{2f};$$

for on making z negative, in which case r is changed into $-s$ and $-s$ into r, then there is produced :

$$ke^{\frac{-s}{2k}} - ke^{\frac{r}{2k}} = -z\sqrt{2f};$$

which equation agrees with the former. Let P be some even function of z, which is not changed, even if $-z$ is put in place of z, and there is put in place :

$$ke^{\frac{-s}{2k}} = -\frac{1}{2}z\sqrt{2f} + P;$$

with which put in place the question is satisfied. And if we make *z* negative; then *–s* changes into *r* and there is obtained :

$$ke^{\frac{r}{2k}} = \frac{1}{2}z\sqrt{2f} + P \text{ and } ke^{\frac{r}{2k}} - ke^{\frac{-s}{2k}} = z\sqrt{2f},$$

as required. For the other equation

$$\int e^{\frac{r}{k}}dt = \int e^{\frac{-s}{k}}dx$$

is now satisfied by this

$$\int e^{\frac{-s}{k}}dx = z^2$$

; for on putting *z* negative and *dt* in place of *dx* and *r* in place of *–s* , there is produced :

$$\int e^{\frac{r}{k}}dt = z^2 = \int e^{\frac{-s}{k}}dx.$$

Therefore from the variable *z*, of which *P* is some even function, the curve sought *AM* of which the continuation is the other *AN*, is thus determined, in order that it becomes

$$e^{\frac{-s}{2k}} = -\frac{z\sqrt{2f}}{2k} + \frac{P}{k} \text{ and } dx = 2e^{\frac{s}{k}}zdz = \frac{8k^2zdz}{(2P - z\sqrt{2f})^2}.$$

Hence it follows that

$$x = 8k^2\int\frac{zdz}{(2P - z\sqrt{2f})^2} \text{ and } s = 2kl\frac{2k}{2P - z\sqrt{2f}}.$$

Let *z* be become

$$z = \frac{u\sqrt{2}}{\sqrt{f}}$$

which enables simpler formulas to be put into effect as well as producing more convenient homogeneous equations, as *u* must be of a single dimension; thus *P* also is an even function of *u* of one dimension. Whereby there is obtained :

$$s = 2kl\frac{k}{P - u} \text{ and } x = \frac{4k^2}{f}\int\frac{udu}{(P - u)^2}.$$

Q.E.I. [p. 444]

Corollary 1.

800. Hence an infinitude of tautochronous curves *MAN* can be found, if from the different values of *P* that can be put in place, all those substituted are even functions of *u*. Now the equation between *x* and *s* is obtained, if from the two equations found

$$s = 2kl\frac{k}{P-u} \text{ and } x = \frac{4k^2}{f}\int\frac{udu}{(P-u)^2}$$

the variable *u*, which also is not present in *P*, can be eliminated.

Corollary 2.

801. Since we have $s = 2kl\frac{k}{P-u}$, then

$$ds = \frac{2k(du-dP)}{P-u}$$

and

$$e^{\frac{s}{2k}} = \frac{k}{P-u} \text{ or } P-u = ke^{\frac{-s}{2k}}$$

and

$$e^{\frac{s}{2k}}ds = \frac{2k^2(du-dP)}{(P-u)^2}.$$

Since with which equation,

$$dx = \frac{4k^2udu}{f(P-u)^2}$$

if it can be combined with the other, there is produced

$$\frac{e^{\frac{s}{2k}}ds}{dx} = \frac{f(du-dP)}{2udu}.$$

Which equation is often the most convenient in the elimination of *u*.

Scholium 1.

802. Since with *s* vanishing, *x* also must vanish, the first to be investigated is that in which *u* itself vanishes for a given value of *s*. Then the integral

$$\int\frac{udu}{(P-u)^2}$$

thus must be taken, in order that it vanishes, if the same value is substituted in place of *u*. And this is to be observed since in the construction of the curve, which can be put in place with the help of the two equations found, then in putting together the equation between *x* and *s*, if indeed from the equation integrated : [p. 445]

$$x = \frac{4k^2}{f}\int\frac{udu}{(P-u)^2}$$

this can be deduced. Otherwise if in place of *u* and *P* some other multiple of these can be put to use, in place of the two equations found these can be used :

$$s = 2kl\frac{c}{P-u} \text{ and } x = \frac{4k^2}{f}\int\frac{udu}{(P-u)^2},$$

where the constant *c* is arbitrary and thus in this manner can be determined, as in the same case that *s* vanishes, in which *x* vanishes. But *s* vanishes if $u = 0$, since

$$\int e^{\frac{-s}{k}}dx = \frac{2u^2}{f}$$

and $\int e^{\frac{-s}{k}}dx$ vanishes on *s* vanishing ; whereby *c* must be equal to the value of *P*, if *u* is put equal to 0 in that. Therefore in the same case *x* must vanish, from which the constant in the integration of the value of *x* is determined. Or even in place of *P* such an even function of *u* must be accepted, which becomes equal to *c*, if *u* is put equal to 0.

Corollary 3.

803. Since the length of the isochronous pendulum in vacuo and with gravity equal to *g* is equal to *f* and the smallest oscillations in a medium with resistance are in agreement with the oscillations in vacuo, the radius of osculation of the curve at $A = f$, if indeed the tangent to the curve at *A* is horizontal.

Example 1.

804. Because *P* must be an even function of *u*, let *P* be the constant $c = k$, from which on putting $u = 0$, making $s = 0$. Hence it follows that

$$k - u = ke^{\frac{-s}{2k}} \text{ or } u = k\left(1 - e^{\frac{-s}{2k}}\right).$$

And on account of $dP = 0$ there is obtained [p. 446]

$$\frac{e^{\frac{s}{2k}}ds}{dx} = \frac{f}{2u} \text{ or } u = \frac{fe^{\frac{-s}{2k}}dx}{2ds}.$$

From which equations there is thus put in place :

$$\frac{fdx}{2} = kds\left(e^{\frac{s}{2k}} - 1\right).$$

Which equation is for the tautochrone of the descents ; which made continuous beyond *A* gives the curve of the ascents ; and all the successive semi–oscillations upon this curve, provided they start from *MA*, are isochronous.

Example 2.

805. Let

$$P = k + \frac{u^2}{a};$$

then *P* keeps the same value on making *u* negative. With this in place, then

$$k - u + \frac{u^2}{a} = ke^{\frac{-s}{2k}}$$

and on account of $dP = \frac{2udu}{a}$ then

$$\frac{e^{\frac{s}{2k}}ds}{dx} = \frac{f(a-2u)}{2au};$$

from which equation there is produced :

$$u = \frac{fadx}{2\left(ae^{\frac{s}{2k}}ds + fdx\right)}.$$

Which value of *u* substituted into the other equation gives :

$$2fa^2e^{\frac{s}{k}}dxds = 4k\left(e^{\frac{s}{2k}} - 1\right)\left(ae^{\frac{s}{2k}}ds + fdx\right)^2 - f^2ae^{\frac{s}{2k}}dx^2$$

and with the root extracted :

$$\frac{ae^{\frac{s}{2k}}ds}{fdx} = \frac{ae^{\frac{s}{2k}}}{4k\left(e^{\frac{s}{2k}}-1\right)} - 1 \pm \sqrt{\frac{ae^{\frac{s}{2k}}}{4k\left(e^{\frac{s}{2k}}-1\right)}\left(\frac{ae^{\frac{s}{2k}}}{4k\left(e^{\frac{s}{2k}}-1\right)} - 1\right)}.$$

In the special case, if $a = 4k$, this equation becomes :

$$\frac{4ke^{\frac{s}{2k}}ds}{fdx} = \frac{1 \pm e^{\frac{s}{4k}}}{e^{\frac{s}{2k}} - 1},$$

in which two equations are contained, of which the one is :

$$fdx = 4ke^{\frac{s}{2k}}ds\left(e^{\frac{s}{4k}} - 1\right)$$

and the other : [p. 447]

$$-fdx = 4ke^{\frac{s}{2k}}ds\left(e^{\frac{s}{4k}} + 1\right).$$

Moreover the latter of these, since *dx* does not vanish on putting *s* = 0 and on account of negative *dx* is useless. Now the first equation gives :

$$fx = 16k^2\left(\frac{e^{\frac{3s}{4k}}}{3} - \frac{e^{\frac{s}{2k}}}{2} + \frac{1}{6}\right)$$

or

$$3fx = 8k^2\left(2e^{\frac{3s}{4k}} - 3e^{\frac{s}{2k}} + 1\right).$$

Which made continuous beyond *A* is expressed by this equation :

$$3ft = 8k^2\left(2e^{\frac{-3r}{4k}} - 3e^{\frac{-r}{2k}} + 1\right).$$

Now through a series this [first] equation is obtained :

$$fx = \frac{s^2}{2} + \frac{5s^3}{24k} + \frac{19s^4}{384k^2} + \text{etc.}$$

and for the other part *AN* this series :

$$ft = \frac{r^2}{2} - \frac{5r^3}{24k} + \frac{19r^4}{384k^2} - \text{etc.}$$

Corollary 4.

806. Since

$$z^2 = \frac{2u^2}{f} = \int e^{\frac{-s}{k}}dx,$$

then

$$u = \sqrt{\frac{1}{2}f\int e^{\frac{-s}{k}}dx}.$$

Now on eliminating *u* there is obtained :

$$P = ke^{\frac{-s}{2k}} + \sqrt{\frac{1}{2}f\int e^{\frac{-s}{k}}dx}.$$

Whereby if that value of *u* is substituted into this equation, then the equation between *s* and *x* is produced at once.

Example 3.

807. We may put in place

$$P = \sqrt{(k^2 + u^2)} \text{ or } P^2 = k^2 + u^2;$$

substituted in place of P and u^2 with the values given above, then

$$k^2 e^{\frac{-s}{k}} + k e^{\frac{-s}{2k}} \sqrt{2f} \int e^{\frac{-s}{k}} dx = k^2 \text{ or } \sqrt{2f} \int e^{\frac{-s}{k}} dx = k\left(e^{\frac{s}{2k}} - e^{\frac{-s}{2k}}\right).$$

Hence with the square taken, there arises :

$$2f \int e^{\frac{-s}{k}} dx = k^2 \left(e^{\frac{s}{2k}} - e^{\frac{-s}{2k}}\right)^2 = k^2 \left(e^{\frac{s}{k}} + e^{\frac{-s}{k}} - 2\right).$$

Now this equation differentiated gives this : [p. 448]

$$2f e^{\frac{-s}{k}} dx = k ds \left(e^{\frac{s}{k}} - e^{\frac{-s}{k}}\right) \text{ or } 2f dx = k ds \left(e^{\frac{2s}{k}} - 1\right),$$

the integral of which is :

$$2fx = \frac{k^2 e^{\frac{2s}{k}}}{2} - ks - \frac{k^2}{2}.$$

Which equation converted into a series gives :

$$fx = \frac{s^2}{2} + \frac{s^3}{3k} + \frac{s^4}{6k^2} + \frac{s^5}{15k^3} + \frac{s^6}{45k^4} + \text{etc.}$$

Scholium 2.

808. Which tautochronous curves we have found in these examples for a medium that resists in the ratio of the square of the speeds, these have thus been composed so that the arcs *MA* and *AN* are dissimilar. Therefore since all the descents must begin on the curve *MA*, the following semi–oscillations, which begin on the curve *NA*, are not tautochrones, and because of that these curves cannot be adapted for oscillatory motion. But a remedy for this inconvenience is produced, if of the curves of this kind, even similar and equal curves *MA* and *NA* are found ; for in this case descents are likewise able to be made on each curve. Also there is no doubt that such a case exists, and the discovery of this, since the two curves perhaps are not continued, pertain rather to the preceding proposition. Clearly the curve of the descents must be investigated, to which there corresponds a similar and equal curve of the ascents ; now this investigation is thus difficult on account of deficiencies in the analysis, as I doubt

that the goal can be reached without the development of some outstanding analysis.
[p. 449] Now this question is thus reduced so that the equation between *s* and *x* of this condition can be investigated, as, if in that equation is put

$$-2kl\left(e^{\frac{s}{2k}}-\frac{1}{k}\sqrt{2f}\int e^{\frac{s}{k}}dx\right)$$

in place of *s* and

$$\int\frac{dx}{\left(1-\frac{1}{k}e^{\frac{-s}{2k}}\sqrt{2f}\int e^{\frac{s}{k}}dx\right)^2}$$

in place of *x*, the same equation can be produced, which was had before [788]. Indeed this condition can be more easily effected in many ways; yet I cannot see how that condition can be satisfied. If the medium should be the rarest, then it is not difficult from what has been reported on, to find the case in which two curves MA et AN are similar and equal to each other. Indeed in the end by leading through this calculation I have found the equation

$$fdx=sds+\frac{s^3ds}{9k^2}\text{ or }fx=\frac{s^2}{2}+\frac{s^4}{36k^2};$$

which curve likewise on being continued beyond A has the branch AN similar and equal to the arc AM; whereby a pendulum oscillating on this curve completes single semi–oscillations in equal times. Moreover we have :

$$s^2=-9k^2+3k\sqrt{(9k^2+4fx)}\text{ and }s=\sqrt{\left(-9k^2+3k\sqrt{(9k^2+4fx)}\right)}.$$

Because now *k* is a very large quantity, then

$$s=\sqrt{2fx}-\frac{fx\sqrt{2fx}}{18k^2}\text{ and }ds=\frac{fdx}{\sqrt{2fx}}-\frac{fdx\sqrt{2fx}}{12k^2}.$$

Hence the equation becomes :

$$dy=dx\sqrt{\frac{f-2x}{2x}}-\frac{f^2dx}{12k^2}\sqrt{\frac{2x}{f-2x}}.$$

On putting $f = 2a$; then

$$y=\int dx\sqrt{\frac{a-x}{x}}-\int\frac{a^2dx}{3k^2}\sqrt{\frac{x}{a-x}}=\int\frac{adx-xdx}{\sqrt{(ax-xx)}}-\int\frac{a^2xdx}{3k^2\sqrt{(ax-xx)}}$$
$$=\left(1+\frac{a^2}{3k^2}\right)\sqrt{(ax-xx)}+\left(1-\frac{a^2}{3k^2}\right)\int\frac{\frac{1}{2}adx}{\sqrt{(ax-xx)}}.$$

Which curve can hence be described in almost the same manner in which the cycloid is described with the help of the rectification of the circle. [p. 450]

Corollary 5.

809. If there is taken $a = k\sqrt{3}$ or $f = 2k\sqrt{3}$, this curve changes into an ellipse, the horizontal axis of which is twice as great as the vertical axis, which is equal to $2k\sqrt{3}$. Hence it can happen, that an ellipse can be the tautochrone in the rarest medium and is more satisfactory than the cycloid.

Scholium 3.

810. Moreover the construction of the tautochrone curve in the rarest medium in the preceding scholium has been given as follows: On the vertical straight line $AB = a = \frac{1}{2} f$ (Fig. 88) the semicircle *AOB* is described, and from this on the base BD the cycloid AFD is described; with the same inverted in place AGD is described. With which accomplished the curve sought AMC is constructed by taking everywhere for the applied lines of this :

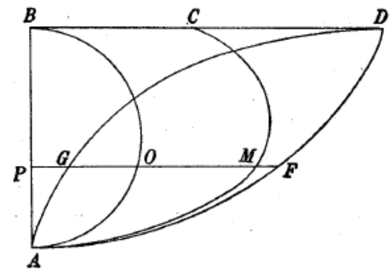

Fig. 88.

$$PM = PF - \frac{a^2}{3k^2} PG;$$

from which ratio endless points of the curve become known. Or it is possible to take

$$PM = \left(1 + \frac{a^2}{3k^2}\right) PO + \left(1 - \frac{a^2}{3k^2}\right) AO,$$

thus so that there is no need for the cycloid. Moreover this curve has a vertical tangent somewhere or the applied line *PM* is maximum, which is found on putting $dy = 0$. Moreover there is produced :

$$\frac{a-x}{x} = \frac{a^2}{3k^2} \text{ or } x = \frac{3ak^2}{a^2 + 3k^2};$$

if *AP* is taken equal to this value, then the maximum applied line is found.

PROPOSITION 89.

Problem. [p. 451]

811. *According to the hypothesis of gravity acting uniformly downwards g with some given curve am* (Fig. 89) *for the descents in vacuo to find the curve AM for the descents in a medium of this innate character with uniform resistance in the ratio of the square of the speeds, in order that all the descents on MA are isochrones with respect to all the descents on ma, if the speeds at the bottom points a and A are equal.*

Solution.

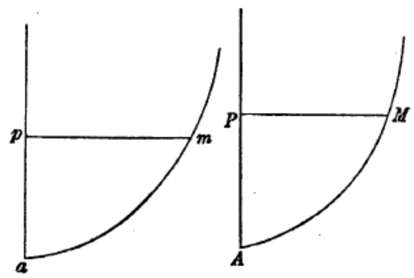

Fig. 89.

Let the abscissa for the curve *am* of the descents *in vacuo* abscissa be $ap = t$, the arc $am = r$; now the exponent of the resistance is put equal to k ; now for the curve of the descents in the medium with resistance let $AP = x$ and $AM = s$; Now two descents are considered on these curves, in which the speeds acquired at A and a are equal and correspond to the height b. Hence the time of the descent *in vacuo* is equal to

$$\int \frac{dr}{\sqrt{(b-gt)}},$$

and after integration there is put in place $gt = b$. But for the time of the descent on the curve *MA* in the medium with resistance there is obtained :

$$\int \frac{ds}{e^{\frac{s}{2k}}\sqrt{\left(b-g\int e^{\frac{-s}{k}}dx\right)}},$$

if likewise after the integration there is put in place :

$$g\int e^{\frac{-s}{k}}dx = b.$$

On account of which [on comparing the integrals] these times are equal, if

$$\frac{ds}{e^{\frac{s}{2k}}} = dr \text{ and } \int e^{\frac{-s}{k}}dx = t;$$

for with these put in place for each time this expression is obtained :

$$\int \frac{dr}{\sqrt{(b-gt)}}.$$

Therefore since we have $\frac{ds}{e^{\frac{s}{2k}}} = dr$, then on integration it becomes :

$$2k\left(1 - e^{\frac{-s}{2k}}\right) = r \text{ and } e^{\frac{-s}{2k}} = \frac{2k - r}{2k};$$

thus there is produced [p. 452]

$$s = 2kl\frac{2k}{2k - r}.$$

Now the other equation $\int e^{\frac{-s}{k}}dx = t$ gives

$$e^{\frac{-s}{k}}dx = dt .$$

Moreover then we have

$$e^{\frac{-s}{k}} = \frac{(2k - r)^2}{4k^2},$$

in which with the value substituted, there is found

$$dx = \frac{4k^2dt}{(2k - r)^2},$$

from which there arises :

$$x = \int \frac{4k^2dt}{(2k - r)^2}.$$

Hence from the given equation between *t* and *r* for the curve *am* with the help of these two equations, from which *s* and *x* are determined by *t* and *r*, the curve sought *AM* can be constructed. Now the equation between *x* et *s* can be conveniently found from the given equation between *t* and *r*, if in that in place of *r* there is substituted $2k(1 - e^{\frac{-s}{2k}})$ and $\int e^{\frac{-s}{k}}dx$ in place of *t*.

Q.E.I.

Corollary 1.

812. Around the lowest point *a*, where *t* and *r* are vanishing quantities, there becomes

$$s = r + \frac{r^2}{4k} \text{ and } x = t + \int \frac{rdt}{k}$$

or

$$ds = dr + \frac{rdr}{2k} \text{ and } dx = dt + \frac{rdt}{k}.$$

Whereby the inclination of to the curve *MA* to the axis at *A* is equal to the inclination of the curve *ma* at *a*.

Corollary 2.

813. Again the radius of osculation at the lowest point *a*, if the tangent is horizontal, is equal to $\frac{rdr}{dt}$ and at *A*, since the tangent also is horizontal, is equal to

$$\frac{sds}{dx} = \frac{rdr + \frac{3r^2dr}{4k}}{dt + \frac{rdt}{k}}.$$

Hence it follows that

$$\frac{sds}{dx} = \frac{rdr}{dt} - \frac{r^2dr}{4kdt} = \frac{rdr}{dt}\left(1 - \frac{r}{4k}\right).$$

Whereby as *r* is infinitely small, then

$$\frac{sds}{dx} = \frac{rdr}{dt}.$$

Corollary 3.

814. If therefore the curve *ma* has a horizontal tangent at *a*, then the tangent of the curve *MA* at *A* and the radius of osculation at *A* is equal to the radius of osculation at *a*. [p. 453]

Corollary 4.

815. Therefore if the curve found *ma in vacuo*, in which the times of the descents have some relation to the speeds acquired at *a*, likewise the problems for the resisting medium can be solved for the curve *MA*, which by the prescribed reason can be constructed from the curve *ma*.

Corollary 5.

816. Therefore if the curve *ma* should be a cycloid or tautochrone *in vacuo*, then *AM* is the tautochrone of the descents in the resisting medium found above. For on putting $r^2 = 2at$ or $rdr = adt$ this equation is produced on substituting the values found in place of *r* and *t* :

$$2ke^{\frac{-s}{2k}}ds\left(1 - e^{\frac{-s}{2k}}\right) = ae^{\frac{-s}{k}}dx \text{ or } adx = 2kds\left(e^{\frac{s}{2k}} - 1\right).$$

Example.

817. Let *am* be some right line inclined in some manner, thus in order that $r = nt$; then the time of the descent, in which a speed is generated corresponding to the height *b*, is equal to

$$\int \frac{ndt}{\sqrt{(b - gt)}} = \frac{2n\sqrt{b}}{g}.$$

Therefore the same property is in place for the curve *MA*, so that the time of each descent in the resisting medium, in which the speed $\sqrt{b}$ is generated, is equal to $\frac{2n\sqrt{b}}{g}$ or in proportion to the speed arising. Moreover since $r = nt$, then $dr = ndt$; in which if the values found are substituted in place of dr and dt, then there is produced : [p. 454]

$$e^{\frac{-s}{2k}}ds = ne^{\frac{-s}{k}}dx \text{ or } ndx = e^{\frac{s}{2k}}ds;$$

which is the equation for the tractrix generated by the thread of length $2k$, such as is shown in Fig. 86, truly the curve *CA*, which has the same inclination at *A* as the right line *ma*.

Scholium 1.

818. To the extent that this curve *MA* has been found, on which all the descents in the resisting medium are completed in the same times as the descents *in vacuo* on the curve *ma*, if the final speeds at *A* and *a* should be equal, thus in the same manner the curve *MA* can be defined, upon which all the ascents in the resisting medium are completed in the same times in which likewise with the same starting speeds of ascent the motions are completed *in vacuo* on the curve *am*. For since for a resisting medium the descent and the ascent can be changed into each other on making k negative, if on putting $AP = x$ and $AM = s$, there is obtained :

$$x = \int \frac{4k^2 dt}{(2k+r)^2} \text{ and } s = 2k\, l\frac{2k+r}{2k}$$

or conversely,

$$t = \int e^{\frac{s}{k}}dx \text{ and } r = 2k\left(e^{\frac{s}{2k}} - 1\right).$$

From which then the curve *AM* can be constructed easily from the equation to be found for that.

Scholium 2.

819. In this problem we have determined the curve of the descent in the resisting medium from the curve of the descent *in vacuo*. Moreover it is readily apparent in turn from the given curve *AM* for the resisting medium the other *am* for the vacuum can be found. For since it is given by :

$$r = 2k\left(1 - e^{\frac{-s}{2k}}\right) \text{ and } t = \int e^{\frac{-s}{k}}dx,$$

the construction of the curve *am* with the help of the two equations is performed. Now the equation for the curve *am* between t and r is found more conveniently from the given equation between x and s, if there in place of x there is substituted $\frac{4k^2dt}{(2k-r)^2}$ and $2kl\frac{2k}{2k-r}$ in place of s. Because here besides what has been said about the descents, likewise is valid for the ascents, but only if k is put negative, as we have advised in scholium 1.

Scholium 3.

820. The devising of one curve from the other of two given curves *am* and *AM* also has a place and is treated here, if the equation of the given curve is not in place, but if it has been drawn by hand in some manner; for the construction can be deduced from the formulas, since it no longer depends on the equation. On this account in the preceding chapter (432), in the case we have considered for the vacuum, in which the curve *cm* (Fig. 90) that we have devised must be joined to the given *ac*, so that all the descents from any point of the curve *cm* as far as to *a* are completed in equal times, now similar examples to these for the resisting medium can be elicited, in which the line composed from the two different curves is a tautochrone. For if the curve *acm* is a tautochrone curve of this kind for the vacuum, from that by the solution of this problem a like curve can be found composed for the resisting medium. Clearly from *ac* by the method proposed the curve *AC* is defined; which is found on putting *bp* = *t*, *cm* = *r* and *BP* = *x* and also *CM* = *s* and besides *ab* = *a*, *ac* = *c* ; *AB* = *A*, and *AC* = *C*, for then, since the equation between *t* and *r* is given, then

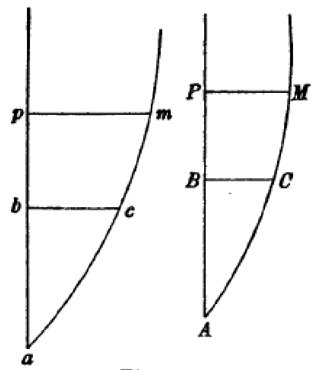

Fig. 90.

$$AP = A + x = \int \frac{4k^2 dt}{(2k - c - r)^2}$$

and

$$AM = C + s = 2kl\frac{2k}{2k - c - r}.$$

[p. 455] But if for the resisting medium the curve *AC* is given and there is required for the other part of this *CM* the property, in order that all the descents on *MCA* are completed in equal times, the solution can be effected in a not dissimilar manner. For from the given curve *AC* for the resisting medium there is found the curve with the same property for the vacuum *ac* by scholium 2. With which devised, there is sought the curve *cm* to be joined to that, which produces all the isochronous descents *in vacuo* (432). Then by the method treated in the manner according to the composite curve *acm* for the vacuum there is sought a like composite curve for the resisting medium *ACM*, of which indeed the part *AC* is now known ; clearly from that line *ac* we have defined. Hence likewise the problem, in which *in vacuo* there was some difficulty, in the resisting medium too can be resolved. Hence finally the chapter ends and I ask the benevolent Reader, that before progressing to the following chapter, it may be wished to repeat what has been presented in chapter I from § 58 to the end of the chapter.

CAPUT TERTIUM

DE MOTU PUNCTI SUPER DATA LINEA IN MEDIO RESISTENTE.

[p. 431]

PROPOSITIO 86.

Problema.

782. *Iisdem positis ut ante invenire casus, quibus duae curvae MA et AN* (Fig. 84), *super quibus descensus et sequentes ascensus aequalibus temporibus absolvuntur, unam curam continuam constituunt.*

Solutio.

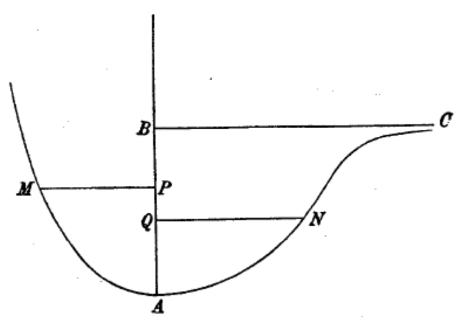

Fig. 84.

Manentibus iisdem denominationibus, quibus in praecendente propositione usi sumus, scilicet $AP = x$, $AM = s$, $AQ = t$ et $AN = r$, praeter duas aequationnes ibi inventas

$$2 = e^{\frac{r}{2k}} + e^{\frac{-s}{2k}} \text{ et } e^{\frac{-s}{k}} dx = e^{\frac{r}{k}} dt$$

effici debet, ut aequationes inter s et x et inter r et t sub eadem aequatione comprehendantur. Sumamus ad hoc novam variabilem z, ex qua punctum M in curva AM determinetur, ita ut , si z fiat negativum, eodem modo obtineatur punctum N in altera curva. Hanc ob rem s huiusmodi esse oportet functionem ipsius z, ut eadem, si loco z ponatur $-z$, det arcum AN, qui ob positionem negativam est $-r$, ita ut s abeat $-r$ posita $-z$ loco z. Ponatur

$$e^{\frac{-s}{2k}} = 1 + Q;$$

erit

$$e^{\frac{r}{2k}} = 1 - Q$$

propter $2 = e^{\frac{r}{2k}} + e^{\frac{-s}{2k}}$ ideoque Q erit functio impar ipsius z, quae in sui negativam abit facto z negativo. [p. 432]

Erit itaque

$$e^{\frac{-s}{k}} = (1+Q)^2 \text{ et } e^{\frac{r}{k}} = (1-Q)^2.$$

Porro autem esse debet

$$dx(1+Q)^2 = dt(1-Q)^2$$

atque x talis esse debet functio ipsius z, quae abeat in t posito z negativo. Ponatur $dx = Mdz$ abeatque M in N facto z negativo; erit ergo $dt = -Ndz$. Quamobrem fiet

$$M(1+Q)^2 = -N(1-Q)^2.$$

Sit ergo

$$M = P(1-Q)^2$$

existente P quoque = functioni impari ipsius z; atque tum fiet

$$N = -P(1+Q)^2$$

ideoque aequalia inter se erunt $M(1+Q)^2$ et $-N(1-Q)^2$, uti requiritur. Sumtis ergo pro lubitu loco P et Q functionibus imparibus ipsius z erit

$$dx = Pdz(1-Q)^2 \text{ seu } x = \int Pdz(1-Q)^2$$

atque

$$s = 2kl\frac{1}{1+Q}.$$

Unde innumerabiles oriuntur curvae MA, quarum partes continuae AN ascensus producunt isochronos respective descensibus super MA factis. Quia autem duae functiones occurrant P et Q, determinatur altera, ut sit $Q = -z$; erit

$$s = 2kl\frac{1}{1-z} \text{ atque } x = \int Pdz(1+z)^2.$$

In quarum posteriore aequatione valor ipsius z ex priore, qui est $1-e^{\frac{-s}{2k}}$, substituatur habebiturque aequatio inter x et s pro curva quaesita. Q.E.I.

Corollarium 1.

783. Si $z = 0$, fit quoque $s = 0$. Quare integrale ipsius $Pdz(1+z)^2$ ita accipi debet, ut evanescat posito $z = 0$. Nam evanescente arcu s abscissa quoque x evanescere debet.

Corollarium 2. [p. 433]

784. Cum sit $s = 2kl\frac{1}{1-z}$, erit

$$ds = \frac{2kdz}{1-z}$$

et quia est $dx = Pdz(1+z)^2$, erit

$$\frac{ds}{dx} = \frac{2k}{P(1-z^2)(1+z)};$$

nisi ergo $P(1-z^2)(1+z)$ maius fuerit quam $2k$, curva realis.

Corollarium 3.

785. In puncto infimo *A*, quia evanescit *z*, erit

$$\frac{ds}{dx}=\frac{2k}{P}.$$

Quare *P* talis esse debet functio impar ipsius z, ut ea, si $z = 0$, minor sit quam $2k$; hoc autem evenire non potest, nisi *P* talis fuerit functio ipsius *z*, quae evanescat posito $z = 0$, hocque casu tangens in *A* erit horizontalis.

Exemplum 1.

786. Quia *P* debet esse functio impar ipsius *z*, ponatur

$$P=\frac{az}{(1-z^2)^2}.$$

Quo posito erit

$$x=\int\frac{azdz}{(1-z)^2}\quad \text{seu}\quad dx=\frac{azdz}{(1-z)^2}.$$

Est vero

$$z=1-e^{\frac{-s}{2k}}\quad \text{et}\quad dz=\frac{1}{2k}e^{\frac{-s}{2k}}ds\quad \text{atque}\quad 1-z=e^{\frac{-s}{2k}}.$$

Quibus substitutis habebitur

$$dx=\frac{ae^{\frac{-s}{2k}}ds\left(1-e^{\frac{-s}{2k}}\right)}{2ke^{\frac{-s}{k}}}=\frac{ads}{2k}\left(e^{\frac{s}{2k}}-1\right).$$

Quae est aequatio pro curva tautochrona supra inventa, super cuius parte *MA* omnes descensus aequalibus absolvuntur temporibus, super parte autem altera *AN* omnes ascensus iisdem temporibus.

Exemplum 2. [p. 434]

787. Sit

$$P=\frac{6az-2az^3}{(1-zz)^2};$$

erit

$$x=\int\frac{6azdz-2az^3dz}{(1-z)^2}=\frac{3az^2+az^3}{1-z}.$$

Cum autem sit

$$z=1-e^{\frac{-s}{2k}}\quad \text{et}\quad 1-z=e^{\frac{-s}{2k}},$$

erit

$$x=\frac{a\left(1-e^{\frac{-s}{2k}}\right)^2\left(4-e^{\frac{-s}{2k}}\right)}{e^{\frac{-s}{2k}}}=a\left(1-e^{\frac{-s}{2k}}\right)^2\left(4e^{\frac{s}{2k}}-1\right)=a\left(4e^{\frac{s}{2k}}-\left(3-e^{\frac{-s}{2k}}\right)^2\right).$$

Quae aequatio in seriem conversa dat

$$\frac{4k^2x}{3a} = ss + \frac{s^3}{6k} - \frac{s^4}{48k^2} + \frac{s^5}{96k^3} - \frac{s^6}{640k^4} + \text{etc.} = bx$$

mutata constante a in $\frac{4k^2}{3b}$.

PROPOSITIO 87.

Problema.

788. *In hypothesi gravitatis uniformis deorsum tendentis et medio uniformi in duplicata celeritatum ratione resistente si detur curva quaecunque MA* (Fig. 87), *super qua corpus descensus absolvat, invenire curvam AN ei iungendam ad ascensus idoneam, ita ut omnes semioscillationes, quae super curva MAN fiunt, aequalibus absolvantur temporibus.*

Solutio.

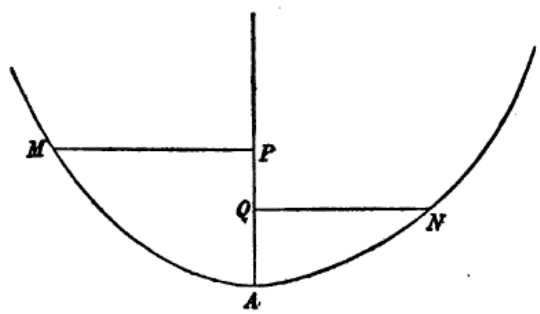

Fig. 87.

Positis ut hactenus potentia sollicitante g et exponente resistantiae k sit curvae datae MA abscissa $AP = x$, arcus $AM = s$, curvae vero quaesitae abscissa $AQ = t$, arcus $AN = r$. [p. 435] Incipiat nunc descensus in quocunque curvae MA puncto sitque celeritas in A acquisita debita altitudini b, qua celeritate corpus sequentem ascensum in curva AN absolvet. His positis erit altitudo celeritati corporis descendentis in M debita =

$$e^{\frac{s}{k}}\left(b - g\int e^{\frac{-s}{k}}dx\right)$$

et altitudo celeritati corporis ascendentis in N debita =

$$e^{\frac{-r}{k}}\left(b - g\int e^{\frac{r}{k}}dt\right).$$

Ex his erit tempus, quo in hac semioscillatione arcus MA et AN percurruntur, =

$$\int \frac{ds}{e^{\frac{s}{2k}}\sqrt{\left(b - g\int e^{\frac{-s}{2k}}dx\right)}} + \int \frac{dr}{e^{\frac{-r}{2k}}\sqrt{\left(b - g\int e^{\frac{r}{2k}}dt\right)}},$$

quae expressio integrum dabit semioscillationis tempus, si post integrationem ponatur

$$g\int e^{\frac{-s}{k}}dx = b \quad \text{atque} \quad g\int e^{\frac{r}{k}}dt = b.$$

Cum igitur hoc tempus debeat semper habere valorem constantem, qui non a quantitate litterae *b* pendeat, ex hac conditione determinari debebit aequatio inter *t* et *r* ope datae aequationis inter *x* et *s*. Ponamus brevitatis gratia

$$g\int e^{\frac{-s}{k}}dx = X \quad \text{et} \quad g\int e^{\frac{r}{k}}dt = T$$

atque

$$\frac{ds}{e^{\frac{s}{2k}}} = dS \quad \text{et} \quad \frac{dr}{e^{\frac{-r}{2k}}} = dR.$$

Quibus substitutis habere debebit

$$\int\frac{dS}{\sqrt{(b-X)}} + \int\frac{dR}{\sqrt{(b-T)}}$$

valorem constantem, si post integrationem ponatur $X = b$ et $T = b$. Fiat ergo generaliter $T = X$, quia T ab X non pendet; habebimus pro tempore hanc expressionem

$$\int\frac{dS+dR}{\sqrt{(b-X)}},$$

quae ita debet esse comparata, ut post integrationem facto $X = b$ littera *b* prorsus ex calculo evanescat. Hoc autem fiet, si fuerit [p. 436]

$$dS + dR = \frac{\alpha dX}{\sqrt{X}};$$

erit enim tempus semioscillationis =

$$\int\frac{\alpha dX}{\sqrt{(bX - XX)}} = \pi\alpha$$

denotante π peripheriam circuli, cuius diameter = 1. Sit

$$\alpha = \frac{\sqrt{2f}}{\sqrt{g}};$$

denotabit *f* longitudinem penduli in vacuo et gravitate = *g* semioscillationes minimas eodem tempore absolventis, quo hae semioscillationes super curvis *MA* et *AN* perguntur (167). Cum igitur sit

$$dS + dR = \frac{dX\sqrt{2f}}{\sqrt{gX}},$$

erit

$$S + R = 2\sqrt{\frac{2fX}{g}} = 2\sqrt{2f}\int e^{\frac{-s}{k}}dx.$$

Est vero

$$S = 2k\left(1 - e^{\frac{-s}{2k}}\right) \quad \text{et} \quad R = 2k\left(e^{\frac{r}{2k}} - 1\right),$$

unde erit

$$ke^{\frac{r}{2k}} - ke^{\frac{-s}{2k}} = \sqrt{2f}\int e^{\frac{-s}{k}}dx$$

seu

$$e^{\frac{r}{2k}} = e^{\frac{-s}{2k}} + \frac{1}{k}\sqrt{2f}\int e^{\frac{-s}{k}}dx$$

ideoque fiet

$$r = 2kl\left(e^{\frac{-s}{2k}} + \frac{1}{k}\sqrt{2f}\int e^{\frac{-s}{k}}dx\right).$$

Huic vero valori ipsius *r* respondens valor ipsius *t* ex hac aequatione determinabitur $T = X$ seu

$$e^{\frac{r}{k}}dt = e^{\frac{-s}{k}}dx.$$

Ex quo invenitur

$$t = \int \frac{dx}{\left(1 + \frac{1}{k}e^{\frac{s}{2k}}\sqrt{2f}\int e^{\frac{-s}{k}}dx\right)^2},$$

ex quibus constructio curvae innotescit. Aequatio autem pro curva quaesita *AN* commodius ex data aequatione inter *x* et *s* obtinebitur, si loco *s* substituatur

$$-2kl\left(e^{\frac{r}{2k}} - \frac{1}{k}\sqrt{2f}\int e^{\frac{r}{k}}dt\right)$$

et loco *x* hic valor

$$\int \frac{dt}{\left(1 - \frac{1}{k}e^{\frac{-r}{2k}}\sqrt{2f}\int e^{\frac{r}{k}}dt\right)^2}.$$

His enim substitutis orietur haec aequatio inter *r* et *t*, quae est pro curva quaesita *AN*. Q.E.I. [p. 437]

Corollarium 1.

789. Cum oscillationes minimae congruant cum oscillationibus in vacuo, si curvae *MA* tangens in *A* fuerit horizontalis vel radius osculi in *A* fuerit infinite parvus, curvae quaesita *AN* radius osculi in *A* erit 4*f*. Erit enim hoc casu tempus descensus minimi = 0 et tempus ascensus $= \frac{\pi\sqrt{2f}}{\sqrt{g}}$.

Corollarium 2.

790. Sin autem curvae *MA* in *A* radius osculi fuerit finitae magnitudinis, scilicet *h*, erit tempus descensus minimi $=\frac{\pi\sqrt{2h}}{\sqrt{g}}$ (166). Quo igitur tempus semioscillationis sit = $\frac{\pi\sqrt{2f}}{\sqrt{g}}$, erit radius osculi curvae *AN* in *A* = $(2\sqrt{f}-\sqrt{h})^2$; debet autem esse $h < 4f$ seu $f > \frac{1}{4}h$, ne curva *AN* fiat imaginaria.

Corollarium 3.

791. Curvae ergo *MA* et *AN* in *A* et tangentem horizontalem et radium osculi commumem habebunt, si fuerit $f = h$. Hoc enim casu curvae *AN* radius osculi in *A* fiet quoque = *h*.

Scholion 1.

792. Quemadmodum hic ex data curva descensuum curvam ascensuum determinavimus, ita perspicitur simili modo ex curva ascensuum data curvam descensuum inveniri posse; si enim detur aequatio inter *t* et *r*, quia est [p. 438]

$$s=-2kl\left(e^{\frac{r}{2k}}-\frac{1}{k}\sqrt{2f}\int e^{\frac{r}{k}}dt\right)$$

atque

$$x=\int\frac{dt}{\left(1-\frac{1}{k}e^{\frac{-r}{2k}}\sqrt{2f}\int e^{\frac{r}{k}}dt\right)^2},$$

his valoribus substituendis aequatio pro curva descensuum inter *s* et *x* obtinebitur.

Corollarium 4.

793. Cum *f* innumerabiles habere possit valores, modo sit $f > \frac{1}{4}h$, ad quamvis curvam sive descensuum sive ascensuum datam innumerae adiungi possunt curvae eiusmodi, ut semioscillationes super iis factae sint omnes isochronae, omnino uti in vacuo fieri potest.

Corollarium 5.

794. Quia in solutione posuimus *T* = *X*, hac aequatione relatio continetur inter quemque arcum descensus integrum et arcum respondentis ascensus. Ita, si arcus descensus fuerit *s*, erit arcus ascensus

$$r=2kl\left(e^{\frac{-s}{2k}}+\frac{1}{k}\sqrt{2f}\int e^{\frac{-s}{k}}dx\right).$$

Exemplum 1.

795. Sit linea descensuum data recta verticalis *PA*, pro qua est $s = x$. Erit ergo quoque $ds = dx$ atque

$$\int e^{\frac{-s}{k}} dx = k\left(1 - e^{\frac{-s}{k}}\right).$$

Unde igitur fiet

$$r = 2kl\left(e^{\frac{-s}{2k}} + \sqrt{\frac{2f}{k}}\left(1 - e^{\frac{-s}{k}}\right)\right)$$

seu

$$e^{\frac{r}{2k}} = e^{\frac{-s}{2k}} + \sqrt{\frac{2f}{k}}\left(1 - e^{\frac{-s}{k}}\right).$$

Porro autem est [p. 439]

$$t = \int \frac{ds}{\left(1 + e^{\frac{s}{2k}}\sqrt{\frac{2f}{k}}\left(1 - e^{\frac{-s}{k}}\right)\right)^2} \quad \text{seu} \quad e^{\frac{r+s}{k}} dt = ds.$$

Eliminato ergo *s* prodibit pro cura ascensuum ista aequatio

$$(2f + k)^2 dt = k(2f - k)dr - \frac{2kf(2f + k)dr - 4fk^2 e^{\frac{r}{k}} dr}{e^{\frac{r}{2k}}\sqrt{\left(4ff + 2fk - 2fke^{\frac{r}{k}}\right)}},$$

in qua variabiles sunt a se invicem separatae, quare ea ad curvam construendam sufficit. Integratio vero huius aequationis a quadratura circuli pendet. Pro vacuo ex hac aequatione elicitur faciendo $k = \infty$ aequatio

$$dt + dr = \frac{dr\sqrt{2f}}{\sqrt{(2f - r)}} \quad \text{seu} \quad t = 4f - r - 2\sqrt{2f(2f - r)},$$

quae aequatio ad eam, quam in capite praecedenta [466] invenimus, reduci potest.

Exemplum 2.

796. Sit linea descensuum data ipsa tautochrona descensuum supra [719] inventa, cuius aequatio est

$$adx = kds\left(e^{\frac{s}{2k}} - 1\right).$$

Erit ergo

$$\int e^{\frac{-s}{k}} dx = \frac{k}{a}\int ds\left(e^{\frac{-s}{2k}} - e^{\frac{-s}{k}}\right) = \frac{2k^2}{a}\left(\frac{1}{2} - e^{\frac{-s}{2k}} + \frac{1}{2}e^{\frac{-s}{k}}\right) = \frac{k^2}{a}\left(1 - e^{\frac{-s}{2k}}\right)^2$$

et

$$\sqrt{\int e^{\frac{-s}{k}}dx} = \frac{k\left(1-e^{\frac{-s}{2k}}\right)}{\sqrt{a}}.$$

Quamobrem fiet

$$e^{\frac{r}{2k}} = e^{\frac{-s}{2k}} + \frac{\sqrt{2f} - e^{\frac{-s}{2k}}\sqrt{2f}}{\sqrt{a}} = \frac{\sqrt{2f} + e^{\frac{-s}{2k}}(\sqrt{a} - \sqrt{2f})}{\sqrt{a}}$$

atque [p. 440]

$$e^{\frac{-s}{2k}} = \frac{e^{\frac{r}{2k}}\sqrt{a} - \sqrt{2f}}{\sqrt{a} - \sqrt{2f}} \quad \text{et} \quad ds = \frac{e^{\frac{r}{2k}}dr\sqrt{a}}{\sqrt{2f} - e^{\frac{r}{2k}}\sqrt{a}}$$

atque

$$adx = \frac{ake^{\frac{r}{2k}}dr\left(e^{\frac{r}{2k}} - 1\right)}{\left(e^{\frac{r}{2k}}\sqrt{a} - \sqrt{2f}\right)^2}.$$

Cum autem porro sit

$$e^{\frac{-s}{k}}dx = e^{\frac{r}{k}}dt,$$

erit

$$ae^{\frac{r}{k}}dt = \frac{ake^{\frac{r}{2k}}dr\left(e^{\frac{r}{2k}} - 1\right)}{(-\sqrt{a} + \sqrt{2f})^2} \quad \text{sive} \quad adt = \frac{akdr\left(1 - e^{\frac{-r}{2k}}\right)}{(-\sqrt{a} + \sqrt{2f})^2}$$

seu

$$(-\sqrt{a} + \sqrt{2f})^2dt = kdr\left(1 - e^{\frac{-r}{2k}}\right),$$

ubi $\sqrt{2f}$ maius esse debet quam $\sqrt{a}$. Haec autem aequatio inventa comprehendit omnes tautochronas ascensuum; quae enim harum cunque cum tautochrona descensuum iungatur, super curva ex iis composita omnes semioscillationes debent esse isochronae. Si sumatur $f = 2a$, aequatio erit haec

$$adt = kdr\left(1 - e^{\frac{-r}{2k}}\right),$$

quae est pro tauteochrona ascensuum, super qua omnes ascensus eodem tempore absolvuntur quo descensus super tautochrona descensuum data, atque ea est continuatio tautochronae descensuum.

Exemplum 3.

797. Sit linea descensuum data *MA* tautochrona ascensuum et quaeratur, quales curvae cum ea iunctae semioscillationes isochronas producant. Aequatio vere pro hac curva *MA* est

$$adx = kds\left(1 - e^{\frac{-s}{2k}}\right).$$

Erit ergo [p. 441]

$$\int e^{\frac{-s}{k}} dx = \frac{k}{a}\int ds\left(e^{\frac{-s}{k}} - e^{\frac{-3s}{2k}}\right) = \frac{k^2}{a}\left(\frac{1}{3} - e^{\frac{-s}{k}} + \frac{2}{3}e^{\frac{-3s}{2k}}\right) = \frac{k^2}{3a}\left(1 - 3e^{\frac{-s}{k}} + 2e^{\frac{-3s}{2k}}\right)$$
$$= \frac{k^2}{3a}\left(1 + 2e^{\frac{-s}{2k}}\right)\left(1 - e^{\frac{-s}{2k}}\right)^2.$$

Ex his oritur

$$e^{\frac{r}{2k}} = e^{\frac{-s}{2k}} + \left(1 - e^{\frac{-s}{2k}}\right)\sqrt{\frac{2f}{3a}\left(1 + 2e^{\frac{-s}{2k}}\right)}$$

atque

$$t = \frac{k}{a}\int \frac{ds\left(1 - e^{\frac{-s}{2k}}\right)}{\left(1 + \left(e^{\frac{s}{2k}} - 1\right)\sqrt{\frac{2f}{3a}\left(1 + 2e^{\frac{-s}{2k}}\right)}\right)^2},$$

ex quibus constructio curvae consequitur.

Scholion 2.

798. Hoc exemplum ideo atttulimus, ut appareat, cum quanam curva tautochronam ascensuum coniunctam esse oporteat, quo semioscillationes omnes aequalibus temporibus absolvantur. Ex formulis autem inventis apparet curvam quaesitam non esse tautochronam descensuum; aequatio enim

$$cdt = kdr\left(e^{\frac{r}{2k}} - 1\right)$$

in illis formulis non continetur, quod periculum facienti statim patebit. Quamobrem si *MA* fuerit tautochrona descensuum et *AN* ascensuum, etiamsi omnes itus per *MAN* iisdem absolvantur temporibus, tamen reditus seu semioscillationes sequentes per *NAM* non erunt isochronae. Pendulum ergo, quod secundum curvas *MA* et *AN* oscillari efficitur, oscillationes non faciet isochronas, etiamsi alternae semioscillationes, in quibus descensus in [p. 442] curva MA incipit, aequalibus peragantur temporibus. Haec consequenter curva composita *MAN* non est idonea ad pendulorum motum in medio resistente aequabilem efficiendum. Optimum vero huic incommodo remedium afferretur, si casus determinaretur, quo curva *AN* similis et aequalis curvae *MA* prodiret.

PROPOSITIO 88.

Problema.

799. *Si curva MA et AN* (Fig. 87) *eam habuerint proprietatem, ut omnes semioscillationes, quae in curva MA incipiunt, sint inter isochronae in medio, quod in duplicata ratione celeritatum resistit, determinare casus, quibus hae duae curvae coniunctae MA et AN unam curvam continuam constuunt.*

Solutio.

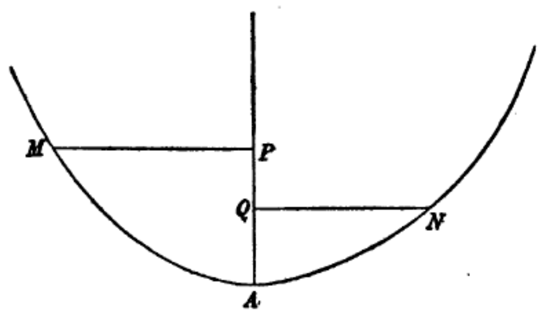

Fig. 87.

Manentibus iisdem denominationibus, quas in praecedente propositione adhibuimus, scilicet *AP* = *x*, *AM* = *s*, *AQ* = *t*, et *AN* = *r* atque *f* = longitudini penduli isochroni in vacuo et gravitate = *g*, invenimus ibi has duas aequationes

$$ke^{\frac{r}{2k}} - ke^{\frac{-s}{2k}} = \sqrt{2f}\int e^{\frac{-s}{k}}dx \quad \text{et} \quad e^{\frac{r}{k}}dt = e^{\frac{-s}{k}}dx,$$

quibus relatio inter utramque curvam continetur. Iam quia curvae *MA* et *NA* duo debent esse rami curvae continuae, aequatio inter *x* et *s* ita debet esse comparata, ut, si *x* abeat *t*, tum s fiat = – *r* propter situm negativum. Ad hoc accipiamus novam variabilem *z*, cuis *s* et *x* sint tales functiones, ut facto *z* negativo *x* abeat *t* et s in –*r* . Sit [p. 443]

$$\int e^{\frac{-s}{k}}dx = z^2;$$

erit

$$ke^{\frac{r}{2k}} - ke^{\frac{-s}{2k}} = z\sqrt{2f};$$

facto enim *z* negativo, quo casu *r* in –*s* et –*s* in *r* transit, prodibit

$$ke^{\frac{-s}{2k}} - ke^{\frac{r}{2k}} = -z\sqrt{2f};$$

quae aequatio cum priore congruit. Sit *P* functio quaecunque par ipsius *z*, quae non mutatur, etiam si loco *z* ponatur –*z* , et ponatur

$$ke^{\frac{-s}{2k}} = -\frac{1}{2}z\sqrt{2f} + P;$$

quo posito quaesito satisfaciet. Namque faciamus *z* negativum; abibit –*s* in *r* atque habebitur

$$ke^{\frac{r}{2k}} = \frac{1}{2}z\sqrt{2f} + P \quad \text{ac} \quad ke^{\frac{r}{2k}} - ke^{\frac{-s}{2k}} = z\sqrt{2f},$$

uti requiritur. Alteri aequationi

$$\int e^{\frac{r}{k}}dt = \int e^{\frac{-s}{k}}dx$$

per hanc

$$\int e^{\frac{-s}{k}}dx = z^2$$

iam satisfacit ; posito enim *z* negativo et *dt* loco *dx* atque *r* loco –*s* prodit

$$\int e^{\frac{r}{k}}dt = z^2 = \int e^{\frac{-s}{k}}dx.$$

Ex variabili ergo *z*, cuius *P* est functio quaecunque par, curva quaesita *AM*, cuius continua est altera *AN*, ita determinatur, ut sit

$$e^{\frac{-s}{2k}} = -\frac{z\sqrt{2f}}{2k} + \frac{P}{k} \quad \text{atque} \quad dx = 2e^{\frac{s}{k}}zdz = \frac{8k^2zdz}{(2P - z\sqrt{2f})^2}.$$

Erit ergo

$$x = 8k^2\int\frac{zdz}{(2P - z\sqrt{2f})^2} \quad \text{et} \quad s = 2kl\frac{2k}{2P - z\sqrt{2f}}.$$

Sit

$$z = \frac{u\sqrt{2}}{\sqrt{f}}$$

tum ad formulas simpliciores efficiendas tum ad homogeneitatem commodius producendam, quia debebit esse *u* unius dimensiones ; erit ergo *P* functio par ipsius *u* unius dimensionis quoque. Quare habebitur

$$s = 2kl\frac{k}{P - u} \quad \text{atque} \quad x = \frac{4k^2}{f}\int\frac{udu}{(P - u)^2}.$$

Q.E.I. [p. 444]

Corollarium 1.

800. Infinitae ergo curvae tautochronae *MAN* invenientur, si infiniti varii valores loco *P*, qui omnes sint functiones pares ipsius *u*, substituantur. Aequatio vero inter *x* et *s* obtinebitur, si ex duabus aequationibus inventis

$$s = 2k\, l\frac{k}{P-u} \quad \text{atque} \quad x = \frac{4k^2}{f}\int\frac{udu}{(P-u)^2}$$

variabilis *u*, quae etiam in *P* inest, eliminetur.

Corollarium 2.

801. Quia est $s = 2kl\frac{k}{P-u}$, erit

$$ds = \frac{2k(du-dP)}{P-u}$$

et

$$e^{\frac{s}{2k}} = \frac{k}{P-u} \quad \text{seu} \quad P-u = ke^{\frac{-s}{2k}}$$

atque

$$e^{\frac{s}{2k}}ds = \frac{2k^2(du-dP)}{(P-u)^2}.$$

Cum qua aequatione si altera

$$dx = \frac{4k^2udu}{f(P-u)^2}$$

coniungatur, prodibit

$$\frac{e^{\frac{s}{2k}}ds}{dx} = \frac{f(du-dP)}{2udu}.$$

Quae aequatio ad eliminandum *u* est saepe commodissima.

Scholion 1.

802. Quia evanescente *s* quoque *x* evanescere debet, primum investigandum est, quo ipsi *u* dato valore *s* evanescat. Deinde integrale

$$\int\frac{udu}{(P-u)^2}$$

ita accipi debet, ut evanescat, si loco *u* idem valor substituatur. Hocque observandum est cum in constructione curvae, quae ope duarum inventarum aequationum perfici potest, tum in concinnatione aequatione inter *x* et *s*, si quidem ea ex aequatione integrata [p. 445]

$$x = \frac{4k^2}{f}\int\frac{udu}{(P-u)^2}$$

deducatur. Ceterum si loco u et P quaelibet eorum multipla adhibeantur, loco duarum inventarum aequationum adhiberi possunt istae

$$s = 2k\,l\frac{c}{P-u} \quad \text{et} \quad x = \frac{4k^2}{f}\int\frac{udu}{(P-u)^2},$$

ubi constans c est arbitraria et idcirco ita determinari potest, ut s eodem casu evanescat, quo evanescit x. At s evanescit, si $u = 0$, quia est

$$\int e^{\frac{-s}{k}}dx = \frac{2u^2}{f}$$

atque $\int e^{\frac{-s}{k}}dx$ evanescit evanescente s; quare c aequale esse debet valori ipsius P, si in eo ponatur $u = 0$. Eodem ergo casu x debet evanescere, ex quo constans in integratione valoris ipsius x determinatur. Vel etiam loco P talis functio par ipsius u accipi debet, quae fiat $= c$, si ponatur $u = 0$.

Corollarium 3.

803. Cum longitudino penduli isochroni in vacuo et gravitate $= g$ sit $= f$ atque oscillationes minimae in medio resistente non discrepent ab oscillationibus in vacuo, erit radius osculi curvae in $A = f$, si quidem tangens curvae in A fuerit horizontalis.

Exemplum 1.

804. Quia P esse debet functio par ipsius u, sit P constans $= c = k$, quo posito $u = 0$ fiat $s = 0$. Erit ergo

$$k - u = ke^{\frac{-s}{2k}} \quad \text{seu} \quad u = k\left(1 - e^{\frac{-s}{2k}}\right).$$

Atque ob $dP = 0$ habebitur [p. 446]

$$\frac{e^{\frac{s}{2k}}ds}{dx} = \frac{f}{2u} \quad \text{seu} \quad u = \frac{fe^{\frac{-s}{2k}}dx}{2ds}.$$

Ex quibus aequationibus conficitur ista

$$\frac{fdx}{2} = kds\left(e^{\frac{s}{2k}} - 1\right).$$

Quae aequatio est pro ipsa tautochrona descensuum; quae ultra A continuata dat tautochronam ascensuum; atque omnes semioscillationes super hac curva continua, si modo MA incipiant, erunt isochronae.

Exemplum 2.

805. Sit

$$P = k + \frac{u^2}{a};$$

retinebit P eundem valorem facto u negativo. Hoc posito erit

$$k - u + \frac{u^2}{a} = ke^{\frac{-s}{2k}}$$

et propter $dP = \frac{2udu}{a}$ erit

$$\frac{e^{\frac{s}{2k}}ds}{dx} = \frac{f(a-2u)}{2au};$$

ex qua aequatione prodit

$$u = \frac{fadx}{2\left(ae^{\frac{s}{2k}}ds + fdx\right)}.$$

Qui ipsius *u* valor in altera aequatione substitutus dat

$$2fa^2e^{\frac{s}{k}}dxds = 4k\left(e^{\frac{s}{2k}} - 1\right)\left(ae^{\frac{s}{2k}}ds + fdx\right)^2 - f^2ae^{\frac{s}{2k}}dx^2$$

atque extracta radice

$$\frac{ae^{\frac{s}{2k}}ds}{fdx} = \frac{ae^{\frac{s}{2k}}}{4k\left(e^{\frac{s}{2k}}-1\right)} - 1 \pm \sqrt{\frac{ae^{\frac{s}{2k}}}{4k\left(e^{\frac{s}{2k}}-1\right)}\left(\frac{ae^{\frac{s}{2k}}}{4k\left(e^{\frac{s}{2k}}-1\right)} - 1\right)}.$$

In case speciali, si fuerit $a = 4k$, ista aequatio abit in hanc

$$\frac{4ke^{\frac{s}{2k}}ds}{fdx} = \frac{1 \pm e^{\frac{s}{4k}}}{e^{\frac{s}{2k}} - 1},$$

quas duas aequationes in se complectitur, quarum altera est

$$fdx = 4ke^{\frac{s}{2k}}ds\left(e^{\frac{s}{4k}} - 1\right)$$

et altera [p. 447]

$$-fdx = 4ke^{\frac{s}{2k}}ds\left(e^{\frac{s}{4k}} + 1\right).$$

Harum autem posterior, quia posito $s = 0$ non evanescit dx et ob valorem ipsius dx negativum, est inutilis. Prior vero integrata dat

$$fx = 16k^2\left(\frac{e^{\frac{3s}{4k}}}{3} - \frac{e^{\frac{s}{2k}}}{2} + \frac{1}{6}\right)$$

seu

$$3fx = 8k^2\left(2e^{\frac{3s}{4k}} - 3e^{\frac{s}{2k}} + 1\right).$$

Quae ultra *A* continuata hac aequatione exprimitur

$$3ft = 8k^2\left(2e^{\frac{-3r}{4k}} - 3e^{\frac{-r}{2k}} + 1\right).$$

Per seriem vero habebitur ista aequatio

$$fx = \frac{s^2}{2} + \frac{5s^3}{24k} + \frac{19s^4}{384k^2} + \text{etc.}$$

et pro altera curvae parte *AN* haec

$$ft = \frac{r^2}{2} - \frac{5r^3}{24k} + \frac{19r^4}{384k^2} - \text{etc.}$$

Corollarium 4.

806. Quia est

$$z^2 = \frac{2u^2}{f} = \int e^{\frac{-s}{k}} dx,$$

erit

$$u = \sqrt{\frac{1}{2} f \int e^{\frac{-s}{k}} dx}.$$

Eliminato vero *u* est

$$P = ke^{\frac{-s}{2k}} + \sqrt{\frac{1}{2} f \int e^{\frac{-s}{k}} dx}.$$

Quare si ille valor ipsius *u* in hac aequatione substituatur, prodibit statim aequatio inter *s* et *x*.

Exemplum 3.

807. Ponamus esse

$$P = \sqrt{(k^2 + u^2)} \quad \text{seu} \quad P^2 = k^2 + u^2;$$

substitutis loco *P* et u^2 valoribus supra datis erit

$$k^2 e^{\frac{-s}{k}} + ke^{\frac{-s}{2k}}\sqrt{2f\int e^{\frac{-s}{k}} dx} = k^2 \quad \text{seu} \quad \sqrt{2f\int e^{\frac{-s}{k}} dx} = k\left(e^{\frac{s}{2k}} - e^{\frac{-s}{2k}}\right).$$

Hinc quadratis sumendis oritur

$$2f\int e^{\frac{-s}{k}} dx = k^2\left(e^{\frac{s}{2k}} - e^{\frac{-s}{2k}}\right)^2 = k^2\left(e^{\frac{s}{k}} + e^{\frac{-s}{k}} - 2\right).$$

Haec vero aquatio differentiata dat hanc [p. 448]

$$2fe^{\frac{-s}{k}}dx = kds\left(e^{\frac{s}{k}} - e^{\frac{-s}{k}}\right) \quad \text{seu} \quad 2fdx = kds\left(e^{\frac{2s}{k}} - 1\right),$$

cuius integralis est

$$2fx = \frac{k^2 e^{\frac{2s}{k}}}{2} - ks - \frac{k^2}{2}.$$

Quae aequatio in seriem conversa dat

$$fx = \frac{s^2}{2} + \frac{s^3}{3k} + \frac{s^4}{6k^2} + \frac{s^5}{15k^3} + \frac{s^6}{45k^4} + \text{etc.}$$

Scholion 2.

808. Quas in his exemplis invenimus curvas tautochronas pro medio, quod resistit in duplicatata ratione celeritatum, eae ita sunt comparatae, ut arcus *MA* et *AN* sint dissimiles. Cum igitur omnes descensus super curva *MA* incipere debeant, sequentes semioscillationes, quae in curva *NA* incipiunt, non erunt tautochronae, id quod in causa est, quod hae curvae ad motum oscillatorium accommodari nequeant. Huic autem incommodo remedium afferretur, si huiusmodi curvarum *MA* et *NA* par inveniretur quae essent inter se similes et aequales; hoc enim casu perinde super utraque curva descensus fieri posset. Dubium quoque nullum est, quin talis casus existat, eiusque inventio, quia hae duae curvae forte non erunt continuae, ad praecedentem propositionem potius pertinet. Indagari scilicet debet curva descensuum, cui respondens curva ascensuum similis et aequalis sit; haec vero investigatio ob defectum analyseos ita est difficilis, ut dubitem, num quisqam ante insignem analyseos promotionem ad hunc scopum pertingere possit. [p. 449]
Haec vero quaestio huc reducitur, ut investigetur aequatio inter *s* et *x* huius conditionis, ut, si in ea ponatur

$$-2kl\left(e^{\frac{s}{2k}} - \frac{1}{k}\sqrt{2f}\int e^{\frac{s}{k}}dx\right)$$

loco *s* et

$$\int \frac{dx}{\left(1 - \frac{1}{k}e^{\frac{-s}{2k}}\sqrt{2f}\int e^{\frac{s}{k}}dx\right)^2}$$

loco *x*, eadem prodeat aequatio, quae habebatur ante [788]. Conditio quidem haec multis modis facilior effici potest; attamen quomodo ei satisfieri possit, non video. Si medium fuerit rarissimum, non difficile est ex allatis casum invenire, quo duae curvae *MA* et *AN* sint inter se similes et aequales. Ego quidem ad finem perducto calculo hanc inveni aequationem

$$fdx = sds + \frac{s^3 ds}{9k^2} \quad \text{seu} \quad fx = \frac{s^2}{2} + \frac{s^4}{36k^2};$$

quae curva simul ultra A continuata ramum habet AN similem et aequalem arcui AM; quare pendulum in hac curva oscillans singulas semioscillationes absolvet aequalibus temporibus. Erit autem

$$s^2 = -9k^2 + 3k\sqrt{(9k^2 + 4fx)} \quad \text{et} \quad s = \sqrt{\left(-9k^2 + 3k\sqrt{(9k^2 + 4fx)}\right)}.$$

Quia vero *k* est quantitas valde magna, erit

$$s = \sqrt{2fx} - \frac{fx\sqrt{2fx}}{18k^2} \quad \text{atque} \quad ds = \frac{fdx}{\sqrt{2fx}} - \frac{fdx\sqrt{2fx}}{12k^2}.$$

Hincque fit

$$dy = dx\sqrt{\frac{f-2x}{2x}} - \frac{f^2dx}{12k^2}\sqrt{\frac{2x}{f-2x}}.$$

Ponatur $f = 2a$; erit

$$y = \int dx\sqrt{\frac{a-x}{x}} - \int \frac{a^2dx}{3k^2}\sqrt{\frac{x}{a-x}} = \int \frac{adx - xdx}{\sqrt{(ax-xx)}} - \int \frac{a^2xdx}{3k^2\sqrt{(ax-xx)}}$$
$$= \left(1 + \frac{a^2}{3k^2}\right)\sqrt{(ax-xx)} + \left(1 - \frac{a^2}{3k^2}\right)\int \frac{\frac{1}{2}adx}{\sqrt{(ax-xx)}}.$$

Quae ergo curva eodem fere modo quo cyclois describi potest ope rectificationis circuli. [p. 450]

Corollarium 5.

809. Si sumatur $a = k\sqrt{3}$ seu $f = 2k\sqrt{3}$, curva haec abit in ellipsin, cuius axis horizontalis est duplo maior quam verticalis, qui est = $2k\sqrt{3}$. Fieri ergo potest, ut ellipsis sit tautochrona in fluido rarissimo atque magis satisfaciat quam cyclois.

Scholion 3.

810. Constructio autem curvae tautochronae in medio rarissimo in praecedente scholio datae est, ut sequitur : Super recta verticali $AB = a = \frac{1}{2}f$ (Fig. 88) describatur semicirculus AOB et ex hoc super basi BD cyclois AFD; quae eadem inverso situ describatur AGD. Quibus factis curva quaesita AMC construetur sumendis ubique eius applicatis

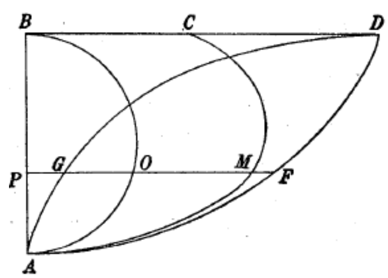

Fig. 88.

$$PM = PF - \frac{a^2}{3k^2}PG;$$

qua ratione curvae infinita puncta cognoscuntur. Vel etiam accipi potest

$$PM = \left(1 + \frac{a^2}{3k^2}\right)PO + \left(1 - \frac{a^2}{3k^2}\right)AO,$$

ita ut cycloide non sit opus. Curva autem haec alicubi habebit tangentem verticalem seu applicatam *PM* maximam, quae invenitur posito $dy = 0$. Prodibit autem

$$\frac{a-x}{x} = \frac{a^2}{3k^2} \quad \text{seu} \quad x = \frac{3ak^2}{a^2+3k^2};$$

cui valori si *AP* aequalis capiatur, invenietur applicata maxima.

PROPOSITIO 89.

Problema. [p. 451]

811. *In hypothesi gravitatis uniformis deorsum tendentis g data curva quacunque am* (Fig. 89) *pro descensibus in vacuo invenire curvam AM pro descensibus in medio resistente uniformi in duplicata ratione celeritatum huius indolis, ut omnes descensus super MA sint isochroni respective omnibus descensibus super ma, si celeritates in punctis imis a et A fuerint aequales.*

Solutio.

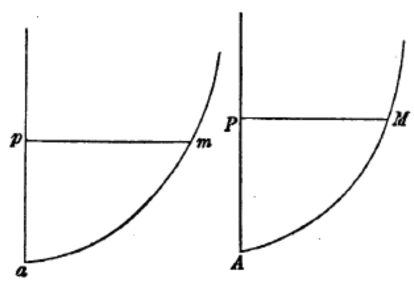

Fig. 89.

Sit pro curva descensuum in vacuo *am* abscissa $ap = t$, arcus $am = r$; pro curva vero descensuum in medio resistente sit $AP = x$ et $AM = s$ resistentiae vero exponens ponatur = k. Iam considerentur bini descensus super his curvis, in quibus celeritates in *A* et *a* acquisitae sint aequales et debitae altitudini b. Erit ergo tempus descensus in vacuo =

$$\int \frac{dr}{\sqrt{(b-gt)}},$$

et post integrationem ponatur $gt = b$. At pro tempore descensus in medio resistente super curva *MA* habebitur

$$\int \frac{ds}{e^{\frac{s}{2k}}\sqrt{\left(b-g\int e^{\frac{-s}{k}}dx\right)}},$$

si item post integrationem ponatur

$$g\int e^{\frac{-s}{k}}dx = b.$$

Quamobrem haec tempora erunt aequalia, si fuerit

$$\frac{ds}{e^{\frac{s}{2k}}} = dr \quad \text{et} \quad \int e^{\frac{-s}{k}}dx = t;$$

his enim positis pro utroque tempore habebitur eadem expressio

$$\int \frac{dr}{\sqrt{(b-gt)}}.$$

Cum igitur sit $\frac{ds}{e^{\frac{s}{2k}}} = dr$, erit integrando

$$2k\left(1 - e^{\frac{-s}{2k}}\right) = r \quad \text{atque} \quad e^{\frac{-s}{2k}} = \frac{2k-r}{2k};$$

unde prodit [p. 452]

$$s = 2kl\frac{2k}{2k-r}.$$

Altera vero aequatio $\int e^{\frac{-s}{k}} dx = t$ dat

$$e^{\frac{-s}{k}} dx = dt.$$

Est autem

$$e^{\frac{-s}{k}} = \frac{(2k-r)^2}{4k^2},$$

quo valore substituto habetur

$$dx = \frac{4k^2 dt}{(2k-r)^2},$$

ex quo oritur

$$x = \int \frac{4k^2 dt}{(2k-r)^2}.$$

Dato ergo aequatione inter *t* et *r* pro curva *am* ope duarum harum aequationum, quibus *s* et *x* per *t* et *r* determinatur, construi poterit curva quaesita *AM*. Aequatio vero inter *x* et *s* commodius invenietur ex data aequatione inter *t* et *r*, si in ea loco *r* substituatur $2k(1-e^{\frac{-s}{2k}})$ et $\int e^{\frac{-s}{k}} dx$ loco *t*.

Q.E.I.

Corollarium 1.

812. Circa punctum infinum *a*, ubi *t* et *r* sunt quantitates evanescentes, fit

$$s = r + \frac{r^2}{4k} \quad \text{et} \quad x = t + \int \frac{r\,dt}{k}$$

seu

$$ds = dr + \frac{r\,dr}{2k} \quad \text{et} \quad dx = dt + \frac{r\,dt}{k}.$$

Quare inclinatio curvae *MA* ad axem in *A* aequalis erit inclinationi curvae *ma* in *a*.

Corollarium 2.

813. Porro radius osculi in puncto infimo *a*, si tangens fuerit horizontalis, est = $\frac{rdr}{dt}$ et in *A*, quia tangens quoque erit horizontalis, =

$$\frac{sds}{dx} = \frac{rdr + \frac{3r^2dr}{4k}}{dt + \frac{rdt}{k}}.$$

Erit ergo

$$\frac{sds}{dx} = \frac{rdr}{dt} - \frac{r^2dr}{4kdt} = \frac{rdr}{dt}\left(1 - \frac{r}{4k}\right).$$

Quare ob *r* infinite parvum erit

$$\frac{sds}{dx} = \frac{rdr}{dt}.$$

Corollarium 3.

814. Si ergo curva *ma* in *a* habuerit tangentem horizontalem, erit curvae *MA* tangens in *A* horizontalis atque radius osculi in *A* aequalis erit radio osculi in *a*. [p. 453]

Corollarium 4.

815. Si igitur in vacuo inventa curva *ma*, in qua tempora descensuum quamcunque habeant relationem ad celeritates in *a* acquisitas, idem problema pro medio resistente solventur curva *MA*, quae praescripta ratione ex curva ma construitur.

Corollarium 5.

816. Si igitur curva ma fuerit cyclois seu tautochrona in vacuo, *AM* erit tautochrona descensuum in medio resistente supra inventa. Posita enim $r^2 = 2at$ seu $rdr = adt$ prodit substitutis loco *r* et *t* inventis valoribus ista aequatio

$$2ke^{\frac{-s}{2k}}ds\left(1 - e^{\frac{-s}{2k}}\right) = ae^{\frac{-s}{k}}dx \quad \text{seu} \quad adx = 2kds\left(e^{\frac{s}{2k}} - 1\right).$$

Exemplum.

817. Sit *am* linea recta utcunque inclinata, ita ut sit $r = nt$; erit tempus descensus, quo celeritas altitudini *b* debita generatur, =

$$\int \frac{ndt}{\sqrt{(b - gt)}} = \frac{2n\sqrt{b}}{g}.$$

Eandem ergo habebit proprietatem curva *MA*, ut tempus cuiusque descensus in medio resistente, quo celeritas $\sqrt{b}$ generatur, sit $= \frac{2n\sqrt{b}}{g}$ seu proportionale ipsi celeritati genitae. Cum autem sit $r = nt$, erit $dr = ndt$; in qua si loco dr et dt valores invent substituantur, prodit [p. 454]

$$e^{\frac{-s}{2k}}ds = ne^{\frac{-s}{k}}dx \quad \text{seu} \quad ndx = e^{\frac{s}{2k}}ds;$$

quae est aequatio pro tractoria filo longitudinis $2k$ generata, qualis repraesentatur in Fig. 86, nempe curva CA, quae in A eam habet inclinionem quam recta data ma.

Scholion 1.

818. Quemadmodum hic curva MA est determinata, super qua omnes descensus in medio resistente iisdem absolvuntur temporibus quibus descensus in vacuo super curva ma, si celeritates ultimae in A et a fuerint aequales, ita eodem modo curva MA potest definire, super qua omnes ascensus in medio resistente iisdem temporibus absolvuntur quibus similes iisdem celeritatibus incepientes ascensu in vacuo super curva am. Nam cum medio resistente descensus in ascensum mutetur facto k negativo, si ponatur $AP = x$ et $AM = s$, havebitur

$$x = \int \frac{4k^2dt}{(2k+r)^2} \quad \text{et} \quad s = 2kl\frac{2k+r}{2k}$$

seu inverse

$$t = \int e^{\frac{s}{k}}dx \quad \text{et} \quad r = 2k\left(e^{\frac{s}{2k}} - 1\right).$$

Ex quibus tum facile curva AM potest constui et aequatio pro ea inveniri.

Scholion 2.

819. In hoc problemate ex curva descensuum in vacuo data determinavimus curvam descensuum in medio resistente. Facile autem apparet vicissim ex data curva AM pro medio resistente alteram am pro vacuo inveniri posse. Cum enim sit

$$r = 2k\left(1 - e^{\frac{-s}{2k}}\right) \quad \text{et} \quad t = \int e^{\frac{-s}{k}}dx,$$

constructio curvae am ope harum duarum aequationum perficitur. Aequatio vero pro curva am inter t et r commodius ex data aequatione inter x et s reperitur, si in ea loco x substituatur $\frac{4k^2dt}{(2k-r)^2}$ et $2kl\frac{2k}{2k-r}$ loco s. Quod hic praeterea de descensibus dictum est, idem de ascensibus valet, si modo k ponatur negativum, uti in scholio 1 monuimus.

Scholium 3.

820. Tradita hic inventio alterius curvae duarum *am* et *AM* ex altera etiam locum habet, si datae curvae non habeatur aequatio, sed si manu utcunque fuerit ducta; ex formulis enim inventis constructio potest deduci, quae ab aequatione non amplius pendeat. Quamobrem cum in capite praecedente (432) in casum inciderimus pro vacuo, quo curvam *cm* (Fig. 90) invenimus cum data ac iungendam, ut omnes descensus ex quovis puncto curvae *cm* usque ad *a* aequalibus absolvantur temporibus, similis exempla ex iis pro medio resistente erui poterunt, quibus linea ex partibus duarum diversarum curvarum composita sit tautochrona. Si enim curva *acm* fuerit huiusmodi curva tautochrona pro vacuo, ex ea per solutionem huius problematis similis curva composita pro medio resistente invenietur. Scilicet ex *ac* methodo tradita curva *AC* definiatur; qua inventa posito $bp = t$, $cm = r$ et $BP = x$ atque $CM = s$ et praeterea $ab = a$, $ac = c$ et $AB = A$ et $AC = C$, tum enim, cum data sit aequatio inter *t* et *r*, erit

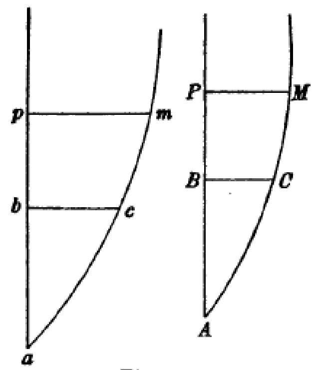

Fig. 90.

$$AP = A + x = \int \frac{4k^2 dt}{(2k - c - r)^2}$$

et

$$AM = C + s = 2kl \frac{2k}{2k - c - r}.$$

[p. 455] At si pro medio resistente data fuerit curva *AC* atque requiratur altera *CM* eius proprietatis, ut omnes descensus super *MCA* aequalibus absolvantur temporibus, solutio non dissimili modo efficietur. Nam ex data curva *AC* pro medio resistente inveniatur curva eiusdem proprietatis pro vacuo *ac* per scholion 2. Qua inventa quaeratur curva cm ei adiungenda, quae omnes descensus in vacuo isochronos producat (432). Denique methodo modo tradita ex curva composita *acm* pro vacuo quaeratur similis curva composita pro medio resistente *ACM*, cuius quidem pars *AC* iam est cognita; quippe ex ea lineam *ac* definivimus. Idem ergo problema, quod in vacuo tantum in se habebat difficultatis, in medio quoque resistente resolvitur. Denique hinc caput finiens Lectorem benevolum rogo, ut, antequam ad caput sequens progrediatur, quae in capite 1 ab § 58 usque ad finem capite tradita sunt, repetere velit.