CHAPTER FOUR

CONCERNING THE MOTION OF A POINT ON A GIVEN SURFACE.

[p. 457]

PROPOSITION 90.

Problem.

821. *For a given path* $Mm\mu$ *on some surface* (Fig. 91) *to find the position of this path with respect to a given plane APQ, and of the radius of osculation of this path at M, as long as neither the position nor the length of the radius lies on the surface.*

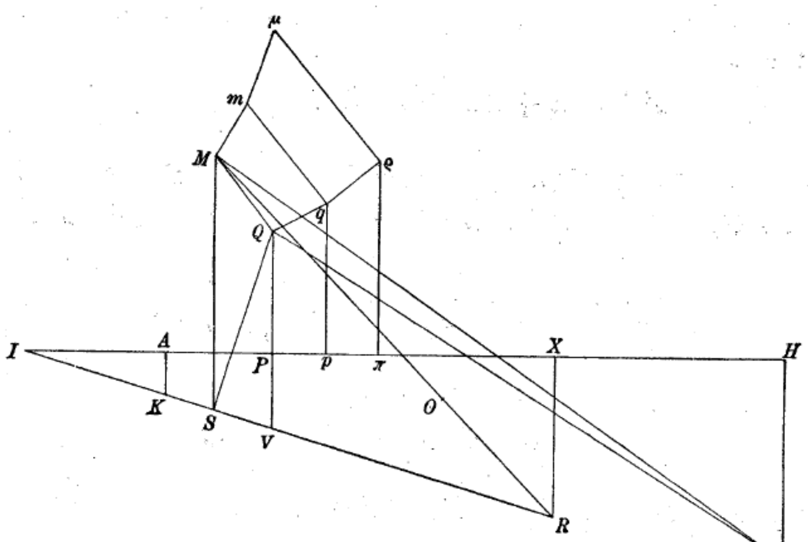


Fig. 91.

Solution.

With the plane *APQ* taken for argument's sake [in the plane of the page] and in that plane the axis *AP* is taken, with respect to which the position of the curve $Mm\mu$ is to be determined ; now from three nearby points *M*, *m* and μ of the given path on the surface the perpendiculars *MQ*, *mq*, $\mu\rho$ are sent to the plane *APQ* and the perpendiculars *QP*, *qp*, and $\rho\pi$ [are dropped] from the points *Q*, *q*, ρ to the axis *AP*. Now the initial position of the abscissa at *A* are $AP = x$, $PQ = y$ and $QM = z$. Again since the given surface is put in place, an equation is given expressing the nature of this between these three variables *x*, *y* and *z*; and this equation is [of the form]:

$$dz = Pdx + Qdy.$$

Since if this equation is connected with another, a certain line present on the surface is expressed; whereby, as the given line $Mm\mu$ is put in place, as well as the equation $dz = Pdx + Qdy$ above, another equation is given, from which the curve $Mm\mu$ can be determined, [p. 458] but there is no need to represent that here. Let the elements of the abscissa $Pp, p\pi = dx$ be equal to each other, or the element is *dx* taken to be constant. [The derivations that follow rely heavily on the section §68 onwards at the end of Ch. 1] Hence there is:

$$pq = y + dy, \quad \pi\varrho = y + 2dy + ddy$$

and

$$qm = z + dz \text{ and } \varrho\mu = z + 2dz + ddz.$$

With these in place, let *MN* be the normal to the surface at the point *M*, and *N* the point at which this normal crosses the plane *APQ* ; the perpendicular NH is sent from N to the axis ; then

$$AH = x + Pz \text{ and } HN = -Qz - y$$

(68).

[For we can write in modern terms : $dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$; hence $P = \frac{\partial z}{dx}$ and $Q = \frac{\partial z}{\partial y}$ at *M*.

If the line *QH'* (not shown) is drawn parallel to *AH* in the *xz*-plane, we have $\frac{QH'}{MQ} = tan(QMH') = \frac{\partial z}{\partial x}$, and hence we have the subnormal $QH' = \frac{z\partial z}{\partial x} = Pz$ as required. Similarly, for the line *QN'* (not shown) is drawn parallel to *HN* in the *yz*-plane, we have $\frac{QN'}{MQ} = tan(QN'M) = \frac{\partial z}{\partial y}$, and hence we have the subnormal $QN' = \frac{z\partial z}{\partial y} = Qz$ as required, and the signs can be taken into account ; see Euler's explanation and the note on page 19. Note also that Euler has in mind very simple surfaces such as those of cylinders, cones, and surfaces of revolution about an axis, so that only one radius of curvature has to be found. You may wish to copy the above figure and annotate it, as this helps greatly in understanding the working.]

Now let *MR* be the position of the line of the radius of osculation of the curve $Mm\mu$ and *R* the point of incidence of this in the plane *APQ*; then with the perpendicular *RX* sent from R to the axis :

$$AX = \frac{zdx(dyddy + dzddz)}{(dx^2 + dy^2)ddz - dydzddy} + x$$

and

$$XR = \frac{zdx^2ddy + zdz(dzddy - dyddz)}{(dx^2+dy^2)ddz - dydzddy} - y$$

(68). Now the length of the radius of osculation , clearly *MO*, is equal to

$$\frac{(dx^2+dy^2+dz^2)^{\frac{3}{2}}}{\sqrt{(dx^2(ddy^2+ddz^2)+(dyddz-dzddy)^2)}}$$

(72). Finally the plane considered, in which the elements $Mm, m\mu$ are in place, is produced until it intersects the plane *APQ*, and let the line of intersection be *RKI*, which the perpendiculars from *A* and P cross at *K* and *V*; it was found above (68) that

$$PV = \frac{zddy}{ddz} - y$$

Now since we have $XR - PV : AX - AP = PV : PI$, then

$$PI = \frac{(AX-AP)PV}{XR-PV}.$$

Now

$$AX - AP = \frac{zdx(dyddy+dzddz)}{(dx^2+dy^2)ddz - dydzddy}$$

and

$$XR - PV = \frac{zddyddz(dz^2-dy^2)+zdydz(ddy^2-ddz^2)}{ddz((dx^2+dy^2)ddz - dydzddy)}.$$

With which in place, we have : [p. 459]

$$PI = \frac{dx(dyddy+dzddz)(zddy-yddz)}{ddyddz(dz^2-dy^2)+dydz(ddy^2-ddz^2)} = \frac{zdxddy - ydxddz}{dzddy - dyddz}$$

and

$$AI = PI - AP = \frac{zdxddy - ydxddz - xdzddy + xdyddz}{dzddy - dyddz}.$$

Hence, it is found that :

$$AK = \frac{PV\cdot AI}{PI} = \frac{zdxddy - ydxddz - xdzddy + xdyddz}{dxddz}.$$

Now the inclination of the plane in which the elements Mm et $m\mu$ are placed to the plane APR can be found by sending the perpendicular *QS* from *Q* to the line of intersection RI; for the tangent of the angle of inclination is equal to $\frac{QM}{QS}$. But since $IV : PI = QV : QS$, then that tangent is equal to :

$$\frac{QM \cdot IV}{PI \cdot QV} = \frac{\sqrt{(dx^2 ddz^2 + (dzddy - dyddz)^2)}}{dx\,ddy}.$$

Now the tangent of the angle *NMR*, that the radius of osculation makes with the normal to the surface, is equal to (71) :

$$\frac{ddy(dx + Pdz) - ddz(Pdy - Qdx)}{(ddz - Qddy)\sqrt{(dx^2 + dy^2 + dz^2)}}$$

Therefore from these everything can be deduced that is required in understanding the position of the curve $Mm\mu$. Q.E.I.

Corollary 1.

822. The projection of the curve $Mm\mu$ in the plane *APQ* is the curve $Qq\rho$, the nature of which is expressed from the equation between *x* and *y*. Whereby this projection is obtained, if with the help of the equations $dz = Pdz + Qdy$ and that by which the curve is determined on the surface, a new equation is formed from the elimination of the variable *z*, which is between *x* and *y* only.

Corollary 2.

823. In a like manner, if *x* is eliminated, in order that an equation is produced between y and *z*, from this equation the projection of the curve $Mm\mu$ is defined in the plane normal to the axis *AX*. [p. 460] And the equation, in which *y* is not present, but only *x* and *z*, gives the projection of the curve $Mm\mu$ in the plane normal to the plane *APQ* cutting the axis AX.

Corollary 3.

824. But the nature of the curve $Mm\mu$ is known distinctly from any two of these normal projections in two of the planes in turn. Such knowledge is also supplied by a single projection together with the surface itself.

Corollary 4.

825. On account of which the curve on the surface is required to be designated by some characters as well as the equation $dz = Pdx + Qdy$, from which surface is determined, and an equation is given involving only two variables for some projection of the curve $Mm\mu$.

Corollary 5.

826. If the surface is cut by a plane, in a like manner to that in which the cone is accustomed to be cut producing the conic sections, then the curve arises from this section is in the same plane. Whereby in these cases as the position of the right line *IR* is constant so the inclination of the plane *IMR* to the plane *APQ*.

Example.

827. Therefore if some surface is given and that is cut by the plane *IMR*, the curve is sought arising from this section. [p. 461] Accordingly, there is put in place

$AI = a,\ AK = b$ and the tangent of the angle of inclination of the plane *IMR* to the plane *APQ* is equal to *m*; then

$$a = \frac{zdxddy - ydxddz}{dzddy - dyddz} - x \text{ and } b = \frac{zdxddy - ydxddz - xdzddy + xdyddz}{dxddz}$$

and

$$m = \frac{\sqrt{(dx^2 ddz^2 + (dzddy - dyddz)^2)}}{dxddy}$$

From which equations joined with $dz = Pdx + Qdy$ the nature of the curve generated by this section can be determined. Now from previously from the two equations there arises

$$\frac{b}{a} = \frac{dzddy - dyddz}{dxddz} \text{ or } ddz : ddy = adz : bdx + ady;$$

and the integral of this equation is :

$$\tfrac{1}{a} ldz = \tfrac{1}{a} l(bdx + ady) - \tfrac{1}{a} lc \text{ or } cdz = bdx + ady$$

and again

$$cz = bx + ay + ff.$$

Now in the first equation if in place of *ddz* and *ddy* the proportionals of these are substituted, there is produced

$$a + x = \frac{bzdx + azdy - aydz}{bdz} \text{ or } abdz + bxdz = bzdx + azdy - aydz,$$

and the integral of this divided by *zz* is this :

$$c - \frac{ab}{z} = \frac{bx + ay}{z} \text{ or } cz = bx + ay + ab;$$

hence what before was *ff*, this is *ab*, or *ff* = *ab*. Now the constant *c* of the third equation can be defined ; moreover then

$$m = \frac{dz\sqrt{(a^2 + b^2)}}{bdx + ady} \text{ or } \frac{dz\sqrt{(a^2 + b^2)}}{m} = bdx + ady.$$

Where the above letter is

$$c = \frac{\sqrt{(a^2 + b^2)}}{m}$$

and in addition the nature of the surface is expressed by this equation :

$$\frac{z\sqrt{(a^2 + b^2)}}{m} = bx + ay + ab,$$

from which the nature of the curve sought can be derived. Moreover because the whole curve sought is in the plane *IMR*, most conveniently that can be expressed from the equation between the orthogonal coordinates taken in the same plane. Hence with *IR* taken for the axis, from *M* to that there is sent the perpendicular *MS* and calling $IS = t$ and $MS = u$. Now we have $IA : AK = IP : PV$ or

$$PV = \frac{ab + bx}{a}$$

and

$$QV = \frac{ab + bx + ay}{a} = \frac{z\sqrt{(a^2 + b^2)}}{ma}.$$

Again we have [p. 462]

$$\sqrt{(a^2 + b^2)} : a = \frac{z\sqrt{(a^2+b^2)}}{ma} : QS;$$

whereby

$$QS = \frac{z}{m} \text{ and } SV = \frac{bz}{ma}.$$

From these there is produced :

$$MS = u = \frac{z\sqrt{(1 + m^2)}}{m} \text{ and } IS = t = \frac{m(a + x)\sqrt{(a^2 + b^2)} - bz}{ma}.$$

From which there arises :

$$z = \frac{mu}{\sqrt{(1 + m^2)}} \text{ and } x = \frac{bu + at\sqrt{(1 + m^2)}}{\sqrt{(a^2 + b^2)(1 + m^2)}} - a$$

and with these values substituted in the equation $\frac{z\sqrt{(a^2+b^2)}}{m} = bx + ay + ab$ there is produced :

$$y = \frac{au - bt\sqrt{(1 + m^2)}}{\sqrt{(1 + m^2)(a^2 + b^2)}}.$$

Therefore with these values substituted in place of *x*, *y* and *z* in the equation of the surface there comes about the equation between *t* and *u*, or the orthogonal coordinates of the curve sought.

Corollary 6.

828. If the intersection of the cutting plane *IR* falls on the axis *AX* and *I* is taken at *A*, then

$$z = \frac{mu}{\sqrt{(1 + m^2)}},\ y = \frac{u}{\sqrt{(1 + m^2)}}, \text{ and } x = t.$$

Corollary 7.

829. If the intersection *IR* of the cutting plane *IMR* with the plane *APQ* is normal to the axis *AX*, then $b = \infty$. Whereby there is produced :

$$z = \frac{mu}{\sqrt{(1+m^2)}}, \quad x = \frac{u}{\sqrt{(1+m^2)}} - a, \text{ and } y = -t.$$

Corollary 8.

830. Since the values to be substituted in place of *z*, *y* and *x* are of one dimension of *t* and *u*, it is evident that the equation between *t* and *u* is not possible to have more dimensions than the equation itself between *z*, *y* and *x*.

Corollary 9. [p. 463]

831. Whereby if the equation between *z, y* and *x* is of two dimensions, there are many surfaces of this kind given in addition to the cone, all the sections made by a plane are conic sections.

Scholium.

832. In that dissertation in Book III of the Commentaries [of the St. Petersburg Ac. of Sc.], in which I have determined the shortest line on a surface, I have pursued three kinds of surfaces, which are the cylinder, the cone, and the surface of revolution.
[L. Euleri Commentatio 9 (E09): *Concerning the shortest line on a surface joining any two points*. Comment. acad. sc. Petrop. 3 (1728), 1732, p. 110; *Opera Omnia* series I, vol. 25.]

Now the general equation $dz = Pdx + Qdy$ gives a cylindrical surface, if *P* vanishes and *Q* depends only on *y* and *z*, thus so that the abscissa *x* does not enter the equation for this kind of surface ; for all the sections are parallel to each other and are equal also ; for these the equation is therefore $dz = Qdy$.

I refer all these surfaces to the genus of confides, which are generated by drawing right lines from some points of an individual curve to a fixed point placed beyond the plane of that curve. Which surfaces have this property, that all parallel sections are similar to each other and the homologous lengths of these are as the distance of the sections from the vertex of the cone. Now equations for the surfaces of this kind, if indeed the vertex of the cone is at *A*, thus are compared, so that *x, y* and *z* everywhere together constitute a number of the same dimensions.

Finally I have turned or rounded surfaces [of revolution], which are generated by the rotation of any curve about an axis ; if *AX* were such an axis, on putting *x* constant, the equation between *y* and *z* gives a circle with centre *P*. Whereby the equation for these has this form : [p. 464]

$$dz = Pdx - \frac{ydy}{z} \text{ or } zdz + ydy = zPdx,$$

where Pz only depends on x ; or $Q = -\frac{y}{z}$ and $P = \frac{X}{z}$ with X present as a function of x.

Moreover as in these turned surfaces all the sections are circles normal to the axis, thus such surfaces can be taken, the sections of which are any similar curves normal to the axis. All such surfaces hold this general property, that any function of x is everywhere equal to a function of y and z of the same number of dimensions. As, if the number of this dimension is n, for this is a property of the equation $Pdx = Rdz + Qdy$ that it is

$$Rz + Qy = n\int Pdx \text{ or } Rdz + Qdy = \frac{zdR + ydQ}{n-1}.$$

[See E044.] From which, or the equation for a surface of this kind can at once be concluded from what has been given.

PROPOSITION 91.

Problem.

833. *In any given surface to determine the line, that the body describes in that motion, acted on by no forces either in a vacuum or in a medium with some kind of resistance.*

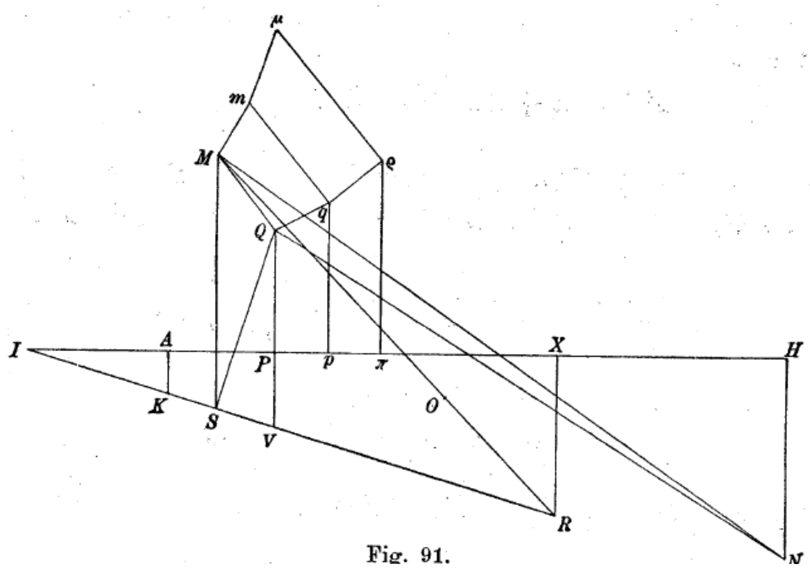


Fig. 91.

Solution. [p. 465]

Because the body put in place is not acted on by any absolute forces, the line described by that on the surface is the shortest line *in vacuo* (62). But the force of resistance in a medium only diminishes the speed of the body and does not affect the direction in any manner; whereby also in a medium with resistance the path described by the body on some surface is equally the shortest. Therefore with the variables in place as before : $AP = x$, $PQ = y$ and $QM = z$, (Fig. 91) let $dz = Pdx + Qdy$ be the equation expressing the nature of the surface and $Mm, m\mu$ any two elements of the shortest line. From these found above (69) for the shortest line, this is the equation:

$$Pdzddy + dxddy = Pdyddz - Qdxddz$$

Thus there arises :

$$ddz = \frac{(Pdz + dx)ddy}{Pdy - Qdx}.$$

But the equation for the surface differentiated gives :

$$ddz = dPdx + Qddy + dQdy,$$

with these connected together there is given :

$$ddy = \frac{(Pdy - Qdx)(dPdx + dQdy)}{dx(1+P^2+Q^2)} \text{ and } ddz = \frac{(Pdz + dx)(dPdx + dQdy)}{dx(1+P^2+Q^2)}.$$

Therefore with the element *Mm* given then the following element $m\mu$ on the shortest line is found; for it is given by :

$$\pi\varrho = PQ + 2dy + ddy \text{ and } \varrho\mu = QM + 2dz + ddz$$

and the values of *ddy* and *ddz* have been found. Whereby hence the position of any following element is determined and the nature of the shortest line by some projection of these is known. Q.E.I.

Corollary 1.

834. If, in the equation for the surface *P* and *Q* are given in terms of *x* and *y* only, then the equation

$$ddy = \frac{(Pdy - Qdx)(dPdx + dQdy)}{dx(1+P^2+Q^2)}$$

denotes the projection of the shortest line in the plane *APQ*. [p. 466]

Corollary 2.

835. Therefore for the shortest line $Mm\mu$, with the elements selected equally from the axis, then :

$$\pi\varrho = y + 2dy + \frac{(Pdy - Qdx)(dPdx + dQdy)}{dx(1+P^2+Q^2)}$$

and

$$\varrho\mu = z + 2dz + \frac{(Pdz + dx)(dPdx + dQdy)}{dx(1+P^2+Q^2)},$$

from which equations the point μ is known from the two preceding points *M* and *m*.

Corollary 3.

836. Because the angle *RMN* vanishes for the shortest line (71), *R* falls on *N*; hence the position of the radius of osculation thus is obtained, in order that $AX = x + Pz$ and $XR = -Qz - y$. Now the length of the radius of osculation (73) is equal to :

$$-\frac{(dx^2 + dy^2 + dz^2)\sqrt{(1+P^2+Q^2)}}{dPdx + dQdy}.$$

Corollary 4.

837. Now the plane *IMR*, in which the shortest elements $Mm\mu$ are situated, is thus determined, as it becomes :

$$AI = -x + \frac{y(dx + Pdz) - z(Pdy - Qdx)}{Qdz + dy}$$

and

$$AK = -y + \frac{z(Pdy - Qdx) + x(dy + Qdz)}{dx + Pdz}.$$

Now the tangent of the angle, that the plane IMR makes with the plane APQ, is equal to :

$$\frac{\sqrt{((dx + Pdz)^2 + (dy + Qdz)^2)}}{Pdy - Qdx}.$$

The secant of this angle is equal to :

$$\frac{\sqrt{(1 + P^2 + Q^2)(dx^2 + dy^2 + dz^2)}}{Pdy - Qdx}$$

or the cosine is equal to :

$$\frac{Pdy - Qdx}{\sqrt{(1 + P^2 + Q^2)(dx^2 + dy^2 + dz^2)}}.$$

Example 1. [p. 467]

838. Let some cylindrical surface have the axis *AP*; the nature of this is expressed by the equation $dz = Qdy$ with *P* vanishing in the general equation $dz = Pdx + Qdy$. Whereby for the projection of the shortest line of this surface in the plane *APQ* on account of $P = 0$ and $dP = 0$ there is obtained this equation:

$$ddy = \frac{-QdQdy}{1 + Q^2}$$

or

$$l\frac{\alpha dx}{dy} = l\sqrt{(1 + Q^2)} \text{ and } \alpha dx = dy\sqrt{(1 + Q^2)},$$

if indeed *Q* is only given in terms of *y* ; but if *Q* is given in terms of *y* and *z*, the variable can be eliminated with the help of the equation $dz = Qdy$. As in the circular cylinder, in which $z^2 + y^2 = a^2$, then

$$z = \sqrt{(a^2 - y^2)} \text{ and } Q = \frac{-y}{\sqrt{(a^2 - y^2)}}.$$

Whereby it follows that :

$$\alpha dx = \frac{a dy}{\sqrt{(a^2 - y^2)}}.$$

Moreover in general, $\int dy\sqrt{(1+Q^2)}$ expresses the arc of the section normal to the axis AP; whereby with the said arc equal to *s* then $\alpha x = s$. From which it is understood, if such a surface is set out on a plane, to be the line of the shortest straight line, as agreed.

Example 2.

839. Let the proposed surface be some cone having the vertex at *A*; the equation for such a surface thus can be adapted, so that *z* is equal to a function of one dimension of *x* and *y*. Whereby in the equation $dz = Pdx + Qdy$ the letters *P* and *Q* are functions of zero dimensions of *x* and *y*. On this account, as now shown elsewhere, it follows that [see E044] :

$$Px + Qy = 0 \text{ or } Q = \frac{-Px}{y};$$

hence [on differentiation] there becomes : [p. 468]

$$dQ = \frac{Pxdy - Pydx - yxdP}{y^2} \text{ and } Pdy - Qdx = \frac{P(ydy + xdx)}{y}$$

and

$$dPdx + dQdy = \frac{y^2dPdx + Pxdy^2 - Pydxdy - yxdPdy}{y^2} = \frac{(ydx - xdy)(ydP - Pdy)}{y^2}$$

and finally :

$$1 + P^2 + Q^2 = \frac{y^2 + P^2y^2 + P^2x^2}{y^2}.$$

With which substituted, there is :

$$ddy = \frac{P(ydy + xdx)(ydx - xdy)(ydP - Pdy)}{ydx(y^2 + P^2y^2 + P^2x^2)}.$$

Put $y = px$; *P* is equal to a certain function of *p* only, because *P* is a function of zero dimensions of *x* and *y*. Now there is :

$$dy = pdx + xdp$$

and

$$ddy = xddp + 2dxdp = -\frac{P(p^2xdx + px^2dp + xdx)(pxdP - Ppdx - Pxdp)x^2dp}{px^3dx(p^2 + P^2p^2 + P^2)}$$

$$= \frac{Pdp(p^2dx + pxdp + dx)(Ppdx + Pxdp - pxdP)}{pdx(p^2 + P^2 + P^2p^2)}.$$

From which equation indeed the projection can hardly be recognized. Moreover how the shortest line in such a surface is to be determined, I have set out in more detail in Comment. III. p. 120, [E09 in this series of translations]. Moreover the same as before is to be noted concerning the shortest line, clearly because that set out from the conical surface becomes a straight line in the plane.

Scholium.

840. I will not tarry here with the determination in a similar manner of the shortest lines on other forms of surfaces, since in the place cited I have set out this material more fully. Hence I progress to the investigation of the lines which are described on a surface by a body acted on by some forces. Now before this, it is necessary that we examine more carefully the effect of each force.

Definition 4.

841. *In the following we call the pressing force that normal force, the direction of which is normal to the surface itself in which the body is moving.*

Corollary. [p. 469]

842. Therefore this pressing force either increases or decreases the centrifugal force, according as the direction of this force falls either opposite to the direction of the radius of osculation of the shortest line, or in that direction (79).

Definition 5.

843. *In the following we call the force of deflection that normal force, the direction of which is on surface in the tangent plane, and perpendicular to the path described by the body.*

Corollary.

844. Hence this force deflects the body from the shortest line that the body describes when acted on by no forces, and either draws the body to this or that side [of this line] as the direction of this force either pulls the body this way or that.

PROPOSITIO 92.

Problem.

845. *To determine the effect of the pressing force on the body moving on any surface, that is not acted on by any additional forces.*

Solution. [p. 470]

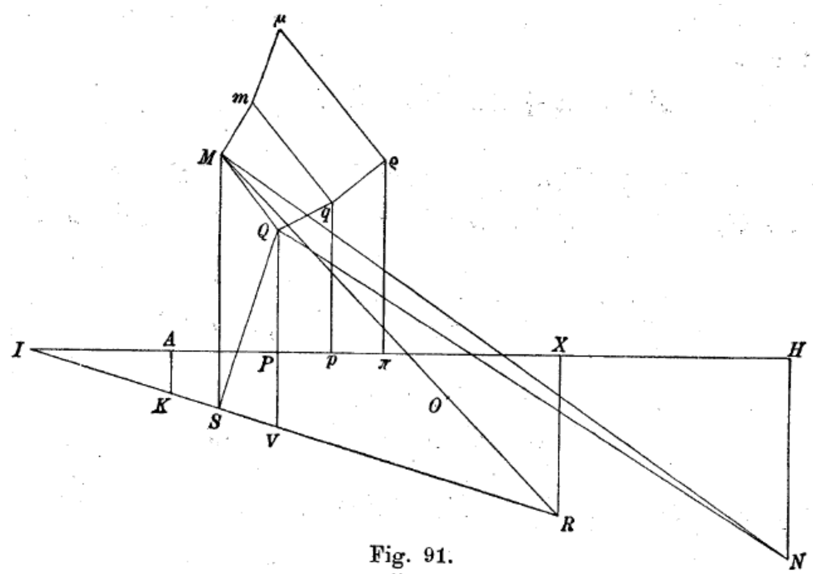


Fig. 91.

Because this pressing force is normal to the surface and thus the direction of this is in the direction *MN*, this affects neither the speed nor the direction, as the whole force is taken up on pressing the surface, and therefore the body progresses on the same line on which it was moving if this force were absent; but this is the shortest line determined in the preceding proposition. Therefore the body is moving on the line $Mm\mu$, and the radius of osculation of this *MO* lies along the normal to the surface *MN*. Therefore let the direction of this pressing force be *MN*, which therefore presses the surface inwards along *MN*. This pressing force is put equal to *M*; by that the surface is pressed on by a force along *MN* equal to *M*. But if the radius of osculation *MO* is put to lie along the same [undirected] line, then the centrifugal force is contrary to the pressing force, and the effect of this is lessened. Since moreover $Mm\mu$ is the shortest line, the radius of osculation is (73) :

$$MO = -\frac{(dx^2+dy^2+dz^2)\sqrt{(1+P^2+Q^2)}}{dPdx+dQdy};$$

if twice the height *v* corresponding to the speed at *M* is divided by which, then the centrifugal force is produced. On this account the force by which the surface is pressed along *MN*, is equal to

$$+ \frac{2v(dPdx + dQdy)}{(dx^2 + dy^2 + dz^2)\sqrt{(1 + P^2 + Q^2)}} + M.$$

[Note that the radius of curvature has been given as negative above for a concave curve and the sign of this has been reversed, to give a greater force pressing into the curve.]
Finally the position of this pressing force has been found previously (68) :

$$AH = x + Pz \text{ and } HN = -Qz - y,$$

obviously on being sent from the point *N*, in which the normal *MN* intersects the plane *APQ*, with the perpendicular *NH* to the axis .Q.E.I.

Corollary 1.

846. Since no another force is deflecting the normal force, neither a tangential force nor a resistive force if that is present affect the force pressing on the surface, [p. 471], and it is evident from any forces besides acting on the body that the pressing force is always to be of such a size as we have assigned here.

Corollary 2.

847. Therefore however great the departure between the path described by the body from the shortest line, the pressing force on the surface is still along the normal to the surface or along the radius of osculation of the shortest line, not along the radius of osculation of the curve described by the body, and neither is the length of this required for the pressing force.

Scholium.

848. For that reason we have used that formula of the radius of curvature of the shortest line, in which differentials of the second order are not present, lest these depend on the positions of the two elements Mm and $m\mu$, through which the body is itself moving. But now the radius of osculation must be known from a single element *Mm* ; for if the body does not describe the shortest line on account of a deflecting force, then differentials of the second order *ddy* and *ddz* must be advance, not present in the radius of osculation of the shortest line.

PROPOSITION 93.

Problem.

849. *To determine the effect on the motion of the body of the tangential force that pulls the body along the tangent line MT* (Fig.92) *on some surface.*

Solution. [p. 472]

Let this tangential force be equal to T and the body is progressing through the element Mm with a speed corresponding to the height v ; since this force diminishes the motion, then

$$dv = -T \cdot Mm = -T\sqrt{(dx^2 + dy^2 + dz^2)}$$

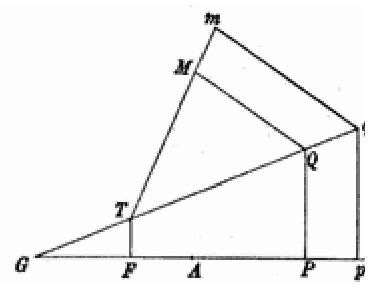


Fig. 92.

with the quantities maintaining the same denominations that we have used previously. Now besides this force does not affect either the pressing force nor by the deviation from the shortest line. Now according to the position of the direction of this force, the tangent MT is produced that then crosses the plane APQ at T, then T is a point on the element qQ produced. Therefore

$$dz : \sqrt{(dx^2 + dy^2)} = z : QT$$

and hence

$$QT = \frac{z\sqrt{(dx^2 + dy^2)}}{dz}.$$

From T the perpendicular TF is sent to the axis; then

$$\sqrt{(dx^2 + dy^2)} : dx = QT : PF;$$

whereby there is obtained :

$$PF = \frac{zdx}{dz} \text{ and } AF = \frac{zdx - xdz}{dz}.$$

Again since $dx : dy = \frac{zdx}{dz} : y - FT$ then $FT = y - \frac{zdx}{dz}$, from which the point T is determined. Q.E.I.

Corollary.

850. Since resistance is to be referred to the tangential force, from these it is understood, how the effect is to be determined. For if the resistance is equal to R, then

$$dv = -(T + R)\sqrt{(dx^2 + dy^2 + dz^2)}.$$

PROPOSITION 94.

Problem.

851. *To determine the effect of the normal deflecting force N on a body moving on any surface.*

Solution. [p. 473]

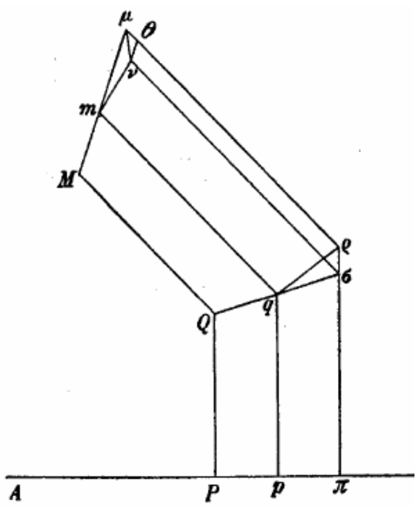


Fig. 93.

On placing as before $AP = x$, $PQ = y$ and $QM = z$ (Fig.93) the nature of the surface is expressed by this equation :

$$dz = Pdx + Qdy$$

and the body is moving with a speed corresponding to the height v through the element Mm; in traversing which, unless the deflecting force is present, it proceeds along the element $m\mu$ following the shortest line and it gives :

$$\pi\varrho = y + 2dy + \frac{(Pdy - Qdx)(dPdx + dQdy)}{dx(1 + P^2 + Q^2)}$$

and

$$\varrho\mu = z + 2dz + \frac{(dx + Pdz)(dPdx + dQdy)}{dx(1 + P^2 + Q^2)}$$

(835). Now the force of the normal deflection N is added, which has the direction against increase. Therefore this force has the effect, that the body in describing the element Mm does not advance to $m\mu$, but is deflected forwards from this direction. Therefore we place it to act along $m\nu$; Mm and $m\nu$ are two elements of the curve described by the body. Whereby with the perpendicular $\nu\alpha$ sent from ν to the plane APQ then :

$$\pi\sigma = y + 2dy + ddy \text{ and } \sigma\nu = z + 2dz + ddz.$$

There is hence obtained :

$$\sigma\varrho = \frac{(Pdy - Qdx)(dPdx + dQdy)}{dx(1 + P^2 + Q^2)} - ddy$$

and

$$\mu\varrho - \nu\sigma = \frac{(dx + Pdz)(dPdx + dQdy)}{dx(1 + P^2 + Q^2)} - ddz.$$

Now on placing for brevity:

$$\frac{(Pdy-Qdx)(dPdx+dQdy)}{dx(1+P^2+Q^2)}=dd\eta \text{ and } \frac{(dx+Pdz)(dPdx+dQdy)}{dx(1+P^2+Q^2)}=dd\zeta$$

the radius of osculation corresponding to the angle between the elements $\mu m\nu$ (72) is equal to :

$$\frac{(dx^2+dy^2+dz^2)^{\frac{3}{2}}}{\sqrt{((dzdd\eta-dzddy-dydd\zeta+dyddz)^2+dx^2(dd\eta-ddy)^2+dx^2(dd\zeta-ddz)^2)}}.$$

Therefore if we call the radius here equal to *r*, then $N=\frac{2v}{r}$ or $2v=Nr$, since here the angle is generated in the same way in which a body in a plane is deflected from a straight line by a normal force. Now it follows that : [p. 474]

$$dzdd\eta-dydd\zeta=-\frac{(dy+Qdz)(dPdx+dQdy)}{1+P^2+Q^2}.$$

And in place of $dd\eta$ and $dd\zeta$ with the due values substituted, the radius becomes :

$$r=\frac{(dx^2+dy^2+dz^2)^{\frac{3}{2}}}{\sqrt{\left(dx^2(ddy^2+ddz^2)+(dzddy-dyddz)^2-\frac{(dx^2+dy^2+dz^2)(dPdx+dQdy)^2}{1+P^2+Q^2}\right)}}.$$

But since through differentiation of the equation $dz=Pdx+Qdy$, there is $dPdx+dQdy=ddz-Qddy$, there is made on substituting this into the equation $dz=Pdx+Qdy$, on calling

$$r=\frac{(dx^2+dy^2+dz^2)^{\frac{3}{2}}\sqrt{(1+P^2+Q^2)}}{-ddy(dx+Pdz)+ddz(Pdy-Qdx)}.$$

Then on this account,

$$ddz(Pdy-Qdx)-ddy(dx+Pdz)=\frac{N}{2v}(dx^2+dy^2+dz^2)^{\frac{3}{2}}\sqrt{(1+P^2+Q^2)}.$$

Q.E.I.

Scholium 1.

852. This formula agrees with that which we have found previously (79) in determining the effect of this kind of force. For the difference is only in the sign of the letter N, as it was apparent that the force everywhere had to be taken as negative. And even here we cannot be certain of the sign, because we have extracted it from the root of a squared quantity, and it can be equally positive or negative. Now this doubt, if the calculus is adapted to this special case, is at once removed, because the formula must be of this kind, so that the point ν falls on this side of μ, if the force N is deflecting the body forwards, in order that the direction should be as we have put in place. From which with the help of an example the sign of the square root can also be determined and hence the formula itself found.

Corollary 1. [p. 475]

853. If the deflecting force *N* vanishes, the body continues moving along its own shortest line ; which is indicated by the equation also. For on putting $N = 0$ there is obtained

$$ddy(dx + Pdz) = ddz(Pdy - Qdx),$$

which is the equation for the shortest line :

Corollary 2.

854. Therefore whatever the pressing force and the tangential force and the force of the resistance acting on the body moving on the surface, only if nothing aids the deflecting force, then the body always moves along the shortest line [now called geodesic curves].

Scholium 2.

855. Moreover as concerning the position of this deflecting force *N*, that can be deduced as follows, since that force is placed on the surface in the tangent plane and likewise it is normal to the curve described, therefore let *MG* (Fig. 94) be the direction of this force and *G* is the point at which it crosses the plane *APQ* , thus so that the force *N* can be thought to pull along the line *MG*, while that force we have put before to be deflecting and pressing forwards. Therefore in the first place it is necessary to determine the intersection of the plane of the tangent of the surface at *M* with the plane *APQ*, which is the right line *TVG* ; now this can be found, if two tangents of the surface can be produced as far as the plane APQ, and the points in which they are incident on the plane *APQ* are joined by a right line. Therefore

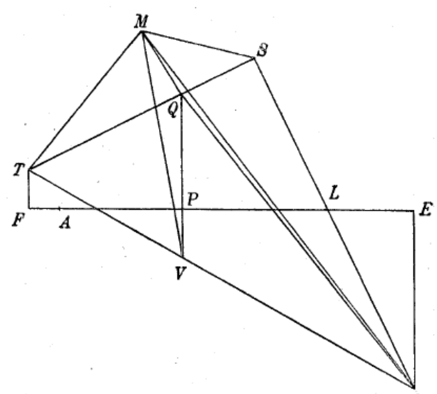


Fig. 94.

let *MT* be the tangent of the described line, which in addition is a tangent of the surface also ; then as we have now established, [p. 476]

$$AF = \frac{zdx}{dz} - x \text{ and } FT = y - \frac{zdy}{dz}$$

(849). Again the surface is understood to be cut by the plane *PQM* and let *MV* be the tangent of this cut ; then we have $QV = \frac{z}{Q}$ from the equation $dz = Pdx + Qdy$ on putting $dx = 0$ [or, from the subtangent]. Therefore the point *V* is known, on account of which the line *TV* produced is the intersection of the tangent plane of the surface at *M* with the plane *APQ*. Therefore the point *G*, in which the line *MG* crosses the plane *APQ*, is placed on the line *TV*. Again there is taken on the line *TQ* :

$$QS = \frac{zdz}{\sqrt{(dx^2 + dy^2)}}$$

and *MS* is the normal described to the element *Mm*. And if the normal *SG* is drawn to *QS*, from this line *SG* all the right lines drawn to *M* are perpendicular to the element *Mm*. [Thus, the element *Mm* is normal to the plane *MSG*.] Whereby since *MG* is also normal to the element described, the point *G* is also placed on the line *SG*. Hence the point *G* is at the intersection of the lines *TV* and *SG*. Now it is the case that :

$$PL = \frac{ydy + zdz}{dx}$$

[See annoted Fig. 94] and ang. *ELG* = ang. *PQT*. Putting *GE = t;* then

$$LE = \frac{tdy}{dx} \text{ and } PE = \frac{ydy+tdy+zdz}{dx}.$$

Finally also on account of the similar triangles, we have $FP : FT + PV = PE : GE - PV$, that is :

$$\frac{zdx}{dz} : \frac{z}{Q} - \frac{zdy}{dz} = \frac{ydy + tdy + zdz}{dx} : t - \frac{z}{Q} + y.$$

Hence there comes about :

$$t = \frac{z(dx + Pdz)}{Qdx - Pdy} - y = GE \text{ and } AE = x + \frac{z(dy + Qdz)}{Qdx - Pdy},$$

and thus the point *G* is determined. Therefore, if the line *QG* is drawn, then

$$QG^2 = \frac{z^2(dx + Pdz)^2}{(Qdx - Pdy)^2} + \frac{z^2(dy + Qdz)^2}{(Qdx - Pdy)^2}$$

and

$$QG = \frac{z\sqrt{(dx^2+dy^2+dz^2+dz^2(1+P^2+Q^2))}}{Qdx-Pdy}$$

and

$$MG = \frac{z(dx^2+dy^2+dz^2)^{\frac{1}{2}}\sqrt{(1+P^2+Q^2)}}{Qdx-Pdy}.$$

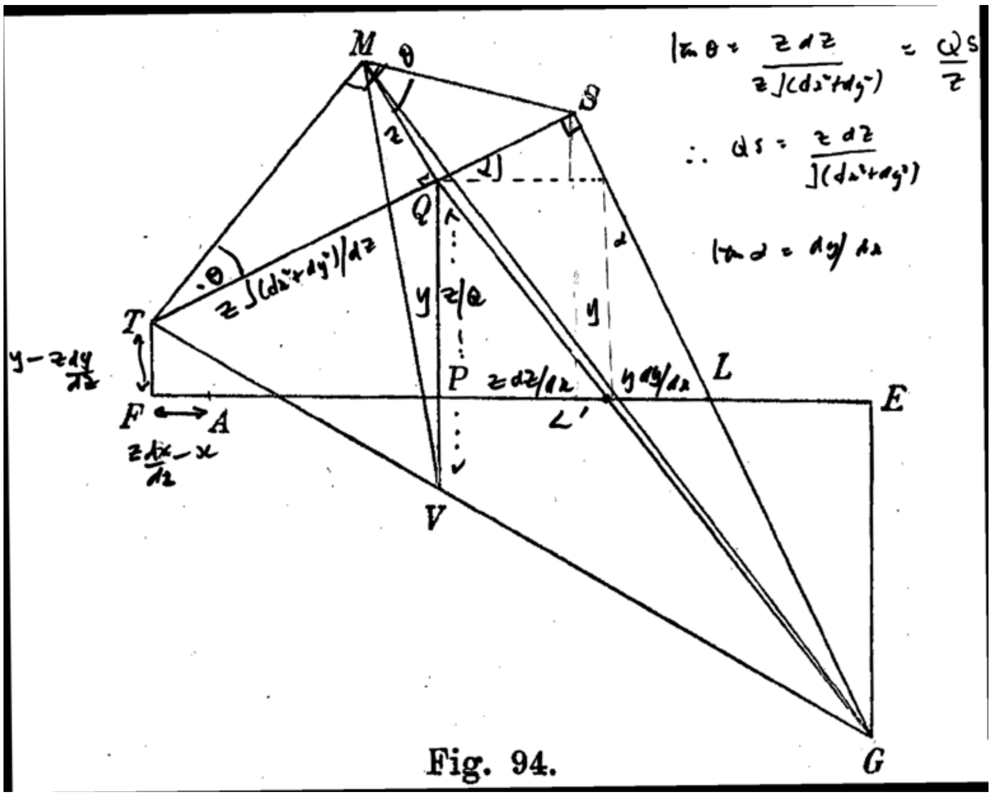


Fig. 94.

CAPUT QUARTUM

DE MOTU PUNCTI SUPER DATA SUPERFICIE

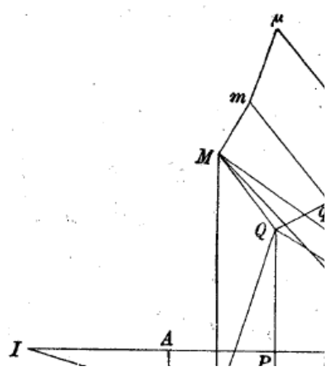
[p. 457]

PROPOSITIO 90.

Problema.

821. *Data via in superficie quacunque* $Mm\mu$ (Fig. 91) *invenire eius positionem respectu plani dati APQ et radii osculi illius viae in M tam positionem quam longitudinem nec non normalis in superficiem situm.*

Solutio.



Sumto pro lubitu plano *APQ* in eoque axe *AP*, quorum respectu positio curvae $Mm\mu$ sit determinanda, ex tribus punctis *M*, *m* et μ datae viae in superficie in planum *APQ* demittantur perpendicula *MQ*, *mq*, $\mu\rho$ atque ex punctis *Q*, *q*, ρ ad axem *AP* perpendicula *QP*, *qp*, et $\rho\pi$. Posito nunc initio abscissarum in *A* sit *AP* = *x*, *PQ* = *y* et *QM* = *z*. Quia porro superficies data ponitur, dabitur aequatio eius naturam exprimens inter tres has variables *x*, *y* et *z*; quae aequatio sit haec

$$dz = Pdx + Qdy.$$

Cum hac aequatione si coniungatur alia aequatio, exprimetur linea quaedem in ista superficie existens; quare, cum linea $Mm\mu$ data ponatur, dabitur praeter aequationem $dz = Pdx + Qdy$ insuper alia aequatio, qua curva $Mm\mu$ determinatur, [p. 458] quam autem hic repraesentare non est opus. Sint elementa abscissae $Pp, p\pi = dx$ inter se aequalia seu sumatur elementum *dx* constans. Erit ergo

$$pq = y + dy, \quad \pi\varrho = y + 2dy + ddy$$

atque

$$qm = z + dz \quad \text{et} \quad \varrho\mu = z + 2dz + ddz.$$

His positis sit *MN* normalis in superficiem in puncto *M* et *N* punctum, quo haec normalis plano *APQ* occurrit; demittatur ex *N* in axem perpendiculum *NH*; erit

$$AH = x + Pz \quad \text{et} \quad HN = -Qz - y$$

(68). Sit nunc *MR* posito radii osculi curvae $Mm\mu$ et *R* incidentia eius in planum *APQ*; erit ex *R* in axem demisso perpendiculo *RX*

$$AX = \frac{zdx(dyddy + dzddz)}{(dx^2 + dy^2)ddz - dydzddy} + x$$

atque

$$XR = \frac{zdx^2ddy + zdz(dzddy - dyddz)}{(dx^2 + dy^2)ddz - dydzddy} - y$$

(68). Longitudino vero radii osculi, scilicet *MO*, =

$$\frac{(dx^2 + dy^2 + dz^2)^{\frac{3}{2}}}{\sqrt{(dx^2(ddy^2 + ddz^2) + (dyddz - dzddy)^2)}}$$

(72). Denique concipiatur planum, in quo sita sunt elementa $Mm, m\mu$, productum, donec planum *APQ* intersecet, sitque intersectio recta *RKI*, cui ex *A* erecta perpendicularis in *K* occurrat et ex *P* in *V*; inventum est supra esse

$$PV = \frac{zddy}{ddz} - y$$

(68). Cum nunc sit $XR - PV : AX - AP = PV : PI$, erit

$$PI = \frac{(AX - AP)PV}{XR - PV}.$$

Est vero

$$AX - AP = \frac{zdx(dyddy + dzddz)}{(dx^2 + dy^2)ddz - dydzddy}$$

atque

$$XR - PV = \frac{zddyddz(dz^2 - dy^2) + zdydz(ddy^2 - ddz^2)}{ddz((dx^2 + dy^2)ddz - dydzddy)}.$$

Quibus substitutis erit [p. 459]

$$PI = \frac{dx(dyddy+dzddz)(zddy-yddz)}{ddyddz(dz^2-dy^2)+dydz(ddy^2-ddz^2)} = \frac{zdxddy-ydxddz}{dzddy-dyddz}$$

atque

$$AI = PI - AP = \frac{zdxddy-ydxddz-xdzddy+xdyddz}{dzddy-dyddz}.$$

Hinc reperitur

$$AK = \frac{PV\cdot AI}{PI} = \frac{zdxddy-ydxddz-xdzddy+xdyddz}{dxddz}.$$

Plani vero, in quo sita sunt elementa Mm et $m\mu$, inclinatio ad planum *APR* invenietur demittenda ex *Q* perpendiculari *QS* ad intersectionem *RI*; erit enim tangens anguli inclinationis = $\frac{QM}{QS}$. At cum sit $IV : PI = QV : QS$, erit illa tangents =

$$\frac{QM\cdot IV}{PI\cdot QV} = \frac{\sqrt{(dx^2ddz^2+(dzddy-dyddz)^2)}}{dxddy}.$$

Anguli vero *NMR*, quaem radius osculi cum normali in superficiem constituit tangens est =

$$\frac{ddy(dx+Pdz)-ddz(Pdy-Qdx)}{(ddz-Qddy)\sqrt{(dx^2+dy^2+dz^2)}}$$

(71). Ex his igitur omnia deduci possunt, quae ad positionem curvae $Mm\mu$ cognoscendum requiruntur. Q.E.I.

Corollarium 1.

822. Curvae $Mm\mu$ proiectio in plano *APQ* est curva $Qq\rho$, cuius natura exprimitur aequatione inter *x* et *y*. Quare ista proiectio habebitur, si ope aequationum $dz = Pdz + Qdy$ et eius, qua ipsa curvaq in superficie ducta determinatur, nova formetur aequatio eliminanda variabili *z*, quae sit inter *x* et *y* tantum.

Corollarium 2.

823. Simili modo, si eliminatur *x*, ut prodeat aequatio inter *y* et *z*, hac aequatione definietur proiectio curvae $Mm\mu$ in plano, quod est normale ad axem *AX*. [p. 460] Atque aequatio, in qua non inest *y*, sed tantum *x* et *z*, dabit proiectionem curvae $Mm\mu$ in plano, quod normaliter planum *APQ* secundum axem AX intersecat.

Corollarium 3.

824. Curvae autem $Mm\mu$ natura ex duabus eius proiectionibus in duobus planis invicem normalibus distincte cognoscitur. Qualem cognitionem quoque suppeditat unica proiectio una cum ipsa superficie.

Corollarium 4.

825. Quamobrem ad curvam in superficie data quamcunque characteribus designandam requiritur, ut praeter aequationem $dz = Pdz + Qdy$, qua superficies determinatur, detur aequatio duas tantum variabiles involvens pro proiectione quapiam curvae $Mm\mu$.

Corollarium 5.

826. Si superficies secitur plano, simili modo, quo conus ad sectiones conicas producendas secari solet, curva ex hac sectione orta erit in eodem plano. Quare his casibus tam posito rectae *IR* erit constans quam plani *IMR* inclinatio ad planum *APQ*.

Exemplum.

827. Si igitur detur superficies quaecunque eaque secetur plano *IMR*, quaeratur curva hac sectione orta. [p. 461] Ad hoc ponatur $AI = a$, $AK = b$ et anguli inclinationis plani *IMR* ad planum *APQ* tangens = *m*; eritque

$$a = \frac{zdxddy - ydxddz}{dzddy - dyddz} - x \text{ et } b = \frac{zdxddy - ydxddz - xdzddy + xdyddz}{dxddz}$$

atque

$$m = \frac{\sqrt{(dx^2ddz^2 + (dzddy - dyddz)^2)}}{dxddy}$$

Ex quibus aequationibus coniunctis cum $dz = Pdz + Qdy$ natura curvae hac sectione genitae determinabitur. Ex prioribus vero duabus aequationibus oritur

$$\frac{b}{a} = \frac{dzddy - dyddz}{dxddz} \text{ seu } ddz : ddy = adz : bdx + ady;$$

cuius aequationis integralis est

$$\frac{1}{a}ldz = \frac{1}{a}l(bdx + ady) - \frac{1}{a}lc \text{ seu } cdz = bdx + ady$$

et porro

$$cz = bx + ay + ff.$$

In prima vero aequatione si loco *ddz* et *ddy* eorum proportionalis substituantur, prodibit

$$a + x = \frac{bzdx + azdy - aydz}{bdz} \text{ seu } abdz + bxdz = bzdx + azdy - aydz,$$

cuius per *zz* divisae integralis est haec

$$c - \frac{ab}{z} = \frac{bx + ay}{z} \text{ seu } cz = bx + ay + ab;$$

quod ergo ante erat *ff*, hic est *ab*, seu *ff* = *ab*. Constantem vero *c* tertia aequatio definiet; erit autem

$$m=\frac{dz\sqrt{(a^2+b^2)}}{bdx+ady}\ \text{seu}\ \frac{dz\sqrt{(a^2+b^2)}}{m}=bdx+ady.$$

Quare erit superior littera

$$c=\frac{\sqrt{(a^2+b^2)}}{m}$$

atque praeter aequationem superficiei naturam exprimentem habetur ista

$$\frac{z\sqrt{(a^2+b^2)}}{m}=bx+ay+ab,$$

ex quibus natura quaesitae curvae est derivanda. Quia autem tota curva quaesit est in plano *IMR*, commodissime ea exprimetur aequatione inter coordinatas orthogonales in eodem plano sumtas. Sumto ergo *IR* pro axe ex *M* in eum demittatur perpendiculum *MS* et vocetur *IS* = *t* et *MS* = *u*. Est vero *IA* : *AK* = *IP* : *PV* seu

$$PV=\frac{ab+bx}{a}$$

et

$$QV=\frac{ab+bx+ay}{a}=\frac{z\sqrt{(a^2+b^2)}}{ma}.$$

Porro est [p. 462]

$$\sqrt{(a^2+b^2)}:a=\frac{z\sqrt{(a^2+b^2)}}{ma}:QS;$$

quare erit

$$QS=\frac{z}{m}\quad\text{et}\quad SV=\frac{bz}{ma}.$$

Ex his prodibit

$$MS=u=\frac{z\sqrt{(1+m^2)}}{m}\quad\text{atque}\quad IS=t=\frac{m(a+x)\sqrt{(a^2+b^2)}-bz}{ma}.$$

Ex quo oritur

$$z=\frac{mu}{\sqrt{(1+m^2)}}\quad\text{et}\quad x=\frac{bu+at\sqrt{(1+m^2)}}{\sqrt{(a^2+b^2)(1+m^2)}}-a$$

et substitutis his valoribus in aequatione $\frac{z\sqrt{(a^2+b^2)}}{m}=bx+ay+ab$ prodibit

$$y=\frac{au-bt\sqrt{(1+m^2)}}{\sqrt{(1+m^2)(a^2+b^2)}}.$$

His igitur valoribus loco *x*, *y* et *z* in aequatione superficiei substitutis proveniet aequatio inter *t* et *u* seu coordinatas orthogonales curvae quaesitae.

Corollarium 6.

828. Si intersectio plani secantis *IR* in ipsum axem *AX* incidat sumaturque *I* in *A*, erit

$$z = \frac{mu}{\sqrt{(1+m^2)}}, \quad y = \frac{u}{\sqrt{(1+m^2)}} \quad \text{et} \quad x = t.$$

Corollarium 7.

829. Si intersectio *IR* plani secantis *IMR* cum plano *APQ* fuerit normalis ad axem *AX*, erit $b = \infty$. Quare prodibunt

$$z = \frac{mu}{\sqrt{(1+m^2)}}, \quad x = \frac{u}{\sqrt{(1+m^2)}} - a \quad \text{et} \quad y = -t.$$

Corollarium 8.

830. Cum valores loco *z, y* et *x* substituendi sint unius dimensionis ipsarum *t* et *u*, perspicuum est aequationem inter *t* et *u* non plures habere posse dimensiones quam ipsam aequationem inter *z, y* et *x*.

Corollarium 9. [p. 463]

831. Quare si aequatio inter *z, y* et *x* fuerit duarum dimensionum, cuiusmodi praeter conicam innumerabiles dantur superficies, omnes sectiones plano factae erunt sectiones conicae.

Scholion.

832. In Comment. Tom. III ea dissertatione, in qua lineam brevissimam in superficie quacunque determinavi, tria praecipue superficierum genera sum persecutus, quae erant cylindrica, conica et tornata seu rotanda.
[L. Euleri Commentatio 9 (E09): *De linea brevissima in superficie quacunque duo quaelibet puncta iungente*. Comment. acad. sc. Petrop. 3 (1728), 1732, p. 110; *Opera Omnia* series I, vol. 25.]

Aequatio vero generalis $dz = Pdz + Qdy$ dat superficies cylindricas, si *P* evanescit et *Q* tantum ab *y* et *z* pendeat, ita ut aequationem pro hoc superficierum genere abscissa *x* non ingrediatur; omnes enim sectiones inter se parallelae sunt quoque adquales; pro his ergo est aequatio $dz = Qdy$.

Ad genus coniodicum refero omnes eas superficies, quae generantur ducendis rectis ex singulis curvae cuiupiam punctis ad punctum fixum extra planum eius curvae situm. Quae superficies hanc habent proprietatem, ut omnes sectiones parallelae sint inter se similes earumque latera homologia ut distantiae sectionum a vertice coni. Aequationes vero pro huiusmodi superficiebus, si quidem vertex coni fuerit in *A*, ita sunt comparatae, ut *x, y* et *z* coniunctim ubique eundem dimensionum numerum constituant.

Superficies denique tornatae seu rotundae mihi sunt, quae generantur conversione cuiuscunque curvae circa axem; qui axis si fuerit *AX*, posito *x* constante aequatio inter *y* et *z* dabit circulum centri *P*. Quare aequatio pro iis hanc habeat formam [p. 464]

$$dz = Pdx - \frac{ydy}{z} \quad \text{seu} \quad zdz + ydy = zPdx,$$

ubi *Pz* ab *x* tantum pendet; seu est

$$Q = -\frac{y}{z} \quad \text{et} \quad P = \frac{X}{z}$$

existente *X* functione ipsius *x*.

Quemadmodum autem in his superficiebus tornatis omnes sectiones axi normales sunt circuli, ita tales superficies concipi possunt, quarum sectiones axi normales sint curvae quaecunque similes. Tales superficies omnes hac continebuntur proprietate generali, ut functio quaecunque ipsius *x* aequalis sit functioni eiusdem ubique dimensionum ipsarum *y* et *z* numeri. Ut si iste dimensionum numerus fuerit *n*, aequationis $Pdx = Rdz + Qdy$ pro haec erit proprietas, ut sit

$$Rz + Qy = n\int Pdx \quad \text{vel} \quad Rdz + Qdy = \frac{zdR + ydQ}{n-1}.$$

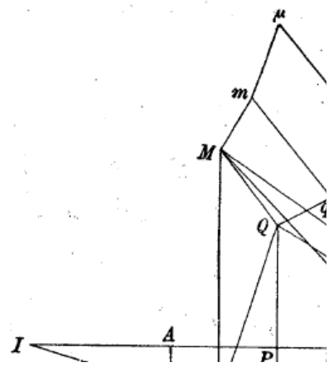
[Vide notam p. 44.] Ex quo, an data aequatio sit ad huiusmodi superficiem, statim concludi potest.

PROPOSITIO 91.

Problema.

833. *In superficie quacunque data lineam determinare, quam corpus in ea motum et a nullis potentis sollicatum describit tam in vacuo quam in medio quocunque resistente.*

Solutio. [p. 465]



Quia corpus a nullis potentiis absolutis sollicari ponitur, linea ab eo in superficie descripta erit linea brevissima in vacuo (62). Medii autem resistentis vis celeritatem corporis tantum imminuit neque directionem ullo modo afficit; quare etiam in medio resistente via a corpore in quavis superficie descripta erit pariter brevissima. Manentibus igitur ut ante $AP = x$, $PQ = y$ et $QM = z$. (Fig. 91) sit $dz = Pdx + Qdy$ aequatio superficiei naturam exprimens atque $Mm, m\mu$ duo lineae brevissimae cuiuspiam elementa. Ex his supra pro linea brevissima inventa est haec aequatio

$$Pdzddy + dxddy = Pdyddz - Qdxddz$$

(69), unde oritur

$$ddz = \frac{(Pdz + dx)ddy}{Pdy - Qdx}.$$

At aequatio ad superficiem differentia dat

$$ddz = dPdx + Qddy + dQdy,$$

ex quibus coniunctis fit

$$ddy = \frac{(Pdy - Qdx)(dPdx + dQdy)}{dx(1 + P^2 + Q^2)} \quad \text{et} \quad ddz = \frac{(Pdz + dx)(dPdx + dQdy)}{dx(1 + P^2 + Q^2)}.$$

Dato ergo elemento Mm sequens $m\mu$ in linea brevissima invenietur; erit enim

$$\pi\varrho = PQ + 2dy + ddy \quad \text{et} \quad \varrho\mu = QM + 2dz + ddz$$

et ipsarum ddy et ddz valores sunt inventi. Quare hinc sequentis cuiusque elementi positio determinatur atque ipsius lineae brevissimae natura per quamcunque eius proiectionem cognoscitur. Q.E.I.

Corollarium 1.

834. Si in aequatione pro superficie P et Q tantum per x et y dantur, aequatio

$$ddy = \frac{(Pdy - Qdx)(dPdx + dQdy)}{dx(1 + P^2 + Q^2)}$$

proiectionem lineae brevissimae in plano *APQ* denotat. [p. 466]

Corollarium 2.

835. Pro linea ergo brevissima $Mm\mu$ sumtis elementis axis aequalibus erit

$$\pi\varrho = y + 2dy + \frac{(Pdy - Qdx)(dPdx + dQdy)}{dx(1 + P^2 + Q^2)}$$

atque

$$\varrho\mu = z + 2dz + \frac{(Pdz + dx)(dPdx + dQdy)}{dx(1 + P^2 + Q^2)},$$

ex quibus aequationibus punctum μ ex duobus praecendentibus M et m cognoscitur.

Corollarium 3.

836. Quia pro linea brevissima angulus *RMN* evanescit (71), incidet R in N; positio ergo radii osculi ita se habebit, ut sit $AX = x + Pz$ et $XR = -Qz - y$. Longitudo vero radii osculi erit (73) =

$$-\frac{(dx^2 + dy^2 + dz^2)\sqrt{(1 + P^2 + Q^2)}}{dPdx + dQdy}.$$

Corollarium 4.

837. Planum vero *IMR*, in quo sita sunt elementa lineae brevissimae $Mm\mu$, ita determinabitur, ut sit

$$AI = -x + \frac{y(dx + Pdz) - z(Pdy - Qdx)}{Qdz + dy}$$

et

$$AK = -y + \frac{z(Pdy - Qdx) + x(dy + Qdz)}{dx + Pdz}.$$

Tangens vero anguli, quem planum IMR cum plano APQ constituit, erit =

$$\frac{\sqrt{((dx + Pdz)^2 + (dy + Qdz)^2)}}{Pdy - Qdx}.$$

Huiusque anguli secans est =

$$\frac{\sqrt{(1+P^2+Q^2)(dx^2+dy^2+dz^2)}}{Pdy-Qdx}$$

seu cosinus =

$$\frac{Pdy-Qdx}{\sqrt{(1+P^2+Q^2)(dx^2+dy^2+dz^2)}}.$$

Exemplum 1. [p. 467]

838. Sit superficies cylindrica quaecunque axem habens *AP*; exprimatur eius natura hac aequatione $dz = Qdy$ evanescente *P* in generali aequatione $dz = Pdx + Qdy$. Quare pro proiectione lineae brevissimae huius superficiei in plano *APQ* habebitur ob $P = 0$ et $dP = 0$ haec aequatio

$$ddy = \frac{-QdQdy}{1+Q^2}$$

seu

$$l\frac{\alpha dx}{dy} = l\sqrt{(1+Q^2)} \quad \text{et} \quad \alpha dx = dy\sqrt{(1+Q^2)},$$

si quidem *Q* tantum per *y* detur; at si *Q* per *y* et *z* detur, variabilis *z* est eliminanda ope aequationis $dz = Qdy$. Uti in cylindro circulari, in quo est $z^2 + y^2 = a^2$, erit

$$z = \sqrt{(a^2 - y^2)} \quad \text{et} \quad Q = \frac{-y}{\sqrt{(a^2-y^2)}}.$$

Quare erit

$$\alpha dx = \frac{ady}{\sqrt{(a^2-y^2)}}.$$

In genere autem $\int dy\sqrt{(1+Q^2)}$ exprimit arcum sectionis ad axem AP normalis; quare dicto hoc arcu = *s* erit $\alpha x = s$. Ex quo intelligitur, si talis superficies in planum explicetur, fore lineam brevissimam rectam; uti constat.

Exemplum 2.

839. Sit superficies proposita conica quaecunque verticem habens in *A*; aequatio pro tali superficie ita poterit adaptari, ut *z* aequatur functioni unius dimensionis ipsarum *x* et *y*. Quare in aequatione $dz = Pdx + Qdy$ litterae *P* et *Q* erunt functiones nullius dimensiones ipsarum *x* et *y*. Hanc ob rem, uti iam alibi ostendi, [vide notam, p.44] erit

$$Px + Qy = 0 \quad \text{seu} \quad Q = \frac{-Px}{y};$$

unde fiet [p. 468]

$$dQ = \frac{Pxdy - Pydx - yxdP}{y^2} \quad \text{et} \quad Pdy - Qdx = \frac{P(ydy + xdx)}{y}$$

atque

$$dPdx + dQdy = \frac{y^2dPdx + Pxdy^2 - Pydxdy - yxdPdy}{y^2} = \frac{(ydx - xdy)(ydP - Pdy)}{y^2}$$

et tandem

$$1 + P^2 + Q^2 = \frac{y^2 + P^2y^2 + P^2x^2}{y^2}.$$

Quibus substitutis erit

$$ddy = \frac{P(ydy + xdx)(ydx - xdy)(ydP - Pdy)}{ydx(y^2 + P^2y^2 + P^2x^2)}.$$

Ponatur $y = px$; aequabitur P functioni cuidam ipsius p tantum, quia P est functio nullius dimensionis ipsarum x et y. Erit vero

$$dy = pdx + xdp$$

et

$$ddy = xddp + 2dxdp = -\frac{P(p^2xdx + px^2dp + xdx)(pxdP - Ppdx - Pxdp)x^2dp}{px^3dx(p^2 + P^2p^2 + P^2)}$$
$$= \frac{Pdp(p^2dx + pxdp + dx)(Ppdx + Pxdp - pxdP)}{pdx(p^2 + P^2 + P^2p^2)}.$$

Ex qua aequatione quidem proiectio difficulter cognoscitur. Quomodo autem linea brevissima in tali superficie sit determinanda, fusius docui in Comment. III. p. 120, [E09]. Ceterum idem de linea brevissima est notandum quod ante, scilicet quod ea in planum explicata superficie conica abeat in rectam.

Scholion.

840. Simili modo in determinandis lineis brevissimis super aliis superficierum speciebus non hic immoror, quia in citato loco hanc materiam plenius exposui. Progredior ergo ad investigationem linearum, quae in superficie a corpore a quibuscunque potentiis sollicitato describuntur. Antea vero necesse est, ut in effectus cuiusque potentiae curatius inquiramus.

Definitio 4.

841. *Vim prementem vocabimus in sequentibus eam vim normalem, cuius directio est normalis ad ipsam superficiem, in qua corpus movetur.*

Corollarium. [p. 469]

842. Haec vis premens ergo vel auget vim centrifugam vel minuit, prout eius directio directioni radii osculi lineae brevissimae vel contraria est vel in eam incidit (79).

Definitio 5.

843. *Vim deflectentem vocabimus in sequentibus eam vim normalem, cuius directio est in plano superficiem tangente et perpendicularis in viam a corpore descriptam.*

Corollarium.

844. Haec ergo vis corpus a linea brevissima, quam a nullis potentiis sollicitatum describeret, deflectit et vel cis vel ultra eam detrahit pro eius directione vel cis vel ultra tendente.

PROPOSITIO 92.

Problema.

845. *Determinare effectum vis prementis in corpus super superficie quacunque motum, quod praeterea a nullin potintiis sollititatur.*

Solutio. [p. 470]

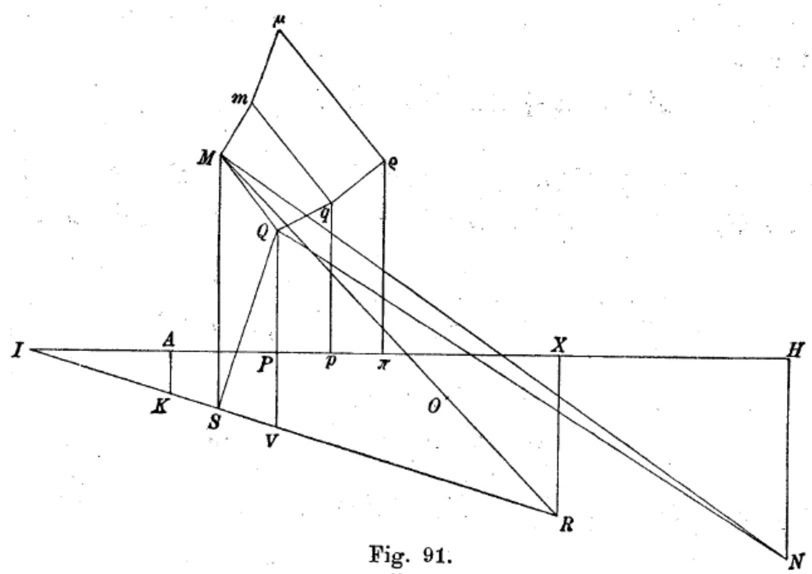


Fig. 91.

Quia haec vis premens est normalis in superficiem ideoque eius directio *MN*, ea neque celeritatem neque directionem afficiet, sed tota in pressione superficiei consumetur; corpus igitur in eadem linea progredietur, in qua, si haec vis abesset, moveretur; quae autem est linea brevissima in propositione praecedente determinata. Movebitur ergo corpus in linea $Mm\mu$, cuius radius osculi *MO* incidet in normalem superficiei *MN*. Sit ergo *MN* directio huius potentiae prementis, quae propterea superficiem versus interiora secundum *MN* premet. Ponatur haec vis premens = *M*; premetur ab ea superficies secundum *MN* vi = *M*. At si radius osculi *MO* in eandem plagam incidere ponatur, vis centrifuga vi prementi erit contraria eiusque effectum minuet. Cum autem $Mm\mu$ sit linea brevissima, est radius osculi (73)

$$MO = -\frac{(dx^2+dy^2+dz^2)\sqrt{(1+P^2+Q^2)}}{dPdx+dQdy};$$

per quem si divitatur dupla altitudo *v* celeritati in *M* debita, prodibit vis centrifuga. Hanc ob rem erit vis, qua superficies secundum *MN* premitur, =

$$+\frac{2v(dPdx+dQdy)}{(dx^2+dy^2+dz^2)\sqrt{(1+P^2+Q^2)}}+M.$$

Positio tandem huius vis prementis per superiora (68) inventa est

$$AH = x+Pz \text{ et } HN = -Qz-y,$$

demisso scilicet ex puncto *N*, in quo normalis *MN* plano *APQ* occurrit, ad axem perpendiculo *NH* .Q.E.I.

Corollarium 1.

846. Cum neque altera vis normalis deflectens neque vis tangentialis neque vis resistentiae, si quae adest, pressionem in superficiem afficiant, [p. 471] perspicitur, a quibuscunque potentiis corpus praeterea sollicitetur, pressionem semper tantam esse, quantam hic assignavimus.

Corollarium 2.

847. Quantumvis igitur via a corpore descripta a linea brevissima discrepet, tamen pressio in superficiem fit secundum normalem in superficiem seu secundum radium osculi lineae brevissimae, non vero secundum ipsius curvae descriptae radium osculi, cuius longitudo etiam ad pressionem non requiritur.

Scholion.

848. Ob hanc causam eam radii osculi lineae brevissimae formulam adhibuimus, in qua differentialia secundi gradus non insunt, ne is pendeat a positione duorum elementorum Mm et $m\mu$, per quae corpus reipsa movetur. Sed iste radius osculi ex unico elemento Mm innotescere debet; si enim corpus propter vim deflectentem non limeam brevissimam describat, differentialia secundi gradus ddy et ddz non amplius in radium osculi linea brevissimae ingredi debent.

PROPOSITIO 93.

Problema.

849. *Vis tangentialis, quae secundum tangentem MT* (Fig.92) *corpus trahit, effectum in corpus in superficie quacunque motum determinare.*

Solutio. [p. 472]

Sit haec vis tangentialis = T corpusque per elementum Mm progrediatur celeritate altitudini v debita; quia haec vis motum diminuit, erit

$$dv = -T\cdot Mm = -T\sqrt{(dx^2+dy^2+dz^2)}$$

manentibus iisdem denominationibus, quibus ante sumus usi. Praeterea vero haec vis neque pressionem neque deviationem a linea brevissima afficit. Ad positionem vero directionis huius vis inveniendam producatur tangens MT, donec occurrat plano APQ in T, erit T punctum in elemento qQ producto. Fint ergo

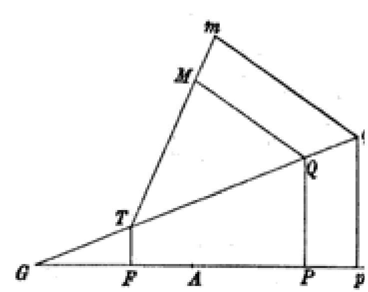


Fig. 92.

$$dz:\sqrt{(dx^2+dy^2)} = z:QT$$

eritque

$$QT = \frac{z\sqrt{(dx^2+dy^2)}}{dz}.$$

Ex T demittatur in axem perpendiculum TF;erit

$$\sqrt{(dx^2+dy^2)}:dx = QT:PF;$$

quare habetur

$$PF = \frac{zdx}{dz} \quad \text{et} \quad AF = \frac{zdx - xdz}{dz}.$$

Porro ob $dx : dy = \frac{zdx}{dz} : y - FT$ erit $FT = y - \frac{zdx}{dz}$, ex quo punctum T determinatur. Q.E.I.

Corollarium.

850. Cum resistentia ad vim tangentialem sit referenda, ex his intelligitur, quomodo resistentia effectus sit determinandus. Ut si fuerit resistentia = *R*, erit

$$dv = -(T + R)\sqrt{(dx^2 + dy^2 + dz^2)}.$$

PROPOSITIO 94.

Problema.

851. *Vis normalis deflectentis N effectum in corpus super superficie quacunque motum determinare.*

Solutio. [p. 473]

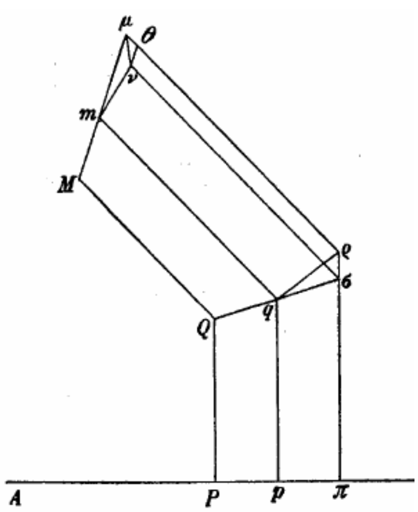


Fig. 93.

Posito ut ante $AP = x$, $PQ = y$ et $QM = z$ (Fig.93) exprimatur superficiei natura hac aequatione $dz = Pdx + Qdy$ et moveatur corpus celeritate altitudine *v* debita per elementum *Mm*; quo percurso nisi vis deflectens adesset, progrederetur per elementum $m\mu$ secundum lineam brevissimam foretque

$$\pi\varrho = y + 2dy + \frac{(Pdy - Qdx)(dPdx + dQdy)}{dx(1 + P^2 + Q^2)}$$

et

$$\varrho\mu = z + 2dz + \frac{(dx + Pdz)(dPdx + dQdy)}{dx(1 + P^2 + Q^2)}$$

(835). Iam accedat vis normalis deflectionis *N*, quae directionem habeat antrorsum. Haec ergo vis efficiet, ut corpus descripto elemento *Mm* non per $m\mu$ incedat, sed ab hac directione antrorsum deflectat. Ponamus igitur pergere per $m\nu$; erunt *Mm* et $m\nu$ duo elementa curvae a corpore descriptae. Quare demisso ex ν in planum *APQ* perpendiculo $\nu\alpha$ erit

$$\pi\sigma = y + 2dy + ddy \quad \text{et} \quad \sigma\nu = z + 2dz + ddz.$$

Hinc ergo habebitur

$$\sigma\varrho = \frac{(Pdy - Qdx)(dPdx + dQdy)}{dx(1 + P^2 + Q^2)} - ddy$$

et

$$\mu\varrho - \nu\sigma = \frac{(dx + Pdz)(dPdx + dQdy)}{dx(1 + P^2 + Q^2)} - ddz.$$

Posito vero brevitatis ergo

$$\frac{(Pdy - Qdx)(dPdx + dQdy)}{dx(1 + P^2 + Q^2)} = dd\eta \quad \text{et} \quad \frac{(dx + Pdz)(dPdx + dQdy)}{dx(1 + P^2 + Q^2)} = dd\zeta$$

invenietur radius osculi angulo elementari $\mu m\nu$ respondens (72) =

$$\frac{(dx^2 + dy^2 + dz^2)^{\frac{3}{2}}}{\sqrt{((dzdd\eta - dzddy - dydd\zeta + dyddz)^2 + dx^2(dd\eta - ddy)^2 + dx^2(dd\zeta - ddz)^2)}}.$$

Hic ergo si dicatur = r, erit $N = \frac{2v}{r}$ seu $2v = Nr$, quia hic angulus eodem modo generatur quo corpous a vi normali in plano a linea recta deflectitur. Est vero [p. 474]

$$dzdd\eta - dydd\zeta = -\frac{(dy + Qdz)(dPdx + dQdy)}{1 + P^2 + Q^2}.$$

Atque loco $dd\eta$ et $dd\zeta$ debitis valoribus substitutis fit

$$r = \frac{(dx^2 + dy^2 + dz^2)^{\frac{3}{2}}}{\sqrt{\left(dx^2(ddy^2 + ddz^2) + (dzddy - dyddz)^2 - \frac{(dx^2 + dy^2 + dz^2)(dPdx + dQdy)^2}{1 + P^2 + Q^2}\right)}}.$$

Cum autem per aequationem $dz = Pdx + Qdy$ sit $dPdx + dQdy = ddz - Qddy$, erit in subsidium hac ipsa aequatione $dz = Pdx + Qdy$ vocata

$$r = \frac{(dx^2 + dy^2 + dz^2)^{\frac{3}{2}}\sqrt{(1 + P^2 + Q^2)}}{-ddy(dx + Pdz) + ddz(Pdy - Qdx)}.$$

Hanc ob rem erit

$$ddz(Pdy - Qdx) - ddy(dx + Pdz) = \frac{N}{2v}(dx^2 + dy^2 + dz^2)^{\frac{3}{2}}\sqrt{(1 + P^2 + Q^2)}.$$

Q.E.I.

Scholion 1.

852. Congruit haec formula cum ea, quam supra (79) pro effectu huiusmodi vis determinando invenimus. Differentia enim tantum est in signo litterae N, quam vim ibi negative accipiendam esse apparet. Atque hic etiam de signo non certi esse potuissemus, quia ex quantitate radix quadrata, quam hic extraximus, aeque potest esse negativa ac affirmativa. Hoc vero dubium, si calculus ad casum specialem accommadetur, statim tollitur, quia formula eiusmodi esse debet, ut punctum ν cis μ cadat, si potentia N antrorsum, ut posuimus, fuerit directa. Ex quo ope exempli etiam signum radicis quadraticae determinavi atque hanc ipsam formulam inveni.

Corollarium 1. [p. 475]

853. Si vis deflectens *N* evanescat, corpus motum suum in linea brevissima continuabit; id quod ipsa aequatio quoque indicat. Posito enim *N* = 0 habebitur

$$ddy(dx + Pdz) = ddz(Pdy - Qdx),$$

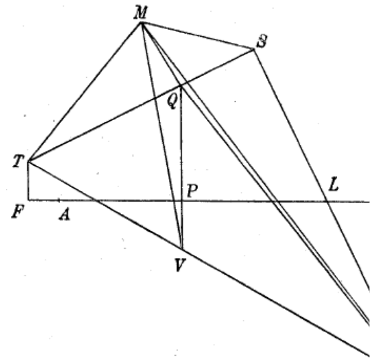
quae aequatio est pro linea brevissima.

Corollarium 2.

854. Quaecunque ergo vis premens et vis tangentialis atque resistentia corpus in superficie motum sollicet, si modo nulla affuerit vis deflectens, corpus semper in linea brevissima movebitur.

Scholion 2.

855. Quod autem ad positionem huius vis deflectentis *N* attinet, ea ex hoc deducetur, quod ea posita sit in plano tangente superficiem atque simul sit normalis curvae descriptae; sit ergo *MG* (Fig. 94) eius directio et *G* punctum, in quo plano *APQ* occurit, ita ut vis *N* secundum *MG* trahere censenda sit, dum eam ante antrorsum urgere posuimus. Primo ergo determinari debet intersectio plani superficiem in *M* tangentis cum plano *APQ*, quae sit recta *TVG;* haec vero invenietur, si duae tangentes superficiem ad planum *APQ* usque producantur atque puncta, in quibus in planum *APQ* incidunt, linea recta iungantur. Sit ergo *MT* tangens lineae descriptae, qua propterea superficiem quoque tanget; erit, ut iam invenimus, [p. 476]



$$AF = \frac{zdx}{dz} - x \quad \text{et} \quad FT = y - \frac{zdy}{dz}$$

(849). Porro superficies secta intelligatur plano *PQM* sitque *MV* sectionis huius tangens; erit $QV = \frac{z}{Q}$ ex aequatione $dz = Pdx + Qdy$ posito $dx = 0$. Innotescit ergo punctum *V*, quocirca recta *TV* producta erit intersectio plani tangentis superficiem in *M* cum plano *APQ*. Punctum ergo *G*, in quo recta *MG* plano *APQ* occurrit, positum erit in recta *TV*. Porro in recta *TQ* sumatur

$$QS = \frac{zdz}{\sqrt{(dx^2 + dy^2)}}$$

eritque *MS* normalis in elementum *Mm* descriptum. Atque si ad *QS* ducatur normalis *SG*, ex hac recta *SG* omnes rectae ad *M* ductae erunt in elementum *Mm* perpendiculares. Quare cum *MG* sit quoque normalis in elementum descriptum, punctum *G* quoque positum erit in recta *SG*. Punctum ergo *G* erit in intersectione rectarum *TV* et *SG*. Est vero

$$PL = \frac{ydy + zdz}{dx}$$

et ang. *ELG* = ang. *PQT*. Ponatur *GE* = *t;* erit

$$LE = \frac{tdy}{dx} \text{ et } PE = \frac{ydy + tdy + zdz}{dx} : t - \frac{z}{Q} + y.$$

Deinde etiam propter triangula similia est $FP : FT + PV = PE : GE - PV$, hoc est

$$\frac{zdx}{dz} : \frac{z}{Q} - \frac{zdy}{dz} = \frac{ydy + tdy + zdz}{dx} : t - \frac{z}{Q} + y.$$

Hinc provenit

$$t = \frac{z(dx + Pdz)}{Qdx - Pdy} - y = GE \quad \text{et} \quad AE = x + \frac{z(dy + Qdz)}{Qdx - Pdy},$$

unde punctum *G* determinatur. Si ergo ducatur recta *QG*, erit

$$QG^2 = \frac{z^2(dx + Pdz)^2}{(Qdx - Pdy)^2} + \frac{z^2(dy + Qdz)^2}{(Qdx - Pdy)^2}$$

et

$$QG = \frac{z\sqrt{(dx^2 + dy^2 + dz^2 + dz^2(1 + P^2 + Q^2))}}{Qdx - Pdy}$$

atque ipsa

$$MG = \frac{z(dx^2 + dy^2 + dz^2)^{\frac{1}{2}}\sqrt{(1 + P^2 + Q^2)}}{Qdx - Pdy}.$$