PREFACE

As I have set out the motion of free bodies acted on by any forces in the first Book, thus in this second Book I have decided to treat the motion of bodies that are constrained in their motion; and for which the difference in the explanation of the motion is of such merit that a whole division of the work is made. For in the free motion of a body, a path is determined for the body and described by the motion that ensues, due to the effects of the absolute and resistive forces acting on the body; since besides the absolute forces and resistance nothing else is present that can determine the motion of the body. And on this account a foremost property of free motion is that the path described by the body is entirely unconstrained; clearly a channel along the path described by the body has exactly the curvature for the passing body to sustain no added force, as the body it transmitted through it freely. But in constrained motion, in addition to the absolute forces and resistance which act upon the body, we put in place a prescribed path, in order that the body is compelled to move thus in this way. Hence this prescribed path can be conveniently considered to be the counterpart of the channel, in which the body is moving, and it is unable to break free from this channel. Therefore since in motion of this kind, the path is given along which the body has to move, it is required to find how much speed the body will have when acted upon by some other forces and resistance at individual points, clearly when this is known then whole motion is perfectly understood. Moreover in addition, since the body, unless it is enclosed in this channel, describes another curve, even on being retained it tries to move along that line, along which it would move if it were free, and the body progresses by pressing on the sides of the channel in the attempt to break free, and if the walls are not firm enough then it can indeed break free. On account of this, besides the speed that the body has at individual places along the channel, the force must also be determined that it exerts on the walls of the channel, [or rather the reaction] and the direction of this force, by which the firmness of the sides of the channel required to retain the body in its course becomes known. Moreover constrained motion of this kind without channels [*i. e.* fixed paths] can be produced by other means that can be observed in pendulums and slings, in which the body likewise is forced to move on a given line. Indeed as Huygens has taught for pendulums, in order that the body can be effective it is forced to move along some prescribed curve, and for pendulums with a simpler suspension the body is forced to move along the arc of a circle, while it is apparent for these which are usually suspended between cycloids, that the body is forced to move along a cycloidal path; and in a like manner it can be put into effect that the body moves along some given curve.

This is therefore the first kind of constrained motion, which is along a given line. Moreover besides this, other kinds of constrained motion are worthy of our attention, in which the motion is not restrained to a certain path, but only a surface on which the body is forced to move is prescribed; therefore the constraint on the motion is less restricting than previously, since in this case the body is able to chose its own path on the given surface. On this account, this kind of constrained motion must thus be treated, in order that in the first place a line can be determined on the given surface that the body describes, acted on by the external forces and the resistance, then secondly in order that the speed of the body at the individual points along this line can be defined, and then thirdly, in order also the pressing force that the body exercises on the surface can be

found. Moreover constrained motion of this kind can also be equally conveniently represented by pendulums in the first place ; for the body of the pendulum, on being given an oblique blow in a direction which is not in the vertical plane, describes curves of various kinds, which moreover are all on the surface of a sphere, the centre of which stands out as the point of suspension. Therefore the examination of this motion is reduced to the point that the first line on the surface of the sphere that the body describes on the surface of the sphere is determined, and then truly the speed at the individual points is found, and in the third place the force pressing that the body exercises on the surface is found. In a like manner it can also be brought about that the body of the pendulum is not restricted to the surface of a sphere, but to some other surface, where the evolved surface is set out around the point of suspension. Therefore this is another kind of constrained motion, in which the motion of the body on a given surface is taken up in a limited manner; and the whole second Book is here completed with an investigation of these two kinds of motion.

Therefore in this treatment, these things are set out which are known to be necessary, and I have set out the fundamentals and principles in the first Chapter, and from which these things which are pertinent to the understanding of both kinds of constrained motion can be derived. Obviously I have explained how a body ought to move with a regular motion either on a line or a surface in the absence of forces acting; moreover on a surface, this is to be the path described by the body of the shortest line which it is possible to draw on that surface. Then I investigate the general laws which any forces, including resistance, observe in accelerated or retarded motion, or in the pressing [contact] forces to be generated. The theory concerning centrifugal forces is also set forth in this discussion, the forces that bodies also exercise upon each other when no external forces are acting, and which arise from the curvilinear motion that the body is made to follow.

In the second and third Chapters, I have considered and examined in great detail the motion of bodies on a given line both in a vacuum and in a medium with resistance. Clearly at first I determine the motion, by which a body moves on a given line or curve with any forces acting, is either by descending or ascending; and if the curve thus is compared, so that it is suitable for producing both ascending and descending motion, then I define the oscillations too, and I compare the ratio of the times between these ; and in this matter I define the innate properties of the oscillations made in both circular and cycloidal motions. Then I tackle the inverse problems, in which mainly I look for the curves with given forces acting, upon which the motion has a given property. Here clearly the problems relate to finding the curves of equal descent or of receding from a given point, and many others of this kind which have now been treated by others, or the problem itself is set up and gone through. Among these, the problems concerned with brachystochrones and tautochrones are the most outstanding, each of which, that hitherto has been constructed by some means or another, I have raised to the highest level of perfection. Indeed I have corrected an error that had arisen concerning the brachistochrone curves for both the vacuum and resistive medium cases, and in place of the principle of Huygens which is in itself true but insufficient, I have substituted another more general principle, in which I have demonstrated that in any medium and for any forces acting for any hypothesis, that in order for the curve upon which the body is thus moving is to be a brachistochrone, the total force pressing must be twice as large as the

centrifugal force. The method I have presented enables in a similar manner new and natural tautochronous curves to be found (indeed tautochronous curves were not found before entirely by some method, but rather were elicited by divination), with the help of this not only the cycloid now recently discovered under the celebrated name of the tautochrone, but besides that I have found innumerable other curves satisfying the quest, among which so far I have observed algebraic curves ; so besides other questions arising in a vacuum as well as the whole business of the treatment of resistive motion in mediums, it is abundantly clear to understood that this is an excellent useful method. Moreover to promote the use the counterpart of this method in Analysis as well as in Mechanics has to be considered, thus too elsewhere in the solutions of the more difficult problems they appear as Analytical aids not to be despised, by which also this science is seems to be advanced more than a little.

In the fourth Chapter I then pursue motion on a given surface, which are treated by a method not used by anyone until now, thus also it is a most difficult treatment on account of the nature and properties of three dimensions, which are not yet evident enough nor able to be recalled for calculations. Therefore before any of this kind of motion could be put in place, it was necessary to explain the method, by which the properties of surfaces and lines drawn on these were elicited and for the calculation to become evident. And thus with the aid of this improved equation containing three variable quantities, with which now as before in the Commentary to Book III in the determination of the shortest line on a given surface, as in the treatment on this in the previous Book that I have used in the investigation of free motion made in a different plane. Finally with these attended to, it is permitted to progress to the effect of the forces on a body with the motion to be defined on surfaces, from which the manner is to be elicited for the path described by the body and well as the symptomatic discovery of the rest of the motion. Truly since the calculation, as long as it is moved around in general terms, becomes exceedingly long and difficult to handle, with the resistance omitted and everything reduced to a vacuum and with ordinary gravity, I have carefully studied in particular the oblique oscillations of pendulums, and I have diligently determined the anomalies of this motion and the progression of the apses.

These are therefore what I have embraced in this second Book, and which I shall give with expediency, in order that, as it becomes permissible to do so, the motion of finite bodies and at first indeed of rigid bodies I will arrange in order and explain by a suitable method.

PRAEFATIO

Quaemadmodum in Tomo primo motus liberos corporum a quibuscunque potentiiis sollicitatorum exposui, ita in hoc Tomo altero motus non liberos pertractare constitui; quae differentia in motus explicatione tanti est momenti, ut ex ea merito totius operis divisio sit facta. In motu enim libero via a corpore descripta cum ex motu iam insito, tum ex potentiis tam absolutis quam resistentia, quibus corpus afficitur, determinatur, quia praeter potentias et resistentiam nihil adesse ponitur, quod corporis motum determinet. Atque idcirco motus liberi haec est primaria proprietas, ut via a corpore descripta omnino non prematur; canalis scilicet secundum viam, quam corpus describere debet, exacte incurvatus a corpore transeunte nullam omnino pressionem sustinebit, sed corpus per eum libere transibit. In motu autem non libero praeter potentias et resistentiam, quibus corpus sollicitatur, viam praescriptam esse ponimus, ita ut corpus sit coactum in hac via moveri. Haec ergo via praescripta ad instar canalis commode considerari potest, in quo corpus movetur, neque ex eo erumpere potest. Cum igitur in istiusmodi motibus data sit via, in qua corpus moveri debet, inquirendum est, quantum corpus a quibuscunque potentiis et resistentia sollicitatum in singulis locis habiturum sit celeritatem, quippe qua cognita totus motus perfecte cognoscitur. Praeterea autem cum corpus, nisi in hoc canali esset inclusum, aliam lineam describeret, retinebit saltem in canali conatum in ea linea, in qua, si liberum esset, moveretur, progrediendi hocque conatu latera canalis premet et, nisi satis habeant firmitatis, reipsa disrumpet. Hanc ob rem praeter celeritatem, quam corpus in singulis canalis locis habebit, determinari debet quoque pressio, quam in latera canalis exercet, eiusque pressionis directio, quo firmitas laterum canalisad corpus retinendum requisita cognoscatur. Huiusmodi autem motus non liberi etiam sine canali aliis modis produci possunt, id quod observare licet in pendulis atque in fundis, quibus corpus itidem in data linea moveri cogitur. Pendulis enim, prout Hugenius docuit, effici potest, ut corpus in quacunque curva praescripta moveri cogatur, quemadmodum in pendulis tum simpliciter suspensis, quibus corpus in linea circulari moveri cogitur, apparet, tum iis, quae intra cycloides suspendi solent, quibus corpusin cycloide moverei cogitur; similique modo effici potest, ut corpus in data quaque curva incedere cogatur.

Haec igitur est prima species motus non liberi, qui fit super data linea. Praeter eam autem alia species motus non liberi attendi meretur, in qua non ipsa quidem via, sed tantum superficies praescribitur, in qua corpus moveri cogitur; minus igitur haec motuum non liberorum species est restricta quam prior, cum in hac corpori adhuc libertas sit relicta sibi viam in data superficie sitam eligendi. Hanc ob rem haec motus non liberi species ita tractari debet, ut primo linea in data superficie determinetur, quam corpus a potentiis et resistentia sollicitatum describet, deinde vero, ut celeritas corporis in singulis huius lineae punctis definiatur, tertio denique, ut etiam pressio, quam corpus in superficiem exercet, investigetur. Huiusmodi autem motus non liberi pariter ac priores pendulis quoque commode repraesentari possunt; corpus enim pendulum oblique impulsum, ut eius directio non sit in plano verticali, lineas curvas varii generiis describet, quae autem omnes sunt in superficie spherica, cuius centrum in ipso suspensionis puncto extat. Inquisitio ideo huius motus huc redit, ut in superficie sphaerica primo linea, quam corpus proiectum describit, determinetur, deinde vero celeritas in singulis locis et tertio

pressio, quam in superficiem exercet. Simili modo etiam perspicitur effici posse, ut corpus pendulum non ad superficiem sphaeriem, sed ad aliam quamque restringatur, dum scilicet circa punctum suspensionis superficies evoluta disponatur. Haec igitur est altera motuum non liberorum species, quae in motu super data superficie determinando occupatur; atque in his duabus motuum non liberorum speciebus indagandis totus hic Tomus secundus absolvitur.

Quo ergo ad hanc tractationem, quae scitu necessaria sunt, praeparentur, in Capite primo fundamento et principia exposui, ex quibus, quae ad cognitionem utriusque speciei motuum non liberorum pertinent, derivari queant. Demonstravi nimirum corpus a nullis potentiis sollicitatum tam super data linea quam super superficie motu aequabili moveri debere; in superficie autem fore viam a corpore descriptam ipsam lineam brevissimam, quae in ea superficie duci potest. Deinde investigavi leges generales, quas quaeque potentiae atque etiam resistentia tum in accelerando vel retardando motum, tum in pressione generanda observant. Ad haec etiam doctrina de vi centrifuga exponitur, quam corpora etiam a nullis potentiis sollicitata exercent quaeque ex motu curvilineo, quo corpus incedere cogitur, ortum habet.

In Capitibus deinde secundo et tertio motus corporum super data linea tam in vacuo quam in medio resistante fuse contemplor et examino. Primo nimirum motum determino, quo corpus a quibuscunque potentiis sollicitatum super data linea sive curva movetur, sive descendendo sive ascendendo; atque si curva ita fuerit comparata, ut tam ad descensus quam ascensus producendos sit idonea, oscillationes quoque definio easque inter se ratione temporum comparo; atque in hac negotio indolem et proprietates oscillationum tam in circulo quam cycloide factarum definio. Deinceps problema tracto inversa, quibus potissimum pro datis potentiis sollicitantibus in curvas inquiro, super quibus motus datam habeat proprietatem. Huc scilicet pertinent problemata de inveniendas curvis aequabilis descensus vel recessus a dato puncto et huiusmodi plura, quae vel ab aliis iam sunt tractata, vel ad quae ipsum institutum perduxit. Inter haec prae ceteris eminent problemata de lineis brachystochronis et tautochronis, quorum utrumque ad ulteriorem, quam adhuc a quoquam est factum, perfectionis gradum evexi. Circa curvas enim brachystochronas errorem, qui a nonnullis tam in vacuo quam medio resistente erat commissus, correxi et loco principii Hugeniani in se quidem veri, sed insufficientis, alius latissime patens substituti, quo demonstravi in quocunque medio et potentiarum sollicitantium hypothesi quacunque eam perpetuo curvam esse brachystochronam, super qua corpus ita moveatur, ut tota pressio duplo sit maior quam vis centrifuga. Simili modo novam atque genuinam curvas tautochronas inveniendi methodum trado (quae enim ante sunt inventae tautochronae, nulla omnino methodo, sied potius divinatione sunt erutae), cuius ope non solum cycloidem iam dudum sub tautochronae nomine celebrem inveni, sed praeter eam innumerabiles alia curvas quaesito satisfacientes elicui, inter quas adeo curvam algebraicam observavi; praeterea tam ex aliis agnatis quaestionibus in vacuo quam ex integra huius negotii tractatione pro medio resistente praestantiam et utilitatem huius methodi abunde intelligere licebit. Ceterum uti haec methodus instar speciminis tam Analyseos quam Mechanicae promotae est censenda, ita quoque passim in difficiorum quorundam problematum solutionibus non contemnenda Analyseos subsidia apparebunt, quibus etiam haec scienta non parum promota esse videatur.

In quarto denique Capite motum super data superficie persequor, quae doctrina, uti a nemine adhuc est tracta, ita quoque tractatu est difficillima propter naturam et proprietates solidorum nondum satis perspectas neque ad calculum revocatas. Antequam igitur de huius modi motu quicquam statui potuerat, methodum exponere necesse erat, qua proprietates superficerum et linearum in iis ductarum erui atque calculo subiici possent. Hoc itaque praestiti ope aequationum tres quantitates variabiles continentium, quibus iam ante tum in Comment. Tomo III ad lineam brevissimam super quavis superficie determinandam, tum in huius Tractatus Tomo praecedente ad motus liberos non in eodem plano factos investigando sum usus. His denique praeparitis progredi licuit ad effectus potentiarum in corpora super superficiebus mota definiendos, ex quibus modum elicui tam viam a corpore descriptam quam reliqua motus symtomata inveniendi. Quum vero calculus, quamdiu in generalibus versamur, nimis fiat prolixus et tractatu difficilis, omissa resistentia omnia ad vacuum et gravitatem ordinariam reduxi atque praecipue motum pendulorum oblique oscillantium sum perscrutatus, cuius motus anomalias et absidum progressiones diligenter determinavi.

Haec igitur sunt, quae in isto Tomo secundo sum complexus, quibus expeditis operam dabo, ut, quam primum licuerit, motus corporum finitorum et primo quidem rigidorum in ordinem reducam atque pari methodo exponam.