CHAPTER THREE

CONCERNING STABILITY, WHEREBY BODIES FLOATING IN WATER MAY REMAIN IN THEIR STATE OF EQUILIBRIUM

PROPOSITION 19

THEOREM

204. *The stability, whereby a body floating in water may persist in a state of equilibrium, is required to be determined from the moment of the restoring forces acting, if the body may have been inclined by a very small angle from its equilibrium state.*

DEMONSTRATION

If a body floating in water may be inclined by a very small amount from its state of equilibrium, then either it will be restored to that state, remain at rest in the new state, or also it may depart further from the state of equilibrium, and as if it may slide to be received into another state of equilibrium. Therefore in the case, where it may remain at rest when inclined from the state of equilibrium, there is no particular stable state to which the body itself may be restored; truly in the case, where it recedes further from the state of equilibrium, it must be agreed not only that there is no state of equilibrium but such a state is lacking. Therefore the ability to return to the state of equilibrium is to be attributed to these only, in which the body if it may be declined a little, is restored to that state of equilibrium. Moreover if the body may be declined minimally from that state by such an angle from the state of equilibrium, the restoration will be made about a horizontal axis passing through the centre of gravity, just as has been demonstrated in proposition 18. Truly the reason for the restitution is the moment of the pressure of the water about that axis, which will be shown below to be proportional to the angle for the same body. Therefore so that, for the same angle of declination from the state of equilibrium for diverse bodies, that moment of the restoration will have been greater, thus the force of restitution will be greater, and therefore from that the greater the force requiring to be acting in the equilibrium state, that I call the stable state. For this reason, by which a body floating on water persists in a state of equilibrium, it is required to be estimated from the moment of the restoring force, if the body may be declined by an infinitely small angle from the equilibrium state.

Q.E.D.

COROLLARY 1

205. Therefore since in the same body the restoration moment shall be proportional to the angle of the declination from the state of equilibrium, and the stability in diverse bodies may be defined by equal angles, in any body the absolute stability will be as the moment of restoring force divided by the angle of declination.

COROLLARY 2

206. Since a certain horizontal axis shall be considered in determining the stability, about which the minimum inclination may be considered to be made, it is evident for the same body and for the same state of equilibrium innumerable estimates of the stability to be found, for innumerable axis, with respect of which the stability is found.

COROLLARY 3

207. Therefore concerning stability, where a given body floating in water continues in a state of equilibrium, at the same time the axis is required to be mentioned, to which the stability is referred ; besides indeed no other quantity can be had to determine the stability.

COROLLARIUM 4

208. Therefore if some body may float in water in a state of equilibrium, the stability will be indicated with respect of some fixed horizontal axis, how much that body may resist being inclined about that axis. So that indeed the more the body floating on the water resists being inclined about some axis, thus its stability is considered greater with respect of the same axis.

COROLLARY 5

209. Therefore where a greater value of the stability may be found with regard to a certain axis, thus the body will resist a greater inclination about this axis. Yet if the value of the stability $=0$ may be produced, then the body cannot be restored, if it may be inclined to that axis for a short time. On the other hand, if the stability were negative, then not only will the body not be minimally restricted but also it will turn around, until it may arrive at a firm and steady state.

SCHOLIUM 1

210. This principle concerned with the stability of bodies floating on water, by which they continue to hold in the place of equilibrium, is of the greatest concern in the construction and loading of ships. Indeed it is accustomed to be required that ships may

persist in their upright positions as firmly as possible, and resist the inclining forces most strongly. On this account thus this same principle is to be set out more carefully in this chapter, so that afterwards from that useful rules for the construction and loading of ships may be able to be set out. Therefore since we have investigated all the situations in the first chapter, by which a body floating on water may be able to persist in equilibrium, here we will inquire into the stability, by which it may remain in a state of equilibrium with respect of each axis. Therefore we will find the different firm and stable states of equilibrium, evidently when a positive value will be obtained for the stability, truly the others to be unstable and prone to slipping, when a negative stability arises. Also sometimes an ambiguous situation may arise, which coincides with vanishing stability; all of which cases will be considered carefully which will be the most conducive precepts of nautical matters requiring to be treated.

SCHOLIUM 2

211. Whatever is required to be known fully about the given equilibrium state, so that the stability may be defined with respect of all the horizontal axis, yet the order of the stability will be understood well enough, if the stability with respect of only two of the axes may be investigated, of which either may provide the maximum stability and the other truly the minimum ; indeed from these it will be allowed to estimate easily the stability with respect of any other axis, and since generally it may suffice to know the limits, between which the stability shall be contained. Thus ships with external forces inclined from the prow or stern are much more difficult than from the sides, and hence the stability of those with respect of the axis drawn along the width is a maximum, truly the stability with respect of the axis drawn along the length of the ship, to be a minimum. Clearly the similar definition of stability depends both on the moments of the forces as well as on inertia, which cannot be assigned absolutely, but always must be referred relative to some axes, about which the inclination is considered to act. Therefore with this approach agreed on, which may be seen to be almost infinite, it will certainly be contracted, so that the matter may be able to be resolved easily. But so that we may begin with the most simple cases, initially we will not consider bodies but only plane surfaces of these, which may sit in place vertically, and motions can be presented around the horizontal axis passing normally to the plane through the centre of gravity of the surface itself. Evidently in this way we will not contemplate the declinations of plane surfaces away from the state of equilibrium, except for which these surfaces may remain vertical.

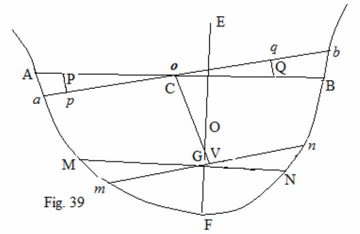


Fig. 39

PROPOSITION 20

PROBLEM

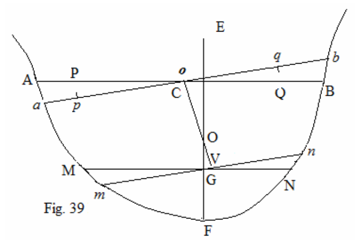


Fig. 39

212. *To find the stability, whereby some plane surface resting on water vertically may persevere in its state of equilibrium.*

SOLUTION

AMFNB (Fig. 39) shall be some surface resting vertically on the surface of the water thus so that the horizontal right line *AB* shall be the section of the water. The centre of gravity of this figure shall be at *G,* and *M* the weight of the figure. A vertical right line *EF* may be drawn through *G,* on which the centre of the magnitude *O* of the submerged part *AMFNB* lies, since the figure is put to be held in this state of equilibrium; and on that account the submerged part *AMFNB* will be so great, in order that the weight of the mass of water equal to the volume of the water shall be $= M$. Now this figure may be inclined minimally from its state of equilibrium, thus so that *ab* may become the section of the water, and through *G* itself the lines *MN* and *mn* may be drawn parallel to *AB* and *ab,* of which *MN* will be horizontal in the state of equilibrium, and *mn* truly will be the horizontal in the inclined state. And the angle of inclination shall be $MGm = dw$. Therefore since the submerged part must always be constant, the area $aMFNb =$ area $AMFNB$: for unless they shall be equal, the centre of gravity will either rise or fall until equality were agreed on, from which the motion of restitution shall be made about the centre of gravity, which alone we may regard here not to be disturbed. Therefore with the intersection of the right lines *AB* and *ab* at *C,* the area *AGa* will be equal to the area *BCb.* Moreover on account of the infinitely small angle of inclination *dw* the area

$$ACa = \frac{AC^2 \cdot dw}{2}$$

and the area

$$BCb = \frac{BC^2 \cdot dw}{2}$$

from which there arises

$$AC = BC = \frac{AB}{2}$$

Now so that the force of restitution from the inclined position to the position of equilibrium may be found, the centre of the magnitude of the submerged part *aMFNb* is required to be found*;* which since it shall be

$$= AMFNB \; + BCb \; - ACa,$$

the centre of the magnitude of the true submerged parts will be able to be found from the centres of gravity of these parts, and thence the moment of the pressure of the water for the restoration of the equilibrium. Truly it will be restored to equilibrium, as long as the right line *mn* will be restored to the horizontal *MN*. Therefore in the first place we will consider the area *AMFNB,* of which the centre of gravity is at *O* and the force acting upwards arising from that $= M$. The vertical *VOo* is drawn through *O* which is the direction of the force *M* acting upwards on the figure; therefore the moment of this force for the restoration is :

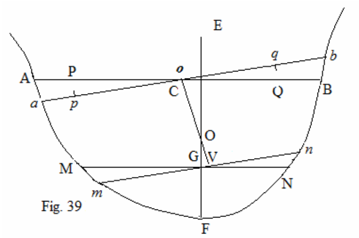


Fig. 39

$$M \cdot GV = M \cdot GO \cdot dw$$

on account of the angle

$$GOV = MGm = dw.$$

Again the area element *CBb* will be considered*,* of which the area is :

$$\frac{BC^2 \cdot dw}{2} = \frac{AB^2 \cdot dw}{8}.$$

Therefore the force arising to push the figure upwards is $= \frac{M \cdot AB^2 \cdot dw}{8 \cdot AMFNB}$; the direction of which passes through the centre of gravity *Q* of the element *BCb;* the normal or vertical line *Qq* shall be drawn from *Q* in *Cb,* there will become

$$Cq = \tfrac{2}{3} Cb = \tfrac{2}{3} CB = \tfrac{1}{3} AB,$$

therefore the moment of this force required for restoration is :

$$= \frac{M \cdot AB^2 dw}{8 \cdot AMFNB}(qo + GV).$$

Finally the element *ACa* in a similar manner will give the force $= \frac{M \cdot AB^2 \cdot dw}{8 \cdot AMFNB}$, and its direction, which is vertical passes through its centre of gravity *P.* Therefore with the vertical *Pp* drawn*,* there will become :

$$pC = \tfrac{2}{3} aC = \tfrac{2}{3} AC \ \left[= \tfrac{1}{3} AB \right].$$

Therefore the moment of this opposite force from before will be:

$$= -\frac{M \cdot AB^2 \cdot dw}{8 \cdot AMFNB}(po - GV).$$

Therefore since this moment taken from the previous must become, on account of

$$aMFNb = AMFNB + BCb - ACa,$$

the moment, by which the pressure of the water exerted in the totally submerged part *aMFNb* restores equilibrium :

$$= M \cdot GO \cdot dw + \frac{M \cdot AB^2 \cdot dw}{8 \cdot AMFNB}(po + qo) = M \cdot GO \cdot dw + \frac{M \cdot AB^3 \cdot dw}{12AMFNB}$$

on account of

$$po + qo = pq = \tfrac{2}{3}AB.$$

Therefore since the moment, by which the figure shall be restored to the equilibrium state, shall become

$$= Mdw\left(GO + \frac{AB^3}{12AMFNB}\right),$$

on being divided by the angle *dw*, the stability with which the figure continues in the state of equilibrium *AMFNB* will be

$$= M\left(GO + \frac{AB^3}{12AMFNB}\right)$$

with *AFB* indicating the area of the submerged part. Q. E. I.

COROLLARY 1

213. Therefore it is evident, as we have asserted above, the restoration force in the state of equilibrium to be proportional to the angle, by which the body is inclined from the state of equilibrium, if indeed the angle were as minimal, and thus if complete stability were required, so that the angle *dw,* by which the inclination may be indicated, can be omitted. Therefore the expression of the moments of the forces for stability thus will be homogeneous, since the product from the force or the force *M* in a certain right line shall be

$$GO + \frac{AB^3}{12AF}.$$

SCHOLIUM 1

214. In the expression of the stability *GO* denotes the distance of the centre of gravity of the figure from the centre of the magnitude of the submerged part, when the figure exists in equilibrium. Therefore when we may have put this centre of gravity in the figure

to fall below the centre of the magnitude, likewise to be evident, if in the other case the centre of gravity G may fall above the centre of the magnitude, then the interval GO must be accepted as negative, thus so that in cases of this kind the stability shall be going to be produced

$$= M\left(\frac{AB^3}{12AFB} - GO\right).$$

Evidently the figures we consider here to be constructed from some heterogeneous material, thus so that the centre of gravity G shall be able to fall both above as well as below the centre of the magnitude O ; but if the figure may be put in place made from a homogeneous material, then by necessity the centre of gravity will be required to fall above the centre of the magnitude of the submerged part. Therefore with cases of this kind, the stability always requires to be estimated, in which GO is to be affected by a negative sign.

COROLLARY 2

215. Therefore as often as the centre of gravity falls below the centre of the magnitude of the submerged part, then the state of equilibrium always will be firm and stable, since the expression of the stability cannot become negative.

COROLLARY 3

216. But if the centre of gravity G falls above the centre of the magnitude O, then the state of the equilibrium will not be stable, unless there were

$$\frac{AB^3}{12 \cdot AFB} > GO.$$

But if there were

$$\frac{AB^3}{12 \cdot AFB} < GO,$$

the state will be unstable or liable to slip, and the figure even minimally inclined from its state of equilibrium will fall, and another state of equilibrium will be sought.

COROLLARY 4

217. Therefore the maximum state of stability will be had, if the centre of gravity were the deepest, moreover if the centre of the magnitude in place were raised to the highest place; and besides if the section of the water or the right line AB were the greatest: evidently with the same weight of the figure M, from which the magnitude of the submerged parts themselves remain invariant.

COROLLARY 5

218. Therefore with bodies floating in water, where the weights may be located more deeply, thus those will acquire greater stability in the equilibrium position. Truly also the stability will be increased more, if the section of the water may be rendered more fully by adding another weight.

SCHOLIUM 2

219. Although this proposition shall be adapted only for plane surfaces sitting vertically in water, yet that itself may be extended more widely, and includes cylindrical bodies. For if a cylindrical body thus shall float on water, so that its longitudinal axis may maintain a horizontal position, then its stability with respect of the horizontal axis of the same is understood to arise from the stability of each transverse section, which is a plane vertical surface. Therefore for these cases *AFB* will be a part of the middle section of the cylindrical body submerged under water, *G* the centre of gravity of the whole body, *O* the centre of the magnitude of the submerged part; *M* indeed the weight of the whole body, and *AFB* as before the part of any section submerged in the water. Besides also from the same proposition conclusions may be drawn, but with regard to these in the following, since we will consider bodies of all kinds from the principles in place, which we will discuss more fully.

PROPOSITION 21

PROBLEM

220. *If a plane figure placed vertically floating on water may be inclined a little from the equilibrium position, to determine the motion by which it may be restored to the state of equilibrium.*

SOLUTION

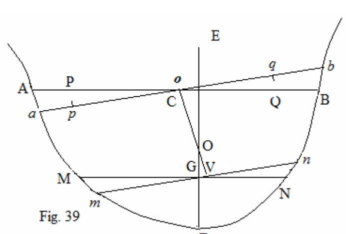


Fig. 39

The plane figure *AFB* (Fig. 39) shall be resting on water in equilibrium, where as well as the right line drawn normally to the plane through the centre of gravity *G* also the right line *MGN* would be horizontal. The weight of the figure shall be $= M$ and *AB* the section of the water, and *O* the centre of magnitude of the submerged part *AFB.* Now the figure may be inclined by a small amount from the situation of equilibrium so that the right line *ab* shall become the section of the water, and the angle *ACa* shall become $= dw$. With these in place, from the preceding proposition, the moment restoring the figure to equilibrium will be obtained, where clearly the figure rotated about the horizontal axis passing through *G* normal to the plane, if the stability found before may be multiplied by the angle of inclination *dw*, and therefore this moment [or torque] tending towards the restoration of the body

$$= Mdw\left(GO + \frac{AB^3}{12AFB} \right).$$

Therefore as often as this expression were positive, the figure will be restored to the position of equilibrium, and the restoration will be made by rotating about the centre of gravity *G,* while meanwhile the centre of gravity *G* itself either rises or falls directly, just as is required by that condition, where an equal part must always be submerged in the water. Therefore since this moment shall be proportional to the angle gone through, the figure will arrive in the same manner in the state of equilibrium, whereby a pendulum on descending approaches to the vertical state. On this account the figure will complete the oscillations in the manner of a pendulum, until the whole motion will have been used up by the resistance. Therefore this motion will become known, if the length of the simple pendulum may be determined, so that it may resolve its oscillations in the same time. But truly for this pendulum requiring to be assigned it is necessary, that the moment of inertia of the figure may be agreed on with respect of the axis about which it turns. Therefore *S* shall be the moment of inertia of the figure, or the sum of all the small parts multiplied by the squares of their distances from the axis of rotation, which axis passes through *G* normally to the figure. Hence the rotational force [*i.e.* the angular acceleration or force, if the mass is retained] will be

$$= \frac{Mdw}{S}\left(GO + \frac{AB^3}{12 \cdot AFB} \right), \frac{1}{2}$$

from which the length of the simple pendulum will be produced, [*c.f.* $l\ddot{\theta} = -g\theta$ for the simple pendulum of length *l*.] so that these isochronous oscillations will be resolved for the oscillations of the figure

$$= \frac{12S \cdot AFB}{12M \cdot GO \cdot AFB + M \cdot AB^3},$$

always indeed isochronous with the length of the simple pendulum, if the angle of inclination may be divided by the rotational force, which is easily deduced from the principles of mechanics. Q. E. I.

COROLLARY 1

221. Therefore the length of the isochronous pendulum is equal to the moment of inertia of the figure with respect of the axis of rotation divided by the stability of the figure with respect of the same axis, indeed just as we have established to express the stability.

COROLLARY 2

222. Therefore with the same stability of the figure remaining in its equilibrium state, thus the oscillations will be faster, when the moment of inertia of the figure will have been less ; but with this moment being the greatest, the oscillations will become the slowest.

COROLLARY 3

223. Moreover, with the same moment of inertia of the figure remaining, there the oscillations will happen faster, where the stability of the figure will have been greater; but with the stability diminished, the oscillations will be performed more slowly.

COROLLARY 4

224. Therefore besides the weight, the figure, and the centre of gravity requiring to be defined for the oscillatory motion, which suffice for the stability requiring to be known, it will be required to know the moment of inertia of the figure with respect of the axis about which the oscillations will be made.

SCHOLIUM

225. So that both the stability as well as the oscillatory motion of figures of this kind floating on water may be understood more clearly, it will help to be considering special cases, in which until now indeterminate quantities will be able to be determined and will be allowed to be compared with each other. Therefore we will consider bounded figures, which shall float on water, where indeed it will suffice for parts of the figure may be put to be submerged, with the figure of the parts arising above the water may not enter into the calculation. Truly from the figure of the submerged parts likewise the centre of its magnitude is given. Moreover it may be agreed only regular figures may be going to be investigated , which shall have equal and similar parts around the vertical *EF* [as in Fig. 39], lest initially there would be a need to find the state of the equilibrium. Therefore we may put the centre of gravity of figures of this kind on the vertical *EF* itself, which is the diameter, the place where equilibrium may be had, if that vertical diameter will have found a place. Therefore here we shall link together some propositions of this kind, before we may progress to these bodies requiring to be examined.

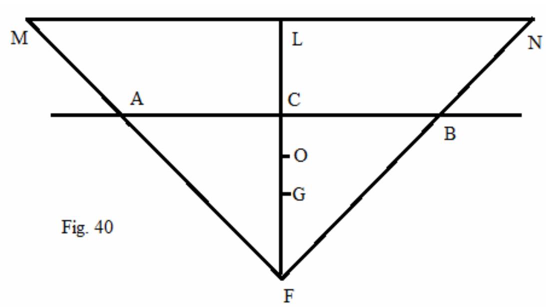


Fig. 40

PROPOSITION 22

PROBLEM

226. *If the submerged part AFB of the figure resting on water were an isosceles triangle* (Fig. 40), *to determine the stability of this state, and the oscillatory motion that the figure will acquire, if it may be inclined a little from this state.*

SOLUTION

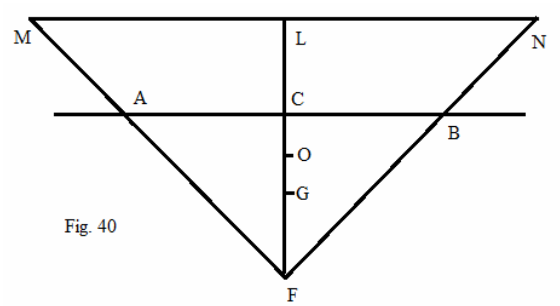


Fig. 40

From the vertex *F* in the base *AB,* which represents the section of the water, the perpendicular *FG* is drawn bisecting the base *AB* at *C.* There may be put $AC = BC = a$; and the perpendicular $FC = b$; the submerged part will be the area $AFB = ab$, and its centre of magnitude at *O,* so that there shall be $CO = \frac{1}{3}b$. Again G will be the centre of gravity of the whole figure, and $CG = h$, there will become

$$GO = CG - CO = h - \tfrac{1}{3}b.$$

Then the mass or total weight of the figure $= M$, and its moment of inertia with respect of the axis normal to the plane *AFB* and passing through *G* is $= S$. Therefore with these in place, the stability of this state of equilibrium will become :

$$M\left(h - \frac{1}{3}b + \frac{2a^2}{3b}\right) = \frac{M\left(2a^2 - b^2 + 3bh\right)}{3b}.$$

[Note that here the above term $\dfrac{AB^3}{12 \cdot AFB}$ becomes $\dfrac{2a^2}{3b}$.]

Truly, the length of the simple isochronous pendulum agreeing with the oscillations of this figure will be

$$= \frac{3bS}{M\left(2a^2 - b^2 + 3bh\right)},$$

if indeed the stability may have a positive value. Q. E. I.

COROLLARY 1

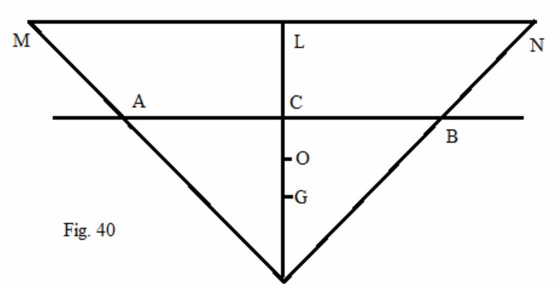


Fig. 40

227. Therefore if there were

$$2a^2 + 3bh > b^2 \text{ or } h > \frac{b^2 - 2a^2}{3b},$$

this state of the equilibrium will be stable, and thus the greater will be the stability, where the greater were the excess.

COROLLARY 2

228. Here again the state of the equilibrium will be stable, if there were

$$2a^2 + 3bh = b^2;$$

but if there were

$$2a^2 + 3bh < b^2;$$

then the state will be unstable, and that figure will not be able to be maintained, but rather will depart from that state on being disturbed a little.

EXAMPLE

229. If the whole figure were the isosceles triangle *MFN,* constructed from a uniform material, the specific gravity of which may hold the ratio $p : q$ to that of water, and there may be put

$$ML = LN = A \text{ and } FL = B,$$

then

$$AC = BC = a \text{ and } FL = b,$$

there will become $ab : AB = p : q$, and on account of $a : b = A : B$ there will become

$$a = A\sqrt{\frac{p}{q}} \text{ and } b = B\sqrt{\frac{p}{q}}.$$

Then truly there will become $LC = \frac{1}{3}B$, and on account of

$$LC = B - B\sqrt{\frac{p}{q}}$$

there will be had

$$CG = h = B\sqrt{\frac{p}{q}} - \frac{2}{3}B.$$

But with the mass or weight of the figure remaining $= M$, the moment will become

$$S = \frac{M(MN^2 + MF^2 + NF^2)}{36}(\S 170) = \frac{\mathrm{M}(3A^2 + B^2)}{18}.$$

With these substituted the stability of the isosceles triangle *MFN* floating on water will be found, thus so that the base *MN* may emerge horizontally above the water,

$$= \frac{\mathrm{M}\left(2(A^2 + B^2)\sqrt{\frac{p}{q}} - 2B^2\right)}{3B}.$$

Which, if it were positive, the length of the simple isochronous pendulum will become

$$= \frac{B\left(3A^2 + B^2\right)}{12(A^2 + B^2)\sqrt{\frac{p}{q}} - 12B^2}.$$

COROLLARY 1

230. Therefore the state of equilibrium of this same triangle floating on water will be stable, if there were

$$(A^2 + B^2)\sqrt{\frac{p}{q}} > B^2 \text{ or } \frac{p}{q} > \frac{B^2}{\left(A^2 + B^2\right)^2}$$

that is,

$$\frac{p}{q} > \frac{LF^2}{MF^2}.$$

Truly it will slip if there were

$$\frac{p}{q} < \frac{LF^2}{MF^2}.$$

COROLLARY 2

231. If the triangle were equilateral, there will become $B = A\sqrt{3}$; and its stability produced in this equilibrium state will be

$$= \frac{2\mathrm{AM}\left(4\sqrt{\frac{p}{q}} - 3\right)}{3\sqrt{3}}.$$

Therefore the length of the isochronous pendulum will be

$$= \frac{A\sqrt{3}}{8\sqrt{\frac{p}{q}} - 6}.$$

COROLLARY 3

232. Therefore in this case the state of equilibrium will be stable, if there were $\sqrt{\frac{p}{q}} > \frac{3}{4}$, that is, if $\frac{p}{q} > \frac{9}{16}$. Therefore with the specific gravity of water may be put to be $q = 1000$, the state will be stable, if there were $p > 562\frac{1}{2}$: but it will be unstable, if the specific gravity of the triangle were less than $562\frac{1}{2}$.

COROLLARY 4.

233. If the angle at *F* were right, so that there shall become $B = A$, the stability will be

$$= \frac{2\mathrm{AM}\left(2\sqrt{\frac{p}{q}} - 1\right)}{3}.$$

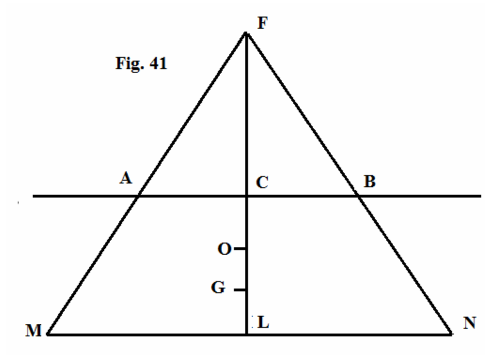
Therefore the state of equilibrium will be stable, if, on putting the specific gravity of water $= 1000$, the specific gravity of water were greater than 250. But if the specific gravity were less than 250, the state of equilibrium will be unstable. Therefore the length of the isochronous pendulum in that case will become

$$= \frac{\mathrm{A}}{6\sqrt{\frac{p}{q}} - 1}.$$

PROPOSITION 23.

PROBLEM

234. *If the figure floating on water were the isosceles triangle FMN* (Fig. 41), *having put the base MN underwater parallel to the horizontal section of the water AB; or rather, if the submerged part AMNB were a trapezium, in which the sides AB and NM are parallel to each other, and the angles at M and N are equal: to determine the stability by which this state of equilibrium may be conserved, and to determine the oscillatory motion, that a figure of this kind will acquire, if it may be disturbed a little from its state of equilibrium.*



SOLUTION

With the right line *CL* drawn to the vertical, which will bisect both the section of the water *AB*, as well as the base *MN*, in this position *O* will be the centre of the submerged part of the magnitude. Whereby it is necessary, that the centre of gravity of the whole figure shall lie on the same right line, which shall be at *G*. There may be put $AC = BC = a,\ ML = LN = c, CL = b$ and $CG = h.$ Truly the centre of the magnitude of the submerged part *O* thus will found, following the precepts of the situation, so that there shall become

$$CO = \frac{b(a+2c)}{3(a+c)};$$

therefore there will become:

$$GO = h - \frac{b(a+2c)}{3(a+c)}.$$

Besides *M* will denote the mass of the whole figure, and *S* the moment of inertia of the same with respect of the axis passing normally through the centre of gravity *G*. Therefore since the immersed part *AMNB* shall be $= b(a+\mathrm{c})$,
there will become:

$$\frac{AB^3}{12AMNB} = \frac{2a^3}{3b(a+c)},$$

from which the stability of the figure produced in this same state of equilibrium will become

$$= M\left(h - \frac{b(a+2c)}{3(a+c)} + \frac{2a^3}{3b(a+c)} \right) = \frac{M(3bh(a+c) - b^2(a+2c) + 2a^3)}{3b(a+c)}.$$

Finally the length of the simple pendulum isochronous with the oscillations of the figure will become

$$= \frac{3b(a+c)S}{M(3bh(a+c) - b^2(a+2c) + 2a^3)}.$$

Q.E.I.

COROLLARY 1

235. Therefore the state of figures of this kind will be stable, the submerged part of which is *AMNB,* if there were

$$3bh(a+c) - b^2(a+2c) + 2a^3 :$$

that is, if there were

$$h > \frac{b^2(a+2c) - 2a^3}{3b(a+c)}.$$

And thus the stability will be greater, where this expression will have a greater value:

$$3bh(a+c) - b^2(a+2c) + 2a^3,$$

or this one:

$$h - \frac{b^2(a+2c) + 2a^3}{3b(a+c)}.$$

COROLLARY 2

236. On the other hand, if there were

$$h < \frac{b^2(a+2c) - 2a^3}{3b(a+c)}$$

the state of the equilibrium of this same figure will be unstable, and liable to be overturned. Truly the state of equilibrium will be neutral, if there were

$$h = \frac{b^2(a+2c) - 2a^3}{3b(a+c)}$$

in which generally no stability will be had.

COROLLARY 3

237. If there shall become $c = a,$ the submerged part will become a rectangle; therefore with these cases there will be stability, in which the figure persists in this case

$$= M\left(h - \frac{b}{2} + \frac{aa}{3b}\right)$$

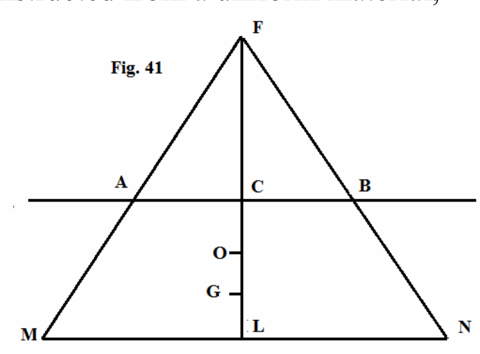
COROLLARY 4

238. If there may be put $c = 0$, the submerged part will become an isosceles triangle, which is the case treated in the preceding proposition ; moreover the stability will be found entirely as before $= M\left(h - \frac{b}{2} + \frac{aa}{3b}\right)$.

EXAMPLE

239. The whole figure floating in water shall be constructed from a uniform material, the specific gravity of which to that of water may hold the ratio $p : q$.
There may be put

$$ML = NL = A;$$



and the perpendicular $FL = B$; therefore there will become $c = A$; and $FG = B - b$. On account of which there will be had

$$FL \cdot ML : FG \cdot AC = q : q - p$$

that is

$$AB : \ a(B-b) = q : q - p$$

and on account of

$$FC : AC = FL : ML$$

there will become

$$B - b : a = B : A \text{ or } a = \frac{A(B-b)}{B};$$

with which value substituted into that ratio there will become

$$AB^2 : \ A(B-b)^2 = q : \ q - p \text{ or } q(B-b)^2 = (q-p)B^2;$$

from which there arises :

$$pB^2 = q\left(2Bb - bb\right),$$

or

$$b = B - \frac{B\sqrt{(q-p)}}{\sqrt{q}},$$

and hence

$$a = \frac{A\sqrt{(q-p)}}{\sqrt{q}}.$$

Truly since there shall be $FG = \frac{2}{3}B$, there will be

$$CG = h = \tfrac{2}{3}B - \frac{B\sqrt{(q-p)}}{\sqrt{q}}.$$

If now all these may be substituted into the stability formula of this situation by trial, this stability will be found

$$= \frac{2M\left(q-p\right)}{3Bp}\left(\left(A^2 + B^2\right)\sqrt{\frac{q-p}{q}} - B^2\right).$$

But the moment of inertia of this figure is as before $= \frac{M\left(3A^2 + B^2\right)}{18}$. On account of which if the stability were positive, the length of the isochronous simple pendulum will be

$$= \frac{Bp\left(3A^2 + B^2\right)}{12\left(q-p\right)\left(A^2 + B^2\right)\sqrt{\frac{q-p}{q}} - 12\left(q-p\right)B^2}.$$

COROLLARY 2

240. Therefore the state of this triangle floating on the water with this ratio will be stable, if there were

$$\left(A^2 + B^2\right)\sqrt{\frac{q-p}{q}} > B^2;$$

or

$$\frac{q-p}{q} > \frac{B^4}{\left(A^2 + B^2\right)^2}$$

that is, if

$$\frac{p}{q} < \frac{A^2\left(A^2+B^2\right)}{\left(A^2+B^2\right)^2}.$$

But this state of equilibrium will be unstable, if there were

$$\frac{p}{q} > \frac{A^2\left(A^2+2B^2\right)}{\left(A^2+B^2\right)^2}.$$

COROLLARY 1

241. If these may be brought together with the preceding proposition (§ 230) it will be apparent twice the same triangle to be able to have a stable position, if there were $A^4+2A^2B^2>B^4$ or $A^2>B^2(\sqrt{2}-1)$, that is, if there were $FM>FL\sqrt[4]{2}$. Which occurs, if ang.$MFL>33°$ or ang.$MFN>66°$. Truly, with this happening, the bounds of $\frac{p}{q}$ themselves will become

$$\frac{FM^4-FL^4}{FL^4} \text{ and } \frac{FL^4}{FM^4};$$

between which the case where $p:q=1:2$ will always be contained; or where the figure is two times as light as water.

COROLLARY 3

242. Again it is understood to be able to happen, so that neither state of equilibrium of the isosceles triangle can happen, where the base is horizontal, may be had stable ; which can happen, if the ang. MFN were greater than $66°$; and $\frac{p}{q}$ may be contained between these limits

$$\frac{FL^4}{FM^4} \text{ and } \frac{FM^4-FL^4}{FM^4},$$

where the first of these is greater, here the other smaller. Again the simple case, where $\frac{p}{q}=\frac{1}{2}$, is contained between these two limits.

COROLLARY 4

243. The isosceles triangle may become equilateral; in which case there becomes $B=A\sqrt{3}.$ Hence on account of this change the stability of the equilateral triangle as expressed in the figure floating on water

$$= \frac{2AM(q-p)}{3p\sqrt{3}}\left(4\frac{\sqrt{(q-p)}}{\sqrt{q}} - 3\right).$$

COROLLARIUM 5

244. Therefore the situation of such an equilateral triangle floating in water will be stable, if there were

$$\sqrt{\frac{q-p}{q}} > \frac{3}{4} \text{ or } \frac{q-p}{q} > \frac{9}{16},$$

which happens if there were $\frac{p}{q} < \frac{7}{16}$. Therefore with the specific gravity of water put $= 1000$; the situation will be stable, if the specific gravity of the triangle were $< 437\frac{1}{2}$.

COROLLARY 6

245. If the angle *MFN* were right, or $B = A$; the stability would be

$$= \frac{2AM(q-p)}{3p}\left(2\sqrt{\frac{(q-p)}{p}} - 1\right),$$

the state of the equilibrium therefore will be stable, if there were

$$\frac{q-p}{q} > \frac{1}{4} \text{ or } \frac{p}{q} < \frac{3}{4}.$$

Therefore this happens, if the specific gravity of the triangle were < 750, with the specific gravity of the water put $= 1000$.

SCHOLIUM

246. Although these propositions evidently shall consider only plane triangular figures, yet these also, as I have observed now, can be adapted to bodies, of which all the transverse sections are equal isosceles triangles ; triangular prisms are of this manner. Therefore with the aid of various prisms of this kind, the bases of which are isosceles triangles of various kinds, the truth of these, which we have deduced from these propositions, will be able to be verified by experiment. Here in the other propositions we have considered only these isosceles triangles in the state of floating on water, in which the horizontal bases maintain a position, with the remaining situations of equilibrium omitted, since for these exceedingly prolix calculations are being avoided, then truly especially, because then it will be allowed to judge well enough with care for the

protracted cases concerned with the stability of these remaining situations. Indeed in the following several equilibrium states will be shown, in which each body is able to float on water, some to be stable, others to be unstable. Whereby since here we shall have determined the stability, concerned with the remaining cases in which isosceles triangles are able to float on water, with the remaining cases judgment will be easy to be made. Indeed if a certain state of equilibrium were unstable, these states, which will occur near each other, certainly will be stable, unless both may coalesce into one, in which case the state of equilibrium is stable.

PROPOSITIO 24

PROBLEM 107

247. *If the submerged part of the figure AIHB* (Fig. 42) *floating on water were a rectangle; to determine the stability, by which the figure may persevere in this state; and the oscillatory motion, by which the figure may sway, if it may be declined a little from this state.*

SOLUTION

The vertical line *GL* may be drawn bisecting the rectangle *AIHB*, at the mid-point *O* of which will be the centre of the submerged magnitude [now called the metacentre or the centre of buoyancy, while *GO* is now called the metacentric height; we shall however, retain Euler's notations]; therefore on the same line will fall the centre of gravity of the whole figure, which shall be at *G*.

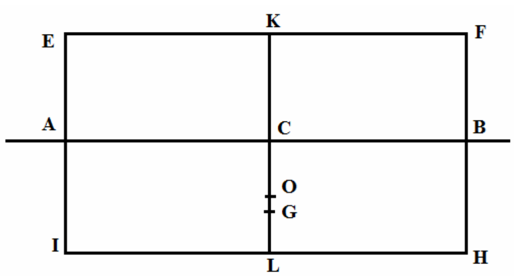


Fig. 42

Now there may be put $AC = BC = \frac{a}{2}$; or $AB = a$ and $AI = BH = CL = b$; and $CG = h$; there will become $CO = \frac{1}{2}b$ and $OG = h - \frac{1}{2}b$; finally the mass of the whole figure shall be $= M$. Therefore the stability, by which this figure persists in this state of equilibrium, which in general is

$$= M\left(GO + \frac{AB^3}{12AIHB}\right),$$

for this case will become

$$= M\left(h - \frac{1}{2}b + \frac{a^2}{12b}\right) = \frac{M(a^2 - 6bb + 12hb)}{12b}.$$

But for the oscillations requiring to be defined, which the figure will perform about this state of equilibrium, the moment of inertia of the figure with respect of the point $G = S$, and hence the length of the simple isochronous pendulum will be found

$$= \frac{12Sb}{M(a^3 - 6bb + 12bh)},$$

from which time may be determined, in which the smallest rocking motions may be performed. Q.E.I.

COROLLARY 1

248. Therefore so that the equilibrium may be preserved, it is necessary that

$$a^2 - 6bb + 12bh$$

shall be a positive quantity, that which arises if there were

$$h > \frac{6bb - aa}{12b}.$$

But if there were

$$h = \frac{6bb - aa}{12b},$$

then the equilibrium state will be constant, and truly with slipping and to be liable to be overturned, there would become,

$$h < \frac{6bb - aa}{12b}.$$

COROLLARY 2

249. Therefore an exceedingly thin stick will be able to stay in a vertical state in water, if there were $h > \frac{b}{2}$, on account of the thickness a to be almost vanishing. Therefore this state will arise, if its centre of gravity falls below the middle of the submerged part, or below its middle part.

COROLLARIUM 3

250. Therefore unless the lower part of the stick shall be noticeably heavier than the upper part, a stick will not be able to maintain a vertical position placed in water. But from the formula found it will be permitted to determine, how great a quantity of lead 0r

other material will required to be added to the lower part, so that it may remain in a vertical situation.

EXAMPLE

251. The whole figure resting shall be the parallelogram *EIHF* constructed from a uniform material, of which the specific gravity to water shall maintain the ratio $p : q,$ and its length shall be

$$EF = IH = A,$$

and its width

$$EI = FH = B;$$

there will be

$$a = A \text{ and } B{:}b = q : p$$

from which there becomes

$$b = \frac{Bp}{q} = CL;$$

but

$$KG = LG = \tfrac{1}{2}B;$$

hence there will be produced

$$CG = CL - LG = \frac{Bp}{q} - \tfrac{1}{2}B = h.$$

With these substituted in the stability formula expressed there will be found :

$$\frac{\mathrm{M}(A^2q^2 - 6B^2pq + 6B^2p^2)}{12Bpl},$$

from which expression, the stability of this state of equilibrium is defined. Truly the moment of inertia of this figure with respect of its centre of gravity G is $= \dfrac{\mathrm{M}(A^2 + B^2)}{12}$, and from which the length of the isochronous simple pendulum will be obtained

$$= \frac{Bpq(A^2 + B^2)}{A^2q^2 - 6B^2pq + 6B^2p^2},$$

from which the oscillations of this state of equilibrium will become known, if indeed it were stable.

COROLLARY 1

252. Therefore so that this state of equilibrium shall be stable, it is required that there shall be

$$A^2q^2 - 6B^2pq + 6B^2p^2 > 0$$

or

$$A > \frac{B}{q}\sqrt{6pq - 6pp}.$$

Therefore with the material given, from which the rectangle is constructed with the specific gravity, hence the ratio of the sides A and B becomes known, which becomes as the rectangle with the side B to the vertical, and with the side A to the horizontal, it may be able to float on the water.

COROLLARIUM 2

253. From these likewise it is understood this same rectangle to be able to float on water with the side A emerging to the vertical, truly B to the horizontal, if there were

$$B > \frac{A}{q}\sqrt{6pq - 6pp}.$$

COROLLARY 3

254. Therefore of these two equilibrium states, either will be able to be stable, if both $\frac{A}{B}$ as well as $\frac{B}{A}$ were greater than $\frac{\sqrt{6pq - 6pp}}{q}$. But this cannot happen, unless there shall be $q > \sqrt{(6pq - 6pp)}$ or $qq > 6pq - 6pp$.
That which can happen in two ways, in the first place clearly if there were

$$\frac{q}{p} > 3 + \sqrt{3},$$

and in the second place, if there were

$$\frac{q}{p} < 3 - \sqrt{3}.$$

COROLLARY 4

255. Therefore so that each side of the rectangle may be able to sit more firmly in the water, the materials from which it is constructed must have a specific gravity either

greater than $788\frac{2}{3}$ or less than $211\frac{1}{3}$ with the weight to water put to be 1000. Of these cases both sides of the rectangle are able to be put to use, so that each may become a state of equilibrium.

COROLLARY 5

256. Therefore if the specific gravity of the material, from which the rectangle is constructed, may be held between these limits $788\frac{2}{3}$ and $211\frac{1}{3}$, then no rectangle can be made, so that for each situation it shall be able to float more firmly.

COROLLARY 6

257. But if, for the given sides of the rectangle *A* and *B,* it was required according to this in order that the rectangle , with the side *A* being horizontal and *B* vertical, may be able to float more firmly in the water, that there shall become either

$$\frac{q}{p} > \frac{3B^2 + B\sqrt{(9B^2 - 6A^2)}}{A^2} \text{ or } \frac{q}{p} < \frac{3B^2 - B\sqrt{(9B^2 - 6A^2)}}{A^2},$$

or just as it is necessary that there shall be either

$$p < \frac{q\left(3B - B\sqrt{(9B^2 - 6A^2)}\right)}{6B} \text{ or } p > \frac{q\left(3B + B\sqrt{(9B^2 - 6A^2)}\right)}{6B}.$$

COROLLARIUM 7

258. Therefore, if the rectangle may become a square, and there shall become $B = A$, then the state of equilibrium will be stable, so that either side is present horizontal and the other vertical, if there were either

$$p < \frac{q\left(3 - \sqrt{3}\right)}{6} \text{ or } p > \frac{q\left(3 + \sqrt{3}\right)}{6}$$

that is if, with 1000 denoting the specific gravity of water, the square of the specific gravity shall be either greater than $788\frac{2}{3}$ or less than $211\frac{1}{3}$.

SCHOLIUM

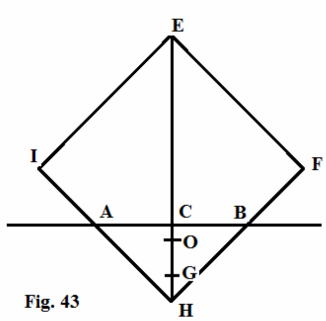
259. Even if these may be seen to pertain only to the plane figures, which have been determined in this proposition, yet they pertain to rectangular parallelepipeds of all kinds; indeed from these, for whatever the proposed parallelepiped, it will be allowed to judge

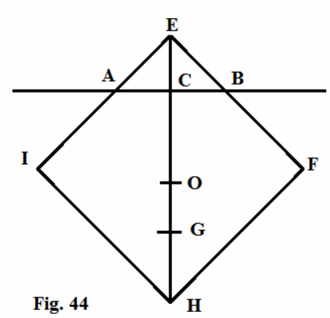
from above, how many faces will be able to float on water with stability. Accordingly the corollary of the last example is extremely well adapted for the floating of cubes on water, if indeed cubes shall be constructed either from a homogeneous material, or perhaps they may have the centre of gravity in its middle position. Moreover it is understood cubes of this kind, where both the faces are horizontal, the remaining faces vertical, are unable to float in water erect, unless the specific gravity of these either were greater than $788\frac{2}{3}$ or smaller than $211\frac{1}{3}$, with the specific gravity of water put to be $=1000$. Therefore as often as the specific gravity of the cube is held between these limits, evidently it is greater than $211\frac{1}{3}$ less than $788\frac{2}{3}$, then by no means will such a cube be able to float erect in water, but will adopt another situation, where either the plane with the diagonal or the diagonals themselves, will occupy either a horizontal or vertical position. Indeed a great number of states of this kind will be allowed to be deduced to become stable from the following proposition, in which indeed not only the square thus to be floating in water as above, but so that others with the diagonals horizontal and with the other vertical may have a place, which I am going to subject to an analysis.

PROPOSITION 25

PROBLEM

260. *To define the stability of the square EIHF* (Fig. 43, 44) *thus so that if it floats on water, its diagonal EH may obtain a vertical state, whereby it may remain in this state, and the oscillatory motion shall be about this state of equilibrium.*





SOLUTION

This case is going to be resolved in a twofold manner, just as the part immersed in water shall be more or less than half , of which the one happens, if there were $p > \frac{1}{2}q$ and truly the other if $p < \frac{1}{2}q$, with $p : q$ denoting the ratio the weights of the square holds to an equal volume of water. Moreover, the side of the square shall be $= A$; and with the centre of gravity of the square at *G,* there shall become $HG = h$. From these premises in kind we will consider the first case (Fig. 43) where there is $p < \frac{1}{2}q$, and the submerged part shall be the triangle *AHB,* with the section of the water being AB. Therefore there will become $AC \cdot CH$ or AC^2 to become $AC^2 : A^2 = p : q$ and thus

$$AC = CH = A\sqrt{\frac{p}{q}},$$

truly the centre of gravity of the submerged part falls at *O,* so that there shall become

$$HO = \tfrac{2}{3}A\sqrt{\frac{p}{q}}.$$

On account of which,

$$GO = -h + \tfrac{2}{3}A\sqrt{\frac{p}{q}}.$$

Hence therefore the stability [*i.e.* the buoyancy force] which will produce this state of equilibrium

$$= M\left(-h + \tfrac{2}{3}A\sqrt{\frac{p}{q}} + \tfrac{2}{3}A\sqrt{\frac{p}{q}}\right) = M\left(\tfrac{4}{3}A\sqrt{\frac{p}{q}} - h\right).$$

Finally with the moment of inertias of the square with respect of the centre of gravity $G = S$, the length of the simple pendulum will become

$$= \frac{S}{M\left(\tfrac{4}{3}A\sqrt{\frac{p}{q}} - h\right)},$$

which performs the isochronous oscillations of the square. Q. E. D. First part.

Now there shall be put $p > \frac{1}{2}q$, in which case (Fig. 44) as more than half the part *AIHFB* is immersed, with the section of the water being *AB*. Therefore there will be on dividing :

$$\mathrm{A}^2 - AC^2 : A^2 = p : q,$$

from which there will be

$$AC^2 : A^2 = q - p : q$$

and

$$AC : CE = A\sqrt{\frac{q-p}{q}}$$

$$CH = A\sqrt{2} - A\sqrt{\frac{q-p}{q}}.$$

From these found the centre of the magnitude of the submerged part at O thus shall be so that

$$HO = A\sqrt{2} - \frac{Aq}{p\sqrt{2}} + \frac{2}{3}\frac{A(q-p)\sqrt{(q-p)}}{p\sqrt{q}}$$

and hence there shall become

$$GO = h + A\sqrt{2} - \frac{Aq}{p\sqrt{2}} + \frac{2A(q-p)\sqrt{(q-p)}}{3p\sqrt{q}}$$

truly the submerged part itself will be $= \frac{A^2 p}{q}$. On account of which the stability of this state of equilibrium will be

$$= M\left(-h + A\sqrt{2} - \frac{Aq}{p\sqrt{2}} + \frac{4A(q-p)\sqrt{(q-p)}}{3p\sqrt{q}}\right)$$

if the moment of the figure with respect of the centre of gravity G shall be divided by that, which shall be S, the length of the simple pendulum, which oscillates around this place of equilibrium will be provided. Q. E. D. Second part.

COROLLARIUM I

261. If the centre of gravity of this square may lie at its midpoint, where the case shall be $h = \frac{A}{\sqrt{2}}$; in the first case where $p < \frac{1}{2}q$ the stability will become

$$= AM\left(\frac{4}{3}\sqrt{\frac{p}{q}} - \frac{1}{\sqrt{2}}\right),$$

truly in the latter case, where $p > \frac{1}{2}q$ the stability will become

$$= AM\left(\frac{4}{3}\sqrt{\frac{q-p}{q}} - \frac{1}{\sqrt{2}}\right).$$

COROLLARIUM 2

262. Therefore a constant square from a uniform material, which is more than twice as heavy as water, will sit firmly in the situation of the vertical diagonal, if there were

$$\frac{4}{3}\sqrt{\frac{p}{q}} > \frac{1}{\sqrt{2}},$$

this is if there were $\frac{p}{q} > \frac{9}{32}$ (Fig. 43). Therefore with the specific gravity of water put to be 1000, the state of equilibrium will have this stability, if the specific gravity were less than 500, truly greater than $281\frac{1}{4}$.

COROLLARY 3

263. Truly a constant square made from a material more than double the weight of water will sit firmly in water with the diagonal vertical, if there were

$$\frac{q-p}{q} > \frac{9}{32} \text{ or } \frac{p}{q} < \frac{23}{32} \text{ (Fig. 44).}$$

Therefore this happens, if its specific gravity were greater than 500, but less than $718\frac{3}{4}$.

COROLLARY 4

264. But before we have found a square cannot float on water, so that two sides may be held horizontally, two placed vertically, if indeed its specific gravity may be contained between the limits $211\frac{1}{2}$ et $788\frac{2}{3}$. On account of which squares of this kind are unable to float on water, the specific gravity of which is contained either between these limits $211\frac{1}{3}$ and $281\frac{1}{4}$, or between these limits $788\frac{2}{3}$ and $718\frac{3}{4}$.

SCHOLIUM

Hence the floating of prisms in water can be considered, made from homogeneous materials the bases of which are squares, if indeed the axes may maintain a horizontal situation or if the bases shall be placed vertically. Indeed prisms of this kind will float on water in three ways, for various ratios of the specific gravity. Clearly in the first place, two faces horizontal, indeed two vertical will be held in place, if the specific gravity of the prism were either smaller than $211\frac{1}{3}$ or greater than $788\frac{2}{3}$. Secondly the position will be of two diagonal planes, the one vertical and the other truly horizontal, if the specific gravity of the prism may be contained between the limits $281\frac{1}{4}$ and $718\frac{3}{4}$. Finally with neither of these treated in this manner, the prism will float in water with each side situated obliquely, if its specific gravity may be contained either between the limits $211\frac{1}{3}$ and $281\frac{1}{4}$, or between these $718\frac{3}{4}$ and $788\frac{2}{3}$. If it were required to establish this from experiments, the prisms will be required to be taken long enough, so that the axes of these will always lie horizontally on the water : for shorter prisms of this kind are less suitable for this investigation, since these may be able to float in water in more than three ways, therefore also since all the axes between themselves shall be able to preserve the flotation state constantly situated horizontally, here such [unwanted] variation is not present in longer prisms.

PROPOSITION 26

PROBLEM

266. *To determine the stability, by which some curvilinear figure AFB* (Fig. 45), *having similar and equal parts on both sides around the axis FC, may sit in a state of equilibrium in water.*

SOLUTION

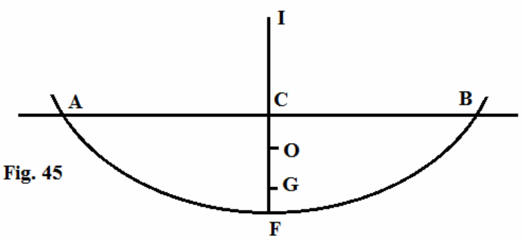


Fig. 45

AFB shall be the part immersed in water, and *AB* the section of the water, *FC* will be a vertical line and likewise a diameter of the orthogonal figure, thus so that there shall be $AC = BC$. M shall specify the mass of the whole figure, and its centre of gravity shall be at *G*, with there being $FG = h$.

Again there may be put

$$FC = x \text{ and } AC = BC = y,$$

thus so that the equation between *x* and *y* may express the nature of the proposed curve. Now *O* shall be the centre of the volume of the submerged area *AFB*, there will become

$$FO = \frac{\int yxdx}{\int ydx}, \text{ and thus } GO = \frac{\int yxdx}{\int ydx} - h.$$

Truly since the whole area immersed in water shall be $= 2\int ydx$, the stability of this equilibrium state will be found

$$= M\left(\frac{\int yxdx}{\int ydx} - h + \frac{y^3}{3\int ydx}\right),$$

which expression often can be transformed into this more convenient form :

$$M\left(\frac{\int y(xdx + ydy)}{\int ydx} - h\right).$$

Q.E.I.

COROLLARY 1

267. Therefore as many times as $\frac{\int y(xdx+ydy)}{\int ydx}$ is greater than *h,* this state of equilibrium will be just as many times as stable, and that more stable, where the excess

$$\frac{\int y(xdx+ydy)}{\int ydx}-h$$

will have been greater.

COROLLARY 2

268. But if $\frac{\int y(xdx+ydy)}{\int ydx}$ were either equal or even smaller than *h,* then in that case the state of the equilibrium will be unchanged, thus indeed here the minimum inclination may be overruled.

EXAMPLE 1

269. The figure *AFB* immersed in water shall be the segment of a circle of which the radius shall be $=a$, there will become

$$y=\sqrt{(2ax-xx)} \text{ and } y^2+x^2=2ax,$$

from which there becomes

$$ydy+xdx=adx.$$

Therefore in this case there will be had

$$\int ydx=\int dx\sqrt{(2ax-xx)}$$

and

$$\int y(xdx+ydy)=\int adx\sqrt{(2ax-xx)}.$$

Whereby the stability of this state of equilibrium will be $=M(a-h)$ which thus will be constant whether the segment of the circle may be immersed either more or less.

COROLLARY 1

270. Therefore provided that the centre of gravity of the figure may fall below the centre of the circle equilibrium will be maintained firmly, and that more, where the centre of gravity shall be placed more deeply.

COROLLARY 2

271. But if the centre of gravity may fall on the centre of the circle, then the state of equilibrium will be neutral, which happens with homogeneous cylinders resting horizontally on water.

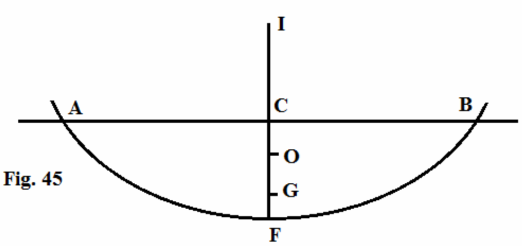


Fig. 45

EXAMPLE 2

272. The figure AFB shall be some conic section immersed in water having the vertex at F and the axis FC; evidently if n were a positive number

$$y=\sqrt{(2ax-nxx)},$$

the curve will be an ellipse, if n were negative a hyperbola, but if $n=0$ then the curve will be changed into a parabola. Therefore there will become

$$y^2+x^2=2ax-(n-1)xx$$

and

$$ydy+xdx=adx-(n-1)xdx.$$

Hence therefore stability will be obtained, where this same state of equilibrium is enjoyed

$$=M\left(\frac{\int dx\left(a-(n-1)x\right)\sqrt{(2ax-nxx)}}{\int dx\sqrt{(2ax-nxx)}}-h\right)=M\left(a-h\frac{(n-1)\int xdx\sqrt{(2ax-nxx)}}{\int dx\sqrt{(2ax-nxx)}}\right).$$

Therefore in the case where $n=0$ and the curve will become a parabola, the stability will become $=M\left(a-h+\frac{3}{5}x\right)$.

COROLLARY 1

273. If the points A, F and B may be considered as fixed, and the stabilities, for which the various conic sections passing through that may be considered between themselves, there may be put

$$CF=c \text{ and } AC=f;$$

on account of

$x = c$ and $y = \sqrt{(2ax - nxx)} = f$; there will become $a = \frac{f^2 + nc^2}{2c}$.

COROLLARY 2

274. Therefore in the case where the curve is a circle the stability will become

$$= M\left(\frac{ff + cc}{2c} - h\right).$$

But in the case where the curve is the parabola, the stability will become

$$= M\left(\frac{ff + \frac{6}{5}cc}{2c} - h\right).$$

Therefore the parabola has greater stability than the circle passing through the same three points.

COROLLARY 3

But generally the equation is satisfied approximately by :

$$\frac{(n-1)\int xdx\sqrt{(2ax - nxx)}}{\int dx\sqrt{(2ax - nxx)}} = \frac{3(n-1)x}{5}$$

On account of which the stability will be

$$= M\left(a - h - \frac{3(n-1)x}{5}\right) = M\left(\frac{ff + \frac{(6-n)}{5}cc}{2c} - h\right).$$

Therefore the stability thus will be greater, where *n* were smaller.

COROLLARY 4

276. But *n* cannot be diminished beyond a certain limit, because *a* must have a positive value, and there is

$$a = \frac{ff + ncc}{2c},$$

therefore for the greatest value there must become

$$n = \frac{-ff}{cc}$$

in which case the conic section will be changed into the isosceles triangle *AFB,* which therefore will conserve this state of equilibrium more firmly, than any other conic section passing through the points *A, F, B*, and having the centre of gravity at the same point *G*.

SCHOLIUM

276. These are able to suffice abundantly for the stability requiring to become known, which is present in any state of equilibrium, if indeed the body floating on the water either is a most tenuous plane figure, or can be considered to be the equal of such a shape. But before I may progress to the stability of bodies requiring to be investigated, I will advance and demonstrate a significant property which several states of equilibrium of the same body maintain between each other.

PROPOSITION 27

THEOREM

277. *If all the states of equilibrium may be considered, by which some given figure* (Fig. 46) *is able to hold in water, then all these states of equilibrium alternately will be stable and unstable.*

DEMONSTRATION

For any state of equilibrium a right line may be considered drawn through the centre of gravity of the figure parallel to the section of the water, and through that same right line drawn through the centre of gravity the state of equilibrium shall become known: indeed with this right line remaining parallel to the horizontal, the figure may be immersed in the water to that point, until the part must be present below the water, with what done the state of equilibrium will be had. Thus in the proposed figure *ACac* the right lines *Aa,Bb, Cc, Dd* drawn through the centre of gravity *G* will designate all the equilibrium states, which are given in this figure; evidently four states of equilibrium are given, in which the sections of the water shall be respectively parallel to the right lines *Aa, Bb, Cc, Dd;* with which in place I say, if the state of equilibrium *Aa* were stable, then also the state arising from this on computing the third *Cc* to be stable ; truly the second *Bb* and the fourth *Dd* to be unstable. Thus in this being required to be demonstrated I shall keep so that between two stable states by necessity one unstable state must be held, and equally between two unstable states there must be one stable state, indeed by this proof the truth of the theorem will prevail. Therefore *Aa* and *Cc* shall be the two states of equilibrium nearest to each other,

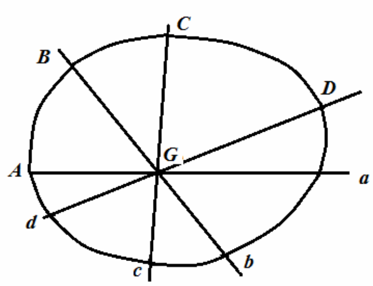


Fig. 46

or such that between which no other stable state may be given. Now if the figure may be inclined from state *Aa* towards the state *Cc* requiring to be converted, then indeed in the first place it will itself have struggled to return to the state *Aa* requiring to be restored, but if it may arrive closer to the state *Cc*, then the figure will have struggled to being received into the state of equilibrium *Cc*. On account of which it is necessary so that between these two stable states *Aa* and *Cc* a single position may exist for example *Bb,* which if the figure maintains a position inclined equally to each position *Aa* and *Cc*, therefore in this position an equilibrium will be given, that indeed is unstable, because the figure inclined a very small amount from that struggles to be either to the state *Aa* or to the state *Cc*; from which it is evident, between two situations of stable equilibrium by necessity there must be contained one situation of unstable equilibrium. In a similar manner if *Bb* and *Dd* were two unstable states, at once following each other, and each will be provided with the property, so that if the figure may be inclined a little towards the other, then there is going to be a struggle in receding from that equilibrium state ; on account of which by necessity between these two unstable states *Bb* and *Dd,* such a stable state as *Cc* must be given*,* in which the figure will avoid each state of equilibrium of the water ; therefore in this situation the figure will hold an equilibrium, and that stable since each figure declined from that prefers to be restored to that state. Therefore since one unstable state arises between two stable states, just as one stable state shall exist between two unstable states, all the states of equilibrium follow each other now stable, then unstable, mutually alternating between each other. Q. E. D.

COROLLARY 1

278. Therefore in each figure floating in water so many stable equilibrium states will be given, just as many unstable states, and because of this the number of all the equilibrium states will be even.

COROLLARY 2

279. Therefore no figure can have fewer than two equilibrium states. Indeed every figure by necessity has one equilibrium state and therefore also one unstable state.

COROLLARY 3

280. Therefore with all the states defined for some figure, by which it may be held in equilibrium in water, if it may be agreed to be from one figure, whether it shall be stable or unstable, likewise the same will be agreed for all the remaining states.

COROLLARY 4

281. Yet meanwhile it can happen, so that the number of equilibrium states in a certain figure actually may be taken to be odd, which will happen if two equilibrium states may be close together with the unstable states all merged into one, where a neutral state arises.

Therefore a neutral state of equilibrium must be considered as the conjunction of two nearby states of equilibrium, and therefore is required to be counted as two.

SCHOLIUM

282. The truth of this proposition extends not only to plane figures but also to all kinds of bodies floating on water. Indeed whichever body may be floating on water, if it may be turned around by the same blow then alternatively from a state of stable equilibrium then by necessity it must arrive at another of unstable equilibrium, just as may be understood from the given demonstration; and the matter is resolve in this way in whatever way the body is turned around by the blow. But all these will be understood more clearly from the following, where I am going to investigate the stability of any body floating on water in equilibrium.

DEFINITION

283. The stability with respect of some given horizontal axis passing through the centre of gravity is the force by which this body of water floating in the situation of equilibrium resists the inclination about the same horizontal axis passing through the centre of gravity.

COROLLARY 1

284. Therefore the stability with respect of a certain given axis drawn through the centre of gravity is required to be estimated from the moment of the pressure of the water, by which the body resists being inclined from the state of equilibrium by an infinitely small angle drawn through the centre of gravity, divided by that infinitely small angle itself.

COROLLARY 2

285. Therefore with bodies floating on water the stability of any equilibrium state is required to be estimated in an infinitude of ways, for the infinite horizontal axes passing through the centre of gravity, around which the body can be displaced on being inclined from its equilibrium state.

COROLLARY 3

286. Therefore it can be done so that the same state of the equilibrium with respect of one of several horizontal axis shall be stable enough, which still is unstable regarding the other remaining axes. But always for each body it will be required to be given one state of equilibrium, respecting which shall be stable with regard to all the axes; indeed otherwise the body may not be able to remain at rest on the water.

COROLLARY 4

287. But if the state of equilibrium of some body were stable in respect of two horizontal axes normal to each other, then this state of equilibrium will be stable with respect of all the remaining axes. Therefore the inclination around the intermediate axes can be resolved into inclinations around these two axes normal to each other, which both with the aforementioned shall be the force for the restitution, which is necessary, so that the state of equilibrium shall be stable with respect of all the axes.

COROLLARY 5

288. Therefore in order that the stability of the body floating on water in equilibrium may become known, it will suffice to have investigated the stability in turn with respect of the two normal axes; since hence the stability with respect of any other axis may depend, and that may be estimated well enough without risk.

SCHOLIUM

289. While we have considered only plane figures floating vertically in water, in a single way we have determined the stability of the equilibrium of each, and that will suffice also, evidently because we have put figures of this kind to be moved normally around a single horizontal axis ; moreover will be easily understood, figures of this kind are stable, however great a magnitude they may be found to have, yet being diverted to the sides to be especially harmful. In a similar manner it is observed, ships being inclined to the prow or stern to be resisted much more strongly than being inclined to the sides, and therefore in that case to have greater stability than in that place. On account of which since now it shall be proposed by us to inquire into the stability, by which some any bodies may partake of floating on water, all the inclinations will be required to be considered, by which bodies can be inclined from their state of equilibrium, and the size of the force they resist to be defined for each inclination. Moreover a body can be inclined from its state of equilibrium in an infinite number of ways, from an infinitude of horizontal axes passing through the centre of gravity, around which motion of the body is present. On this account, the account is sought, when concerning the stability, by which some body may be held in water in a state of equilibrium, that cannot be determined absolutely, but certainly is required to be determined for a certain inclination, which the stability itself may exert; which finally presents that definition of determining the stability, in which the stability for a certain horizontal axis passing through the centre of gravity to be implied. But nevertheless with this difficult sum agreed on concerned with the stability of floating bodies to be determined with certainty, which is to be put in place, since an infinitude of axes must be considered, and to be assigned with respect of each stability, yet now to be noted there is no need for an insurmountable amount of labor of this kind, but to suffice, if the stability may be defined with respect of only two axes normal to each other in turn. Indeed the motion of the inclinations around some other axis can be viewed as composed from two motions of inclination around these two normal

axes in turn, for each of which if the stability were known, then the stability with respect of any other axis will be able to be deduced. On account of which therefore, it is going to be shown, for any state of equilibrium in any body the stability of the flotation must be investigated with respect of any axis.

PROPOSITION 28

PROBLEM

290. *To determine the stability of the body ACFDB* (Fig. 47), *floating in water in equilibrium, with respect of certain given axis cd passing through the centre of gravity G of the body, and to determine the oscillatory motion of which this body is capable about this axis.*

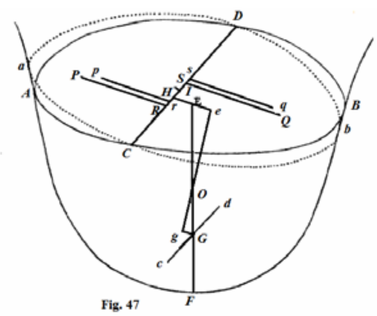


Fig. 47

SOLUTION

ACBD shall be the section of the water, and *AFB* the part of the body immersed, of which the centre of magnitude *O* will be placed on the vertical right line *EF* passing through the centre of gravity *G* of the body, thus so that the body is put in equilibrium. The total mass or weight of the body shall be $=M$, and the whole volume of its submerged part $=V$, and the moment of inertia of the whole body with respect of the axis $cd=S$ .Now the body may be considered for a short time to be inclined about the axis *cd* with the centre of gravity meanwhile either ascending or descending, so that an equal part of the water may remain immersed. Truly the inclination may be made through an infinitely small angle *dw,* with the whole sine put $=1$; and after this inclination the section of the water shall be *aCbD* cutting the first section of the water with the right line *CD* parallel to the axis *cd,* and the angle, which this new section of the water makes with the former section angle, equally $=dw$. Moreover because in each case an equal volume of the body shall be turned under the water, the segment *ACDa* shall be equal to the segment *BCDb.* Therefore the centre of gravity of the area *ACD* may be put at *p,* moreover the area of the centre of gravity *BCD* at *q,* and from *p* and *q* to *CD* the perpendiculars *pr* and *qs* shall be drawn, the whole of the segment $ACDa=ACD\cdot pr\cdot dw$, truly the whole of the segment *BCDb* will be $=BCD\cdot qs\cdot dw$; hence therefore there will be had $ACD\cdot pr=BCD\cdot qs$.
Besides the centre of the magnitude of the segment *ACDa* will fall at *P,* truly that of the segment *BCDb* at *Q* and from *P* and *Q* the normals *PR* and *QS* shall be drawn to *CD.*
Now through the point *O,* *eOg* shall be drawn perpendicular to the plane *aCbD,* which inclination will be vertical in the resting state of the body, and then from *G* on this right line, and then from *e* through *E* on *CD* the normals *Gg* and *eEH* shall be drawn, and there will become

$$Gg=GO\cdot dw \text{ and } Ee=EO\cdot dw.$$

Therefore so that the force requiring to be found from which the body is restored to its initial state of equilibrium from the state of inclination, the part of the body immersed in water is required to be considered, which is

$$= ACFDB + ACDa - BCDb\,,$$

from which individual members the forces are required to be to be defined for the body being restored, or required to be rotated around the axis *cd.* Moreover the moment of the pressure of the water, which the part *ACFDB* sustains for the restoration of the body, is

$$= M \cdot Gg = M \cdot GO \cdot dw\,.$$

Now it shall become so that as the submerged volume V to $ACDa$, thus the weight M to the force arising from the segment *ACDa*, which hence will be

$$\frac{M \cdot ACDa}{V} = \frac{M \cdot ACD \cdot pr \cdot dw}{V},$$

which expression will prevail equally for the pressure of the water in the segment *BCDb.* Moreover since the centre of the magnitude of the segment *ACDa* shall be at *P,* the moment thence arising for restoring the body will become

$$= \frac{M \cdot ACD \cdot pr \cdot dw}{V}(PR + He + Gg)\,;$$

truly the moment arising from the force of the segment *BGDb* will tend towards inverting the body, and thus will be negative and

$$= \frac{-M \cdot ACD \cdot pr \cdot dw}{V}(PR - He - Gg).$$

Of these three moments the two prior ones are required to be added, and the latter are required to be subtracted from the sum, with which done the total moment for the restoration of the body tending towards the restoration of the body to its former state

$$= M \cdot dw\left(GO + \frac{ACD \cdot pr}{V}(PR + QS) \right);$$

which divided by the angle of inclination *dw* will give the stability of this state of equilibrium with respect of the axis *cd*

$$= M\left(GO + \frac{ACD \cdot pr(PR + QS)}{V} \right).$$

The moment of the matter shall be divided by this expression of the stability, or the moment of inertia of the whole body with respect of the axis *cd,* which is *S,* and the length of the simple pendulum will be produced isochronous with the oscillations of the body itself around the axes *cd* with the restoration of the equilibrium, the length of which pendulum thence will be

$$=\frac{SV}{M\left(GO\cdot V+ACD\cdot pr(PR+QS)\right)}.$$

Q.E.I.

COROLLARY 1

291. Because $ACD\cdot pr=BCD\cdot qs$, and p and q are the centres of gravity of the areas *ACD* and *BCD,* it follows that the right line *CD* passes through the centre of gravity of the water section *ACBD.*

COROLLARY 2

292. Therefore if the centre of gravity *ACBD* of the section of the water were found at *I,* and the stability of this state of equilibrium were found with respect of the axis *cd,* then a right line *CD* may be drawn in the section of the water *ACBD* through the centre of gravity *1* parallel to the axis *cd* itself, from which the desired stability may become known from the calculation.

COROLLARY 3

293. Moreover, just as *p* and *q* are the centres of gravity of the parts *ACD* and *BCD* of the section of the water, thus it is understood to permit the points *P* and *Q* to be the centres of oscillation of the same parts oscillating about the axis *CD*.

COROLLARY 4

294. Therefore after the section of the water has been divided into two parts by a right line passing through the centre of gravity, both the centre of gravity of each part as well as the centre of the oscillation must be found, with which done the desired stability will be able to be found without being in respect of the angle of inclination *dw*.

COROLLARY 5

295. Since during the oscillation the centre of gravity of the whole body *G* must ascend and descent along a right line, so that always a part of the body must remain submerged in the water ; it is evident that this kind of motion of the centre of gravity to become minimal, if the right line passing through the centre of gravity of the whole body likewise shall pass through the centre of gravity of the section of the water.

COROLLARY 6

296. Moreover it is understood where the expression

$$M\left(GO+\frac{ACD\cdot pr(PR+QS)}{V}\right),$$

shall be greater, there the body to be going to persist more firmly in its state of equilibrium, evidently if it may be acted on to be inclined around the axis cd ; but if this expression may become negative, then the body is going to be inverted by the smallest inclination.

SCHOLIUM

297. Therefore the stability has been determined conveniently and neatly enough, by which any body floating on water persists in equilibrium, that which may be considered initially may be able to be seen with great difficulty. In addition also that rule is extremely simple and easy, with the help of these oscillations, where the body removed from equilibrium resolves its own restoration, from which the worth and usefulness of this theory is understood very well ; hence indeed it will be particularly suitable to be applied to sailing, if the explanation of this proposition were required to be deduced from inextricable calculations. But which for the stability requiring to be determined for some body it will be required to know, besides the weight of the whole body it will depend on the magnitude of the submerged parts, the distance between the centre of gravity of the whole body and the centre of the magnitude of the submerged parts [*i.e.* the centre of the buoyancy force] and especially the section of the water which is required to be placed below for the calculation of this sum. Therefore it will be agreed for the sections of various figures to be considered, and these expressions, which it will be required to know, to be defined, where according to this way, it shall be easier for a judgment to be considered, concerned with the floating of some body on water. This in the end I am going to accomplish in the following proposition with the expression found from an analytical calculation.

PROPOSITION 29

PROBLEM

298. *If the section of the water were some curve ACBD* (Fig. 48) [see (Fig. 47], *the nature of which is given by an equation, to define the stability of the body floating on water with respect of some axis, by an analytical calculation.*

SOLUTION

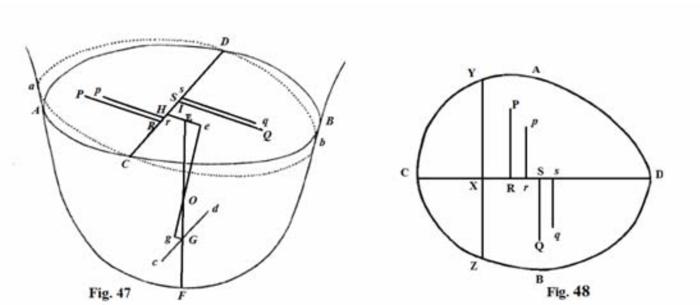


Fig. 47 Fig. 48

M shall be the mass or weight of the body floating on water, and *V* the volume of the submerged part, and *GO* shall express the distance between the centre of gravity of the body and the centre of the magnitude of the submerged part, with the centre of gravity *G* put in the lower location ; for with the centre of gravity *G* put above the centre of the magnitude of the magnitude *O* then in place of $+GO$ there must be written $-GO$. Now the right line *CD* shall be drawn parallel to that axis through the centre of gravity, with respect of which the stability is sought; and the orthogonal ordinates *YXZ* are referred to that line as axis ; and there may be put $CX = x$; $XY = y$ and $XZ = z$. Again *p* and *q* shall be the centre of gravity of the areas *CAD* and *CBD,* and *P* and *Q* the centres of oscillation with respect of the axis *CD* ; and from these points the normals *pr, qs, PR* and *QS* may be drawn normal to the axis *CD.* With these in place there shall be

$$pr = \frac{\int yydx}{2\int ydx};\ qs = \frac{\int zzdz}{2\int zdz};$$

and also

$$PR = \frac{2\int y^3dx}{3\int y^2dx};\ \text{ and }\ QS = \frac{2\int z^3dz}{3\int zzdx},$$

thus with these integrals taken so that they shall vanish on putting $x = 0$; and then on putting x in place of *CD.* Thus the integral $\int ydx$ will express the area *CAD,* and $\int zdx$ the area *CBD.* Truly since there shall become $CAD \cdot pr = CBD \cdot qr$, there will be

$$\int yydx = \int zzdx,$$

on account of

$$ACD \cdot pr = \tfrac{1}{2}\int yydx \text{ and } CBD \cdot qr = \tfrac{1}{2}\int zzdx .$$

Finally moreover there will be had

$$PR + QS = \frac{2\int y^3 dx}{3\int y^2 dx} + \frac{2\int z^3 dx}{3\int z^2 dx} = \frac{2\int \left(y^3 + z^3\right) dx}{3\int y^2 dx};$$

from which with these substituted into the formula found above, the stability of the body in this state of equilibrium found above

$$= M\left(GO + \frac{\int \left(y^3 + z^3\right) dx}{3V}\right).$$

COROLLARY 1

299. Therefore for the stability requiring to be obtained with the help of the integral calculus, with the integral of the formula $\int \left(y^3 + z^3\right) dx$, thus so that it may vanish on putting $x = 0$, and after the integration performed, there must be put $x = CD$.

COROLLARY 2

300. If the right line *CD* shall divide the section of the water into two equal and similar parts, there will be $z = y$ everywhere and in this case the stability will become

$$= M\left(GO + \frac{2\int y^3 dx}{3V}\right).$$

COROLLARY 3

301. If the centre of gravity of the whole body *G* shall fall above the centre of magnitude of the submerged part *O*, then it will be required to take the distance *GO* as negative, and in these cases the stability will become

$$= M\left(\frac{\int \left(y^3 + z^3\right) dx}{3V} - GO\right).$$

COROLLARY 4

302. Therefore unless G shall fall above O, the state of equilibrium with respect of all the axes will be stable; because $\int\left(y^3+z^3\right)dx$ always holds a positive value. But with the point G raised higher than O, then it can happen, that the state shall become unstable, which happens if there were

$$GO > \frac{\int\left(y^3+z^3\right)dx}{3V}.$$

SCHOLIUM

303. Moreover thus so that these formulas shall be allowed to be applied more easily to various kinds of bodies floating in water, I shall substitute some figures requiring to be determined in place of the water sections *ACBD*, and how I will investigate the stability of bodies of this kind floating on water. Moreover, not only will I determine the stability with respect of a single axis, but also with respect of two normal to each other, where from this twofold stability it may be possible to estimate the stability with respect of any other axis. To this end the figures chosen of this kind shall be most suitable, which may have a place either in sailing, or also for experiments requiring to be put in place, so that both the use as well as the usefulness of this theory may be presented most clearly for inspection. Therefore for this business requiring to be resolved I have assigned the following propositions, with which in place this chapter will be completely exhausted.

PROPOSITION 30

PROBLEM

304. *If the section of the body floating on water were the rectangle EFHK* (Fig. 49), *to find its stability both with respect of the axis CD, as well as of the axis AB normal to this.*

SOLUTION

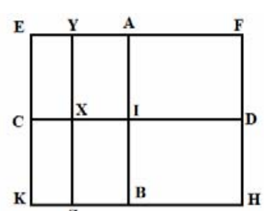


Fig. 49

First the axis *CD* will be considered parallel to the sides *EF* and *KH,* and there shall become $EF = KH = A$, $EK = B$, and the mass or weight of the body $= M$, and the volume of the submerged part $= V$.
There may be put
$CX = x$ there will become $XY = XZ = y = z = \frac{1}{2}B$.
Therefore

$$\int y^3dx = \frac{B^3x}{8} \text{ and } \int\left(y^3+z^3\right)dx = \frac{B^3x}{4} = \frac{A\cdot B^3}{4}$$

with the integral taken through the whole axes *CD.* Therefore the stability with respect of the axis *CD* will be

$$= M\left(GO + \frac{A \cdot B^3}{12V}\right).$$

Moreover, in a similar manner the stability with respect of the other axis *AB* will be

$$= M\left(GO + \frac{A^3 \cdot B}{12V}\right).$$

Q.E.I.

COROLLARY 1

305. Therefore if both $GO + \frac{A \cdot B^3}{12V}$ as well as $GO + \frac{A^3 \cdot B}{12V}$ were positive quantities, then the state of equilibrium will be in respect of some other axis.

COROLLARY 2

306. Therefore the stability with respect of the axis *CD* thus will be greater, where the side $EK = B$ were greater *;* and always the stability will be the greatest with respect of the shorter axis, which indeed is itself evident.

SCHOLIUM

307. If in a similar manner by integration the stability may be computed with respect of each of the diagonals *EH* and *FK,* then there will be found :

$$\int \left(x^3 + y^3\right) dx = \frac{A^3 \cdot B^3}{2\left(A^2 + B^2\right)}.$$

And the stability of the body itself with respect of this axis will become

$$= M\left(GO + \frac{A^3 \cdot B^3}{6V\left(A^2 + B^2\right)}\right).$$

Which expression is in the middle between the expressions found before for the axes *CD* and *AB.* And if there were $A = B$ then the stability will be equal both with respect of the axes *AB* and *CD* as well as with respect of the diagonals. From which it is more easily understood, for the state of the body floating on water requiring to be understood, to suffice to have determined the state with respect of two axes normal to each other.

Moreover two axis of this kind are required to be taken, which are particular in the section of the water, and of which one shall have the maximum and the other the minimum stability, just as we have done in the proposed case.

EXAMPLE 1

308. If the whole body were the parallelogram *MPQNRTVS* (Fig. 50) thus floating on water, so that *EKHF* shall be the section of the water ; and its weight shall be had to the weight of an equal volume of water as p to q. Then the length shall be $MN = a$; the width $MP = b$; and the height $PT = c$; in the section of the water there will be $A = a$ and $B = b$.

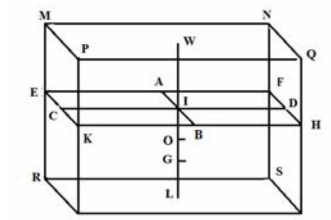


Fig. 50

Moreover, there will be from this state of equilibrium,

$$q : p = c : KT,$$

from which there becomes $KT = \dfrac{pc}{q}$; and the volume of the submerged part

$$= \frac{pabc}{q} = V;$$

the centre of magnitude of which O falls in the middle between I and L of the vertical right line *WL* drawn through middle of the parallelepiped, thus so that there shall be $LO = \dfrac{pc}{2q}$. Truly because this is put to be the state of equilibrium provided, it is necessary, that the centre of gravity of the whole body shall fall on the same vertical right line *LW;* therefore it shall be at *G,* with there being

$$LG = h, \text{ there will become } GO = \frac{pc}{2q} - h.$$

Therefore with these substituted the stability with respect of the axis of the length *CD,* by which the inclination around this axis is restored

$$= M\left(\frac{pc}{2q} - h + \frac{qb^2}{12pc}\right)$$

truly the stability with respect of the axis of the width *AB* will become

$$= M\left(\frac{pc}{2q} - h + \frac{qa^2}{12pc}\right)$$

in which expressions the letter *M* denotes the weight of the parallelepiped, and *p* to *q* the ratio of the specific gravity of the body to water. Therefore from these formulas the stability of this state of equilibrium with respect of any axis can be deduced.

COROLLARY 1

309. Therefore so that this state of equilibrium shall be stable, there will be required to be both

$$h < \frac{pc}{2q} + \frac{qb^2}{12pc}$$

as well as

$$h < \frac{pc}{2q} + \frac{qa^2}{12pc}.$$

Therefore if there shall be $a > b$, provided that there were

$$h < \frac{pc}{2q} + \frac{qb^2}{12pc}$$

the state of equilibrium with respect of all the axes will be stable.

COROLLARY 2

310. If the parallelepiped may be constructed from a uniform material, then its centre of gravity *G* falls in the middle between *L* and *W,* and there will become $h = \frac{1}{2}c$. Therefore in this case the stability will be with respect of the axis *CD.* Truly with respect of the other axis *AB* the stability will be

$$= M\left(\frac{qb^2}{12pc} - \frac{(q-p)c}{2q}\right)$$

COROLLARY 3

311. If the base of the parallelepiped shall be square or $a = b$; the stability shall be

$$= M\left(\frac{qa^2}{12pc} - \frac{(q-p)c}{2q}\right).$$

Therefore in order that this state of equilibrium shall be stable, it is necessary that there shall be

$$c < \frac{qa}{\sqrt{6p(q-p)}}.$$

EXAMPLE 2

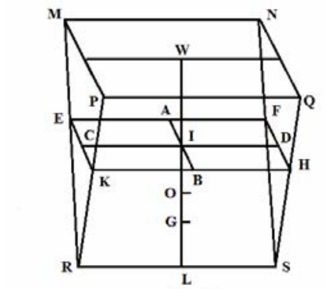


Fig. 51

312. If the body floating in water shall be of the wedge-shaped form *MRPQSN* (Fig. 51) sitting in an erect position, so that the section of the water shall be *EKHF* parallel to the rectangle *MPQN* of the base. Moreover the weight of this body shall be had to the weight of an equal volume of water as p ad q. There may be put $MN = PQ = a$; $MP = NQ = b$; and the height of the wedge *WL* shall $= c$; on which vertical line *WL* both the centre of gravity *G* as well as of the magnitude *O* shall be place, and with $LG = h$. Now for the remaining

$$EF = KH = A;\ EK = FH = B$$

there will be

$$A : B = a : b \text{ and } B : b = c : IL;$$

from which there becomes

$$IL = \frac{Bc}{b}.$$

But from the specific gravity it follows that $q : p = b^2 : B^2$, from which there is produced

$$B = b\sqrt{\frac{p}{q}};\text{ and } A = a\sqrt{\frac{p}{q}};\text{ and } IL = c\sqrt{\frac{p}{q}}.$$

From these there is found

$$LO = \tfrac{2}{3}LI = \tfrac{2}{3}c\sqrt{\frac{p}{q}};\text{ and } GO = \tfrac{2}{3}c\sqrt{\frac{p}{q}} - h.$$

Finally the volume of the submerged part *V* will become

$$= \frac{ABc}{2}\sqrt{\frac{p}{q}} = \frac{abcp\sqrt{p}}{2q\sqrt{q}}.$$

With which values substituted the stability with respect of the axis *CD* will emerge

$$= M\left(\tfrac{2}{3}c\sqrt{\frac{p}{q}} - h + \frac{bb}{6c}\sqrt{\frac{p}{q}}\right).$$

And with regard to the other axis *AB* the stability will become

$$=M\left(\tfrac{2}{3}c\sqrt{\frac{p}{q}}-h+\frac{a^2}{6c}\sqrt{\frac{q}{p}}\right).$$

COROLLARY 1

313. If the wedge has been made from a uniform material, there will be $h=\tfrac{2}{3}c$; and in this case the stability with respect of the axis *CD* will become

$$=M\left(\frac{bb}{6c}\sqrt{\frac{p}{q}}-\tfrac{2}{3}c+\tfrac{2}{3}c\sqrt{\frac{p}{q}}\right);$$

truly with respect of the axis *AB* the stability will become

$$=M\left(\frac{aa}{6c}\sqrt{\frac{q}{p}}-\tfrac{2}{3}c+\tfrac{2}{3}c\sqrt{\frac{p}{q}}\right).$$

COROLLARY 2

314. If there were

$$\sqrt{\frac{p}{q}}>\frac{4cc}{a^2+4cc},$$

and also

$$\sqrt{\frac{p}{q}}>\frac{4cc}{b^2+4cc},$$

then the state of this equilibrium will be stable; truly in the contrary cases the product will be unstable and will be turned upside down.

COROLLARY 3

315. If generally *h* may retain the same value, each expression

$$\frac{a^2}{6c}+\tfrac{2}{3}c \text{ and } \frac{bb}{6c}+\tfrac{2}{3}c$$

shall become infinitely large if $c=0$ while if $c=\infty$ it will adopt a minimum value, if there were $a=2c$ or also $b=2c$. Therefore with these cases stability produced will be a minimum with all else being equal.

EXAMPLE 3

316. The body resting on the water shall be the right pyramid *MNLPQ* (Fig. 52) of which the base *MNPQ* shall be horizontal and rectangular ; the water section *EFHK* of which therefore will be parallel and equally a rectangle. There shall be $MN = PQ = a$; $MP = NQ = b$, and the height $WL = c$; and the centre of gravity may be seen at *G*, so that there shall be $LG = h$. Moreover the weight of this pyramid shall be *M*, so that the volume of water of equal weight may be had, to be as *p* to *q*. Now there will be

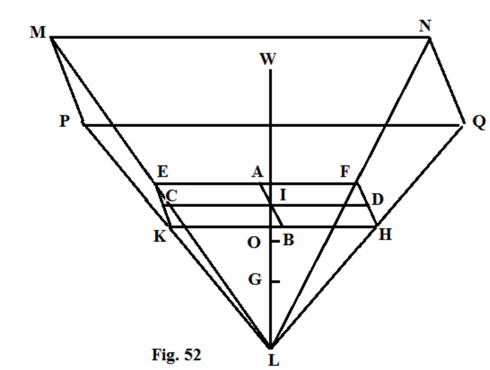


Fig. 52

$$a : b = A : B, \text{ and } a^3 : A^3 = q : p;$$

thus so that there shall be $A = a\sqrt[3]{\frac{p}{q}}$; and $B = b\sqrt[3]{\frac{p}{q}}$,

and likewise,

$$LI = c\sqrt[3]{\frac{p}{q}}.$$

Moreover the centre of the magnitude of the submerged part shall fall at *O* so that there shall become

$$LO = \tfrac{3}{4}c\sqrt[3]{\frac{p}{q}}, \text{ from which there will become } GO = \tfrac{3}{4}c\sqrt[3]{\frac{p}{q}} - h.$$

But the volume of the submerged part will be

$$= \frac{ABc}{3}\sqrt[3]{\frac{p}{q}} = \frac{pabc}{3q}.$$

With these substituted the stability will become, with respect of the axis *CD*

$$= M\left(\frac{3}{4}c\sqrt[3]{\frac{p}{q}} - h + \frac{b^2}{4c}\sqrt[3]{\frac{p}{q}}\right).$$

But with respect of the axis *AB,* the stability will become

$$= M\left(\frac{3}{4}c\sqrt[3]{\frac{p}{q}} - h + \frac{a^2}{4c}\sqrt[3]{\frac{p}{q}}\right).$$

COROLLARY 1

317. Therefore with both h as well as $p{:}q$ remaining the same, the stability with respect of the axis CD will be a minimum, if there were $b = c\sqrt{3}$. Truly the stability AB will be a minimum with respect of the axis, if there were $a = c\sqrt{3}$.

COROLLARY 2

318. Whereby therefore pyramids of this kind will be situated most firmly floating erect in water, it is to be avoided especially, in those requiring to be made, that neither a nor b become nearly equal to $c\sqrt{3}$.

COROLLARY 3

319. If this pyramid shall be agreed to be made from a uniform material, then there will become $h = \frac{3}{4}c$. Therefore the stability of such a pyramid with respect of the axis CD will be

$$= M\left(\frac{b^2}{4c}\sqrt[3]{\frac{p}{q}} + \frac{3}{4}c\sqrt[3]{\frac{p}{q}} - \frac{3}{4}c\right),$$

but with respect of the axis AB the stability will be

$$= M\left(\frac{a^2}{4c}\sqrt[3]{\frac{p}{q}} + \frac{3}{4}c\sqrt[3]{\frac{p}{q}} - \frac{3}{4}c\right).$$

COROLLARY 4

320. Therefore where it is necessary that a pyramid of this kind may be seated more firmly in water it is necessary that both

$$\sqrt[3]{\frac{p}{q}} > \frac{3cc}{bb+3cc} \quad \text{as well as} \quad \sqrt[3]{\frac{p}{q}} > \frac{3cc}{a^2+3cc}.$$

Therefore if there were $a < b$, it would be required that there shall be

$$\frac{p}{q} > \frac{27 \cdot cc}{\left(a^2+3c^2\right)^3}.$$

COROLLARY 5

321. If there were $a = b = c$; the position of such a pyramid will not be able to be preserved, unless there shall be

$$\frac{p}{q} > \frac{27}{64};$$

that is, unless the specific gravity of the pyramid shall be greater than $421\frac{7}{8}$, with the specific gravity of water put $= 1000$.

PROPOSITION 31

PROBLEM

322. *If the section of the body floating were the rhombus ACBD* (Fig. 53), *to determine its stability with respect of each of the diagonals CD and AB.*

SOLUTION

Initially an axis may be considered parallel to the diagonal *CD* passing through the centre of gravity of the body; and some ordinates *YXZ* may be drawn; and by calling

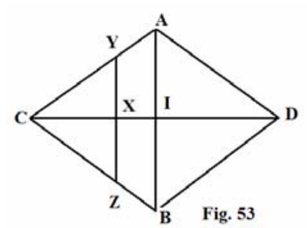


Fig. 53

$CI = DI = A$; $AI = BI = B$; $CX = x$; $XY = XZ = y$,
there will become

$$A : B = x : y \text{ and } y = \frac{Bx}{A} = z;$$

and the side of the rhombus *AC* will become $\sqrt{\left(A^2 + B^2\right)}$. With these in place there will become

$\int y^3 dx = \frac{B^3 x^4}{4A^3}$; and on putting $x = A$ the value of this expression will be had for the part $CIA = \frac{1}{4} A \cdot B^3$, which taken four times will correspond to the whole rhombus *CBDA,* for which hence there will become :

$$\int \left(y^3 + z^3\right) dx = A \cdot B^3.$$

Now if the weight of the whole body may be put $= M$; and the distance of the centre of magnitude above the centre of gravity may be put $= GO$, and the volume of the submerged part $= V$, the stability [*i.e.* the buoyancy force] with respect of the axis *CD*

$$=M\left(GO+\frac{A\cdot B^{3}}{3V}\right).$$

Moreover, in a similar manner the stability with respect of the axis *AB* will be found :

$$=M\left(GO+\frac{A^{3}\cdot B}{3V}\right).$$

From which two expressions the stability with respect of any other axis will be able to be deduced.
Q. E. I.

COROLLARY 1

323. If therefore the diagonals are unequal, the body has less resistance to changes of inclination around the longer axis than around the shorter axis ; which rules are to be found in almost all water sections, where the axes normal to each other are unequal.

COROLLARY 2

324. Therefore so that this state of equilibrium may be stable, it is necessary that both so that both

$$GO+\frac{A\cdot B^{3}}{3V}\quad\text{as well as}\quad GO+\frac{A^{3}\cdot B}{3V}$$

may have a positive value, that which happens, only if the smaller expression were positive.

COROLLARY 3

325. If the side AC of the rhombus may be put $=C$, and the sine of the angle $ACB=m$; truly the cosine of the angle $ACB=n$; and the cosine of the angle $CAD=-n$. Hence there is found

$$B=C\sqrt{\frac{1-n}{2}}\quad\text{and}\quad A=\frac{mC}{\sqrt{2(1-n)}}.$$

Whereby the stability with respect of the axis CD will be

$$=M\left(GO+\frac{m(1-n)C^{4}}{12V}\right)$$

but with respect of the axis *AB*

$$=M\left(GO+\frac{m(1+n)C^{4}}{12V}\right)$$

COROLLARY 4

326. If the rhombus shall be changed into a square, there will become $m=1$ and $n=0$; and in this case the stability with respect of each diagonal will be the same, evidently

$$= M\left(GO + \frac{C^4}{12V}\right);$$

which expression is found from the previous proposition also, with the application made of the parallelogram to a square.

EXAMPLE

327. The part of the body submerged in water is terminated by the right horizontal line *RS* parallel to the diagonal *CD* (Fig. 54), and with the right lines *BL, AL* drawn to the midpoint *L* of the right line *RS,* and likewise with the vertical lines *CR* and *DS* drawn, thus so that the individual horizontal sections shall be rhombi.
There may remain:

$$CI = DI = A;\ AI = BI = B;$$

and there shall be

$$CR=LI=DS=D;$$

the volume of the submerged part $V = ABD$; and its centre of magnitude at *O* so that there shall be $LO = \frac{2}{3}D$. Therefore the centre of gravity of the whole body shall fall at *G,* and there shall be called $LG = h$; therefore $GO = \frac{2}{3}D - h$. Therefore from these the stability of this state of equilibrium is found with respect of the axis

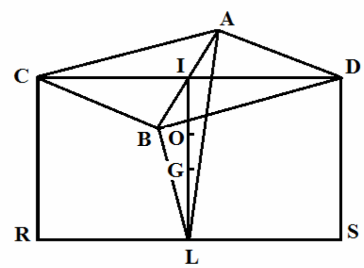


Fig. 54

$$CD = M\left(\frac{2}{3}D - h + \frac{B^2}{3D}\right)$$

But with respect of the axis *AB,* the stability will be

$$= M\left(\frac{2}{3}D - h + \frac{A^2}{3D}\right).$$

COROLLARY 1

328. Therefore so that the first state of equilibrium shall be stable, it is necessary that there shall be :

$$h < \frac{A^2 + 2D^2}{3D},$$

and likewise also:

$$h < \frac{B^2 + 2D^2}{3D},$$

Therefore if there were $B<A$,it will suffice for the stability of the body requiring to be prepared, to become :

$$h < \frac{B^2 + 2D^2}{3D}.$$

COROLLARY 2

329. Therefore unless there shall be $B > D$, necessarily the centre of gravity of the body must fall below the surface of the water, if indeed the state of equilibrium shall be stable.

PROPOSITION 23

PROBLEM

330. *If the section of the body floating on the water were the isosceles triangle ECF* (Fig. 55), *to determine the stability of the floating body both with respect of the axis CD as well as of the axis AB normal to that, and passing through the centre of gravity I of the water section.*

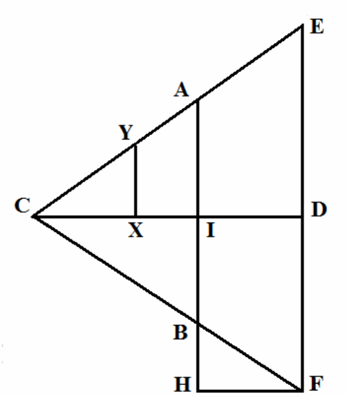


Fig. 55

SOLUTION

With the weight of the body put $= M$, with the volume of the submerged part $= V$, and the distance between the centres of gravity of the body and of the magnitude of the submerged part $= GO$, there shall be $CD = A$, and $DE = DF = B$; there will become $CX(x) : XY(y) = A : B$, from which there shall become $y = \frac{Bx}{A}$. On account of which there will be obtained $\int y^3 dx = \frac{B^3 x^4}{4A^3} = \frac{A \cdot B^3}{4}$, on putting $x = CD = A$. Therefore for the whole section of the body in the water, there will become $\int\left(y^3+z^3\right)dx = \frac{A\cdot B^3}{2}$, from which the stability with respect of the axis

$$CD = M\left(GO + \frac{A\cdot B^3}{6V}\right).$$

Now the axis AB may be considered, in which there is

$$AI = BI = \frac{2}{3}B \text{ and } CI = \frac{2}{3}A;$$

and there will be the integral $\int y^3dx$, arising from the area ACB

$$= \frac{AI\cdot CI^3}{4} = \frac{4A^3\cdot B}{81};$$

therefore the same formula $\frac{8A^3\cdot B}{81}$ will have arisen from the whole triangle ACB. Now for the other part the whole area $IDFH$ will be considered, which is rectangular, with there being

$$IH = DF = B \text{ and } DI = FH = \frac{1}{3}A;$$

and from that there will be produced :

$$\int y^3dx = DI^3\cdot IH = \frac{A^3\cdot B}{27}.$$

From which that value must be subtracted, which arises from the triangle BFH, which is equal to

$$\frac{BH\cdot FG^3}{4} = \frac{A^3\cdot B}{4\cdot 81},$$

and the value of $\int y^3dx$ itself is left for the trapezium $IDBF$, $=\frac{11A^3B}{4\cdot 81}$.
Therefore the value of the trapezium $ABFE$ will correspond to the value of

$$\int y^3dx = \frac{11A^3\cdot B}{2\cdot 81}.$$

On account of which, the total value of this integral itself with respect of the axis AB, will become

$$\int\left(y^3+z^3\right)dx=\frac{8A^3\cdot B}{81}+\frac{11A^3\cdot B}{2\cdot 81}=\frac{A^3\cdot B}{6}.$$

From which the stability of this state of equilibrium with respect of the axis *AB*

$$=M\left(GO+\frac{A^3\cdot B}{18V}\right).$$

Q.E.I.

COROLLARY 1

331. Therefore the stability with respect of the axis *CD* will be greater than the stability with respect of the axis *AB,* if there were

$$A\cdot B^3>\frac{A^3\cdot B}{3},$$

that is, if there were

$$\frac{B}{A}>\frac{1}{\sqrt{3}}.$$

Truly on the other hand, if there were

$$\frac{B}{A}<\frac{1}{\sqrt{3}}$$

then the stability with respect of the axis *AB* will be greater than with respect of the axis *CD.*

COROLLARY 2.

332. Since $\frac{B}{A}$ is the tangent of the angle *DCE* or the tangent of half the angle *ECF,* it is evident if the angle *ECF* were greater than 60°, then the stability with respect of the axis *CD* to exceed the stability with respect of the axis *AB;* truly the opposite, if the angle *ECF* shall be smaller than 60°.

COROLLARY 3

333. Therefore if the triangle *ECF* becomes equilateral, then the stability with respect of each axis will be the same. But on account of $A=B\sqrt{3}$, in this case the stability will become *AB*,

$$=M\left(GO+\frac{A^4}{2V\sqrt{3}}\right),$$

which will prevail for all the remaining axes.

COROLLARY 4

334. If the area of the equilateral triangle may be put $=E$ there will become $B^2\sqrt{3} \ =E;$ from which the state of the stability will be the state of the equilibrium

$$=M\left(GO+\frac{E^2}{6V\sqrt{3}}\right).$$

COROLLARY 5

335. But if the water section is the square of which the area shall be equally $=E$, then from above the stability found will be

$$=M\left(GO+\frac{E^2}{12\cdot V}\right).$$

Whereby, since there shall be

$$\frac{1}{6\sqrt{3}}>\frac{1}{12},$$

it follows the section of the water which is an equilateral triangle to produce a more stable state than the square of the same area, with all else being equal.

EXAMPLE 1

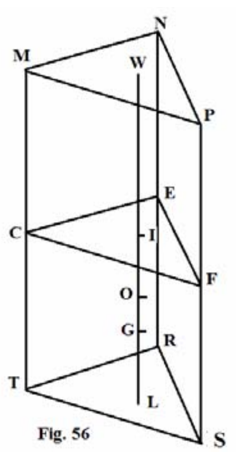


Fig. 56

336. The body floating on water shall be the triangular prism *MNPTRS* (Fig. 56), the horizontal sections of which shall be the equilateral triangles *MNP, CEF, TRS,* the sides of which shall be $=a$; truly the area $=bb$, or $bb=\frac{aa\sqrt{3}}{4}$. The weight of this prism shall be put $=M$, and its specific gravity to water as *p* to *q,* and the total height $MT=WL=c$. Now since *CEF* shall be the section of the water, there will become

$CT=\frac{pc}{q}$, [equating the water and prism pressures at L] and

$LO=\frac{pc}{2q}$, [the centre of buoyancy]; truly the volume of the submerged part

$$V=\frac{pb^2c}{q},\text{[from equating the upthrust to the weight of the prism.]}$$

Again the centre of gravity of the whole prism shall be at *G,* with there being $LG = h$; therefore

$$\mathrm{GO} = \frac{pc}{2q} - h.$$

Therefore from these, the stability of this equilibrium position with respect of each axis will become

$$= M\left(\frac{pc}{2q} - h + \frac{qb^2}{6pc\sqrt{3}}\right) = M\left(\frac{pc}{2q} - h + \frac{qa^2}{24pc}\right).$$

COROLLARY 1

337. If the prism were constructed from a uniform material, there will become $h = \frac{1}{2}c$, and the stability of this state of equilibrium will become

$$= M\left(\frac{qb^2}{6pc\sqrt{3}} - \frac{(q-p)c}{2q}\right) = M\left(\frac{qa^2}{24pc} - \frac{(q-p)c}{2q}\right).$$

COROLLARY 2

338. Therefore so that this state of equilibrium shall be stable, it will be required so that there shall become

$$c < \frac{qb}{\sqrt{3p(q-p)\sqrt{3}}},$$

or so that the same may be rendered

$$c < \frac{qa}{2\sqrt{3p(q-p)}}.$$

Hence therefore it becomes known, how long a part must be cut off from a triangular prism of indefinite length, so that it shall be able to float erect on water.

COROLLARY 3

339. If a square prism may be constructed from the same material, the bases of which shall be squares $= bb$, the length c of these must be less than $\frac{qb}{\sqrt{6p(q-p)}}$, with which in place they shall be able to float erect in water. Therefore to this end, longer triangular prisms will be allowed to be taken, than square ones.

EXAMPLE 2

340. The body floating on the water shall be the triangular pyramid *MNPL* (Fig. 57), of which the base *MNP* shall stand out above the water. Any side of the base *MNP*, which shall be an equilateral triangle, may be put $=a$, and the same base may be put $=bb$, thus so that the area shall become $b^2=\frac{a^2\sqrt{3}}{4}$. Again the height of the pyramid *WL* shall be $=c$, and its weight *M* may be had to an equal volume of water as p ad q; and thus the section *CFE* of the water which equally will be an equilateral triangle, the area of which shall be $=E$. Now there will be $q:p=b^3:E\sqrt{E}$ or $\sqrt{E}=b\sqrt[3]{\frac{p}{q}}$ and $E=b^2\sqrt[3]{\frac{p^2}{q^2}}$.

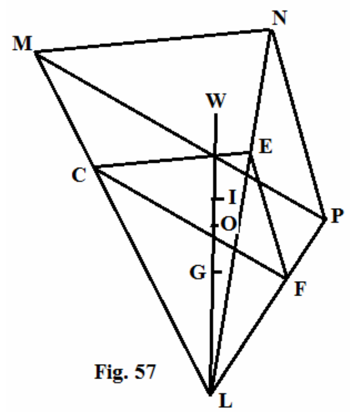


Fig. 57

And in a similar manner there will be

$$LI=c\sqrt[3]{\frac{p}{q}} \text{ and } LO=\tfrac{3}{4}c\sqrt[3]{\frac{p}{q}}.$$

But the volume *V* of the submerged part will be $=\frac{pb^2c}{3q}$. Finally there shall be $LG=h$; the stability of this equilibrium state, which the proposed pyramid holds,

$$=M\left(\frac{3}{4}c\sqrt[3]{\frac{p}{q}}-h+\frac{bb}{2c\sqrt{3}}\sqrt[3]{\frac{p}{q}}\right)=M\left(\frac{3}{4}c\sqrt[3]{\frac{p}{q}}-h+\frac{a^2}{8c}\sqrt[3]{\frac{p}{q}}\right).$$

COROLLARY 1

341. If the pyramid may be understood to be made from a uniform material, there will become $h=\frac{3}{4}c$. Therefore in this case the stability of this equilibrium state will be had

$$=M\left(\frac{bb}{2c\sqrt{3}}\sqrt[3]{\frac{p}{q}}-\frac{3}{4}c+\frac{3}{4}c\sqrt[3]{\frac{p}{q}}\right)=M\left(\frac{aa}{8c}\sqrt[3]{\frac{p}{q}}-\frac{3}{4}c+\frac{3}{4}c\sqrt[3]{\frac{p}{q}}\right).$$

COROLLARY 2

342. If the above pyramid may be changed into a tetrahedron or a regular pyramid, $c = a\sqrt{\frac{2}{3}}$; therefore the stability of the tetrahedron with the angle of the tetrahedron downwards floating on water will become

$$= Ma\left(\frac{\sqrt{3}}{8\sqrt{2}}\sqrt[3]{\frac{p}{q}} - \frac{\sqrt{3}}{2\sqrt{2}} + \frac{\sqrt{3}}{2\sqrt{2}}\sqrt[3]{\frac{p}{q}}\right) = \frac{Ma\sqrt{3}}{8\sqrt{2}}\left(5\sqrt[3]{\frac{p}{q}} - 4\right).$$

COROLLARY 3

343. Therefore so that a tetrahedron of this kind may be able to maintain such a state of equilibrium in water, it is necessary that there shall be

$$\sqrt[3]{\frac{p}{q}} > \frac{4}{5} \text{ or } \frac{p}{q} > \frac{64}{125}.$$

Therefore its specific gravity must be greater than 512; with the specific gravity of water put $= 1000$.

SCHOLIUM

344. Thus far I have set out sections of bodies in water which are rectilinear figures, and the three cases treated are able to suffice according to our principles. And thus I shall progress to curvilinear figures, and for these especially, which are able to be approved easily by experiments, I shall make sections with water, so that from several bodies thence it may be able to be judged, which shall be able to have a place established in water, and how many shall persist in some state of equilibrium.

PROPOSITION 33

PROBLEM

345. *If the section of the body floating on water in equilibrium were the circle ACBD* (Fig. 58), *to determine the stability with respect of some axis CD, since the stability is the same everywhere, by which this state of equilibrium will be enjoyed.*

SOLUTION

The radius of the circle may be put to be $CI = a$, and some applied line *XY* drawn in the quadrant *CIA,* and there may be called :

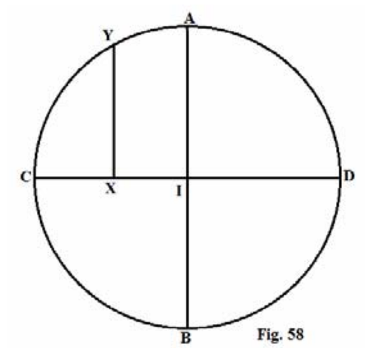


Fig. 58

$IX = x$ and $XY = y$ and there will become

$$y = \sqrt{(a^2 - x^2)},$$

from which there will become

$$\int y^3 dx = \int dx\left(a^2 - x^2\right)^{\frac{3}{2}}.$$

But with the reduction of integral formulas to simpler ones there shall become :

$$\int dx\left(a^2 - x^2\right)^{\frac{3}{2}} = x\left(a^2 - x^2\right)^{\frac{3}{2}} + \frac{3a^2x\sqrt{(a^2 - x^2)}}{8} + \frac{3a^4}{8}\int \frac{dx}{\sqrt{(a^2 - x^2)}}.$$

There may be put $x = a$; there will be given

$$\int \frac{dx}{\sqrt{(a^2 - x^2)}} = \tfrac{1}{2}\pi$$

on putting $\pi : 1$ for the ratio of the periphery to the diameter. With which done for the quadrant *CIA* there will be found:

$$\int y^3 dx = \frac{3\pi a^4}{16},$$

and thus for the whole circle there will become

$$\int \left(y^3 + z^3\right) dx = \frac{3\pi a^4}{4}.$$

Now if the weight of the body shall be $= M$, and the volume of the part submerged in the water $= V$, and *GO* will denote the distance between the centres of gravity and the magnitude of the submerged part, the stability with respect of any axis

$$=M\left(GO+\frac{\pi a^4}{4V}\right).$$

Q.E.I.

COROLLARY 1

346. Because the diameter itself is had to the periphery as 1 to π, πa^2 will express the area of the circle. Therefore if the area of the circle may be put $=bb$, there will become $a^2=\frac{bb}{\pi}$ and the stability will be expressed thus so that there shall be

$$=M\left(GO+\frac{b^4}{4\pi V}\right).$$

COROLLARY 2

347. If the section of the water is a square of area $=b^2$, then the stability found becomes

$$=M\left(GO+\frac{b^4}{12V}\right).$$

and if the section is an equilateral triangle, of which the area of the same is $=bb$, then the stability was

$$=M\left(GO+\frac{b^4}{6V\sqrt{3}}\right).$$

From which it is understood the stability of the circle to be a minimum, truly of the triangle a maximum.

COROLLARY 3

348. Therefore hence it is allowed to deduce, if the section of the water were a regular polygon, the stability being produced thus to become smaller, where the polygon shall contain more sides, evidently with all else being equal.

COROLLARY 4

349. Therefore for the maximum stability of a body floating in water, with respect of all the axes required to be taken together, and it will be agreed bodies of this kind to give the figure, so that the section of the water shall become an equilateral triangle.

EXAMPLE 1

350. The body floating erect in water shall be the right cylinder *MNRS* (Fig. 59), of which the horizontal sections shall be the equal circles *MN, CD,* and *RS*, the radius of which shall be $=a$. Moreover the weight of this cylinder shall be *M,* which it may have equal to the volume of the water displaced as p is to q. Whereby with the height of the whole cylinder put $WL = c$ the height of the submerged part will become

$$IL = \frac{pc}{q},$$

and the volume of the submerged part

$$V = \frac{\pi pa^2c}{q},$$

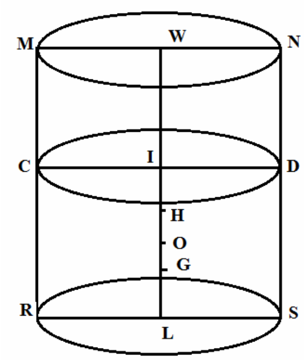


Fig. 59

of which the centre of the displaced magnitude falls at *O,* so that there shall become

$$LO = \frac{pc}{2q}.$$

Moreover the centre of gravity of the whole body shall be at *G,* with there being $LG = h$. Therefore with these substituted, the stability of the cylinder in this erect state of equilibrium will be found

$$= M\left(\frac{pc}{2q} - h + \frac{qa^2}{4pc}\right).$$

COROLLARY 1

351. Therefore the cylinder will persevere firmly in such a state if there were

$$h < \frac{pc}{2q} + \frac{qa^2}{4pc}.$$

That is, if there should become $LI : \frac{1}{2}CI = \frac{1}{2}CI : OH$,and then the point *G* shall fall below *H*.

COROLLARY 2

352. If the cylinder were made from a uniform material, there will become $h = \frac{c}{2}$; therefore in this case the stability

$$= M\left(\frac{qa^2}{4pc} - \frac{(q-p)c}{2q}\right).$$

Therefore in order that this state shall be stable, it is necessary that there shall be

$$c < \frac{qa}{\sqrt{2}p\left(q-p\right)}.$$

COROLLARY 3

353. If a whole cylinder were given, from the specific gravity it will become known whether or not it may be able to float in an erect position. For it will float if $\frac{p}{q}$ were either greater than $\frac{c+\sqrt{(cc-2aa)}}{2c}$, or less than $\frac{c-\sqrt{(cc-2aa)}}{2c}$.

COROLLARY 4

354. Therefore it is observed if there were $c < a\sqrt{2}$, then the cylinder to be floating in an erect position always, whatever were the ratio of the specific gravities.

EXAMPLE 2

355. The solid right cone *MLN* (Fig. 60) shall be floating in water with the vertex turned downwards, of which the radius of the base $WM = WN = a$; and the height $WL = c$. The specific gravity of that to water shall be as p ad q; and the radius of the section *CD* of the water will be

$$IC = a\sqrt[3]{\frac{p}{q}}, \text{ and } \; IL = c\sqrt[3]{\frac{p}{q}}.$$

And since the area of the base *MN* shall be $= \pi a^2$, the area of the water section

$$= \pi a^2 \sqrt[3]{\frac{p^2}{q^2}},$$

from which the volume V of the submerged part will be $= \frac{\pi pa^2 c}{3q}$, and

$$LO = \frac{3}{4}c\sqrt[3]{\frac{p}{q}}.$$

Now on putting $LG = h$, the stability of this state of equilibrium will be

$$= M\left(\frac{3}{4}c\sqrt[3]{\frac{p}{q}} - h + \frac{3q \cdot CI^4}{4pa^2c}\right) = M\left(\frac{3}{4}c\sqrt[3]{\frac{p}{q}} - h + \frac{3a^2}{4c}\sqrt[3]{\frac{p}{q}}\right).$$

COROLLARY 1

356. Therefore if the cone may be agreed to be from a homogeneous material, there will become $h = \frac{3}{4}c$; in this case the stability will be

$$= \frac{3}{4} M \left(\frac{\left(a^2 + c^2\right)}{c} \sqrt[3]{\frac{p}{q}} - c \right).$$

COROLLARY 2

357. Whereby therefore, it is necessary for this situation, so that there shall become

$$\frac{p}{q} > \frac{c^6}{\left(a^2 + c^2\right)^3}.$$

Because if this were not the case, the cone will acquire another state, so that it may float on the water.

PROPOSITION 34

PROBLEM

358. *CMLMD shall be the round solid part of the body submerged under water, arising from the rotation of the figure LMC about the vertical axis LI* (Fig. 61), *and the section of the body with the water, or the upper horizontal section of the solid of revolution, shall be the circle CD.*

SOLUTION

The radius of the horizontal section of the water shall be $IC = a$, and the length of the axis $IL = c$, on which shall be situated both the centre of gravity of the whole body *G,* as well as the centre of the magnitude of the submerged part *O.* Now with the weight of the body put $= M$ and with the volume of the submerged part $= V$, the stability $= M\left(GO + \frac{\pi a^4}{4V} \right)$, with π denoting the periphery of the circle, the diameter of which is $= 1$. But both the volume *V* as well as the point *O* will be required to determined from the nature of the curve *CML*: For which in the first place the abscissa will be required to be called $LP = x$, and with the corresponding applied line $PM = y$, and the volume of the solid will be had from the rotation of the part *MLP* arising $\pi \int y^2 dx$, in which, if there may be put $x = c$, in

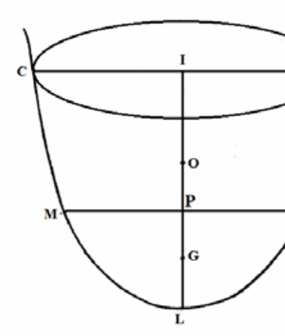


Fig. 61

which case there will become $y = a$,the total volume V of the submerged part will be produced. Therefore with the integral extended through the whole figure *LMC,* there will become $V = \pi\int y^2dx$. In a similar manner the position of the centre of the magnitude O of the submerged part will become :

$$LO = \frac{\int y^2xdx}{\int y^2dx},$$

[*i.e.* the centre of buoyancy] with each of the integrals extended as far as to the section of the water. Therefore if there may be put $LG = h$, clearly the interval will not depend on the nature of the curve *CML,* but on the characteristics of the whole body, the stability [*i.e.* the restoring moment] of this state of equilibrium

$$= M\left(\frac{\int y^2xdx}{\int y^2dx} - h + \frac{a^4}{4\int y^2dx}\right) = M\left(\frac{a^4 + 4\int y^2xdx}{4\int y^2dx} - h\right).$$

Q.E.I.

COROLLARY 1

359. Therefore for the stability of a body of this kind requiring to be found, a twofold integration is required to be put in place ; indeed these two differential formulas must be integrated y^2dx and y^2xdx.

COROLLARY 2

360. Therefore as many of these two algebraic formulas are allowed to be integrated, it will be able for just as many stable states to be expressed algebraically. But the quadratures of the curve will be thrown into confusion, if either one or both of the integrations shall be unable to be done.

SCHOLIUM

361. Moreover it will be deduced from the equation, which will be had between x and y, by which the nature of the curve *LMC* will be expressed, whether each of the formulas y^2dx and y^2xdx shall be integrable algebraically, or may depend on quadratures. Moreover here it will be agreed all the algebraic equations between x and y to be indicated, which may render each integrable formula algebraically, where it is understood generally, which algebraic curves for the generating curve *LMC* assumed may produce a stable algebraic expression. Therefore for this requiring to be investigated I assume some two algebraic quantities P and Q, of which either one shall be an algebraic function of the other, or both shall be algebraic functions of some the third variable such as z; and I make

$$\int yydx = P \text{ and } \int yyxdx = Q.$$

From these therefore there will become

$$y^2 = \frac{dP}{dx} = \frac{dQ}{xdx},$$

from which there is found

$$x = \frac{dQ}{dP}, \text{ and } y^2 = \frac{dP^3}{dPddQ - dQddP},$$

which are the general algebraic values for x et y, which in the first place will provide an algebraic equation between x and y, and then they will produce a stable algebraic expression, evidently which will be

$$= M\left(\frac{a^4 + 4Q}{4P} - h\right).$$

But it will help to illustrate the stability generally found in the solution of problems by a few examples.

EXAMPLE

362. The immersed part *CLD* shall be a portion of a sphere, the radius of which shall be *b;* there will become: $b - c = \sqrt{(bb - aa)}$, and hence $b = \frac{aa + cc}{2c}$. Therefore since *LMC* shall be the arc of the circle of radius *b,* there will be $y^2 = 2bx - xx$, and thus

$$\int y^2 dx = bxx - \frac{1}{3}x^3.$$

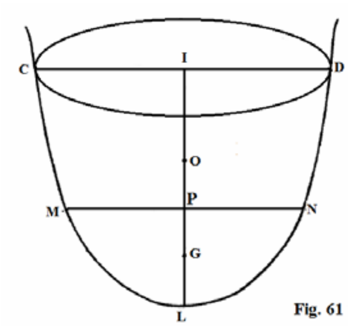


Fig. 61

Therefore with $x = c$ put in place, there will become:

$$\int yydx = bcc - \frac{1}{3}c^3 = \frac{c(3aa + cc)}{6}.$$

Then we will have

$$\int yyxdx = \frac{2bx^3}{3} - \frac{1}{4}x^4 = \frac{2bc^3}{3} - \frac{1}{4}c^4$$

on putting $x = c$, but with the value substituted through a and c substituted in place of b by definition there will become:

$$\int yyxdx = \frac{cc(4aa + cc)}{12}.$$

Therefore from these integrals found the stability sought will become

$$= M\left(\frac{3a^4 + 4aacc + c^4}{2c(3aa + cc)} - h\right) = M\left(\frac{aa + cc}{2c} - h\right) = M\left(b - h\right).$$

COROLLARY 1

363. Since the stability shall be found $= M(b-h)$, that will be proportional to the interval, by which the centre of gravity G of the whole body falls below the centre of gravity of the sphere O. Therefore a body of this kind will tend to be seated more firmly, if the centre of gravity of the spherical part, of which the submerged part is a portion, may fall below the centre of gravity of the whole body [*i.e.* there is a restoring couple acting about G].

COROLLARY 2

364. But if the centre of gravity G of the whole body may fall on the centre of gravity of the sphere, then the state of equilibrium will be neutral, which happens in spheres made from a uniform material; all which floating on the water shall have the pleasing property of equilibrium, moreover neither to be stable nor unstable but indifferent to all states of instability.

EXAMPLE 2

365. The curve LMC shall be a parabola of some kind of ordinates, evidently

$$y^m = b^{m-n}\, x^n.$$

Therefore there will become

$$a^m = b^{m-n}c^n,\ b = \frac{a^{\frac{m}{m-n}}}{c^{\frac{n}{m-n}}}.$$

Moreover, since there shall become

$y^2 = b^{\frac{2m-2n}{m}}\, x^{\frac{2n}{m}}$, there will become $\int y^2 dx = \dfrac{mb^{\frac{2m-2n}{m}}\, x^{\frac{2n+m}{m}}}{2n+m} = \dfrac{mb^{\frac{2m-2n}{m}}\, c^{\frac{2n+m}{m}}}{2n+m}$

on putting c in place of x, where the integral shall pertain to the water as far as the section CD. Also in a similar manner, there will become:

$$\int yyxdx = \frac{mb^{\frac{2m-2n}{m}}c^{\frac{2n+2m}{m}}}{2m+2n}.$$

From which the stability sought will be obtained from the integrals

$$= M\left(\frac{a^4 + \frac{2m}{m+n}b^{\frac{2m-2n}{m}}c^{\frac{2n+2m}{m}}}{\frac{4m}{2n+m}b^{\frac{2m-2n}{m}}c^{\frac{2n+2m}{m}}} - h\right).$$

Moreover with the value assigned in place of b

$\frac{a^{\frac{m}{m-n}}}{c^{\frac{n}{m-n}}}$, from which there shall be $b^{\frac{2m-2n}{m}} = a^2c^{\frac{-2n}{m}}$,

the stability produced

$$= M\left(\frac{a^2 + \frac{2m}{m+n}c^2}{\frac{4m}{2n+m}c} - h\right).$$

COROLLARY 1

366. Therefore the stability can be expressed in this manner, if a may be eliminated in the formula an b can be expressed in this manner, so that it shall be

$$= M\left(\frac{2n+m}{2n+2m}c + \frac{2n+m}{4m}b^{\frac{2m-2n}{m}}c^{\frac{2n-m}{m}} - h\right).$$

From which formula with b and c given, the radius of the water section is determined at once.

COROLLARY 2

367. Moreover it is evident from these expressions, the stability thus to become greater, where the fraction $\frac{m}{n}$ were smaller. Indeed if m may become $= 0$, then the stability will become infinitely great, but neither this case nor other nearby ones find a place in the nature of things .

COROLLARY 3

368. If the same quantity of the height *h* may remain, the stability will become infinite whether $c = 0$ of if $c = \infty$, therefore the stability will be a minimum if there were

$$aa = \frac{2m}{m+n} cc \text{ or } c = a\sqrt{\frac{m+n}{2m}}.$$

Therefore in the case of the conic parabola, where $m = 2,\ n = 1$, the stability will be a minimum if there shall be $c = a\sqrt{\frac{3}{4}}$

PROPOSITION 35

PROBLEM

369. *If the section of the body resting on water in equilibrium were the section of the ellipse ACBD* (Fig. 62), *to determine the stability of this situation with respect of each the major axis CD and minor axis AB.*

SOLUTION

The semi major axis may be put $CI = a$; the semi minor axis $IA = b$; the abscissa will be put $IX = x$, and the applied line
$XY = y$, there will be this equation between *x* and *y* :
$y = \frac{b}{a}\sqrt{\left(a^2 - x^2\right)}.$
Hence there will become

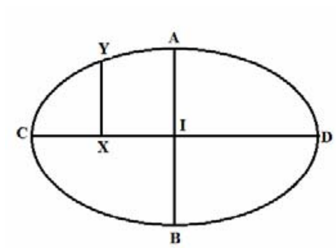


Fig. 62

$$\int y^3 dx = \frac{b^3}{a^3}\int dx\left(a^2 - x^2\right)^{\frac{3}{2}},$$

which integral on putting $x = a$, and with *π* denoting the periphery of the circle of which the diameter is $= 1$, will be changed into $\frac{3\pi ab^3}{16}$ which therefore taken four times will give the whole section of the water

$$\int\left(y^3 + z^3\right)dx = \frac{3\pi ab^3}{4}.$$

Now if the weight of the body shall be $= M$, the volume of the submerged part *V,* and *GO* will indicate the distance between the centres of gravity and of the magnitude [of the buoyancy force], the stability of the body with respect of the axis *CD* will become

$$= M\left(GO + \frac{\pi ab^3}{4V}\right).$$

Moreover with the semi axes *a* and *b* interchanged between themselves, the stability with respect of the minor axis will be produced.
Q.E.I.

COROLLARY

370. Therefore the stability, by which the body resists being inclined about the major axes, is smaller than the stability with respect of the minor axis. Whereby if the situation were stable with respect of the major axis , thus it will be more stable with respect of the minor axis.

COROLLARY 2

371. The whole area of the ellipse is πab; therefore if the area of the ellipse may be put to be $= E$, the stability with respect of the major axis CD shall be

$$= M\left(GO + \frac{Eb^2}{4V}\right),$$

truly the stability with respect of the minor axis *AB* shall be

$$= M\left(GO + \frac{Ea^2}{4V}\right).$$

SCHOLIUM

372. I believe it would be superfluous to illustrate this proposition with examples, since the above examples given for the circle shall be able to be adapted easily for this, and in addition equally to be suitable both for the experiments to be put in place, as well as may be able to be derived by a fuller understanding of the following. On account of which with these sent, by which the section of the water may be considered especially, I will progress to the various figures the submerged parts of which are required to be considered, where by calculation I am going to inquire both into the volume of the submerged part as well as into its centre of magnitude, certainly which matters serve especially towards an understanding not only in the first place of the section of the water but also for the stability. Moreover before all else, I am going to consider the shapes of the submerged parts of this kind, which shall have a certain similarity with ships and for the remaining vessels, which are accustomed to be used for motion on water, so that thence treatments may follow certainly to be applied not only suitable for sailing ships. Therefore I shall place the figure of the submerged part below the end of the right line to the horizontal, which in ships is accustomed to be called the keel, and to that all the

transverse sections are determined to be made vertically. But the shape of the submerged part is determined from these transverse sections, which are vertical and normal to the keel. On account of which both from the section of the water, as well as from the transverse sections of this kind, I will show how the stability shall be able to be defined.

PROPOSITION 36

PROBLEM

373. *If the section of the water were some curve ANBMA with the diameter AB given* (Fig. 63), *truly the part submerged may be terminated both by the horizontal keel EF placed under the axis AB, as well as by the vertices at M of the conic parabola with the sides MQ and having the axes AB horizontal to the normals; to find the stability of such a state of equilibrium of a body held in water with respect of the axis AB.*

SOLUTION

The section of some submerged vertical part MQM may be considered (Fig. 64) , and for

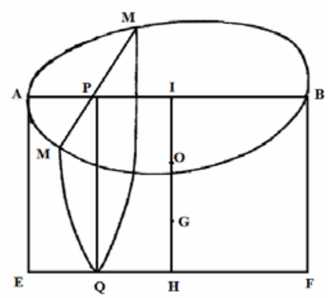


Fig. 63

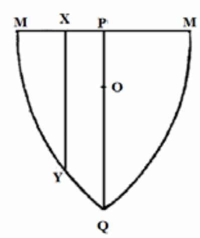


Fig. 64

the abscissa normal to the diameter AB may be called

$$AP = x,\ MP = MP = y,$$

and the depth being the constant $PQ = AE = c$. Now we may consider the section MQM separately, in which the curves MQ and MQ are the parabolas of Appollonius having the vertices M and the common axis MM. Since now there becomes $PM = y$ and $PQ = c$, the parameter of each parabola $= \frac{c^2}{y}$. Whereby if there may be called

$$MX = t \text{ and } XY = u\text{, there will become } u^2 = \frac{c^2 t}{y},$$

and the area

$$MXY = \frac{2}{3}tu = \frac{2ct}{3}\sqrt{\frac{t}{y}},$$

from which the whole area *MQM* on putting $t = y$ will become $= \frac{4}{3}cy.$
Moreover the centre of gravity *o* of the area *MQM* will be found by taking the integral $\int \frac{1}{2}uudt$ and by dividing that by $\int udt$; truly there becomes $\int \frac{1}{2}uudt = \frac{cct}{4y}$, which divided by $\int udt = \frac{2ct}{3}\sqrt{\frac{t}{y}}$, gives $\frac{3c}{8}\sqrt{\frac{t}{y}}$, thus so that on putting $t = y$ there shall become $Po = \frac{3}{8}c.$ Therefore since the centre of gravity of all the sections of this kind *MQM* shall fall at the same distance from the diameter *AB*, the centre of magnitude of the whole submerged part will be situated at *O* so that there shall become $IO = \frac{3}{8}c.$
Again the area of the section *MQM,* $\frac{4}{3}cy$, may be multiplied by *dx,* and the integral $\frac{4}{3}c\int ydx = \frac{2}{3}c \cdot MAM$ will give the volume *AEQMM,* on account of which if the area of the whole section of the water *AMBM* may be put $= E$, the volume of the submerged part will be $\frac{2}{3}Ec.$ Now the weight of the whole body $= M$, and its centre of gravity *G* shall be drawn on the right vertical line *IH* through the centre of magnitude *O*, the stability of this state of equilibrium with respect of the axis *AB* will be

$$= M\left(IG - \frac{3}{8}c + \frac{\int y^3dx}{Ec}\right).$$

Indeed by proposition 29 treated, there is $z = y$, and $V = \frac{2}{3}Ec.$ Therefore on putting $IG = h$, the stability sought will become

$$= M\left(h - \frac{3}{8}c + \frac{\int y^3dx}{Ec}\right).$$

Q.E.I.

COROLLARY 1

374. Hence also the distance of the vertical line *IH* from the point *A* will be found by obtaining the integral of $\frac{4}{3}cyxdx$, and on dividing that by $\int \frac{4}{3}cydx$ thus so that there shall be going to become

$$AI = \frac{\int yxdx}{\int ydx}.$$

COROLLARY 2

375. Therefore from this formula it is observed the right line *HI* through the centre of gravity, to be passing through the section of the water *I* itself, thus so that in this case three centres of gravity shall be situated on the same vertical line, namely that of the whole body, that of the part submerged, and that of the section of the water.

COROLLARY 3

376. Therefore for the given water section for bodies of this kind, from which both its area *E* as well as $\int y^3 dx$ shall become known, the stability of the equilibrium state will be able to be defined easily.

COROLLARY 4

377. Therefore since for such a body the centre of gravity of the water section *I* lies directly above the centre of gravity of the whole body *G*, on being required to oscillate the centre of gravity can neither rise nor fall, and on this account the oscillatory motion will be of the greatest tranquility.

SCHOLIUM

378. Therefore this parabolic form, which is granted to the transverse sections of ships, is suitable enough, since through these that will be acquired conveniently, so that both the centre of magnitude of the submerged part shall lie on the same vertical line, as well as the centre of gravity of the water section. Hence indeed it arises, as we have seen above, so that while oscillations may be performed by the ship, only they shall be minimal, if the centre of gravity may remains at rest, which for the most part allows that motion of the greatest tranquility to be produced. But not only the parabolic figure is adapted for producing this effect, but besides all parabolas of any order likewise are outstanding, and innumerable other curves, which are prepared thus so that the areas of those *MQM* shall be proportionals of the ordinates of the water sections themselves *MPM*; if indeed the keel of the body floating on water is horizontal. But if the whole keel is not a straight line, but shall be either a whole curve, or of such a size to be erected vertically at the prow and stern ends, then there is need for singular curves in order to obtain the same advantage. On account of which in the first place I shall provide parabolas of a higher order than for the case where the keel is horizontal, and then I will investigate suitable curves for keels which are not right lines.

PROPOSITION 37

PROBLEM

379. *If the section of the water with the body were some curve AMBMA with the given diameter AB, below which the horizontal right keel EF* (Fig. 63) *shall be present in the vertical plane to which the part of the body immersed in the water shall be terminated by a parabola of some order MQ, with the vertices being present at M, and with the horizontal axes MM; to determine the stability with respect of the axis AB.*

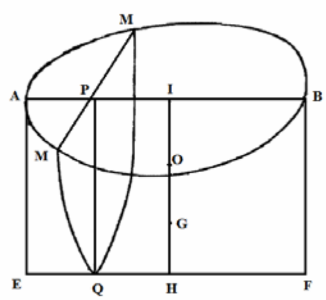


Fig. 63

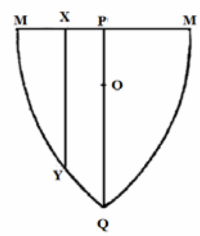


Fig. 64

SOLUTION

As before on putting $AP = x$, $PM = y$ et $AE = PQ = c$, and the transverse section MQM may be considered separately (Fig. 64), in which the abscissa MX shall be taken $= t$ and the applied line $XY = u$; truly the nature of this parabola may be expressed by this equation :

$$u = \frac{t^n}{p^{n-1}},$$

with the parameter being p. But since by making

$$t = MP = y, \text{ there shall be } u = PQ = c$$

there will become

$$c = \frac{y^n}{p^{n-1}} \text{ and } p^{n-1} = \frac{y^n}{c}.$$

But the area MXY will become

$$= \frac{t^{n+1}}{(n+1)p^{n-1}} = \frac{ct^{n+1}}{(n+1)y^n}$$

from which the area of the whole section *MQM* will become $=\dfrac{2cy}{(n+1)}$. Then the centre of gravity of this section will be at *o* so that there shall become:

$$Po=\frac{\int uudt}{2\int udt},$$

on putting $t=y$ after the integration. But there becomes :

$$\int u^2dt=\frac{\int t^{2n}dt}{p^{2n-2}}=\frac{t^{2n+1}}{(2n+1)p^{2n-2}}=\frac{c^2t^{2n+1}}{(2n+1)y^{2n}}=\frac{c^2y}{2n+1}$$

on putting $t=y$. Whereby since there shall become

$$2\int udt=\frac{2cy}{n+1},\text{ there will become } Po=\frac{(n+1)c}{2(2n+1)}.$$

Whereby since the centre of gravity of all the transverse sections shall fall on the same horizontal right line, the centre of magnitude of the submerged parts will be placed at *O,* so that there shall become

$$IO=\frac{(n+1)c}{2(2n+1)}.$$

Moreover the capacity of the submerged part will become

$$=\int\frac{2cydx}{n+1}$$

$=\dfrac{c}{n+1}$ into the area *AMBMA* (Fig. 63), therefore if the surface of the section of the water may be called $=\dfrac{cE}{n+1}$. Finally the centre of gravity of the whole body shall be placed at *G,* so that there shall become $IG=h$, and the weight of the whole body $=M$, there will become

$$GO=h-\frac{(n+1)c}{2(2n+1)},$$

and in the general proposition (§ 298) there will become

$$\int(y^3+z^3)dx=2\int y^3dx,$$

on account of

$$z=y,\text{ and } V=\frac{cE}{n+1}.$$

Hence therefore the stability of this state of equilibrium will arise with respect of the axis

$$AB = M\left(h - \frac{(n+1)c}{2(2n+1)} + \frac{2(n+1)\int y^3 dx}{3Ec}\right).$$

Q.E.I.

COROLLARY 1

380. Since of any transverse section *MQM* shall be proportional to the ordinate of the water section *MM*, it is evident the centre of gravity of the water section *I,* and the centre of the magnitude of the submerged part *O*, to lie on the same vertical line *IH*.

COROLLARY 2

381. Therefore with the centre of gravity *I* of the water section given in a body of this kind, likewise the position of the centre of the magnitude *O* becomes known; and also it is necessary that the centre of gravity *G* of the whole body shall be put in place on the vertical line *IOH*.

COROLLARY 3

382. If there shall become $n = 1$, all the transverse sections will become triangles, and the lines *MQ* become right. Therefore in this case the volume of the submerged parts $V = \frac{cE}{2}$ and the stability will be produced

$$= M\left(h - \frac{c}{3} + \frac{4\int y^3 dx}{3Ec}\right).$$

COROLLARY 4

383. But if there shall be $n < 1$, but yet $n > 0$, the tangents *M Q* at *M* will be vertical, and the submerged part of the figure will become bulging or convex. But if $n > 1$ the figure will become concave.

COROLLARY 5

384. If the stability may be required in the formula with respect of any other horizontal axis passing through *I*, nothing will be required to be changed, except for the expression

$4\int y^3dx$, which will have to be adapted for that axes. Indeed all the rest do not depend on the position assumed for the axis *AB*.

SCHOLIUM

385. The same property is agreed for innumerable other curves, which consists of the conical parabolas as well as all of the remaining parabolas of each order, which will be able to be put in place around the same successive transverse sections *MQ* . For all the curves are to be satisfied in the same way, which have been prepared thus, so that the area of each *MQM* which will correspond to equal abscissa, shall themselves be proportional to the ordinates *MM*, certainly from which it happens, that the centre of magnitude of the submerged parts *O* shall fall vertically below the centre of gravity of the section *I*. Therefore an equation must be prepared between *u* and *t* from these curves, so that at first there shall become $u = 0$, on making $t = 0$, and so that secondly there may become $u = c$ on putting $t = y$. Truly in the third place the area $\int udt$, if there may be put $t = y$, must adopt such a form $\frac{mcy}{n}$. Moreover these will be obtained in the following manner: in general *T* shall be some function of zero dimensions of $t = y$, or some function of $\frac{t}{y}$ which shall vanish on making $t = 0$. Therefore this function T on putting $t = y$ will be changed into a numerical constant, which shall be *n,* with which performed this equation will become $u = \frac{cT}{n}$, satisfying the curve sought. For indeed on making $t = 0$, there will become $u = 0$, and on putting $t = y$ there becomes $u = c$.
Finally there will become:

$$\int udt = \int \frac{cTdt}{n} = \frac{cy}{n}\int \frac{Tdt}{y}.$$

But $\int \frac{Tdt}{y}$ will give a function of $\frac{t}{y}$ itself, which thus will be changed into a constant number for example *m* on making $t = y$; from which the area of the transverse section *MQM* will arise $= \frac{2ncy}{n}$.

Truly besides also the interval *Po,* so that the centre of gravity *o* of any transverse section falling below the horizontal will be constant. Indeed since there shall become

$$Po = \frac{\int uudt}{2\int udt}$$

by putting $t = y$ after the integration, there will become:

$$\int uudt = \frac{cc}{nn}\int T^2dt = \frac{c^2y}{nn}\int \frac{T^2dt}{y}.$$

But $\int \frac{T^2dt}{y}$, will give a function of $\frac{t}{y}$, which on making $t = y$ will go into a numerical constant, which shall be *K,* thus so that there shall become:

$$\int uudt = \frac{Kc^2y}{nn},$$

which expression divided by

$$2\int udt = \frac{2mcy}{n}, \text{ will give } Po = \frac{Kc}{2mn},$$

to which the expression consequently is equal to some interval *IO*.

PROPOSITION 38

PROBLEM

386. *AMBMA* (Fig. 65) *shall be some curved section of the water with the predetermined diameter AB, below which in the vertical plane the submerged part will be terminated to some to the keel EHF curved in some manner, to find a suitable figure for the transverse sections, in order that the centre of the magnitude of the submerged parts O shall fall vertically below the centre of gravity I of the section of the body with the water.*

SOLUTION

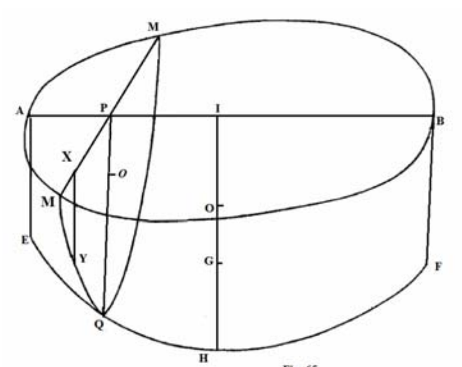


Fig. 65

With $AP = x$, $PM = PM = y$, and $PQ = z$ put in place, there will be given on account of the section of the water *y* given by *x,* and on account of the figure of the keel *EHF* equally given *z* by *x.* Moreover the question will be satisfied most conveniently, if the figure may be attributed to the individual transverse sections *MQM* of this kind, so that the areas of these shall become proportional to the ordinates *MM* or to *y* itself. According to this being done, some applied line *XY* drawn in the transverse section, there shall be $MX = t$ and $XY = u$ and this equation may be assumed for expressing the nature of the indefinite curve *MQ* :

$$u = \frac{At^{n-1}}{y^{n-1}} + \frac{Bt^{m-1}}{y^{m-1}} + \frac{Ct^{k-1}}{y^{k-1}},$$

in which *n, m,* and *k* shall be numbers greater than unity, with which done $t = 0$ shall become $u = 0$. Now since by making $t = y$, there must become $u = z$, there will become $z = A + B + C$. Again the area $\int udt$ may be sought, which will become

$$= \frac{At^{n}}{ny^{n-1}} + \frac{Bt^{m}}{my^{m-1}} + \frac{Ct^{k}}{ky^{k-1}};$$

which, since on putting $t = y$, must become proportional to *y* itself, may be put $= cy$, and there will be had

$$c = \frac{A}{n} + \frac{B}{m} + \frac{C}{k}.$$

From these conditions it follows:

$$B = \frac{kmnc - mnz - m(k-n)A}{n(k-m)}, \text{ and } C = \frac{knc - kmnz + m(k-n)A}{n(k-m)};$$

on account of which the following equation will be had for the curve sought :

$$u = \frac{At^{n-1}}{y^{n-1}} + \frac{(kmnc - mnz - m(k-n)A)t^{m-1}}{n(k-m)y^{m-1}} - \frac{(kmnc - knz - k(m-n)A)t^{k-1}}{n(k-m)y^{k-1}}:$$

in which besides the exponents *k, m, n* the quantity *A* will be allowed to be assumed arbitrarily. But regarding the quantity *A* requiring to be chosen, it will be necessary to attend especially , so that the applied line *u* may increase continually, on progressing from *M* to *Q*, and so that the curve shall be convex everywhere between the points *M* and *Q*, or so that $\frac{du}{dt}$ may decrease continually, but first we must we must understand that if $\frac{du}{dt}$ may retain a positive value from *M* as far as *Q*; and thus it shall be positive at *Q*. At *Q* truly there will become :

$$\frac{du}{dt} = \frac{1}{y}\left(\frac{(n-1)A + (m-1)}{n(k-m)}(kmnc - mnz - m(k-n)A) - \frac{(k-1)}{n(k-m)}(kmnc - knz - k(m-n)A)\right)$$
$$= \frac{1}{y}\left(\frac{(m-n)(k-n)}{n}A - kmc + (k+m-1)z\right).$$

Whereby there will have to become:

$$A > \frac{kmnc}{(m-n)(k-n)},$$

where $\frac{du}{dt}$ shall remain positive, even if z may become a minimum. But if other circumstances are not permitted, so that the figure may induce individual transverse sections of this kind, then there will be just as many greater or smaller transverse sections from one side of the point I, with just as many greater or smaller transverse sections will have to be made from the other side also, so that nevertheless the centre of the magnitude of the submerged part shall lie on the right line IH. Q. E. I.

COROLLARY 1

387. If n may be put to be the minimum, m the mean, and k the maximum of the numbers n, m, and k, the position of the tangent of the transverse section at M will be known from the number n. For if $n-1$ were >1 then the tangent will be horizontal, but if $n-1<1$ then it will be vertical, but if $n=2$ then the angle will be oblique.

COROLLARY 2

388. Therefore if there may be put $n=\frac{3}{2}$, the tangent at M not only will become vertical, but also the radius of osculation at M will be finite. Whereby if again there may be put $m=\frac{5}{2}$ and $k=\frac{7}{2}$ this equation will be had for the curve

$$u=\frac{A\sqrt{t}}{\sqrt{y}}+\frac{(105c-30z-40A)}{12}\frac{t\sqrt{t}}{y\sqrt{y}}-\frac{(105c-42z-28A)t^2\sqrt{t}}{12y^2\sqrt{y}}.$$

COROLLARY 3

389. Since there must become

$$A>\frac{kmnc}{(m-n)(k-n)}-\frac{n(k+m-1)x}{(m-n)(k-n)}.$$

We shall see whether it may be safe for the third term C to vanish ; but hence from this condition there becomes :

$$A=\frac{mnc\ \ -nz}{m-n}.$$

Therefore there must become

$$(m+n-1)z>mnc.$$

COROLLARY 4

390. Therefore the keel has been prepared thus so that z may decrease as far as 0, then it will not be possible for $(m+n-1)z > mnc$ everywhere, and on that account it will be required from these three cases for a trinomial function of t to u to be used.

SCHOLIUM

391. But it is evident cases of this kind can occur, for which z shall be of so many different large values, so that the area of the transverse sections may not be able to be reduced entirely to proportionals of y only ; if indeed figures of these may be desired not entirely dissimilar, which are accustomed to be used in ships. For either where the height z is present greater, there the transversals must be exceedingly close together, or where z becomes exceedingly small, there the area of the section must become so great, so that it may not be able to be contained within the prescribed limits. Therefore with cases of this kind it will be agreed to bring forth that remedy, of which I have made mention in the solution, so that the transverse sections, which may arise exceedingly deformed by the rule, may be either augmented or diminished from each part equally, so that the position of the common centre of gravity may be preserved. But lest there shall be a need of such a less well known convenient correction, then it will be better to adapt both the figure of the water section, as well as of the keel to a suitable form of the transverse section. Whereby in the end I put the areas of the transverse sections to hold a ratio composed from the size of the water section and the depth, or to be everywhere as yz; for indeed with this in place which figure in a single case will be both suitable and practical, it will have the same place in everything remaining. Moreover curves of this kind with the aforementioned property are able to show this property for innumerable transverse sections, which all will be contained in the following general equation, T shall be some function of $\frac{t}{y}$ vanishing on putting $t=0$, which on making $t=y$ shall become the constant number n. Then there shall become $u=\frac{yT}{n}$. Indeed from this equation there becomes $u=0$, if $t=0$ and $u=z$ if $t=y$; and finally there will become

$$\int udt = \frac{\int zTdy}{n} = \frac{Zy}{n}\int\frac{Tdy}{y} \text{ ; but } \int\frac{Tdy}{y}$$

with the factor $t=y$ will be changed into the constant number m thus so that the total area becomes $=\frac{2m^2yz}{n}$.

PROPOSITION 39

PROBLEM

392. *If the area of the transverse sections MQM* (Fig. 65) *were in a ratio composed of the bases MM and of the depths PQ, then to find suitable figures EHF both for the section of the water AMBM as well as for the keel, so that the centres of gravity of the sections of the water I and of the volumes of the submerged parts O may lie on the same vertical line IH.*

SOLUTION

The length of the diameter of the water section shall be $AB = a$; and since both parts of the water section around *AB* most be similar and equal *AB*, and the curve *AMBM* to be agreed to be concave towards *AB*, with the same equation for these $y = (A + Bx)\sqrt{(ax - xx)}$. But truly this equation may be taken for the keel

$$z = \frac{(\alpha + \beta x)(ax - xx)}{A + Bx},$$

where that shall intersect the section of the water both at *A* as well as at *B*, and that at oblique angles just as is accustomed to be done in ships. Here evidently there is put as before $AP = x$, $PM = y$ and $PQ = z$. Now so that the points I and O may fall on the same vertical right line, after the integration performed on making $x = a$ to become

$$\frac{\int yxdx}{\int ydx} = \frac{\int yzxdx}{\int yzdx}.$$

For these integrations requiring to be taken will be attended to by the application of this theorem, at least for the case $x = a$:

$$\int \left(I + Kx + Lx^2 + Mx^3 + \text{etc.}\right) dx \left(ax - xx\right)^n$$
$$= \left(I + \frac{(n+1)Ka}{(2n+2)} + \frac{(n+1)(n+2)}{(2n+2)(2n+3)} La^2 + \frac{(n+1)(n+2)(n+3)}{(2n+2)(2n+3)(2n+4)} Ma^3 + \text{etc.} \right) \int dx \left(ax - xx\right)^n.$$

Hence there will become:

$$\frac{\int yxdx}{\int ydx} = \frac{\int \left(Ax + Bx^2\right) dx \sqrt{(ax - xx)}}{\int \left(A + Bx\right) dx \sqrt{(ax - xx)}} = \frac{\frac{1}{2} Aa + \frac{5}{16} Ba}{A + \frac{1}{2} Ba} = AI.$$

But since there shall become

$yz = (\alpha + \beta x)(ax - xx)^{\frac{3}{2}}$, there will become $\dfrac{\int yzxdx}{\int yzdx} = \dfrac{\frac{1}{2}\alpha a + \frac{7}{24}\beta a^2}{\alpha + \frac{1}{2}\beta a}$.

Which values equated to each other give:

$$\frac{A + \frac{5}{8}Ba}{A + \frac{1}{2}Ba} = \frac{\alpha + \frac{7}{12}\beta a}{\alpha + \frac{1}{2}\beta a},$$

from which there becomes

$$B\beta a = 4A\beta - 6B\alpha, \text{ or } \beta = \frac{6B\alpha}{4A - Ba}.$$

On account of which if this equation may be assumed for the water section:

$$y = \left(m + \frac{nx}{a}\right)\sqrt{(ax - xx)},$$

then this equation will be required to be assumed for the keel:

$$z = \frac{\alpha\left(\left((4m - n)a + 6nx\right)(ax - xx)\right)}{(4m - n)(ma + nx)}.$$

Q.E.I.

COROLLARY 1

393. Since there shall become

$$AI = \frac{\frac{1}{2}Aa + \frac{5}{16}Ba^2}{A + \frac{1}{2}Ba}, \text{ on account of } A = m \text{ and } B = \frac{n}{a}$$

there will become:

$$AI = \frac{a\left(\frac{1}{2}m + \frac{5}{16}n\right)}{m + \frac{1}{2}n}.$$

On account of which there will be had

$$AI = \tfrac{1}{2}a + \frac{na}{16m + 8n}.$$

Therefore as often as n is a positive number or m and n numbers of the same sign, there will be $\mathrm{AI} > \frac{1}{2}\mathrm{AB}$.

COROLLARY 2

394. If $n=0$, the equation for the water section will become $y= m\sqrt{(ax-xx)}$, whereby in the case the water section will be an ellipse, to which the figure of the keel will correspond

$$z=\frac{\alpha}{m}(ax-xx),$$

which therefore will be a parabola. Moreover in this case the point I is present at the midpoint of AB.

COROLLARY 3

395. If there may be put $m=0$, so that the water section may be expressed by this equation

$$y=\frac{nx}{a}\sqrt{(ax-xx)}$$

the figure of the spine will become

$$z=\frac{\alpha(a-x)(a-6x)}{nx},$$

but which figure is unsuitable, on account of $z=0$, if there is $x=\frac{1}{6}a$.

COROLLARY 4

396. Therefore lest anywhere between A and B there may become $z=0$ it is necessary that there shall be

$$a+\frac{6nx}{4m-n}>0,$$

if indeed x may be held between the limits 0 and a. But if there shall become

$$a+\frac{6nx}{4m-n}=0,$$

if there is $x=\frac{a(n-4m)}{6n}$; whereby $\frac{n-4m}{6n}$ either must be less than 0, or greater than *I*.

COROLLARY 5,

397. There may become $n=4m$, which is had with a single case, which shall be for the section of the water

$$y=\left(m+\frac{4mx}{a}\right)\sqrt{(ax-xx)},$$

there shall become for the keel, $z = \dfrac{\alpha xx\left(a-x\right)}{a(a\ +4x)}$, of which therefore in AI the tangent will be horizontal. But the interval produced $= \frac{1}{2}a + \frac{1}{12}a = \frac{7}{12}a.$

SCHOLION 1

398. This proposition extends the most generally, and almost all the figures, which commonly are accustomed to be used in the construction of ships, are included in it. For it is suitable for boundless figures of cross-sections, just as can be seen from § 391, and in addition it contains innumerable figures of cross-sections suitable for use; thus so that from that the construction of ships can be judged, as well as suitable forms of new ships may be able to be found, which indeed shall be in agreement with the principles set forth up to this point. Indeed in a wider sense, if there had been a need, we would have elaborated on the solution, for if for the section of the water were an equation of this kind

$$y = \left(A + Bx + Cx^2 + Da^3 + \text{etc.}\right)\sqrt{\left(ax - xx\right)},$$

truly we would have assumed this equation for the keel

$$z = \frac{\left(\alpha + \beta x + \gamma x^2 + \delta a^3 + \text{etc.}\right)\left(ax - xx\right)}{\left(A + Bx + Cx^2 + Da^3 + \text{etc.}\right)};$$

then indeed several indeterminate letters would have remained in the figure of the keel, from the determination of which many more figures would have been able to be produced. Moreover this equation would have been produced for the given section of the water being adapted for the figure of the keel :

$$\frac{\frac{1}{2}A + \frac{5}{16}Ba^2 + \frac{7}{32}Ca^3 + \frac{21}{128}Da^4 + \text{etc.}}{A + \frac{1}{2}Ba + \frac{5}{16}Ca^2 + \frac{7}{32}Da^3 + \text{etc.}} = \frac{\frac{1}{2}\alpha a + \frac{1}{24}\beta a^2 + \frac{3}{16}\gamma a^3 + \frac{3}{256}\delta a^4 + \text{etc.}}{\alpha + \frac{1}{2}\beta a + \frac{7}{24}\gamma a^2 + \frac{3}{16}\delta a^3 + \text{etc.}} = AI.$$

From which the relation of the coefficients A, B, C, D etc. and α, β, γ, δ, etc. can be defined.

SCHOLIUM 2

399. Everything has been set out well enough in this chapter, which pertain to the understanding and judgment of the stability of bodies resting on water in some state of equilibrium, I myself am seen to have explained, nor is anything seen to be lacking, which could be wished for further in this instruction. On account of which I impose an end to this chapter thus preceding to that where, what has been treated here, will bring great usefulness. Indeed in the following chapter I shall examine more closely the effect, which any forces acting in some manner on some body or ship floating on water produce,

where it is understood, what the force of the wind and of the oars, as well as of the rudder and of the striking of the water itself on the ship may bring about. But since the progressive motion shall be unable become known without the calculation of the resistance, which I have put in place to be set out more fully, here I will be content to consider only action required for that sane motion and momentary acceleration. Truly in this I shall depend especially, so that I may define accurately, how great the magnitudes of any forces will disturb the ship from its state of equilibrium, and to this end for the three axes mentioned above, which are considered to be acting, I shall examine especially, around which the inclination and declination of everything is considered to be acting.

CAPUT TERTIUM

DE STABILITATE, QUA CORPORA AQUAE INSIDENTIA IN SITU AEQUILIBRII PERSISTUNT

PROPOSITIO 19

THEOREMA

204. *Stabilitas, qua corpus aquae innatans in situ aequilibrii perseverat, aestimanda est ex momento potentiae restituentis, si corpus dato angulo infinite parvo ex situ aequilibrii fuerit declinatum.*

DEMONSTRATIO

Si corpus aquae innatans aliquantillum ex situ aequilibrii declinetur, tum vel restituetur, vel quiescet, vel etiam magis a situ aequilibrii recedet, et quasi prolabetur se in alium situm aequilibrii recipiendo. In casu igitur, quo ex situ aequilibrii declinatum quiescit, stabilitas est nulla, cum etiam corpus sibi relictum non restituatur; casu vero, quo magis recedit a situ aequilibrii, stabilitas non solum nulla sed negativa adeo est censenda. Stabilitas ergo iis tantum aequilibrii sitibus est tribuenda, in quos corpus si aliquantillum declinetur, restituitur. Si autem corpus minimo tantum angulo e situ aequilibrii declinetur, restitutio fiet circa axem horizontalem per centrum gravitatis transeuntem, prout in propositione 18 est monstratum. Causa vero restitutionis est momentum pressionis aquae circa illum axem, quod in eodem corpore ipsi angulo proportionale esse infra ostendetur. Quo ergo in diversis corporibus eodem angulo e situ aequilibrii declinatis maius fuerit momentum illud restitutionis, eo fortior erit vis restitutionis, eoque propterea maior vis perseverandi in situ aequilibrii, quam stabilitatem voco. Hancobrem stabilitas, qua corpus aquae innatans in situ aequilibrii persistit, aestimanda est ex momento potentiae restituentis, si corpus angulo infinite parvo e situ aequilibrii declinetur.

Q.E.D.

COROLLARIUM 1

205. Cum igitur in eodem corpore momentum restitutionis angulo declinationis a situ aequilibrii sit proportionalis, atque in diversis corporibus stabilitas per aequales angulos definiatur, erit incorpore quocunque stabilitas absolute ut momentum restitutionis per angulum declinationis divisum.

COROLLARIUM 2

206. Quia in stabilitate determinanda axis quidam horizontalis consideratur, circa quem minima inclinatio fieri concipitur, manifestum est pro eodem corpore eodemque aequilibrii situ infinitas inveniri stabilitatis aestimationes, pro infinitis axibus, quorum respectu stabilitas definitur.

COROLLARIUM 3

207. Quando ergo de stabilitate, qua datum corpus aquae insidens in dato aequilibrii situ persistit, sermo est, axis simul erit indicandus ad quem stabilitas refertur; alioquin enim stabilitas determinatam quantitatem habere nequit.

COROLLARIUM 4

208. Si ergo corpus quodpiam aquae in situ aequilibrii insidat, stabilitas respectu cuiusdam axis fixi horizontalis indicabit, quantum illud corpus inclinationi circa illum axem resistat. Quo magis enim corpus aquae insidens inclinationi circa quempiam axem reluctatur, eo maior censetur eius stabilitas respectu eiusdem axis.

COROLLARIUM 5

209. Quo ergo maior reperietur valor stabilitatis respectu cuiusdam axis, eo magis corpus inclinationi circa hunc axem resistet. Atque si stabilitatis valor prodeat $= 0$, tum corpus ne quidem restituetur, si parumper circa illum axem inclinetur. At si stabilitas fuerit negativa, tum corpus vel minime circa axem inclinatum non solum non restituetur, sed subvertetur, donec in situm aequilibrii firmum et stabilem perveniat.

SCHOLION 1

210. Doctrina haec de stabilitate corporum aquae innatantium, qua in situ aequilibrii quem tenent, perseverant, maximi est momenti in constructione et oneratione navium. Maxime enim in navigatione requiri solet, ut naves in situ suo recto quam firmissime persistant, atque viribus inclinantibus vehementissime resistant. Hancobrem istam doctrinam eo accuratius in hoc capite evolvere constitui, quo postmodum ex ea utiles regulae pro construendis et onerandis navibus elici queant. Cum igitur in primo capite omnes situs investigavimus, quibus corpus aquae insidens in aequilibrio persistere possit, hic in stabilitatem inquiremus, qua in quovis aequilibrii situ respectu cuiusque axis perseveret. Inveniemus ergo alios aequilibrii situs firmos et stabiles, quando scilicet stabilitas affirmativum obtinebit valorem, alios vero instabiles ac labiles, quando stabilitas prodit negativa. Habebuntur etiam nonnunquam situs ambigui, in quos stabilitas

evanescens competit; quos casus omnes diligenter perpendere ad praecepta rei nauticae tradenda maxime conducet.

SCHOLION 2

211. Quamvis ad stabilitatem dati aequibrii situs perfecte cognoscendam requiratur, ut stabilitas respectu omnium axium horizontalium definiatur, tamen satis commode gradus stabilitatis intelligitur, si stabilitas tantum respectu duorum axium investigetur, quorum alter stabilitatem maximam alter vero minimam praebeat; ex his enim licebit stabilitatem respectu cuiusvis alius axis facile aestimare, cum plerumque sufficiat limites nosse, inter quos stabilitas contineatur. Sic naves a viribus externis multo difficilius proram puppemve versus inclinantur quam ad latera, earumque stabilitas proinde respectu axis secundum latitudinem ducti maxima est, stabilitas vero respectu axis navim secundum longitudinem traiicientis, minima. Stabilitatis scilicet definitio similis est momentorum tam virium quam inertiae, quae absolute assignari non possunt, sed semper ad axem quempiam, circa quem inclinatio fieri concipitur, referri debent. Hoc igitur pacto ista tractatio, quae fere infinita videatur, maxime contrahetur, ut facili negotio absolvi queat. Quo autem a casibus simplicioribus inchoemus, primo non corpora sed tantum superficies planas considerabimus, quae aquae in situ verticali insideant, atque circa axem horizontalem per centrum gravitatis superficiei ad ipsius planum normaliter transeuntem mobiles existant. In huius modi scilicet superficiebus planis alias a situ aequilibrii declinationes non contemplabimur, nisi quibus ipsae superficies maneant verticales.

PROPOSITIO 20

PROBLEMA

212. *Invenire stabilitatem, qua superficies quaecunque plana aquae verticaliter insidens in situ aequilibrii perseverat.*

SOLUTIO

Sit *AMFNB* (Fig. 39) superficies quaecunque plana aquae verticaliter insidens ita ut recta horizontalis *AB* sit sectio aquae. Sit centrum gravitatis huius figurae in *G,* et pondus figurae *M.* Ducatur per *G* recta verticalis *EF,* in quam cadat centrum magnitudinis *O* partis submersae *AMFNB* quia figura in hoc situ aequilibrium tenere ponitur; atque ob eandem rationem pars submersa *AMFNB* tanta erit, ut massae aqueae volumine ipsi aequalis pondus sit $= M$. Inclinetur iam figura haec quam minime ex statu aequilibrii, ita ut *ab* fiat sectio aquae, atque per *G* ipsis *AB* et *ab* ducantur parallelae *MN*, et *mn,* quarum *MN* horizontalis erit in situ aequilibrii, *mn* vero horizontalis in situ inclinato. Sitque angulus

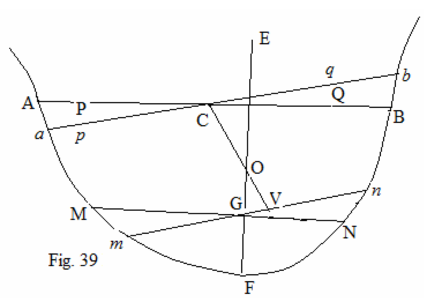


Fig. 39

inclinationis $MGm = dw$. Cum igitur pars submersa perpetuo constans esse debeat, erit area $aMFNb =$ areae $AMFNB$: nisi enim aequalis esset, centrum gravitatis vel ascenderet vel descenderet donec aequalitas fuerit comparata, quo motu ipse restitutionis motus circa centrum gravitatis factus, quem hic solum spectamus non turbatur. Posita ergo intersectione rectarum *AB* et *ab* in *C,* erit area $AGa =$ areae BCb. Ob angulum autem inclinationis *dw* infinite parvum erit area

$$ACa = \frac{AC^2 \cdot dw}{2}$$

et area

$$BCb = \frac{BC^2 \cdot dw}{2}$$

unde prodit

$$AC = BC = \frac{AB}{2}$$

Quo nunc vis restitutionis ex situ inclinato in situm aequilibrii inveniatur, quaerendum est centrum magnitudinis partis submersae *aMFNb;* quae cum sit

$$= AMFNB + BCb - ACa,$$

ex harum partium centris gravitatis reperiri poterit verae partis submersae centrum magnitudinis, indeque pressionum aquae momentum ad restitutionem aequilibrii Restituetur vero in aequilibrium, dum recta *MN* in situm horizontalem *mn* reducetur. Consideremus igitur primo aream *AMFNB* cuius centrum gravitatis est in *O* et vis sursum urgens ab ea orta $= M$. Per *O* ducatur verticalis *VOo* quae est directio vis *M* figuram sursum urgentis; momentum ergo huius vis ad restituendum est

$$M \cdot GV = M \cdot GO \cdot dw$$

ob angulum

$$GOV = MGm = dw.$$

Porro consideretur areola *CBb,* cuius area est

$$\frac{BC^2 \cdot dw}{2} = \frac{AB^2 \cdot dw}{8}.$$

Vis igitur hinc orta figuram sursum urgens est $= \dfrac{M \cdot AB^2 \cdot dw}{8 \cdot AMFNB}$; cuius directio transit per centrum gravitatis *Q* elementi *BCb;* ex Q in *Cb* ducatur normalis seu verticalis *Qq,* erit

$$Cq = \tfrac{2}{3} Cb = \tfrac{2}{3} CB = \tfrac{1}{3} AB,$$

momentum igitur huius vis ad restituendum est

$$=\frac{M\cdot AB^2dw}{8\cdot AMFNB}\left(qo+GV\right).$$

Denique elementum *ACa* pari modo dabit vim $=\frac{M\cdot AB^2dw}{8\cdot AMFNB}$, eiusque directio, quae est verticalis transit per eius centrum gravitatis *P*. Ducta ergo verticali *Pp,* erit

$$pC=\tfrac{2}{3}aC=\tfrac{2}{3}AC\ \left[=\tfrac{1}{3}AB\right].$$

Momentum ergo huius vis contrarium erit prioribus ideoque

$$=-\frac{M\cdot AB^2\cdot dw}{8\cdot AMFNB}\left(po-GV\right).$$

Quia ergo hoc momentum a prioribus subtrahi debet ob

$$aMFNb=AMFNB+BCb-ACa,$$

erit momentum, quo pressio aquae in totam partem submersam *aMFNb* exerta aequilibrium restituit

$$=M\cdot GO\cdot dw-\frac{M\cdot AB^2\cdot dw}{8\cdot AMFNB}\left(po+qo\right)=M\cdot GO\cdot dw+\frac{M\cdot AB^3\cdot dw}{12AMFNB}$$

ob

$$po+qo=pq=\tfrac{2}{3}AB.$$

Cum igitur momentum, quo figura in situm aequilibrii restituitur sit

$$=Mdw\left(GO+\frac{AB^3}{12AMFNB}\right)$$

erit per angulum *dw* dividendo stabilitas, qua figura in situ aequilibrii *AMFNB* perseverat

$$=M\left(GO+\frac{AB^3}{12AMFNB}\right)$$

designante *AFB* aream partis submersae. Q. E. I.

COROLLARIUM 1

213. Patet ergo quod supra asservimus, vim restituentem in situm aequilibrii proportionalem esse angulo, quo corpus ex situ aequilibrii est declinatum, si quidem angulus fuerit quam minimus, ideoque si stabilitas absolute requirat, ut angulum *dw,* quo inclinatio indicatur, omitti oportere. Sic igitur expressio stabilitatis momentis virium erit homogenea, cum sit productum ex vi seu potentia M in lineam quandam rectam

$$GO+\frac{AB^3}{12AF}.$$

SCHOLION 1

214. In expressione stabilitatis denotat GO distantiam centri gravitatis figurae a centro magnitudinis partis submersae, quando figura in aequilibrio existit. Cum igitur posuerimus in figura hac centrum gravitatis infra centrum magnitudinis cadere, per se patet, si in alio casu centrum gravitatis G supra centrum magnitudinis cadat, tum intervallum GO negative accipi debere, ita ut in huius modi casibus stabilitas proditura sit

$$=M\left(\frac{AB^3}{12AFB}-GO\right).$$

Hic scilicet figuras ex materia utcunque heterogenea constantes consideramus, ita ut centrum gravitatis G tam supra quam infra centrum magnitudinis O incidere possit; sin autem figura ex materia homogenea confecta ponatur, tum necessario centrum gravitatis supra centrum magnitudinis partis submersae cadere oportet. Huius modi igitur casibus stabilitas semper ex hac posteriore formula erit aestimanda, in qua GO negativo signo afficitur.

COROLLARIUM 2

215. Quoties ergo centrum gravitatis infra centrum magnitudinis partis submersae cadit, tum situs aequilibrii semper erit firmus et stabilis, quia expressio stabilitatis negativa fieri nequit.

COROLLARIUM 3

216. Sin autem centrum gravitatis G supra centrum magnitudinis O cadit, tum situs aequilibrii non erit stabilis, nisi fuerit

$$\frac{AB^3}{12\cdot AFB}>GO.$$

At si fuerit

$$\frac{AB^3}{12\cdot AFB}<GO,$$

situs erit instabilis seu labilis, et figura vel minime ex situ aequilibrii declinata prolabetur, aliumque situm aequilibrii quaeret.

COROLLARIUM 4

217. Maximam igitur situs aequilibrii habebit stabilitatem, si centrum gravitatis profundissime, centrum magnitudinis autem in loco maxime elevato fuerit situm; atque praeterea si sectio aquae seu recta *AB* fuerit maxima: manente scilicet eodem figurae pondere *M,* quo ipso magnitudo partis submersae etiam invariata manet.

COROLLARIUM 5

218. In corporibus ergo aquae innatantibus, quo profundius pondera collocentur, eo maiorem ea stabilitatem in situ aequilibrii acquirent. Magis vero etiam stabilitas augebitur, si alis adiungendis sectio aquae amplior reddatur.

SCHOLION 2

219. Quamvis haec propositio tantum ad superficies planas aquae verticaliter insidentes sit accommodata, tamen ea latius patet, et corpora cylindrica in se complectitur. Si enim corpus cylindricum aquae ita innatet, ut eius axis longitudinalis horizontalem situm teneat, tum eius stabilitas respectu axis horizontalis eiusdem ex stabilitate cuiusque sectionis transversalis, quae est superficies plana verticalis, cognoscetur. His igitur casibus *AFB* erit sectionis mediae corporis cylindrici pars aquae submersa, *G* totius corporis centrum gravitatis, *O* centrum magnitudinis partis submersae; *M* vero pondus totius corporis, et *AFB* ut ante cuiusque sectionis pars aquae submersa. Praeterea etiam ex eadem propositione pro corporibus alius figurae consectaria deduci possent, sed de iis in sequentibus, cum omnis generis corpora ex instituto comtemplabimur, fusius tractabimus.

PROPOSITIO 21

PROBLEMA

220. *Si figura plana aquae in situ verticali insidens ex situ aequilibrii aliquantillum declinetur, determinare motum, quo sese in situm aequilibrii restituet.*

SOLUTIO

Sit figura plana *AFB* (Fig. 39) aquae insistens in aequilibrio, cum praeter rectam per centrum gravitatis *G* ad planum figurae normaliter ductam etiam recta *MGN* fuerit horizontalis. Sit pondus figurae $= M$ atque *AB* sectio aquae et *O* centrum magnitudinis partis submersae *AFB.* Inclinetur nunc aliquantillum figura ex situ aequilibrii ut recta *ab* fiat sectio aquae, et angulus ACa fiat $= dw$. His positis ex propositione praecedente momentum restituens figuram in aequilibrium, quo scilicet figura circa axem horizontalem per G ad planum ipsius normaliter ductum circumvertetur, obtinebitur, si stabilitas ante inventa per angulum inclinationis dw multiplicetur, eritque propterea hoc momentum ad corporis restitutionem tendens

$$= Mdw\left(GO + \frac{AB^3}{12AFB}\right).$$

Quoties igitur haec expressio fuerit affirmativa, figura in situm aequilibrii restituetur, atque restitutio fiet rotando circa centrum gravitatis G, dum interea ipsum centrum gravitatis G recta vel ascendit vel descendit, prout conditio ea, qua semper aequalis pars aquae debet esse submersa, requirit. Cum ergo hoc momentum angulo percurrendo sit proportionale, figura eodem modo in statum aequilibrii perveniet, quo pendulum descendendo ad situm verticalem accedens. Hancobrem figura oscillationes instar penduli perficiet, donec totus motus a restistentia fuerit consumtus. Iste motus oscillatorius ergo cognoscetur, si longitudo penduli simplicis determinetur, quod suas oscillationes aequalibus temporibus absolvat. Ad hoc vero pendulum assignandum necesse est, ut momentum inertiae figurae respectu axis circa quem gyratur constet. Sit igitur S momentum inertiae figurae seu aggregatum omnium particularum per quadrata distantiarum suarum ab axe rotationis multiplicatarum, qui axis ad figuram normaliter per G transit. Hinc igitur erit vis gyratoria

$$= \frac{Mdw}{S}\left(GO + \frac{AB^3}{12 \cdot AFB}\right),$$

ex qua prodibit longitudo penduli simplicis, quod oscillationes isochronas cum oscillationibus figurae absolvit

$$= \frac{12S \cdot AFB}{12M \cdot GO \cdot AFB + M \cdot AB^3}.$$

enim perpetuo longitudo penduli simplicis isochroni, si angulus inclinationis per vim gyratoriam dividatur, id quod facile ex principiis mechanicis colligitur. Q. E. I.

COROLLARIUM 1

221. Longitudo ergo penduli isochroni aequatur momento inertiae figurae respectu axis gyrationis diviso per stabilitatem figurae respectu eiusdem axis, prout quidem stabilitatem exprimere constituimus.

COROLLARIUM 2

222. Manente igitur eadem figurae stabilitate in suo aequilibrii situ oscillationes eo erunt celeriores, quo minus fuerit momentum inertiae figurae; maximo autem existente hoc momento oscillationes tardissimae fient.

COROLLARIUM 3

223. Manente autem eodem figurae momento inertiae oscillationes eo crebriores evenient; quo maior fuerit figurae stabilitas; minuta autem stabilitate, oscillationes segniores perficientur.

COROLLARIUM 4

224. Ad motum oscillatorium ergo definiendum praeter pondus et figuram et centrum gravitatis, quae ad stabilitatem cognoscendam sufficiunt, nosse oportet momentum inertiae figurae respectu axis, circa quem oscillationes fiunt.

SCHOLION

225. Quo tam stabilitas quam motus oscillatorius huiusmodi figurarum aquae innatantium clarius cognoscatur, iuvabit casus speciales considerasse, in quibus quantitates adhuc indeterminatas determinari et inter se comparari licebit. Determinatas igitur figuras contemplabimur, quae aquae insidant, ubi quidem sufficiet figuram partis submersae posuisse, cum figura partis supra aquam eminentis in computum non ingrediatur. Ex figura vero partis submersae simul centrum eius magnitudinis datur. Conveniet autem tantum figuras regulares, quae circa verticalem *EF* partes similes et aequales habeant, investigasse, ne ante opus sit situm aequilibrii invenire. Ponemus igitur centrum gravitatis huiusmodi figurarum in ipsa verticali *EF,* quae est diameter, situm, quo aequilibrium habeatur, si ista diameter verticalem situm obtinuerit. Eiusmodi ergo propositiones aliquot hic subiungemus, antequam ad ipsa corpora examinanda progrediamur.

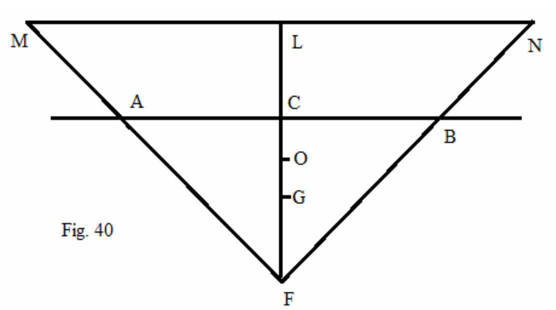


Fig. 40

PROPOSITIO 22

PROBLEMA

226. *Si figurae aquae insidentis pars submersa AFB fuerit triangulum isosceles* (Fig. 40), *determinare stabilitatem huius situs, atque motum oscillatorium, quem figura, si ex hoc situ aliquantillum declinetur, acquiret.*

SOLUTIO

Ex vertice *F* in basin *AB,* quae sectionem aquae repraesentat, ducatur perpendicularis *FG* basin *AB* bifariam secans in *G.* Ponatur $AG = BG = a$*;* et perpendiculum $FC = b$*;* erit pars submersa $AFB = ab$*,* eiusque centrum magnitudinis in *O,* ut sit $CO = \frac{1}{2}b$. Sit porro *G* centrum gravitatis totius figurae, atque $CG = h$*,* erit

$$GO = CG - CO = h - \tfrac{1}{3}b.$$

Vocetur deinde massa seu pondus totius figurae $= M$*,* et momentum eius respectu axis normaliter ad planum *AFB* per *G* transeuntis $= S$. His igitur positis erit stabilitas huius situs aequilibrii

$$M\left(h - \frac{1}{3}b + \frac{2a^2}{3b}\right) = \frac{M\left(2a^2 - b^2 + 3bh\right)}{3b}.$$

Penduli vero simplicis isochroni cum oscillationibus huius figurae oscillantis longitudo erit $= \dfrac{3bS}{M\left(2a^2 - b^2 + 3bh\right)}$, si quidem stabilitas affirmativum habuerit valorem. Q. E. I.

COROLLARIUM 1

227. Si ergo fuerit

$$2a^2 + 3bh > b^2 \quad \text{seu} \quad h > \frac{b^2 - 2a^2}{3b} \ ,$$

situs iste aequilibrii erit stabilis, eoque maior erit stabilitas, quo maior fuerit iste excessus.

COROLLARIUM 2

228. Hic porro situs aequilibrii erit indifferens, si fuerit

$$2a^2 + 3bh = b^2;$$

sin autem fuerit

$$2a^2 + 3bh > b^2;$$

tum situs erit labilis, eumque figura tenere non poterit, sed vel tantillum ex eo deturbata prolabetur.

EXEMPLUM

229. Si integra figura fuerit triangulum isosceles *MFN,* constans ex materia uniformi, cuius ad aquam gravitas specifica teneat rationem $p : q$, ponanturque

$$ML = LN = A \text{ et } FL = B,$$

tum

$$AC = BC = a \text{ et } FL = b,$$

erit $ab : AB = p : q$, atque ob $a : b = A : B$ erit

$$a = A\sqrt{\frac{p}{q}} \text{ et } b = B\sqrt{\frac{p}{q}}.$$

Deinde vero erit $LC = \frac{1}{3}B$, atque ob

$$LC = B - B\sqrt{\frac{p}{q}}$$

habebitur

$$CG = h = B\sqrt{\frac{p}{q}} - \frac{2}{3}B.$$

Manente autem masse seu pondere figurae $= M$, erit momentum

$$S = \frac{M(MN^2 + MF^2 + NF^2)}{36}(\S 170) = \frac{M(3A^2 + B^2)}{18}.$$

His substitutis reperietur stabilitas trianguli isoscelis *MFN* aquae innatantis; ita ut basis *MN* horizontaliter extra aquam emineat,

$$= \frac{M\left(2(A^2 + B^2)\sqrt{\frac{p}{q}} - 2B^2\right)}{3B}.$$

Quae si fuerit affirmativa, erit longitudo penduli simplicis isochroni

$$=\frac{B\left(3A^2+B^2\right)}{12(A^2+B^2)\sqrt{\frac{p}{q}}-12B^2}.$$

COROLLARIUM 1

230. Stabilis ergo erit iste trianguli aquae insidentis situs aequilibrii, si fuerit

$$(A^2+B^2)\sqrt{\frac{p}{q}}>B^2 \text{ seu } \frac{p}{q}>\frac{B^4}{\left(A^2+B^2\right)^2}$$

hoc est

$$\frac{p}{q}>\frac{LF^4}{MF^4}.$$

Labilis vero erit si

$$\frac{p}{q}<\frac{LF^4}{MF^4}.$$

COROLLARIUM 2

231. Si triangulum fuerit aequilaterum, erit $B=A\sqrt{3}$; atque stabilitas eius in hoc aequilibrum situ prodibit

$$=\frac{2\text{AM}\left(4\sqrt{\frac{p}{q}}-3\right)}{3\sqrt{3}}.$$

Longitudo vero penduli isochroni erit

$$=\frac{A\sqrt{3}}{8\sqrt{\frac{p}{q}}-6}.$$

COROLLARIUM 3

232. Hoc ergo casu situa aequilibrii erit stabilis, si $\sqrt{\frac{p}{q}}>\frac{3}{4}$,

hoc est si $\frac{p}{q} > \frac{9}{16}$. Posita ergo gravitate specifica aquae $q = 1000$, situa erit stabilis, si fuerit $p > 562\frac{1}{2}$: instabilis autem erit, si gravitas specifica trianguli minor est quam $562\frac{1}{2}$.

COROLLARIUM 4.

233. Si angulus ad *F* fuerit rectus, ut sit $B = A$, erit stabilitas

$$= \frac{2\mathrm{AM}\left(2\sqrt{\frac{p}{q}} - 1\right)}{3}.$$

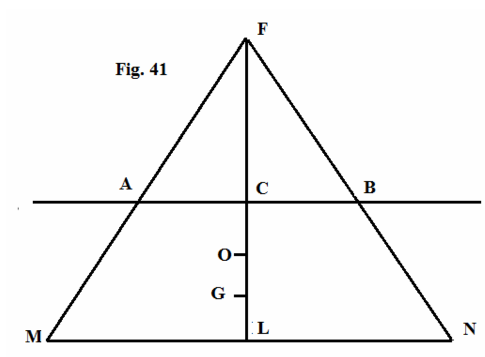
Situa ergo aequilibrii erit stabilis, si, posita aquae gravitate specifica $= 1000$, trianguli gravitas specifica maior fuerit quam 250. Sin autem trianguli gravitas minor sit quam 250, situa aequilibrii erit instabilis. Longitudo vero penduli isochroni illo casu fiet

$$= \frac{\mathrm{A}}{6\sqrt{\frac{p}{q}} - 1}.$$

PROPOSITIO 23.

PROBLEMA

234. *Si figura aquae insidens fuerit triangulum isosceles FMN* (Fig. 41) *basem MN sub aqua in situ horizontali sectioni aquae AB parallelo habens positam;seu potius, si pars submersa AM NB fuerit trapezium, in quo latera AB et NM sunt inter se parallela, angulique ad M et N aequales: determinare stabilitatem qua iste situs aequilibrii conservatur, motumque oscillatorium, quem eiusmodi figura, si aliquantillum ex situ aequilibriî deturbetur acquiret.*



SOLUTIO

Ducta recta verticali *CL,* quae tam sectionem aquae *AB,* quam basin *MN* bisecet, in hac positum erit centrum magnitudinis partis submersae *O.* Quare necesse est, ut in eandem rectam centrum gravitatis totius figurae incidat, quod sit in *G.* Ponatur

$AC = BC = a,\ ML = LN = c, CL = b$ et $CG = h.$ Centrum magnitudinis vero partis submersae O ita secundum praecepta statica reperietur ut sit

$$CO = \frac{b(a+2c)}{3(a+c)};$$

erit ergo

$$GO = h - \frac{b(a+2c)}{3(a+c)}.$$

Denotet praeterea M massam totius figurae, atque S eiusdem momentum inertiae respectu axis ad figuram normaliter per centrum gravitatis G transeuntis.
Cum ergo pars immersa $AMNB$ sit $= b(a+\mathrm{c})$,
erit

$$\frac{AB^3}{12AMNB} = \frac{2a^3}{3b(a+c)},$$

unde prodit stabilitas figurae in isto situ aequilibrii

$$= M\left(h - \frac{b(a+2c)}{3(a+c)} + \frac{2a^3}{3b(a+c)}\right) = \frac{M(3bh(a+c) - b^2(a+2c) + 2a^3)}{3b(a+c)}.$$

Longitudo denique penduli simplicis cum oscillationibus figurae isochroni erit

$$= \frac{3b(a+c)S}{M(3bh(a+c) - b^2(a+2c) + 2a^3)}.$$

Q.E.I.

COROLLARIUM 1

235. Huiusmodi ergo figurae, cuius pars aquae submersa est *AMNB,* situs erit stabilis, si fuerit

$$3bh(a+c) - b^2(a+2c) + 2a^3 > 0 \ :$$

hoc est si fuerit

$$h > \frac{b^2(a+2c) - 2a^3}{3b(a+c)}$$

Atque eo maior erit stabilitas, quo maiorem habuerit valorem ista expressio

$$3bh(a+c) - b^2(a+2c) + 2a^3$$

seu ista

$$h-\frac{b^2(a+2c)+2a^3}{3b\left(a+c\right)}.$$

COROLLARIUM 2

236. Contra vero si fuerit

$$h<\frac{b^2(a+2c)-2a^3}{3b(a+c)}$$

figurae situs iste aequilibrii erit instabilis et subversioni obnoxius. Indifferens vero erit aequilibrii situs, si fuerit

$$h=\frac{b^2(a+2c)-2a^3}{3b(a+c)}$$

in quo nulIam omnino habebit stabilitatem.

COROLLARIUM 3

237. Si fiat $c = a,$ pars figurae submersa erit rectangulum; his igitur casibus erit stabilitas, qua figura in hoc situ persistit

$$=M\left(h-\frac{b}{2}+\frac{aa}{3b}\right)$$

COROLLARIUM 4

238. Si ponatur $c=0$, pars submersa fiet triangulum isosceles, qui est casus in praecedente propositione tractatus; reperitur autem stabilitas omnino ut ante

$$=M\left(h-\frac{b}{2}+\frac{aa}{3b}\right).$$

EXEMPLUM

239. Sit tota figura aquae innatans triangulum isosceles ex materia uniformi constans, cuius gravitas specifica ad aquam teneat rationem $p : q$.
Ponatur

$$ML = NL = A;$$

et perpendiculum $FL = B;$ erit ergo $c = A;$ et $FG = B-b$. Quamobrem habebitur

$$FL\cdot ML : FG\cdot AC = q : q-p$$

hoc est

$$AB:\ a(B-b)=q:q-p$$

atque ob

$$FC:AC=FL:ML$$

erit

$$B-b:a=B:A \text{ seu } a=\frac{A(B-b)}{B};$$

quo valore in illa analogia substituto erit

$$AB^2:\ A(B-b)^2=q:\ q-\ p \text{ seu } q(B-b)^2=(q-\ p)B^2;$$

unde oritur

$$pB^2=q(2Bb-bb),$$

sive

$$b\ =B-\frac{B\sqrt{(q-p)}}{\sqrt{q}},$$

hincque

$$a\ =\frac{A\sqrt{(q-p)}}{\sqrt{q}}.$$

Cum vero sit $FG=\frac{2}{3}B$, erit

$$CG=h=\tfrac{2}{3}B-\frac{B\sqrt{(q-p)}}{\sqrt{q}}.$$

Si nunc haec omnia substituantur in formula stabilitatem huius situs experimente, reperietur ipsa stabilitas

$$=\frac{2M(q-p)}{3Bp}\left((A^2+B^2)\sqrt{\frac{q-p}{q}}-B^2\right).$$

Momentum autem inertiae huius figurae est ut ante $=\dfrac{M(3A^2+B^2)}{18}$. Quocirca si stabilitas fuerit affirmativa, erit longitudo penduli simplicis isochroni

$$=\frac{Bp(3A^2+B^2)}{12(q-p)(A^2+B^2)\sqrt{\frac{q-p}{q}}-12(q-p)B^2}.$$

COROLLARIUM 2

240. Huius igitur trianguli hac ratione aquae insidentis situs erit stabilis, si fuerit

$$\left(A^2+B^2\right)\sqrt{\frac{q-p}{q}}>B^2;$$

seu

$$\frac{q-p}{q}>\frac{B^4}{\left(A^2+B^2\right)^2}$$

hoc est si

$$\frac{p}{q}<\frac{A^2\left(A^2+B^2\right)}{\left(A^2+B^2\right)^2}.$$

Instabilis autem erit situs iste aequilibrii, si fuerit

$$\frac{p}{q}>\frac{A^2\left(A^2+2B^2\right)}{\left(A^2+B^2\right)^2}.$$

COROLLARIUM 1

241. Si haec conferantur cum propositione praecedente (§ 230) apparebit idem triangulum duplicem aequilibrii situm habere posse stabilem, si fuerit $A^4+2A^2B^2>B^4$ seu $A^2>B^2(\sqrt{2}-1)$, hoc est si fuerit $FM>FL\sqrt[4]{2}$. Quod evenit, si fuerit ang.$MFL>33°$ seu ang.$MFN>66°$. Hoc vero accidente erunt ipsius $\frac{p}{q}$ limites isti

$$\frac{FM^4-FL^4}{FL^4} \text{ et } \frac{FL^4}{FM^4};$$

inter quos semper continetur casus quo $p:q=1:2$; seu quo figura duplo levior est quam aqua.

COROLLARIUM 3

242. Intelligitur porro fieri posse, ut neuter situs aequilibrii trianguli isoscelis, quo basis est horizontalis, habeat stabilitatem; quod accidit si fuerit ang. $MFN>66°$; atque $\frac{p}{q}$ inter hos limites

$$\frac{FL^4}{FM^4} \text{ et } \frac{FM^4 - FL^4}{FM^4}$$

contineatur, quorum ille est maior, hic vero minor. Inter hos autem limites continetur iterum semper casus, quo $\frac{p}{q} = \frac{1}{2}$.

COROLLARIUM 4

243. Abeat triangulum isosceles in aequilaterum; quo casu fit $B = A\sqrt{3}$. Hanc ob rem erit stabilitas trianguli aequilateri modo in figura expresso aquae insidentis

$$= \frac{2AM(q-p)}{3p\sqrt{3}}\left(4\frac{\sqrt{(q-p)}}{\sqrt{q}} - 3\right).$$

COROLLARIUM 5

244. Huius ergo trianguli aequilateri tali situ aquae insidentis situs erit stabilis, si fuerit

$$\sqrt{\frac{q-p}{q}} > \frac{3}{4} \text{ seu } \frac{q-p}{q} > \frac{9}{16},$$

id quod accidit si fuerit $\frac{p}{q} < \frac{7}{16}$. Posita igitur gravitate specifica aquae $= 1000$; situs erit stabilis, si trianguli gravitas specifica fuerit $< 437\frac{1}{2}$.

COROLLARIUM 6

245. Si angulus *MFN* fuerit rectus, seu $B = A$; erit stabilitas

$$= \frac{2AM(q-p)}{3p}\left(2\sqrt{\frac{(q-p)}{p}} - 1\right),$$

situs aequilibrii ergo stabilis erit, si fuerit

$$\frac{q-p}{q} > \frac{1}{4} \text{ seu } \frac{p}{q} < \frac{3}{4}.$$

Hoc ergo accidit, si gravitas specifica trianguli fuerit < 750, posita aquae gravitate specifica $= 1000$.

SCHOLION

246. Quamvis hae propositiones figuras tantum planas scilicet triangula respiciant, tamen eae quoque, uti iam nota vi, ad corpora accommodari possunt, quorum omnes sectiones transversales sunt triangula aequalia isoscelia; cuius modi sunt prismata triangularia. Ope variorum igitur huiusmodi prismatum, quorum bases sunt triangula isoscelia varii generis veritas eorum, quae ex istis propositionibus deduximus, per experientiam comprobari poterit. In hic ceterum propositionibus eos tantum triangulorum isoscelium aquae insidentium situs examinavimus, in quibus bases horizontalem obtinent situm, praetermissis reliquis aequilibrii sitibus, cum ad calculos nimis prolixos evitandos, tum vero praecipue, quia de stabilitate horum reliquorum situum ex casibus pertractatis satis tuto iudicare licet. In sequentibus enim demonstrabitur interplures aequilibrii situs, quibus corpus quodque aquae innatare potest, alternos esse stabiles, alternos instabiles. Quare cum hic eorum casuum praecipuorum, quibus triangula isoscelia aquae innatare possunt, stabilitatem determinaverimus, de reliquis casibus facile iudicium ferri poterit. Si enim status quidam aequilibrii fuerit instabilis, ii situs, qui utrinque proxime occurrent, certo erunt stabiles, nisi duo in unam coalescant, quo casu situs aequilibrii erit indifferens.

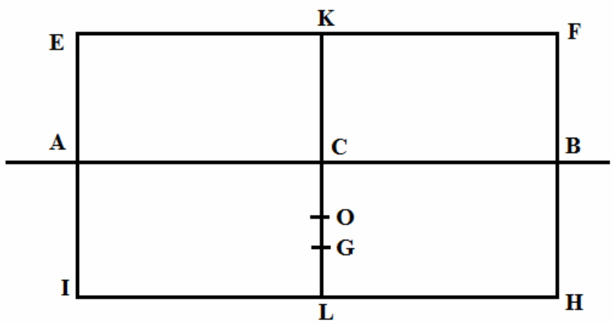


Fig. 42

PROPOSITIO 24

PROBLEMA 107

247. *Si figurae aquae insidentis pars submersa AIHB* (Fig. 42) *fuerit quadrilaterum rectangulum; determinare stabilitatem, qua figura in hoc aequilibrii situ perseverat; atque motum oscillatorium, quo figura ex hoc situ parumper declinata nutabit.*

SOLUTIO

Ducatur verticalis GL parallelogrammum rectangulum $AIHB$ bifariam secans, in cuius puncto medio O situm erit centrum magnitudinis partis submersae; in eandem igitur cadet centrum gravitatis totius figurae, quod sit in G.

Ponatur nunc $AC = BC = \frac{a}{2}$; seu $AB = a$ atque

$$AI = BH = CL = b;\text{et } CG = h;$$

erit $CO = \frac{1}{2}b$ atque $OG = h - \frac{1}{2}b$; massa denique totius figurae sit $= M$.

Stabilitas igitur, qua figura in hoc aequilibrii situ persistit, quae generaliter est

$= M\left(GO + \frac{AB^3}{\text{l2AIHB}}\right)$, erit pro hoc casu

$$= M\left(h - \tfrac{1}{2}b + \frac{a^2}{12b}\right) = \frac{M(a^2 - 6bb + 12hb)}{12b}.$$

At ad 0scillationes definiendas, quas figura circa hunc aequilibrii situm absolvet, sit momentum inertiae figurae respectu puncti $G = S$, hincque reperietur longitudo penduli simplicis isochroni

$$= \frac{12Sb}{M(a^3 - 6bb + 12bh)},$$

ex quo tempus, quo minimae vacillationes absolvuntur, determinatur. Q.E.I.

COROLLARIUM 1

248. Quo ergo iste aequilibrii situs conservetur, necesse est ut

$$a^2 - 6bb + 12bh$$

sit quantitas positiva, id quod evenit si fuerit

$$h > \frac{6bb - aa}{12b}.$$

At si fuerit

$$h = \frac{6bb - aa}{12b}.$$

tum aequilibrii situs erit indifferens, labilis vero et subversioni obnoxius erit, si fuerit

$$h < \frac{6bb - aa}{12b}.$$

COROLLARIUM 2

249. Bacillus igitur admodum gracilis aquae in situ verticali insidere poterit, si fuerit $h > \frac{b}{2}$ ob crassitiem a fere evanescentem. Hoc ergo veniet, si eius centrum gravitatis in inferiorem partis submersae medietatem seu infra punctum medium eius cadit.

COROLLARIUM 3

250. Nisi ergo inferior bacilli pars sit notabiliter ponderosior, quam pars superior, bacillus in aqua situm verticalem tenere non poterit. Ex formula autem inventa

determinari licebit, quantum plumbi seu alius materiae aqua gravioris bacillo in inferiore parte sit adiungendum, quo situs verticalis subsistat.

EXEMPLUM

251. Sit tota figura aquae insidens parallelogrammum *EIHF* ex materia uniformi constans, cuius gravitas specifica ad aquam teneat rationem $p : q$, sitque eius longitudo

$$EF = IH = A,$$

et latitudo

$$EI = FH = B;$$

erit

$$a = A \text{ et } B{:}b = q : p$$

unde fit

$$b = \frac{Bp}{q} = CL;$$

at

$$KG = LG = \tfrac{1}{2}B;$$

hinc igitur prodibit

$$CG = CL - CG = \frac{Bp}{q} - \tfrac{1}{2}B = h.$$

His in formula stabilitatem exprimente substitutis reperietur

$$\frac{\mathrm{M}(A^2q^2 - 6B^2pq + 6B^2p^2)}{12Bpl}$$

qua expressione stabilitas huius aequilibrii situs definitur. Momentum inertiae vero huius figurae respectu eius centri gravitatis G est $= \dfrac{\mathrm{M}(A^2 + B^2)}{12}$ ex quo obtinebitur longitudo penduli simplicis isochroni

$$= \frac{Bpq(A^2 + B^2)}{A^2q^2 - 6B^2pq + 6B^2p^2},$$

unde oscillationes huius aequilibrii situs, si quidem fuerit stabilis, cognoscentur.

COROLLARIUM 1

252. Quo ergo iste aequilibrii situs sit stabilis requiritur ut sit

$$A^2q^2 - 6B^2pq + 6B^2p^2 > 0$$

seu

$$A > \frac{B}{q}\sqrt{6pq - 6pp}.$$

Data ergo materiae, ex qua rectangulum constat gravitate specifica, hinc ratio laterum A et B innotescit, qui fit ut rectangulum latere B verticali, et latere A horizontali aquae innatare queat.

COROLLARIUM 2

253. Ex his simul intelligitur idem hoc rectangulum aquae innatare posse latere A existente verticali, B vero horizontali, si fuerit

$$B > \frac{A}{q}\sqrt{6pq - 6pp}.$$

COROLLARIUM 3

254. Horum duorum igitur aequilibrii situum uterque poterit esse stabilis, si tam $\frac{A}{B}$ quam $\frac{B}{A}$ maius fuerit quam $\frac{\sqrt{6pq - 6pp}}{q}$. Hoc autem accidere nequit, nisi sit
$q > \sqrt{(6pq - 6pp)}$ seu $qq > 6pq - 6pp$.
Id quod accidit duplici modo, primo scilicet si fuerit

$$\frac{q}{p} > 3 + \sqrt{3},$$

secundo si fuerit

$$\frac{q}{p} < 3 - \sqrt{3}.$$

COROLLARIUM 4

255. Quo ergo parallelogrammum rectangulum utroque situ aquae firmiter innatare possit, materiae ex qua constat gravitas specifica vel maior esse debebit quam $788\frac{2}{3}$ vel minor quam $211\frac{1}{3}$ posita aquae gravitate 1000. Horum casuum utroque latera parallelogrammi ita inter se adornari possunt, ut uterque aequilibrii situs fiat stabilis.

COROLLARIUM 5

256. Si igitur materiae, ex qua rectangulum conficitur, gravitas specifica contineatur inter hos limites $788\frac{2}{3}$ et $211\frac{1}{3}$, tum nullum rectangulum confici potest, quod utroque situ aquae firmiter insidere queat.

COROLLARIUM 6

257. At si data fuerint rectanguli latera A et B requiritur ad hoc ut rectangulum, latere A existente horizontali et B verticali, aquae firmiter insidere queat, ut sit vel

$$\frac{q}{p} > \frac{3B^2 + B\sqrt{(9B^2 - 6A^2)}}{A^2} \text{ vel } \frac{q}{p} < \frac{3B^2 - B\sqrt{(9B^2 - 6A^2)}}{A^2},$$

sive necesse est ut sit vel

$$p < \frac{q\left(3B + B\sqrt{(9B^2 - 6A^2)}\right)}{6B} \text{ vel } p > \frac{q\left(3B + B\sqrt{(9B^2 - 6A^2)}\right)}{6B}.$$

COROLLARIUM 7

258. Si ergo rectangulum abeat in quadratum, fiatque $B = A$, tum situs aequilibrii, quo alterum latus horizontale alterum verticale existit, erit stabilis,
si fuerit vel

$$p < \frac{q\left(3 - \sqrt{3}\right)}{6} \text{ vel } p > \frac{q\left(3 + \sqrt{3}\right)}{6}$$

hoc est, si denotante 1000, gravitatem aquae, gravitas specifica quadrati vel maior fuerit quam $788\frac{2}{3}$ vel minor quam $211\frac{1}{3}$.

SCHOLION

259. Quae in hac propositione sunt determinata, etiamsi ad figuras tantum planas pertinere videantur, tamen ad 0mnis generis parallelepipeda rectangula pertinent; ex iis enim, quouis parallelepipedo proposito, diiudicare licet, quanta stabilitate super quaque hedra aquae innatare possit. Deinde ultimum exempli corollarium ad cuborum natatum super aqua investigandum est apprime accommodatum, si quidem cubi vel ex materia homogenea sint confecti, vel saltem centrum gravitatis in sui medio habeant situm. Intelligitur autem eiusmodi cubos aquae in situ erecto, quo binae hedrae sunt

horizontales, reliquae verticales, innatare non posse, nisi eorum gravitas specifica vel maior fuerit quam $788\frac{2}{3}$ vel minor quam $211\frac{1}{3}$, posita aquae gravitatespecifica 1000. Quoties ergo cubi gravitas specifica inter hos limites continetur, maior scilicet est quam $211\frac{1}{3}$ minor tamen quam $788\frac{2}{3}$, tum talis cubus situ erecto aquae neutiquam insidere poterit, sed situm induet alium, quo vel planum diagonale, vel ipsa diagonalis situm horizontalem vel verticalem occupabit. Quanta vero in istius modi sitibus futura sit stabilitas, ex sequente propositione colligere licebit, in qua quidem tantum quadratum aquae ita insidens, ut altera diagonalis horizontalem, altera verticalem situm habeat, examini sum subiecturus.

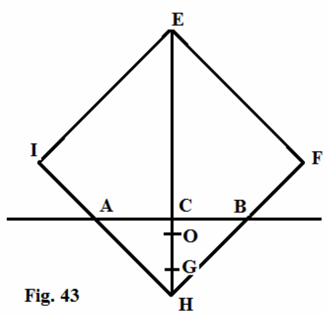


Fig. 43

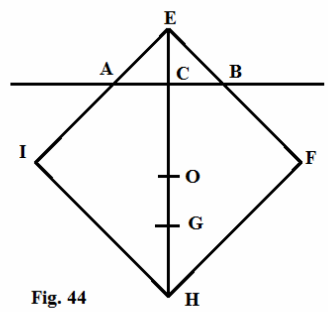


Fig. 44

PROPOSITIO 25

PROBLEM

260. *Quadrati EIHF* (Fig. 43, 44) *quod aquae ita insidit, ut eius diagonalis EH situm verticalem obtineat, stabilitatem definire, qua in hoc situ perseverat,atque motum oscillatorium circa hunc aequilibrii situm.*

SOLUTIO

Duplex hic casus est evoluendus, prout vel maior vel minor pars quam dimidia aquae immergitur, quorum illud accidit, si fuerit $p > \frac{1}{2}q$ hoc vero si $p < \frac{1}{2}q$, denotante $p : q$ rationem quam tenet pondus quadrati ad pondus aequalis voluminis aquae. Sit autem latus quadrati $= A$; et posito quadrati centro gravitatis in G sit $HG = h$. His praemissis in genere consideremus primo casum (Fig. 43) quo est $p < \frac{1}{2}q$, atque pars submersa fit triangulum *AHB,* sectione aquae existente *AB.* Erit ergo $AC \cdot CH$ seu $AC^2 : A^2 = p : q$ ideoque

$$AC = CH = A\sqrt{\frac{p}{q}},$$

centrum gravitatis partis submersae vero cadet in *O,* ut sit

$$HO = \tfrac{2}{3}A\sqrt{\frac{p}{q}}.$$

Quamobrem erit

$$GO = -h + \tfrac{2}{3}A\sqrt{\frac{p}{q}}.$$

Hinc igitur prodibit stabilitas huius situs aequilibrii

$$= M\left(-h + \tfrac{2}{3}A\sqrt{\frac{p}{q}} + \tfrac{2}{3}A\sqrt{\frac{p}{q}}\right) = M\left(\tfrac{4}{3}A\sqrt{\frac{p}{q}} - h\right).$$

Denique posito momento inertiae quadrati respectu centri gravitatis $G = S$, erit longitudo penduli simplicis

$$= \frac{S}{M\left(\frac{4}{3}A\sqrt{\frac{p}{q}} - h\right)},$$

quod oscillationes isochronas vacillationibus quadrati absolvet. Q. E. Alterum.
Sit nunc $p > \frac{1}{2}q$, quo casu (Fig. 44) maior pars *AIHFB* quam dimidia aquae immergitur existente *AB* sectione aquae. Erit ergo et dividendo

$$\mathrm{A}^2 - AC^2 : A^2 = p : q,$$

unde fit

$$AC^2 : A^2 = q - p : q$$

atque

$$AC : CE = A\sqrt{\tfrac{q-p}{q}}$$

$$CH = A\sqrt{2} - A\sqrt{\tfrac{q-p}{q}}.$$

Ex his invenitur centrum magnitudinis partis submersae in *O* ita ut sit

$$HO = A\sqrt{2} - \frac{Aq}{p\sqrt{2}} + \frac{2}{3}\frac{A(q-p)\sqrt{(q-p)}}{p\sqrt{q}}$$

hincque erit

$$GO = h + A\sqrt{2} - \frac{Aq}{p\sqrt{2}} + \frac{2A(q-p)\sqrt{(q-p)}}{3p\sqrt{q}}$$

ipsa vero pars submersa erit $= \dfrac{A^2 p}{q}$. Quocirca stabilitas huius aequilibrii situs erit

$$= M\left(-h + A\sqrt{2} - \frac{Aq}{p\sqrt{2}} + \frac{4A(q-p)\sqrt{(q-p)}}{3p\sqrt{q}}\right)$$

per quam si dividatur momentum figurae respectu centri gravitatis *G,* quod sit *S,* prodibit longitudo penduli simplicis, quod oscillationes circa hunc aequilibrii situm indicabit. Q. E. Alterum.

COROLLARIUM I

261. Si centrum gravitatis quadrati cadat in eius punctum medium, quo casu fit $h = \frac{A}{\sqrt{2}}$; erit casu priore quo $p < \frac{1}{2}q$ stabilitas

$$= AM\left(\frac{4}{3}\sqrt{\frac{p}{q}} - \frac{1}{\sqrt{2}}\right),$$

posteriore vero casu, quo $p > \frac{1}{2}q$ stabilitas erit

$$= AM\left(\frac{4}{3}\sqrt{\frac{q-p}{q}} - \frac{1}{\sqrt{2}}\right).$$

COROLLARIUM

262. Quadratum ergo ex materia uniformi constans, quod plusquam duplo levius est quam aqua, in situ diagonalis verticali aquae firmiter insidebit, si fuerit

$$\frac{4}{3}\sqrt{\frac{p}{q}} > \frac{1}{\sqrt{2}}$$

hoc est si fuerit $\frac{p}{q} > \frac{9}{32}$ (Fig. 43). Posita ergo aquae gravitate specifica 1000, stabilitatem habebit iste aequilibrii situs, si fuerit quadrati gravitas specifica minor quam 500, maior-vero quam $281\frac{1}{4}$.

COROLLARIUM 3

263. Quadratum vero ex materia plus quam duplo graviore quam aqua constans in situ diagonalis verticali firmiter aquae innatabit, si fuerit

$$\frac{q-p}{q} > \frac{9}{32} \text{ seu } \frac{p}{q} < \frac{23}{32} \text{ (Fig. 44).}$$

Hoc ergo accidit, si eius gravitas specifica fuerit maior quam 500, minor vero quam $718\frac{3}{4}$.

COROLLARIUM 4

264. Ante autem invenimus quadratum aquae ita innatare non posse, ut bina latera teneant horizontalem, bina vero verticalem situm, si eius gravitas specifica contineatur intra limites $211\frac{1}{2}$ et $788\frac{2}{3}$. Quamobrem eiusmodi quadrata, quorum gravitas specifica continetur vel intra hos limites $211\frac{1}{3}$ et $281\frac{1}{4}$ vel intra hos $788\frac{2}{3}$ et $718\frac{3}{4}$, neque situ erecto neque diagonali verticaliter posita aquae innatare possunt.

SCHOLION

. Hinc diiudicari possunt natationes prismatum ex materia homogenea confectorum, quorum bases sunt quadrata, in aqua si quidem axes situm teneant horizontalem sive bases verticaliter sint positae. Triplici enim modo eiusmodi prismata aquae insidebunt, pro varia gravitatis, specificae ratione. Primo scilicet hedrae binae horizontalem, binae vero verticalem situm tenebunt, si prismatis gravitas specifica vel minor fuerit quam $211\frac{1}{3}$ vel maior quam $788\frac{2}{3}$. Secundo duorum planorum diagonalium alterum verticaliter alterum vero horizontaliter erit positum, si prismatis gravitas specifica contineatur inter limites $281\frac{1}{4}$ et $718\frac{3}{4}$. Neutro denique horum modo, sed situ ad utrumque obliquo prisma aquae innatabit, si eius gravitas specifica contineatur vel inter hos limites $211\frac{1}{3}$ et $281\frac{1}{4}$, vel inter hos $718\frac{3}{4}$ et $788\frac{2}{3}$. Si quis hoc experimentis comprobare voluerit, prismata satis longa adhiberi oportet, quo eorum axes semper horizontaliter aquae incumbant: breviora enim huiusmodi prismata ad istud negotium minus sunt idonea, cum ea pluribus quam tribus dictis modis aquae innatare queant, eo quod alii etiam axes inter natandum situm horizontalem constanter servare possint, quae varietas in longioribus locum non habet.

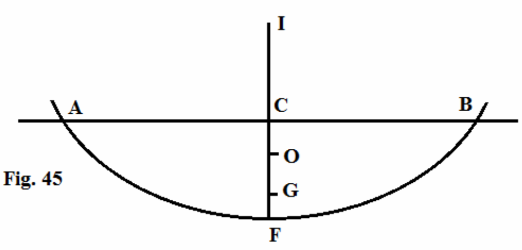


Fig. 45

PROPOSITIO 26

PROBLEMA

266. *Determinare stabilitatem, qua figura quaecunque curvilinea AFB* (Fig. 45) *circa axem FC utrinque partes similes et aequales habens in situ aequilibrii aquae insidet.*

SOLUTIO

Sit *AFB* pars aquae immersa et *AB* sectio aquae, erit *FC* linea verticalis et simul diameter orthogonalis figurae, ita ut sit $AC = BC$. Denotet M totius figurae massam, eiusque centrum gravitatis sit in *G,* existente $FG = h$.
Ponatur porro

$$FC = x \text{ et } AC = BC = y,$$

ita ut aequatio inter x et y naturam curvae propositae exprimat. Sit iam O areae *AFB* aquae submersae centrum magnitudinis, erit

$$FO = \frac{\int yxdx}{\int ydx}, \text{ideoque } GO = \frac{\int yxdx}{\int ydx} - h.$$

Quia vero tota area aquae immersa est $= 2\int ydx$, reperietur stabilitas huius aequilibrii situs

$$= M\left(\frac{\int yxdx}{\int ydx} - h + \frac{y^3}{3\int ydx}\right),$$

quae expressio in hanc commodiorem saepius potest transmutari

$$M\left(\frac{\int y(xdx + ydy)}{\int ydx} - h\right).$$

Q.E.I.

COROLLARIUM

267. Quoties ergo $\frac{\int y(xdx + ydy)}{\int ydx}$ maius est quam *h,* toties iste aequilibrii situs erit stabilis, eoque stabilior, quo maior fuerit excessus,

$$\frac{\int y(xdx + ydy)}{\int ydx} - h$$

COROLLARIUM 2

268. At si fuerit $\frac{\int y(xdx+ydy)}{\int ydx}$ vel aequale vel etiam minus quam *h,* tum illo casu aequilibrii situs erit indifferens, hoc vero adeo instabilis ut minimum declinatus subvertatur.

EXEMPLUM 1

269. Sit figura aquae immersa *AFB* segmentum circuli cuius radius sit $= a$, erit

$$y = \sqrt{(2ax - xx)} \text{ et } y^2 + x^2 = 2ax,$$

unde fit

$$ydy + xdx = adx.$$

Hoc ergo casu habebitur

$$\int ydx = \int dx\sqrt{(2ax - xx)}$$

et

$$\int y(xdx + ydy) = \int adx\sqrt{(2ax - xx)}.$$

Quocirca stabilitas huius aequilibrii situs erit $= M(a-h) = M(a-h)$ quae ideo erit constans sive maius sive minus segmentum circuli aquae immergatur.

COROLLARIUM 1

270. Dummodo ergo figurae centrum gravitatis infra centrum circuli cadat aequilibrium firmiter conservabitur, idque eo magis, quo profundius situm erit centrum gravitatis.

COROLLARIUM 2

271. Sin autem centrum gravitatis in centrum circuli incidat, tum situs aequilibrii erit indifferens, quod evenit in cylindris homogeneis aquae horizontaliter incumbentibus.

EXEMPLUM 2

272. Sit figura aquae immersa *AFB* sectio conica quaecunque verticem in *F* et axem *FC* habens; erit

$$y = \sqrt{(2ax - nxx)}$$

scilicet si n fuerit numerus affirmativus, curva erit ellipsis, si n negativus hyperbola, at si $n=0$ tum curva abibit in parabolam. Erit ergo

$$y^2+x^2=2ax-(n-1)xx$$

et

$$ydy+xdx=adx-(n-1)xdx.$$

Hinc igitur obtinebitur stabilitas, qua iste aequilibrii situs gaudet

$$=M\left(\frac{\int dx(a-(n-1)x)\sqrt{(2ax-nxx)}}{\int dx\sqrt{(2ax-nxx)}}-h\right)=M\left(a-h\frac{(n-1)\int xdx\sqrt{(2ax-nxx)}}{\int dx\sqrt{(2ax-nxx)}}\right).$$

Casu autem quo $n=0$ atque curva in parabolam abit, erit stabilitas $=M\left(a-h+\frac{3}{5}x\right)$.

COROLLARIUM 1

273. Si puncta A, F et B tanquam fixa considerentur, atque stabilitates, quas variae sectiones conicae per ea transeuntes inter se comparentur, ponatur

$$CF=c \text{ et } AC=f;$$

ob

$$x=c \text{ et } y=\sqrt{(2ax-nxx)}=f; \text{ erit } a=\frac{f^2+nc^2}{2c}.$$

COROLLARIUM 2

274. Casu ergo quo curva est circulus stabilitas erit

$$=M\left(\frac{ff+cc}{2c}-h\right).$$

Casu autem quo curva est parabola, erit stabilitas

$$=M\left(\frac{ff+\frac{6}{5}cc}{2c}-h\right).$$

Maiorem ergo habet stabilitatem parabola quam circulus per eadem tria puncta transiens.

COROLLARIUM 3

Est autem generaliter satis prope

$$\frac{(n-1)\int xdx\sqrt{(2ax-nxx)}}{\int dx\sqrt{(2ax-nxx)}}=\frac{3(n-1)x}{5}$$

Quamobrem stabilitas erit

$$=M\left(a-h-\frac{3(n-1)x}{5}\right)=M\left(\frac{ff+\frac{(6-n)}{5}cc}{2c}-h\right).$$

Stabilitas ergo eo erit maior, quo minor fuerit n.

COROLLARIUM 4

276. At n non ultra datum limitem diminui potest, quia a affirmativum habere debet valorem, estque

$$a=\frac{ff+ncc}{2c},$$

ergo ad summum fieri potest

$$n=\frac{-ff}{cc}$$

quo casu sectio conica abit in triangulum isosceles *AFB,* quod ergo hunc aequilibrii situm firmius conservabit, quam ulla alia sectio conica per eadem puncta *A, F, B* transiens, atque centrum gravitatis in eodem puncto *G* habens.

SCHOLION

276. Abunde haec sufficere possunt ad stabilitatem, quae in quolibet aequilibrii situ inest, cognoscendam, si quidem corpus aequae innatans vel est figura plana tenuissima, vel instar talis considerari potest. Antequam autem ad stabilitatem corporum indagandam progrediar, proprietatem insignem quam plures eiusdem corporis aequilibrii situs ratione stabilitatis inter se tenent, proferam et demonstrabo.

PROPOSITIO 27

THEOREMA

277. *Si omnes situs, quibus figura data quaecunque* (Fig. 46) *in aqua aequilibrium tenere potest, considerentur, tum isti aequilibrii situs alternatim erunt stabiles, et instabiles.*

DEMONSTRATIO

Pro quovis aequilibrii situ concipiatur per figurae centrum gravitatis ducta recta parallela sectioni aquae atque per hanc ipsam rectam per centrum gravitatis ductam innotescet aequilibrii situs: manente enim ista recta horizontali parallela, figura aquae eousque immergatur, donec pars debita sub *c:* aqua existat, quo facto habebitur situs aequilibrii Ita in figura proposita *ACac* designent rectae *Aa,Bb, Cc, Dd* per centrum gravitatis *G* ductae omnes aequilibrii situs, qui in hac figura dantur, dentur scilicet quatuor aequilibrii situs, in quibus sectiones aquae respective sint parallelae rectis *Aa, Bb, Cc, Dd;* quibus positis dico, si situs aequilibrii *Aa* fuerit stabilis, tum quoque situm ab hoc computando tertium *Cc* fore stabilem secundum vero *Bb* et quartum *Dd* fore instabiles. In hoc demonstrando ita versabor ut ostendam inter duos situs stabiles necessario unum situm instabilem contineri debere, pariter ac inter duos situs instabiles unum stabilem, hoc enim probato veritas theorematis erit evicta. Sint igitur *Aa* et *Cc* duo aequilibrii situs stabiles inter se proximi, seu tales inter quos non detur alius situs stabilis. Si nunc figura ex situ *Aa* versus situm *Cc* convertendo declinetur, tum primo quidem nisum habebit sese in situm *Aa* restituendi, at si propius ad situm *Cc* pervenietur, tum figura nisum habebit sese in situm aequilibrii *Cc* recipiendi. Quamobrem necesse est ut inter duos hos situs stabiles *Aa* et *Cc* una existat positio puta *Bb,* quam si figura tenet aequaliter ad utrumque situm *Aa* et *Cc* propendeat, in hoc igitur situ dabitur aequilibrium, id vero instabile, quia figura tantillum ex eo declinata vel ad aequilibrii situm *Aa* vel ad *Cc* nititur; ex quo manifestum est, inter duos situs aequilibrii stabiles necessario unum aequilibrii situm instabilem contineri debere. Simili modo si sint *Bb* et *Dd* duo aequilibrii situs instabiles, se immediate insequentes, uterque ea praeditus erit proprietate, ut figura si ex uno situ versus alterum declinetur, tum nisum habitura sit recedendi ab illo aequilibrii situ; quamobrem necessario dabitur inter istos duos aequilibrii situs instabiles *Bb* et *Dd,* talis situs uti *Cc,* in quo figura utrumque illum aequilibrii situm aquae aversabitur; in hoc igitur situ figura aequilibrium tenebit, idque stabile quia figura utrinque ex eo declinata nisu gaudet sese in illum restituendi. Cum igitur tam inter duos situs stabiles unus instabilis, quam inter duos instabiles unus situs

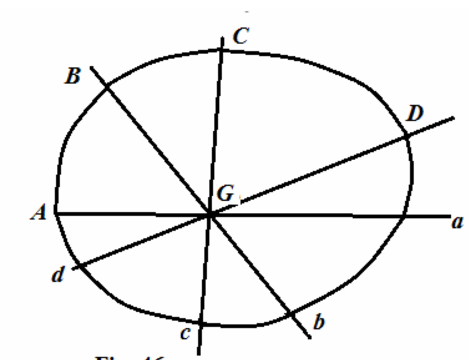


Fig. 46

stabilis existat, situs aequilibrii omnes tum stabiles tum instabiles se mutuo alternatim excipient. Q. E. D.

COROLLARIUM 1

278. In unaquaque ergo figura aquae insidente tot dabuntur aequilibrii situs stabiles, quot instabiles, et hanc ob rem omnium aequilibrii situum numerus erit par.

COROLLARIUM 2

279. Nulla igitur figura pauciores duobus aequilibrii situs habere potest. Omnis enim figura unum necessario habet situm stabilem et propterea unum quoque instabilem.

COROLLARIUM 3

280. Definitis ergo pro quapiam figura omnibus sitibus, quibus in aqua aequilibrium tenet, si de unico constet, utrum stabilis sit an instabilis, simul de omnibus reliquis idem constabit.

COROLLARIUM 4

281. Interim tamen fieri potest, ut numerus aequilibrii situum in quapiam figura actu deprehendatur impar, id quod eveniet si duo aequilibrii situs proximi stabilis et instabilis in unum confundantur, quo situs oritur, indifferens. Situs aequilibrii igitur indifferens spectari debet tanquam coniunctio duorum aequilibrii situum proximorum, ideoqeu pro duobus est numerandus.

SCHOLION

282. Veritas huius propositionis non solum ad figuras planas sed etiam ad 0mnis generis corpora aquae innatantia patet. Quodcunque enim corpus aquae innatans, si in eandem plagam circumagatur tum alternatim ex situ aequilibrii stabili ad instabilem necessario pervenire debet, prout ex demonstratione data intelligi licet; hocque modo res se habet in quamcunque plagam corpus circumagatur. Sed haec omnia clarius percipientur ex sequentibus, ubi stabilitatem corporum quorumcunque aquae in aequilibrio insidentium sum investigaturus.

DEFINITIO

283. Stabilitas respectu axis cuiusdam dati horizontalis per centrum gravitatis transeuntis est vis qua hoc corpus aquae in situ aequilibrii insidens inclinationi circa eundem axem horizontalem per centrum gravitatis transeuntem resistit.

COROLLARIUM 1

284. Stabilitas igitur respectu axis cuiusdam dati horizontalis per centrum gravitatis ducti aestimanda est ex momento pressionis aquae, quo corpus angulo infinite parvo circa

istum axem ex situ aequilibrii declinatum restituitur, diviso per ipsum illum angulum infinite parvum.

COROLLARIUM 2

285. In corporibus ergo aquae innatantibus stabilitas cuiusvis aequilibrii situs infinitis modis est aestimanda, pro infinitis axibus horizontalibus per centrum gravitatis corporis transeuntibus, circa quos corpus inclinando ex situ aequilibrii depelli potest.

COROLLARIUM 3

286. Fieri igitur potest ut idem aequilibrii situs respectu unius pluriumque axium horizontalium sit satis stabilis, qui tamen respectu reliquorum axium est instabilis. Semper autem in unoquoque corpore oportet dari unum aequilibrii situm, qui respectu omnium axium sit stabilis; alioquin enim corpus super aqua quiescere non posset.

COROLLARIUM 4

287. Si autem corporis cuiuspiam aquae insidentis aequilibrii situs fuerit stabilis respectu duorum axium horizontalium inter se normalium, tum iste aequilibrii situs respectu omnium reliquorum axium erit stabilis. Inclinatio enim circa axes intermedios resolvi potest in inclinationes binas circa illos axes inter se normales, quae ambae cum praeditae sint vi restituente, necesse est, ut iste aequilibrii situs respectu omnium axium sit stabilis.

COROLLARIUM 5

288. Ad corporis igitur aquae in aequilibrio insidentis stabilitatem cognoscendam, sufficiet respectu duorum axium invicem normalium stabilitatem investigasse; cum inde stabilitas respectu cuiusvis alius axis pendeat, atque satis tuto aestimari queat.

SCHOLION

289. Quamdiu figuras tantum planas aquae verticaliter innatantes sumus contemplati, unico modo stabilitatem cuiusque aequilibrii situs determinavimus, atque id etiam sufficiebat, quia eiusmodi figuras circa unicum axem horizontalem normalem scilicet ad planum figurae, mobiles posuimus; quilibet autem facile intelliget, huiusmodi figuras, quantumvis eae magnam habere inventae sunt stabilitatem, tamen subversioni ad latera maxime esse obnoxias. Simili modo perspicuum est, naves inclinationi versus proram puppimve multo fortius resistere quam inclinationi ad latera, illoque proinde casu maiorem habere stabilitatem quam isto. Quamobrem cum nunc nobis sit propositum in stabilitatem, qua corpora quaecunque aquae insidentia gaudent, inquirere, omnes inclinationes, quibus corpora ex situ aequilibrii declinari possunt, considerari oportet, atque definiri quanta vi cuique inclinationi resistant. Infinitis autem modis corpus ex situ aequilibrii declinari potest, pro infinitis axibus horizontalibus per centrum gravitatis

transeuntibus, circa quos corpus mobile existit. Hanc ob rationem quando de stabilitate; qua corpus quodpiam in aqua situm aequilibrii tenet, est quaestio, id absolute definiri nequit, sed determinanda est certa inclinatio, in qua stabilitas sese exerat; quem in finem istam stabilitatis determinatam definitionem praemisi, in qua stabilitatem ad certum quendam axem horizontalem per centrum gravitatis transeuntem alligavi. Quamvis autem hoc pacto summe difficile videatur de stabilitate corporum aquae innatantium certi quid statuere, cum infiniti axes deberent considerari, et respectu cuiusque stabilitas assignari, tamen iam notavi eiusmodi insuperabili labore non esse opus, sed sufficere, si respectu duorum tantum axium invicem normalium stabilitas definiatur. Motus enim inclinatorius circa quemvis alium axem spectari potest tanquam compositus ex duobus motibus inclinatoriis circa illos axes invicem normales, pro quorum utroque si stabilitas fuerit cognita, inde stabilitas respectu alius cuiusque axis poterit colligi. Quemadmodum igitur pro quovis aequilibrii situ in corpore quocunque aquae innatante stabilitas respectu cuiusvis axis horizontalis debeat investigari, in sequente propositione docebitur.

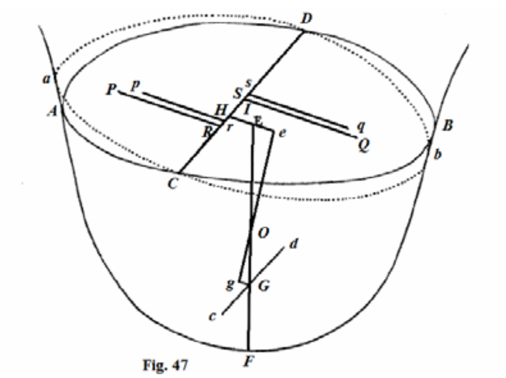


Fig. 47

PROPOSITIO 28

PROBLEMA

290. *Corporis ACFDB* (Fig. 47) *aquae in aequilibrio insidentis determinare stabilitatem respectu axis dati horizontalis cd per centrum gravitatis corporis G transeuntis, atque motum oscillatorium cuius corpus circa hunc axem est capax.*

SOLUTIO

Sit *ACBD* sectio aquae, et *AFB* pars corporis aquae immersa, cuius centrum magnitudinis *O* situm erit in recta verticali *EF* per centrum gravitatia corporis *G* transeunte, eo quod corpus in aequilibrio est positum. Sit totius corporis massa seu pondus $= M$, eiusque partis submersae soliditas seu volumen $= V$, atque momentum inertiae totius corporis respectu axis $cd = S$. Concipiatur nunc corpus paulisper inclinari circa axem *cd* centro gravitatis interea vel ascendente vel descendente, quo aequalis pars aquae maneat immersa. Fiat vero inclinatio per angulum infinite parvum *dw,* posito sinu toto = 1; atque post hanc inclinationem sit *aCbD* sectio aquae priorem aquae sectionem secans recta *CD,* parallela axi *cd,* eritque angulus, quem haec nova sectio aquae cum priore constituit pariter $= dw$. Quia autem utroque casu aequale corporis volumen sub aqua versatur, erit segmentum *ACDa* aequale segmento *BCDb.* Ponatur areae *ACD* centrum gravitatis in *p,* areae autem *BCD* centrum gravitatis in *q,* atque ex *p* et *q* in *CD* ducantur perpendiculares *pr* et *qs,* erit soliditas segmenti $ACDa = ACD\cdot pr\cdot dw$, segmenti vero *BCDb* soliditas erit $= BCD\cdot qs\cdot dw$; hinc igitur habebitur $ACD\cdot pr = BCD\cdot qs$.
Praeterea ipsius segmenti *ACDa* centrum magnitudinis cadat in *P,* segmenti

vero *BCDb* in *Q,* atque ex *P* et *Q* in *CD* ducantur normales *PR* et *QS.*
Iam per punctum *O* ducatur ad planum *aCbD* perpendicularis *eOg,* quae in situ corporis inclinato erit verticalis, atque tum ex *G* in hanc rectam, tum ex *e* per *E* in *GD* ducantur normales *Gg* et *eEH,* erit

$$Gg = GO \cdot dw \text{ et } Ee = EO \cdot dw.$$

Quo igitur vim inveniamus qua corpus ex situ hoc inclinato in pristinum situm aequilibrii restituitur, pars corporis aquae immersa est consideranda quae est

$$= ACFDB + ACDa - BCDb,$$

ex quibus singulis membris vires sunt definiendae ad corpus restituendum, vel convertendum circa axem *cd.* Pressionis autem aquae, quam pars *ACFDB* sustinet, momentum ad corpus restituendum est

$$= M \cdot Gg = M \cdot GO \cdot dw.$$

Nunc fiat ut *V* ad *ACDa* ita pondus M ad vim ex segmento *ACDa* ortam, quae proinde erit

$$\frac{M \cdot ACDa}{V} = \frac{M \cdot ACD \cdot pr \cdot dw}{V},$$

quae expressio pariter valebit pro pressione aquae in segmentum *BCDb.* Cum autem segmenti *ACDa* centrum magnitudinis sit in *P,* erit momentum inde oriundum ad corpus restituendum

$$= \frac{M \cdot ACD \cdot pr \cdot dw}{V}(PR + He + Gg);$$

momentum vero ortum ex vi segmenti *BGDb* tendet ad subversionem, eritque adeo negativum et

$$= \frac{-M \cdot ACD \cdot pr \cdot dw}{V}(PR - He - Gg).$$

Horum trium momentorum duo priora sunt addenda et a summa postremum subtrahendum, quo facto prodibit momentum totale ad restitutionem corporis in pristinum aequilibrii situm tendens

$$= M \cdot dw\left(GO + \frac{ACD \cdot pr}{V}(PR + QS) \right);$$

quod divisum per angulum inclinationis *dw* dabit stabilitatem huius aequilibrii situs respectu axis *cd*

$$= M\left(GO + \frac{ACD \cdot pr(PR + QS)}{V} \right).$$

Dividatur per hanc stabilitatis expressionem momentum materiae seu inertiae totius corporis respectu axis *cd,* quod est *S,* et prodibit longitudo penduli simplicis isochroni cum oscillationibus corporis sese circa axem *cd* in aequilibrium restituentis quae penduli longitudo proinde erit

$$=\frac{SV}{M\left(GO\cdot V+ACD\cdot pr(PR+QS)\right)}.$$

Q.E.I.

COROLLARIUM 1

291. Quia est $ACD\cdot pr=BCD\cdot qs$ atque *p* et *q* sunt centra gravitatis arearum *ACD* et *BCD,* sequitur rectam *CD* transire per centrum gravitatia sectionis aquae *ACBD.*

COROLLARIUM 2

292. Si ergo sectionis aquae *ACBD* centrum gravitatis repertum fuerit in *I,* atque stabilitas huius aequilibrii situs requiratur respectu axis *cd,* tum in sectione aquae *ACBD* per centrum gravitatis *1* ducatur recta parallela *CD* ipsi axi *cd,* qua inventa stabilitas desiderata per calculum innotescet.

COROLLARIUM 3

293. Quemadmodum autem *p* et *q* sunt centra gravitatis partium *ACD* et *BCD* sectionis aquae, ita intelligere licet puncta *P* et *Q* esse centra oscillationis earundem partium circa axem *CD* oscillantium.

COROLLARIUM 4

294. Postquam igitur sectio aquae recta per centrum gravitatis transeunte divisa est in duas partes, utriusque partis tam centrum gravitatis quam centrum oscillationis debet indagari, quo facto sine ullo ad angulum inclinationis *dw* habito respectu stabilitas desiderata poterit inveniri.

COROLLARIUM 5

295. Quoniam inter oscillandum centrum gravitatis totius corporis *G* recta vel ascendit vel descendit, ut perpetuo debita corporis pars maneat aquae submersa; perspicuum est huiusmodi motum centri gravitatis fore minimum, si recta verticalis per centrum gravitatis totius corporis transiens simul per centrum gravitatis sectionis aquae transeat.

COROLLARIUM 6

296. Intelligitur ceterum quo maior sit expressio

$$=M\left(GO+\frac{ACD\cdot pr(PR+QS)}{V}\right),$$

eo firmius corpus in suo aequilibrii situ esse permansurum, si scilicet circa axem *cd* ad inclinandum sollicitetur; sin autem haec expressio fiat negativa, tum corpus minime declinatum iri subversum.

SCHOLION

297. Determinata igitur est satis commode et concinne stabilitas, qua unumquodque corpus aquae iusidens in aequilibrio persistit, id quod primo intuitu summopere difficile videri potuisset. Praeterea etiam ea regula est perquam simplex et facilis, cuius ope ipsae oscillationes, quas corpus ex situ aequilibrii depulsum sese restituens absolvit, quo ipso dignitas et utilitas huius theoriae abunde intelligitur; facile enim erit hinc insignia commoda ad navigationem derivare, quod fieri non potuisset, si enodatio harum propositionum ad inextricabiles calculos deducta fuisset. Quae autem ad stabilitatem determinandam pro quoque corpore nosse oportet, sunt praeter pondus totius corporis a quo quantitas partis submersae pendet, intervallum inter centrum gravitatis corporis et centrum magnitudinis partis submersae atque imprimis sectio aquae quae ad hoc negotium calculo est subiicienda. Sectiones igitur aquae variarum figurarum conveniet considerari, atque eas expressiones, quas ad stabilitatem definiendam nosse oportet, definiri quo post modum facilius sit de quoque corpore aquae innatante iudicium ferre. Hunc in finem in sequente propositione iuventam expressionem calculo analytico sum persecuturus.

PROPOSITIO 29

PROBLEMA

298. *Si sectio aquae fuerit curva quaecunque ACBD* (Fig. 48), *cuius natura per aequationem est data, definire stabilitatem corporis aquae insidentis respectu axis cuiusvis, per calculum analyticum.*

SOLUTIO

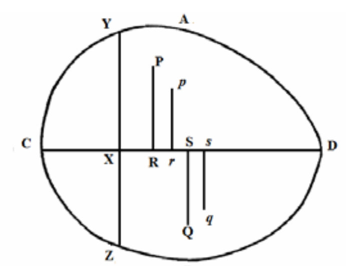


Fig. 48

Sit M massa seu pondus corporis aquae insidentis, et V volumen partis submersae, atque GO exprimat intervallum inter centrum gravitatis corporis et centrum magnitudinis partis submersae, posito centro gravitatis G in loco humiliore; posito enim centro gravitatis G supra centrum magnitudinis O tum loco + GO scribi debet GO. Iam per centrum gravitatis sectionis aquae ducta sit recta CD parallela illi axi, cuius respectu stabilitas quaeritur; et ad hanc rectam tanquam axem referantur orthogonaliter ordinatae YXZ ponaturque $CX = x$; $XY = y$; et $XZ = z$. Sint porro p et q centra gravitatis arearum CAD et CBD, atque P et Q earundem centra oscillationis respectu axis CD; et ex his punctis ad axem CD ducantur normales pr, $q s$, PR et QS. His positis erit

$$pr=\frac{\int yydx}{2\int ydx};\ qs=\frac{\int zzdz}{2\int zdz};$$

atque

$$PR=\frac{2\int y^3dx}{3\int y^2dx};\ \text{ et }\ QS=\frac{2\int z^3dz}{3\int zzdx}$$

integralibus his ita acceptis ut evanescant posito $x=0$; atque tum loco x posito *CD.* Ita exprimet $\int ydx$ aream *CAD,* atque $\int zdx$ aream *CBD.* Quia vero est $CAD\cdot pr=CBD\cdot qr$, erit

$$\int yydx=\int zzdx,$$

ob

$$ACD\cdot pr=\tfrac{1}{2}\int yydx\ \text{ et }\ CBD\cdot qr=\tfrac{1}{2}\int zzdx\ .$$

Denique autem habebitur

$$PR+QS=\frac{2\int y^3dx}{3\int y^2dx}+\frac{2\int z^3dx}{3\int z^2dx}=\frac{2\int\left(y^3+z^3\right)dx}{3\int y^2dx};$$

quibus in formula supra inventa substitutis reperietur stabilitas corporis in isto aequilibrii situ

$$=M\left(GO+\frac{\int\left(y^3+z^3\right)dx}{3V}\right).$$

COROLLARIUM 1

299. Ad stabilitatem igitur obtinendam ope calculi integralis sumendum est integrale formulae $\int\left(y^3+z^3\right)dx$, ita ut evanescat positis $x=0$, atque post integrationem peractam, poni debet $x=CD$.

COROLLARIUM 2

3oo. Si recta *CD* sectionem aquae in duas partes similes et aequales dividat, erit ubique $z=y$ hoc ergo casu stabilitas erit

$$=M\left(GO+\frac{2\int y^3dx}{3V}\right).$$

COROLLARIUM 3

301. Si centrum gravitatis totius corporis G supra centrum magnitudinis O partis submersae cadat, tum intervallum GO negative accipi oportet, eritque his casibus stabilitas

$$= M\left(\frac{\int\left(y^3+z^3\right)dx}{3V} - GO\right).$$

COROLLARIUM 4

302. Nisi ergo G supra O cadat, aequilibrii situs respectu omnium axium erit stabilis; quia $\int\left(y^3+z^3\right)dx$ semper affirmativum tenet valorem. At existente puncto G magis elevato quam 0*,* tum fieri potest, ut situs aequilibrii sit instabilis, id quod accidit si fuerit

$$GO > \frac{\int\left(y^3+z^3\right)dx}{3V}.$$

SCHOLION

303. Quo autem has formulas eo facilius ad varia corporum aquae innatantium genera accommodare liceat, figuras nonnullas determinatas loco sectionis aquae *ACBD* substituam, et quomodo se habeat stabilitas eiusmodi corporum aquae insidentium investigabo. Non solum autem stabilitatem respectu unici axis determinabo, sed respectu binorum inter se normalium, quo ex hac duplici stabilitate respectu cuiusvis alius axis stabilitas possit aestimari. In hunc finem eiusmodi elegi figuras, quae vel in navigatione locum habeant, vel etiam ad experimenta instituenda sint maxime accomodatae, ut tam usus quam utilitas huius theoriae clarissime ob oculos ponatur. Huic igitur negotio absolvendo sequentes destinavi propositiones, quibus institutum huius capitis penitus exhaurietur .

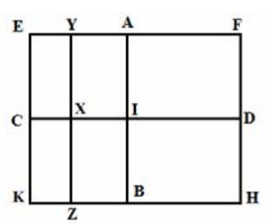


Fig. 49

PROPOSITIO 30

PROBLEMA

304. *Si corporis aquae insidentis sectio aquae fuerit parallelogrammum rectangulum EFH K*(Fig. 49), *invenire eius stabilitatem tum respectu axis CD, tum axis ad hunc normalis AB.*

SOLUTIO

Consideretur primo axis *CD* parallelus lateribus *EF* et *KH,sitque* $EF = KH = A,\ EK = B$, et massa seu pondus corporis $= M$, volumenque partis submersae $= V$.
Ponatur

$$CX = x \text{ erit } XY = XZ = y = z = \frac{1}{2}B.$$

Ergo

$$\int y^3 dx = \frac{B^3 x}{8} \text{ et } \int \left(y^3 + z^3\right) dx = \frac{B^3 x}{4} = \frac{A \cdot B^3}{4}$$

sumto integrali per totum axem *CD*. Erit igitur stabilitas respectu axis

$$CD = M\left(GO + \frac{A \cdot B^3}{12V}\right).$$

Simili autem modo stabilitas respectu alterius axis *AB* erit

$$= M\left(GO + \frac{A^3 \cdot B}{12V}\right).$$

Q.E.I.

COROLLARIUM 1

305. Si ergo tam $GO + \frac{A \cdot B^3}{12V}$ quam $GO + \frac{A^3 \cdot B}{12V}$ fuerint quantitates affirmativae, tum situs aequilibrii corporis erit stabilis respectu cuiusvis axis alius.

COROLLARIUM 2

306. Stabilitas igitur respectu axis *CD* eo erit maior, quo maius fuerit latus $EK = B$; atque semper stabilitas erit maxima respectu axis brevioris, quod quidem per se est planum.

SCHOLION

307. Si pari modo per integrationem computetur stabilitas respectu diagonalis alterutrius *EH* vel *FK* tum reperietur

$$\int \left(x^3 + y^3\right) dx = \frac{A^3 \cdot B^3}{2\left(A^2 + B^2\right)}.$$

Atque ipsa stabilitas corporis respectu huius axis erit

$$= M\left(GO + \frac{A^3 \cdot B^3}{6V\left(A^2 + B^2\right)}\right).$$

Quae expressio media est inter expressiones ante inventas pro axibus *CD* et *AB*. Atque si fuerit *AB* tum stabilitas erit aequalis tam respectu axium *AB* et *CD* quam respectu diagonalium. Ex quo facilius intelligitur, ad statum corporum aquae innatantium cognoscendum sufficere stabilitatem respectu duorum axium inter se normalium determinasse. Eiusmodi autem bini axes sunt accipiendi, qui in sectione aquae sunt praecipui, et quorum alter maximam alter vero minimam habeat stabilitatem, quemadmodum in casu proposito fecimus.

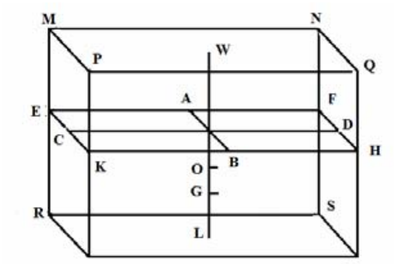


Fig. 50

EXEMPLUM 1

308. Si totum corpus fuerit parallelepipedum *MPQNRTVS* (Fig. 50) aquae ita insidens, ut *EKHF* sit sectio aquae; atque pondus eius se habeat ad aequalis voluminis aquei pondus ut p ad q. Deinde sit longitudo $MN = a$; latitudo $MP = b$; et altitudo $PT = c$; erit in sectione aquae $A = a$ et $B = b$.
Habebitur autem ex statu aequilibrii

$$q : p = c : KT,$$

unde est $KT = \frac{pc}{q}$; atque volumen partis submersae

$$= \frac{pabc}{q} = V;$$

cuius centrum magnitudinis O cadet in medio inter I et L rectae verticalis WL per medium parallelepipedi ductae, ita ut sit $LO = \frac{pc}{2q}$. Quoniam vero hic situs aequilibrio praeditus ponitur, necesse est, ut centrum gravitatis totius corporis cadat in eandem rectam verticalem LW; sit ergo in G, existente

$$LG = h, \text{ erit } GO = \frac{pc}{2q} - h.$$

His igitur substitutis erit stabilitas respectu axis longitudinalis CD, qua inclinationi circa hunc axem resistitur

$$= M\left(\frac{pc}{2q} - h + \frac{qb^2}{12pc}\right)$$

stabilitas vero respectu axis latitudinalis AB erit

$$= M\left(\frac{pc}{2q} - h + \frac{qa^2}{12pc}\right)$$

in quibus expressionibus littera M denotat pondus parallelepipedi, atque p ad q rationem gravitatis specificae corporis ad aquam. Ex his igitur formulis stabilitas huius aequilibrii situs respectu cuiusvis axis colligi poterit.

COROLLARIUM 1

309. Quo ergo iste aequilibrii situs sit stabilis, oportet esse tam

$$h < \frac{pc}{2q} + \frac{qb^2}{12pc}$$

quam

$$h < \frac{pc}{2q} + \frac{qa^2}{12pc}.$$

Si ergo sit $a > b$, dummodo fuerit

$$h < \frac{pc}{2q} + \frac{qb^2}{12pc}$$

situs aequilibrii respectu omnium axium erit stabilis.

COROLLARIUM 2

310. Si parallelepipedum constet ex materia uniformi, tum eius centrum gravitatis G cadet in medio inter L et W, eritque $h = \frac{1}{2}c$. Hoc ergo casu stabilitas erit respectu axis CD. Respectu vero alterius axis AB erit stabilitas

$$= M\left(\frac{qb^2}{12pc} - \frac{(q-p)c}{2q}\right)$$

COROLLARIUM 3

311. Sit basis parallelepipedi quadratum seu $a = b$; erit stabilitas

$$= M\left(\frac{qa^2}{12pc} - \frac{(q-p)c}{2q}\right).$$

Quo ergo iste aequilibrii situs sit stabilis, necesse est ut sit

$$c < \frac{qa}{\sqrt{6p(q-p)}}.$$

EXEMPLUM 2

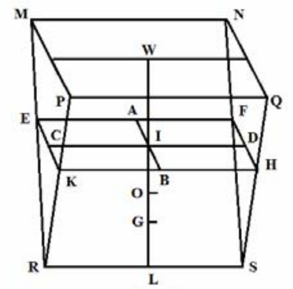


Fig. 51

312. Sit corpus aquae innatans cuneiforme *MRPQSN* (Fig. 51) aquae in situ erecto insidens ut sectio aquae *EKHF* sit rectangulum basi *MPQN* parallelum. Pondus autem huius corporis se habeat ad pondus aequalis voluminis aquae ut p ad q. Ponatur $MN = PQ = a; MP = NQ = b$; atque altitudo cunei WL sit $= c$; in qua recta verticali WL ambo centra tam gravitatis G quam magnitudinis O sint sita, atque $LG = h$. Iam manentibus

$$EF = KH = A;\ EK = FH = B$$

erit

$$A : B = a : b \text{ atque } B : b = c : IL;$$

unde fiet

$$IL = \frac{Bc}{b}.$$

At ex gravitate specifica sequitur $q : p = b^2 : B^2$, unde prodit

$$B = b\sqrt{\frac{p}{q}};\ \text{atque}\ A = a\sqrt{\frac{p}{q}};\ \text{et}\ IL = c\sqrt{\frac{p}{q}}.$$

Ex his reperietur

$$LO = \tfrac{2}{3}LI = \tfrac{2}{3}c\sqrt{\frac{p}{q}};\ \text{atque}\ GO = \tfrac{2}{3}c\sqrt{\frac{p}{q}} - h.$$

Volumen denique partis submersae V erit

$$= \frac{ABc}{2}\sqrt{\frac{p}{q}} = \frac{abcp\sqrt{p}}{2q\sqrt{q}}.$$

Quibus valoribus substitutis emerget stabilitas respectu axis CD

$$= M\left(\tfrac{2}{3}c\sqrt{\frac{p}{q}} - h + \frac{bb}{6c}\sqrt{\frac{p}{q}}\right).$$

Atque respectu alterius axis AB erit stabilitas

$$= M\left(\tfrac{2}{3}c\sqrt{\frac{p}{q}} - h + \frac{aa}{6c}\sqrt{\frac{p}{q}}\right).$$

COROLLARIUM 1

313. Si cuneus iste ex materia uniformi est confectus, erit $h = \frac{2}{3}c$; atque hoc casu stabilitas respectu axis CD erit

$$= M\left(\frac{bb}{6c}\sqrt{\frac{p}{q}} - \tfrac{2}{3}c + \tfrac{2}{3}c\sqrt{\frac{p}{q}}\right);$$

respectu axis AB vero erit stabilitas

$$M\left(\frac{aa}{6c}\sqrt{\frac{p}{q}} - \tfrac{2}{3}c + \tfrac{2}{3}c\sqrt{\frac{p}{q}}\right).$$

COROLLARIUM 2

314. Si ergo fuerit

$$\sqrt{\frac{p}{q}} > \frac{4cc}{a^2 + 4cc},$$

atque etiam

$$\sqrt{\frac{p}{q}} > \frac{4cc}{b^2 + 4cc},$$

tum situs iste aequilibrii erit stabilis; casibus vero contrariis situs prodibit instabilis et ad subversionem proclivis.

COROLLARIUM 3

315. Si generaliter h retineat valorem eundem, utraque expressio

$$\frac{a^2}{6c} + \tfrac{2}{3}c \text{ et } \frac{bb}{6c} + \tfrac{2}{3}c$$

fit infinite magna tam si $c = 0$ quam si $c = \infty$ minimum igitur valorem induet, si fuerit $a = 2c$ vel etiam $b = 2c$. His igitur casibus stabilitas prodibit minima ceteris paribus.

EXEMPLUM 3

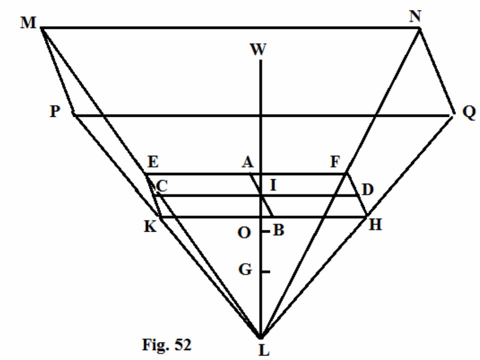


Fig. 52

316. Sit corpus aquae insidens pyramis recta *MNLPQ* (Fig. 52) cuius basis *MNPQ* sit horizontalis et parallelogrammum rectangulum, ; cui ergo sectio aquae *EFHK* erit parallela pariterque parallelogrammum rectangulum. Sit $MN = PQ = a$; $MP = NQ = b$, et altitudo $WL = c$; et centrum gravitatis extet in G, ut sit $LG = h$. Pondus autem huius pyramidis sit M, quod, se habeat ad pondus aequalis voluminis aquae ut p ad q. Iam erit $a : b = A : B$, atque $a^3 : A^3 = q : p$; ita ut sit $A = a\sqrt[3]{\frac{p}{q}}$; et $B = b\sqrt[3]{\frac{p}{q}}$, similiterque

$$LI = c\sqrt[3]{\frac{p}{q}}.$$

Centrum magnitudinis autem partis submersae cadet in O ut sit

$$LO = \tfrac{3}{4}c\sqrt[3]{\frac{p}{q}}, \text{ unde erit } GO = \tfrac{3}{4}c\sqrt[3]{\frac{p}{q}} - h.$$

At volumen partis submersae erit

$$= \frac{ABc}{3}\sqrt[3]{\frac{p}{q}} = \frac{pabc}{3q}.$$

His substitutis erit stabilitas respectu axis

$$CD = M\left(\frac{3}{4}c\sqrt[3]{\frac{p}{q}} - h + \frac{b^2}{4c}\sqrt[3]{\frac{p}{q}}\right).$$

At respectu axis *AB* stabilitas erit

$$= M\left(\frac{3}{4}c\sqrt[3]{\frac{p}{q}} - h + \frac{a^2}{4c}\sqrt[3]{\frac{p}{q}}\right).$$

COROLLARIUM 1

317. Manentibus igitur tam *h* quam ratione *p*:*q* iisdem, stabilitas respectu axis *CD* erit minima, si fuerit $b = c\sqrt{3}$. Respectu axis *AB* vero stabilitas erit
minima, si fuerit $a = c\sqrt{3}$.

COROLLARIUM 2

318. Quo igitur huiusmodi pyramis firmissime situ erecto aquae innatet, in ea conficienda imprimis est evitandum ne sit vel a vel b prope aequale ipsi $c\sqrt{3}$

COROLLARIUM 3

319. Si ista pyramis ex materia uniformi constet tum erit $h = \frac{3}{4}c$
Stabilitas ergo talis pyramidis respectu axis *CD* erit

$$= M\left(\frac{b^2}{4c}\sqrt[3]{\frac{p}{q}} + \frac{3}{4}c\sqrt[3]{\frac{p}{q}} - \frac{3}{4}c\right),$$

at respectu axis *AB* erit stabilitas

$$= M\left(\frac{a^2}{4c}\sqrt[3]{\frac{p}{q}} + \frac{3}{4}c\sqrt[3]{\frac{p}{q}} - \frac{3}{4}c\right).$$

COROLLARIUM 4

320. Quo igitur eiusmodi pyramis aquae firmiter insideat necesse est ut
sit tam

$$\sqrt[3]{\frac{p}{q}} > \frac{3cc}{bb+3cc} \quad \text{quam} \quad \sqrt[3]{\frac{p}{q}} > \frac{3cc}{a^2+3cc}.$$

Si ergo fuerit $a < b$, oportet ut sit

$$\frac{p}{q} > \frac{27 \cdot cc}{\left(a^2+3c^2\right)^3}.$$

COROLLARIUM 5

321. Si fuerit $a = b = c$; talis pyramis situm in figura representatum conservare non poterit nisi sit

$$\frac{p}{q} > \frac{27}{64};$$

hoc est, nisi pyramidis gravitas specifica sit maior quam $421\frac{7}{8}$, posita aquae gravitate specifica $= 1000$.

PROPOSITIO 31

PROBLEMA

322. *Si corporis natantis sectio aquae fuerit rhombus ACBD* (Fig. 53), *determinare eius stabilitatem respectu utriusque diagonalis CD et AB.*

SOLUTIO

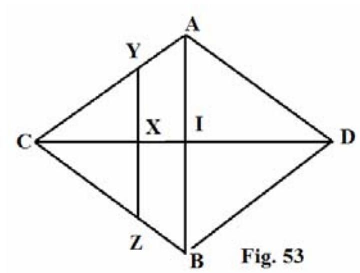


Fig. 53

Consideretur primo axis per centrum gravitatis corporis transiens parallelus diagonali *CD;* ducaturque ordinata quaecunque *YXZ;* atque vocatis
$CI = DI = A$; $AI = BI = B$; $CX = x$; $XY = XZ = y$,
erit

$$A : B = x : y \text{ et } y = \frac{Bx}{A} = z;$$

atque latus rhombi AC erit $\sqrt{\left(A^2 + B^2\right)}$. His positis erit

$\int y^3 dx = \frac{B^3 x^4}{4A^3}$; positoque $x = A$ habebitur valor huius expressionis pro parte $CIA = \frac{1}{4} A \cdot B^3$ qui quater sumtus respondebit toti rhombo *CBDA,* pro quo proinde erit

$$\int \left(y^3 + z^3\right) dx = A \cdot B^3.$$

Si nunc corporis pondus ponatur $= M$; et intervallum centri magnitudinis super centro gravitatis $= GO$ atque volumen partis submersae $= V$, erit stabilitas respectu axis

$$CD = M\left(GO + \frac{A \cdot B^3}{3V}\right).$$

Simili autem modo reperietur stabilitas respectu axis

$$AB = M\left(GO + \frac{A^3 \cdot B}{3V}\right).$$

Ex quibus duabus expressionibus stabilitas respectu cuiusvis alius axis poterit colligi. Q. E. I.

COROLLARIUM 1

323. Si igitur diagonales sunt inaequales, corpus inclinationi circa longiorem minus resistit, quam circa breviorem; quae regula fere in omnibus sectionibus aquae locum habet, ubi axes inter se normales sunt inaequales.

COROLLARIUM 2

324. Quo ergo iste aequilibrii situs sit stabilis, necesse est ut tam

$$GO+\frac{A\cdot B^3}{3V} \quad \text{quam} \quad GO+\frac{A^3\cdot B}{3V}$$

habeat valorem affirmativum, id quod accidit, si tantum minor expressio fuerit affirmativa.

COROLLARIUM 3

325. Si latus rhombi AC ponatur $=C$, atque anguli ACB sinus $=m$; anguli ACB cosinus vero $=n$; erit anguli CAD cosinus $=-n$. Hinc reperitur

$$B=C\sqrt{\frac{1-n}{2}} \quad \text{et} \quad A=\frac{mC}{\sqrt{2}(1-n)}.$$

Quare stabilitas respectu axis CD erit

$$=M\left(GO+\frac{m(1-n)C^4}{12V}\right)$$

respectu axis AB autem

$$=M\left(GO+\frac{m(1+n)C^4}{12V}\right).$$

COROLLARIUM 4

326. Si rhombus abit in quadratum, fiet $m=1$ et $n=0$; hocque casu stabilitas respectu utriusque diagonalis erit eadem scilicet

$$=M\left(GO+\frac{C^4}{12V}\right);$$

quae ipsa expressio quoque inventa est ex praecedente propositione, facta applicatione parallelogrammi ad quadratum.

EXEMPLUM

327. Terminetur pars corporis aquae submersa in recta horizontali *RS* parallela diagonali *CD* (Fig. 54), atque rectis *BL, AL* ad punctum medium *L* rectae *RS* ductis itemque verticalibus *CR* et *DS,* ita ut singulae sectiones horizontales sint rhombi. Maneant

$$CI = DI = A;\ AI = BI = B;$$

sitque

$$CR=LI=DS=D;$$

erit partis submersae volumen $V = ABD$; eiusque centrum magnitudinis in *O* ut sit $LO = \frac{2}{3}D$. Totius vero corporis centrum gravitatis cadat in *G,* dicaturque $LG = h$; erit $GO = \frac{2}{3}D - h$. Ex his igitur reperietur stabilitatis huius aequilibrii situs respectu axis

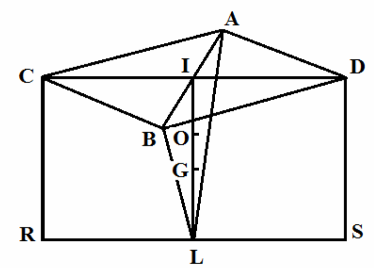


Fig. 54

$$CD = M\left(\frac{2}{3}D - h + \frac{B^2}{3D}\right)$$

At respectu axis *AB* erit stabilitas

$$= M\left(\frac{2}{3}D - h + \frac{A^2}{3D}\right).$$

COROLLARIUM 1

328. Quo igitur iste aequilibrii situs sit stabilis necesse est ut sit

$$h < \frac{A^2 + 2D^2}{3D},$$

simulque etiam

$$h < \frac{B^2 + 2D^2}{3D},$$

Si ergo fuerit $B<A$ sufficiet ad stabilitatem corpori comparandam esse

$$h < \frac{B^2 + 2D^2}{3D}.$$

COROLLARIUM 2

329. Nisi ergo sit $B > D$, necessario centrum gravitatis corporis infra superficiem aquae cadere debet, si quidem situs aequilibrii debeat esse stabilis.

PROPOSITIO 23

PROBLEMA

330. *Si sectio aquae fuerit triangulum isosceles ECF* (Fig. 55), *determinare stabilitatem corporis aquae insidentis tum respectu axis CD tum respectu axis AB ad illum normalis et per centrum gravitatis I sectionis aquae transeuntis.*

SOLUTIO

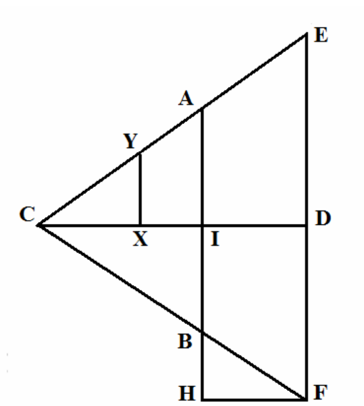


Fig. 55

Positis pondere corporis $= M$, volumine partis submersae $= V$, et distantia inter centra gravitatis corporis et magnitudinis partis submersae $= GO$, sit $CD = A$, et $DE = DF = B$; erit $CX(x) : XY(y) = A : B$, unde sit $y = \frac{Bx}{A}$. Quamobrem habebitur

$$\int y^3 dx = \frac{B^3 x^4}{4A^3} = \frac{A \cdot B^3}{4},$$

posito $x = CD = A$. Pro tota ergo sectione aquae erit $\int (y^3 + z^3) dx = \frac{A \cdot B^3}{2}$, unde fiet stabilitas respectu axis

$$CD = M\left(GO + \frac{A \cdot B^3}{6V}\right).$$

Consideretur nunc axis *AB,* in quo est

$$AI = BI = \frac{2}{3}B \text{ et } CI = \frac{2}{3}A;$$

eritque $\int y^3 dx$, ortum ex area

$$ACB = \frac{AI \cdot CI^3}{4} = \frac{4A^3 \cdot B}{81};$$

ergo eadem formula ex toto triangulo *ACB* orta erit $\frac{4A^3 \cdot B}{81}$. Nunc ex altera parte consideretur area tota *IDFH,* quae est rectangulum, existente

$$IH = DF = B \text{ et } DI = FH = \frac{1}{3}A;$$

prodibitque ex ea.

$$\int y^3 dx = DI^3 \cdot IH = \frac{A^3 \cdot B}{27}.$$

A quo valore subtrahi debet is qui oritur ex triangulo *BFH* qui est

$$= \frac{BH \cdot FG^3}{4} = \frac{A^3 \cdot B}{4 \cdot 81},$$

et relinquetur valor ipsius $\int y^3 dx$ pro trapezio $IDBF = \frac{11A^3B}{4 \cdot 81}$.

Trapezio ergo *ABFE* respondebit valor ipsius

$$\int y^3 dx = \frac{11A^3 \cdot B}{2 \cdot 81}.$$

Quocirca respectu axis *AB* erit totalis valor ipsius

$$\int \left(y^3 + z^3 \right) dx = \frac{8A^3 \cdot B}{81} + \frac{11A^3 \cdot B}{2 \cdot 81} = \frac{A^3 \cdot B}{6}.$$

Ex quo erit stabilitas huius aequilibrii situs respectu axis

$$AB = M\left(GO + \frac{A^3 \cdot B}{18V} \right).$$

Q.E.I.

COROLLARIUM 1

331. Stabilitas igitur respectu axis *CD* maior erit, quam stabilitas respectu axis *AB* si fuerit

$$A \cdot B^3 > \frac{A^3 \cdot B}{3},$$

hoc est si fuerit

$$\frac{B}{A} > \frac{1}{\sqrt{3}}.$$

Contra vero si fuerit

$$\frac{B}{A} < \frac{1}{\sqrt{3}}$$

tum stabilitas respectu axis *AB* maior erit quam respectu axis *CD.*

COROLLARIUM 2.

332. Quia $\frac{B}{A}$ est tangens anguli *DCE* seu tangens dimidii anguli *ECF,* manifestum est si fuerit angulus *ECF* maior quam 60°, tum stabilitatem respectu axis *CD* excedere stabilitatem respectu axis *AB;* contrarium vero evenire, si angulus *ECF* minor sit quam 60°.

COROLLARIUM 3

333. Si ergo triangulum *ECF* fit aequilaterum, tum stabilitas respectu utriusque axis erit eadem. Sed ob $A = B\sqrt{3}$ erit stabilitas hoc casu

$$AB = M\left(GO + \frac{A^4}{2V\sqrt{3}}\right),$$

quae pro omnibus reliquis axibus valebit.

COROLLARIUM 4

334. Si trianguli aequilateri area ponatur $= E$ erit $B^2\sqrt{3} = E;$ unde stabilitas situs erit aequilibrii

$$= M\left(GO + \frac{E^2}{6V\sqrt{3}}\right).$$

COROLLARIUM 5

335. At si sectio aquae est quadratum cuius area sit pariter $= E$, tum ex supra inventis stabilitas erit

$$= M\left(GO + \frac{E^2}{12\cdot V}\right).$$

Quare cum sit

$$\frac{1}{6\sqrt{3}} > \frac{1}{12},$$

sequitur sectionem aquae quae est triangulum aequilaterum stabiliorem situm producere quam quadratum eiusdem areae ceteris paribus.

EXEMPLUM 1

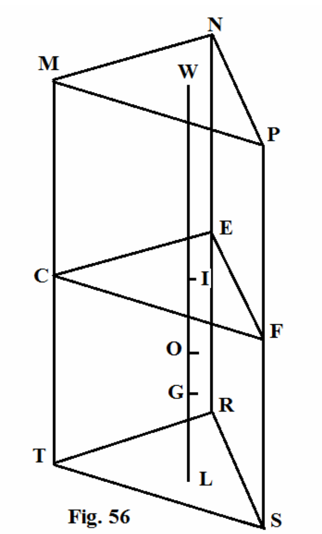


Fig. 56

336. Sit corpus aquae innatans prisma triangulare *MNPTRB* (Fig. 56), cuius sectiones horizontales sint triangula aequilatera *MNP, CEF, TRS,* quorum latera sint $= a$; area vero $= bb$, seu $bb = \frac{aa\sqrt{3}}{4}$. Ponatur pondus huius prismatis $= M$, eiusque gravitas specifica ad aquam ut *p* ad *q,* atque tota altitudo $MT = WL = c$. Cum nunc *CEF* sit sectio aquae, erit $CT = \frac{pc}{q}$, atque $LO = \frac{pc}{2q}$; volumen vero partis submersae

$$V = \frac{pb^2c}{2q}.$$

Totius porro prismatis centrum gravitatis sit in *G,* existente $LG = h$; erit

$$\mathrm{GO} = \frac{pc}{2q} - h.$$

Ex his igitur fiet stabilitas huius aequilibrii situs respectu cuiusque axis

$$= M\left(\frac{pc}{2q} - h + \frac{qb^2}{6pc\sqrt{3}}\right) = M\left(\frac{pc}{2q} - h + \frac{qa^2}{24pc}\right).$$

COROLLARIUM 1

337. Si prisma ex materia uniformi fuerit consectum, erit $h = \frac{1}{2}c$, atque stabilitas huius aequilibrii situs erit

$$= M\left(\frac{qb^2}{6pc\sqrt{3}} - \frac{(q-p)c}{2q}\right) = M\left(\frac{qa^2}{24pc} - \frac{(q-p)c}{2q}\right).$$

COROLLARIUM 2

338. Quo ergo iste situs aequilibrii sit stabilis oportet ut sit

$$c < \frac{qb}{\sqrt{3}p(q-p)\sqrt{3}}$$

sive quod eodem redit

$$c < \frac{qa}{2\sqrt{3}p(q-p)}.$$

Hinc igitur innotescit, quam longa pars a prismate triangulari indefinitae longitudinis debeat abscindi, ut situ erecto aquae innatare queat.

COROLLARIUM 3

339. Si ex eadem materia prisma quadrangulare conficiatur, cuius bases sint quadrata $= bb$, longitudo eorum c minor esse debet quam $\frac{qb}{\sqrt{6}p(q-p)}$ quo situ erecto aquae innatare possint. Longiora igitur in hunc finem licebit accipere prismata triangularia, quam quadrata.

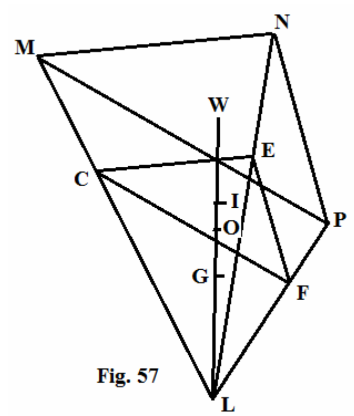


Fig. 57

EXEMPLUM 2

340. Sit corpus aquae innatans pyramis triangularis *MNPL* (Fig. 57), cuius basis *MNP* horizontaliter extra aquam emineat. Ponatur basis *MNP* quae sit triangulum aequilaterum, latus quodlibet $= a$, basisque eiusdem $= bb$ ita ut sit $b^2 = \frac{a^2\sqrt{3}}{4}$. Pyramidis porro altitudo *WL* sit, eiusque $= c$ pondus *M* se habeat ad pondus aequalis voluminis aquae ut p ad q; sitque *CFE* sectio aquae quae pariter erit triangulum aequilaterum, cuius area sit $= E$. Iam erit

$$q : p = b^3 : E\sqrt{E} \text{ seu } \sqrt{E} = b\sqrt[3]{\frac{p}{q}} \text{ et } E = b^2\sqrt[3]{\frac{p^2}{q^2}}.$$

Similique modo erit

$$LI = c\sqrt[3]{\frac{p}{q}} \text{ et } LO = \tfrac{3}{4}c\sqrt[3]{\frac{p}{q}}.$$

Volumen autem partis submersae V erit $= \frac{pb^2c}{3q}$. Sit denique $LG = h$; erit stabilitas huius aequilibrii situa, quem pyramis proposita tenet

$$= M\left(\tfrac{3}{4}c\sqrt[3]{\frac{p}{q}} - h + \frac{bb}{2c\sqrt{3}}\sqrt[3]{\frac{p}{q}}\right) = M\left(\tfrac{3}{4}c\sqrt[3]{\frac{p}{q}} - h + \frac{a^2}{8c}\sqrt[3]{\frac{p}{q}}\right).$$

COROLLARIUM 1

341. Si pyramis ex materia uniformi constet, erit $h = \frac{3}{4}c$. Hoc igitur casu habebitur stabilitas istius aequilibrii situs

$$= M\left(\frac{bb}{2c\sqrt{3}}\sqrt[3]{\frac{p}{q}} - \tfrac{3}{4}c + \tfrac{3}{4}c\sqrt[3]{\frac{p}{q}}\right) = M\left(\frac{aa}{8c}\sqrt[3]{\frac{p}{q}} - \tfrac{3}{4}c + \tfrac{3}{4}c\sqrt[3]{\frac{p}{q}}\right).$$

COROLLARY 2

342. Si pyramis insuper abeat in tetraedron seu pyramidem regularem, $c = a\sqrt{\frac{2}{3}}$; stabilitas igitur tetraedri angulo deorsum verso aquae insidentis erit

$$= Ma\left(\frac{\sqrt{3}}{8\sqrt{2}}\sqrt[3]{\frac{p}{q}} - \frac{\sqrt{3}}{2\sqrt{2}} + \frac{\sqrt{3}}{2\sqrt{2}}\sqrt[3]{\frac{p}{q}}\right) = \frac{Ma\sqrt{3}}{8\sqrt{2}}\left(5\sqrt[3]{\frac{p}{q}} - 4\right).$$

COROLLARIUM 3

343. Quo ergo huiusmodi tetraedron in aqua talem situm aequilibrii servare queat, necesse est ut sit

$$\sqrt[3]{\frac{p}{q}} > \tfrac{4}{5} \text{ seu } \frac{p}{q} > \frac{64}{125}.$$

Eius igitur gravitas specifica maior esse debet quam 512; posita aquae gravitate specifica = 1000 .

SCHOLION

344. Evolui hactenus eiusmodi sectiones aquae quae sunt figurae rectilineae, atque tres casus tractati sufficere possunt ad nostrum institutum. Progrediar itaque ad figuras curvilineas, ex iisque praecipuas, quae facillime experimentis comprobari queant, faciam sectiones aquae, ut de plurimis corporibus inde iudicari queat, quemnam situm aquae imposita sint habitura, et quanta stabilitate in quoque aequilibrii situ persistant.

PROPOSITIO 33

PROBLEMA

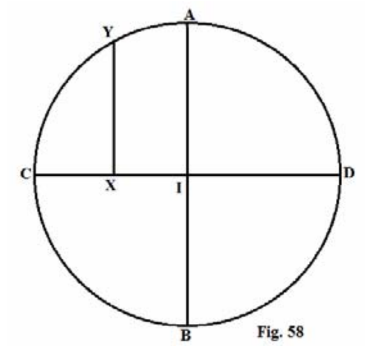


Fig. 58

345. *Si corporis aquae in aequilibrio insidentis sectio aquae fuerit circulus ACBD* (Fig. 58), *determinare stabilitatem respectu cuiuscunque axis GD, quia ubique stabilitas est eadem, qua iste aequilibrii status gaudebit.*

SOLUTIO

Ponatur radius circuli $CI = a$, et ducta in quadrante *CIA* quacunque applicata *XY* vocetur

$$IX = x \text{ et } XY = y \text{ erit } \quad y = \sqrt{\left(a^2 - x^2\right)},$$

unde fiet

$$\int y^3 dx = \int dx\left(a^2 - x^2\right)^{\frac{3}{2}}.$$

At per reductionem formularum integralium ad simpliciores fit

$$\int dx\left(a^2 - x^2\right)^{\frac{3}{2}} = x\left(a^2 - x^2\right)^{\frac{3}{2}} + \frac{3a^2 x\sqrt{\left(a^2 - x^2\right)}}{8} + \frac{3a^4}{8}\int \frac{dx}{\sqrt{\left(a^2 - x^2\right)}}.$$

Ponatur $x = a$; dabit

$$\int \frac{dx}{\sqrt{\left(a^2 - x^2\right)}} = \tfrac{1}{2}\pi$$

posita $\pi : 1$ ratione peripheriae ad diametrum. Quo facto pro quadrante *CIA* habebitur

$$\int y^3 dx = \frac{3\pi a^4}{16},$$

adeoque pro toto circulo erit

$$\int \left(y^3 + z^3\right) dx = \frac{3\pi a^4}{4}.$$

Si nunc corporis pondus sit $= M$, et volumen partis aquae submersae $= V$, atque *GO* denotet intervallum inter centra gravitatis et magnitudinis, erit stabilitas respectu cuiusvis axis

$$= M\left(GO + \frac{\pi a^4}{4V}\right).$$

Q.E.I.

COROLLARIUM 1

346. Quia diameter se habet ad peripheriam ut 1 ad π, exprimet πa^2 aream circuli. Si ergo area circuli ponatur $= bb$, erit $a^2 = \frac{bb}{\pi}$ atque stabilitas ita exprimetur ut sit

$$= M\left(GO + \frac{b^4}{4\pi V}\right).$$

COROLLARIUM 2

347. Si sectio aquae est quadratum areae $= b^2$, tum stabilitas inventa est

$$= M\left(GO + \frac{b^4}{12V}\right).$$

et si sectio aquae est triangulum aequilaterum, cuius area itidem est $= bb$, tum stabilitas erat

$$= M\left(GO + \frac{b^4}{6V\sqrt{3}}\right).$$

Unde intelligitur stabilitatem circuli esse minimam, trianguli vero maximam.

COROLLARIUM 3

348. Colligere ergo hinc licet, si sectio aquae fuerit polygonum regulare, stabilitatem prodituram esse eo minorem, quo plura latera polygonum contineat, ceteris scilicet paribus.

COROLLARIUM 4

349. Ad maximam igitur corpori aquae innatanti stabilitatem respectu omnium axium conciliandam, conveniet corpori eiusmodi dare figuram, ut sectio aquae fiat triangulum aequilaterum.

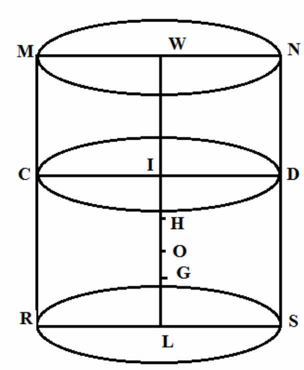


Fig. 59

EXEMPLUM 1

350. Sit corpus aquae innatans cylindrus rectus *MNRS* (Fig. 59), in aqua erectus, cuius sectiones horizontales sint circuli *MN, CD,* et *RS,* aequales, quorum radius sit $= a$. Pondus autem huius cylindri sit *M,* quod se habeat ad pondus aequalis voluminis aquae ut *p* ad *q*. Quare posita totius cylindri altitudine $WL = c$ erit altitudo partis aquae submersae

$$IL = \frac{pc}{q},$$

atque volumen partis submersae

$$V = \frac{\pi p a^2 c}{q},$$

cuius centrum magnitudinis cadet in O ut sit

$$LO = \frac{pc}{2q}.$$

Sit autem centrum gravitatis totius corporis in G, existente $LG = h$. His igitur substitutis invenietur stabilitas cylindri in isto erecto aequilibrii situ

$$= M\left(\frac{pc}{2q} - h + \frac{qa^2}{4pc}\right).$$

COROLLARIUM 1

351. Cylindrus igitur in tali situ firmiter perseverabit si fuerit

$$h < \frac{pc}{2q} + \frac{qa^2}{4pc}.$$

Hoc est si fiat $LI : \frac{1}{2}CI = \frac{1}{2}CI : OH$, tumque punctum G infra punctum H cadat.

COROLLARIUM 2

352. Si cylindrus ex materia uniformi constet, erit $h = \frac{c}{2}$; hoc igitur casu stabilitas

$$= M\left(\frac{qa^2}{4pc} - \frac{(q-p)c}{2q}\right).$$

Quo igitur hic situs sit stabilis, necesse est ut sit

$$c = \frac{qa}{\sqrt{2}p(q-p)}.$$

COROLLARIUM 3

353. Si totus cylindrus fuerit datus, ex gravitate specifica cognoscetur an situ erecto natare queat. Natabit enim si fuerit $\frac{p}{q}$ vel maius quam $\frac{c+\sqrt{(cc-2aa)}}{2c}$ vel minus quam $\frac{c-\sqrt{(cc-2aa)}}{2c}$.

COROLLARIUM 4

354. Perspicitur ergo si fuerit $c < a\sqrt{2}$, tum cylindrum semper situ erecto esse nataturum, quaecunque fuerit ratio gravitatum specificarum.

EXEMPLUM 2

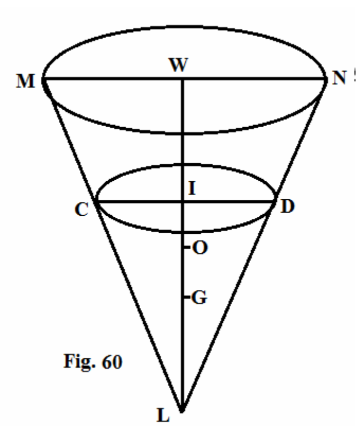


Fig. 60

355. Sit corpus conus rectus *MLN* (Fig. 60) vertice deorsum verso aquae innatans, cuius basis radius $WM = WN = a$; et altitudo $WL = c$. Sit eius gravitas specifica ad aquam ut p ad q; erit sectionis aquae *CD* radius

$$IC = a\sqrt[3]{\frac{p}{q}}, \text{ et } IL = c\sqrt[3]{\frac{p}{q}}.$$

Atque cum basis *MN* area sit $= \pi a^2$, erit area sectionis aquae

$$= \pi a^2 \sqrt[3]{\frac{p^2}{q^2}},$$

unde volumen partis submersae V erit $= \dfrac{\pi pa^2 c}{3q}$, atque

$$LO = \frac{3}{4}c\sqrt[3]{\frac{p}{q}}.$$

Posito nunc $LG = h$, erit stabilitas huius situs aequilibrii

$$= M\left(\tfrac{3}{4}c\sqrt[3]{\frac{p}{q}} - h + \frac{3q \cdot CI^4}{4pa^2c}\right) = M\left(\tfrac{3}{4}c\sqrt[3]{\frac{p}{q}} - h + \frac{3a^2}{4c}\sqrt[3]{\frac{p}{q}}\right).$$

COROLLARIUM 1

356. Si ergo conus ex materia homogenea constet, erit $h = \tfrac{3}{4}c$; hoc ergo casu stabilitas erit

$$= \frac{3}{4}M\left(\frac{\left(a^2 + c^2\right)}{c}\sqrt[3]{\frac{p}{q}} - c\right).$$

COROLLARIUM 2

357. Quo ergo iste situs aequilibrii sit stabilis necesse est ut sit

$$\frac{p}{q} > \frac{c^6}{\left(a^2+c^2\right)^3}.$$

Quod nisi fuerit, conus alium quaeret situm, quo aquae innatet.

PROPOSITIO 34

PROBLEMA

358. *Sit corporis pars aquae submersa CMLMD solidum rotundum, genitum conversione figurae LMC circa axem verticalem LI* (Fig. 61), *atque sectio aquae sit circulus CD seu suprema solidi rotundi sectio horizontalis. Determinare huius corporis aquae insidentis stabilitatem.*

SOLUTIO

Sit sectionis aquae semidiameter $IC = a$, atque longitudo axis $IL = c$, in quo positum sit tum centrum gravitatis totius corporis *G,* tum centrum magnitudinis partis submersae *O.* Positis nunc pondere corporis $= M$ et volumine partis submersae $= V$, erit stabilitas

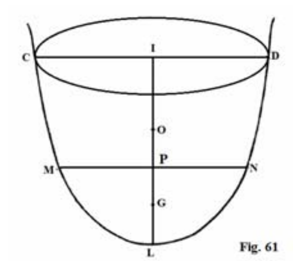


Fig. 61

$= M\left(GO + \frac{\pi a^4}{4V}\right)$, denotante π peripheriam circuli, cuius diameter est $= 1$. At tam volumen *V* quam punctum *O* ex natura curvae *CML* determinari oportet: Ad quod praestandum vocetur abscissa $LP = x$, respondensque applicata $PM = y$, et habebitur volumen solidi ex conversione partis *MLP* orti $\pi\int y^2dx$, in quo integrali si ponatur $x = c$, quo casu fiet $y = a$, prodibit totum partis submersae volumen *V.* Integrali ergo extenso per totam figuram *LMC* erit $V = \pi\int y^2dx$. Simili vero modo reperietur positio centri magnitudinis *O* partis submersae erit scilicet

$$LO = \frac{\int y^2xdx}{\int y^2dx},$$

utroque integrali usque ad sectionem aquae extenso. Si ergo ponatur $LG = h$, quippe quod intervallum non a natura curvae *CML,* sed ab indole totius corporis pendet, erit stabilitas istius situs aequilibrii

$$= M\left(\frac{\int y^2xdx}{\int y^2dx} - h + \frac{a^4}{4\int y^2dx}\right) = M\left(\frac{a^4 + 4\int y^2xdx}{4\int y^2dx} - h\right).$$

Q.E.I.

COROLLARIUM 1

359. Ad stabilitatem ergo huiusmodi corporum inveniendam, duplex integratio est instituenda; integrari enim debent hae duae formulae differentiales y^2dx et y^2xdx.

COROLLARIUM 2

360. Quoties igitur hae duae formulae algebraicam admittunt integrationem, toties stabilitas algebraice exprimi poterit. Ad quadraturas curvarum autem erit confugiendum, si vel alterutra vel utraque integrari nequeat.

SCHOLION

361. Ex aequatione autem, quae habebitur inter x et y, qua curvae LMC natura exprimitur colligetur, utrum formulae y^2dx et y^2xdx. sint algebraice integrabiles, an a quadraturis pendeant. Hic autem conveniet omnes aequationes algebraicas inter x et y indicari, quae utramque formulam reddant algebraice integrabilem, quo generatim intelligatur, quaenam curvae algebraicae pro curva generatrice LMC assumtae producant stabilitatem algebraice expressam. Ad hoc igitur investigandum assumo duas quascunque quantitates algebraicas P et Q, quarum vel altera alterius sit functio algebraica, vel ambae functiones algebraicae tertiae cuiusdam variabilis puta z; et facio

$$\int yydx = P \text{ et } \int yyxdx = Q.$$

Ex his igitur erit

$$y^2 = \frac{dP}{dx} = \frac{dQ}{xdx},$$

unde reperitur

$$x = \frac{dQ}{dP}, \text{ atque } y^2 = \frac{dP^3}{dPddQ - dQddP},$$

qui sunt generales valores algebraici pro x et y, qui primo praebebunt aequationem inter x et y algebraicam, et deinde stabilitatem producent algebraice expressam, quippe quae erit

$$= M\left(\frac{a^4 + 4Q}{4P} - h\right).$$

Sed stabilitatem in solutione problematis generaliter inventam expediet exemplis nonnullis illustrare.

EXEMPLUM

362. Sit corporis pars aquae immersa *CLD* portio sphaerae, cuius radius sit *b;* erit $b-c=\sqrt{(bb-aa)}$, hincque $b=\frac{aa+cc}{2c}$.
Cum igitur *LMC* sit arcus circuli radii *b,* erit $y^2=2bx-xx$, ideoque

$$\int y^2dx=bxx-\frac{1}{3}x^3.$$

Facto ergo $x=c$, erit

$$\int yydx=bcc-\frac{1}{3}c^3=\frac{c\left(3aa+cc\right)}{6}.$$

Deinde habebitur

$$\int yyxdx=\frac{2bx^3}{3}-\tfrac{1}{4}x^4=\frac{2bc^3}{3}-\tfrac{1}{4}c^4$$

posito $x=c$, substituto autem loco *b* valore per *a* et *c* definito erit

$$\int yyxdx=\frac{cc\left(4aa+cc\right)}{12}.$$

His igitur integralibus inventis prodibit stabilitas quaesita

$$=M\left(\frac{3a^4+4aacc+c^4}{2c(3aa+cc)}-h\right)=M\left(\frac{aa+cc}{2c}-h\right)=M\left(b-h\right).$$

COROLLARIUM 1

363. Cum stabilitas sit inventa $=M(b-h)$, erit ea proportionalis intervallo, quo centrum gravitatis *G* infra centrum sphaerae cadit. Eiusmodi igitur corpus firmiter suum situm tenebit, si centrum gravitatis infra sphaerae, cuius pars submersa est portio, centrum cadat.

COROLLARIUM 2

364. Sin autem centrum gravitatis *G* in ipsum sphaerae centrum cadat, tum situs aequilibrii erit indifferens, id quod accidit in globis ex materia uniformi confectis; qui

aquae insidentes omnes situs habebunt aequilibrii proprietate gaudentes, nullum autem neque stabilem, neque instabilem sed omnes indifferentes.

EXEMPLUM 2

365. Sit curva *LMC* parabola cuiuscunque ordinis, scilicet

$$y^m = b^{m-n}\ x^n.$$

Erit ergo

$$a^m = b^{m-n}c^n,\ b = \frac{a^{\frac{m}{m-n}}}{c^{\frac{n}{m-n}}}.$$

Cum autem sit

$$y^2 = b^{\frac{2m-2n}{m}}\ x^{\frac{2n}{m}},\ \text{erit}\ \int y^2 dx = \frac{mb^{\frac{2m-2n}{m}}\ x^{\frac{2n+m}{m}}}{2n+m} = \frac{mb^{\frac{2m-2n}{m}}\ c^{\frac{2n+m}{m}}}{2n+m}$$

posito c loco x, quo integrale ad sectionem aquae *CD* usque pertingat. Simili autem modo erit

$$\int yyxdx = \frac{mb^{\frac{2m-2n}{m}}\ c^{\frac{2n+2m}{m}}}{2m+2n}.$$

Ex quibus integralibus obtinebitur stabilitas quaesita

$$= M\left(\frac{a^4 + \frac{2m}{m+n} b^{\frac{2m-2n}{m}}\ c^{\frac{2n+2m}{m}}}{\frac{4m}{2n+m} b^{\frac{2m-2n}{m}}\ c^{\frac{2n+2m}{m}}} - h\right).$$

Substituto autem loco b valore assignato

$$\frac{a^{\frac{m}{m-n}}}{c^{\frac{n}{m-n}}}\ \text{ex quo sit}\ b^{\frac{2m-2n}{m}} = a^2 c^{\frac{-2n}{m}},$$

prodibit stabilitas

$$= M\left(\frac{a^2 + \frac{2m}{m+n} c^2}{\frac{4m}{2n+m} c} - h\right).$$

COROLLARIUM 1

366. Stabilitas igitur haec, si in formula eliminetur *a* eiusque loco introducatur *b* hoc modo exprimi potest, ut sit

$$= M\left(\frac{2n+m}{2n+2m}c + \frac{2n+m}{4m}b^{\frac{2m-2n}{m}}c^{\frac{2n-m}{m}} - h\right).$$

Ex qua formula datis *b* et *c,* radius sectionis aquae sponte determinatur.

COROLLARIUM 2

367. Manifestum autem est ex istis expressionibus stabilitatem eo fore maiorem, quo minor fuerit fractio $\frac{m}{n}$. Si enim *m* esset $=0,$ tum stabilitas prodiret infinite magna, nec hic autem casus nec alii finitimi in rerum natura locum inveniunt .

COROLLARIUM 3

368. Si maneat altitudo *h* eiusdem quantitatis, stabilitas fiet infinita sive sit $c=0$ sive $c=\infty$, minima ergo erit stabilitas si fuerit

$$aa = \frac{2m}{m+n}cc \text{ sive } c = a\sqrt{\frac{m+n}{2m}}.$$

In casu ergo parabolae conicae, quo $m=2,\ n=1$ stabilitas erit minima si sit $c = a\sqrt{\tfrac{3}{4}}$

PROPOSITIO 35

PROBLEMA

369. *Si corporis aquae insidentis in aequilibrio sectio aquae fuerit ellipsis ACBD* (Fig. 62), *determinare stabilitatem huius aequilibrii situs respectu utriusque axis maioris CD et minoris AB.*

SOLUTIO

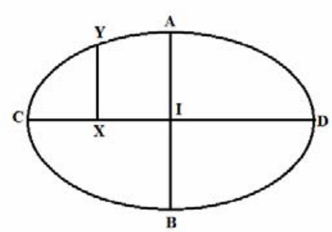


Fig. 62

Ponatur semiaxis maior $CI = a$; semiaxis minor $IA = b$; erit posita abscissa $IX = x$, et applicata $XY = y$, inter *x* et *y* haec aequatio

$y=\frac{b}{a}\sqrt{\left(a^2-x^2\right)}.$
Hinc igitur fiet

$$\int y^3dx=\frac{b^3}{a^3}\int dx\left(a^2-x^2\right)^{\frac{3}{2}},$$

quod integrale posito $x=a$, et denotantae π peripheriam circuli cuius diameter est $=1$, abibit in $\frac{3\pi ab^3}{16}$ quod proinde quater sumtum dabit pro tota sectione aquae

$$\int\left(y^3+z^3\right)dx=\frac{3\pi ab^3}{4}.$$

Si nunc pondus corporis sit $=M$, volumen partis submersae *V,* atque *GO* indicet intervallum inter centra gravitatis et magnitudinis, erit stabilitas corporis respectu axis

$$CD=\left(GO+\frac{3\pi ab^3}{4}\right).$$

Commutatis autem inter se semiaxibus *a* et *b* prodibit stabilitas respectu axis minoris Q.E.I.

COROLLARIUM

370. Stabilitas igitur, qua corpus inclinationi circa axem maiorem resistit, minor est quam stabilitas respectu axis minoris. Quare si situs fuerit stabilis respectu axis maioris, eo stabilior erit respectu axis minoris.

COROLLARIUM 2

371. Tota ellipsis area est πab; si ergo ponatur area ellipsis $=E$, erit stabilitas respectu axis maioris

$$=M\left(GO+\frac{Eb^2}{4V}\right),$$

stabilitas vero respectu axis minoris *AB* erit

$$CD=\left(GO+\frac{Ea^2}{4V}\right).$$

SCHOLION

372. Superfluum fore arbitror hanc propositionem exemplis illustrare, cum superiora exempla pro circulo data huc facillime possint accomodari, atque insuper parum commodi tam ad experimenta instituenda, quam ad pleniorem intelligentiam sequentium derivari queat. Quamobrem missis his, quibus sectio aquae praecipue spectatur, ad varias figuras ipsius partis submersae considerandas progrediar, ubi per calculum sum inquisiturus tum in ipsum partis submersae volumen, tum etiam in eius centrum magnitudinis, quippe quae res praeter sectionem aquae imprimis ad stabilitatem cognoscendam inserviunt. Eiusmodi autem conformationes partis submersae prae aliis sum contemplaturus, quae quandam habeant similitudinem cum navibus reliquisque vasis, quae ad motum super aqua adhiberi solent, quo inde non contemnenda commoda ad navigationem solide tractandam consequantur. Figuram igitur partis submersae infra terminatam ponam linea recta horizontali, quae in navibus spina dici consuevit, et ad quam omnes sectiones transversales verticaliter factae finiuntur. His autem sectionibus transversalibus, quae sunt verticales et ad spinam normales figura partis aquae submersae determinatur. Quamobrem quomodo tum ex sectione aquae, tum ex huius modi sectionibus transversalibus stabilitatem definiri oporteat, docebo.

PROPOSITIO 36

PROBLEMA

373. *Si sectio aquae fuerit curva quaecunque ANBMA diametro AB praedita* (Fig. 63), *pars vero submersa terminetur tum infra spina horizontali EF sub axe AB posita, tum ad latera parabolis conicis MQ vertices in M et axes horizontales ad AB normales habentibus; invenire stabilitatem corporis talem aequilibrii situm in aqua tenentis, respectu axis AB.*

SOLUTIO

Consideretur (Fig. 64) sectio partis submersae quaecunque *MQM* verticalis

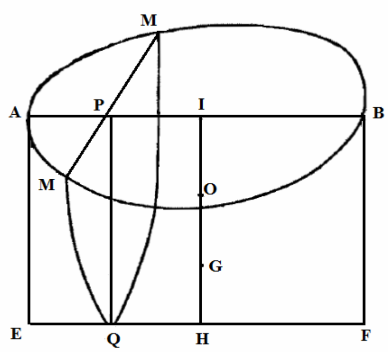


Fig. 63

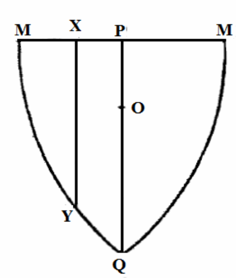


Fig. 64

et ad diametrum AB normalis voceturque abscissa
$AP = x,\ MP = MP = y,$ et profunditas constans $PQ = AE = c$. Iam seorsim contemplemur sectionem MQM, in qua curva MQ et MQ sunt parabolae Appollonianae vertices in M et axem MM communem habentes. Cum nunc fit $PM = y$ et $PQ = c$, erit parameter utriusque parabolae $= \frac{c^2}{y}$. Quare si dicatur

$$MX = t \text{ et } XY = u, \text{ erit } u^2 = \frac{c^2 t}{y},$$

et area

$$MXY = \tfrac{2}{3} tu = \frac{2ct}{3}\sqrt{\frac{t}{y}},$$

unde area tota MQM posito $t = y$ fiet $= \tfrac{4}{3} cy.$
Centrum gravitatis autem o areae MQM reperietur sumendo integrale $\int \tfrac{1}{2} uudt$ idque dividende per $\int udt$; est vero $\int \tfrac{1}{2} uudt = \frac{cct}{4y}$, quod divisum per $\int udt$ $= \frac{2ct}{3}\sqrt{\frac{t}{y}}$, dat $\frac{3c}{8}\sqrt{\frac{t}{y}}$, ita ut posito $t = y$ futurum sit $Po = \tfrac{3}{8} c.$ Cum igitur omnium sectionum eiusmodi MQM centrum gravitatis in eandem a diametro AB distantiam cadat, totius partis submersae centrum magnitudinis situm erit in O ut sit $IO = \tfrac{3}{8} c.$
Multiplicetur porro sectionis MQM area $\tfrac{4}{3} cy$ per dx, atque integrale $\tfrac{4}{3} c \int ydx = \tfrac{2}{3} c \cdot MAM$ MAM dabit soliditatem $AEQMM$, quamobrem si area totius sectionis aquae $AMBM$ ponatur $= E$, erit soliditas partis submersae $\tfrac{2}{3} Ec.$ Ponatur nunc pondus totius corporis $= M$, sitque eius centrum gravitatis G in recta verticali IH per centrum magnitudinis O ducta, erit stabilitas huius aequilibrii situs respectu axis

$$AB = M\left(IG - \tfrac{3}{8} c + \frac{\int y^3 dx}{Ec} \right).$$

Est enim propositione 29 huc traducta $z = y$, atque $V = \tfrac{2}{3} Ec.$ Posito ergo $IG = h$, erit stabilitas quaesita

$$= M\left(h - \tfrac{3}{8} c + \frac{\int y^3 dx}{Ec} \right).$$

Q.E.I.

COROLLARIUM 1

374. Hinc etiam distantia rectae verticalis *IH* a puncto *A* invenietur sumendo integrale ipsius $\frac{4}{3}cyxdx$, idque dividendo per $\int\frac{4}{3}cydx$ ita ut futurum sit

$$AI = \frac{\int yxdx}{\int ydx}.$$

COROLLARIUM 2

375. Ex hac igitur formula perspicuum est rectam *HI* per ipsum centrum gravitatis sectionis aquae *I* esse transituram, ita ut hoc casu tria centra gravitatia scilicet totius corporis, partis submersae, et sectionis aquae in eadem recta verticali sint sita.

COROLLARIUM 3

376. Data ergo pro huiusmodi corporibus sectione aquae, ex qua tam eius area *E* quam $\int y^3dx$ innotescat, stabilitas situs aequilibrii facile definiri poterit.

COROLLARIUM 4

377. Quia ergo in tali corpore centrum gravitatis sectionis aquae *I* verticaliter imminet centro gravitatis *G* totius corporis, inter oscillandum centrum gravitatis neque ascendet neque descendet, et hancobrem motus oscillatorius erit maxime tranquillus.

SCHOLION

378. Satis igitur idonea est haec forma parabolica, quae sectionibus navium transversalibus tribuatur, cum per eas id commodi acquiratur, ut et centrum magnitudinis partis submersae in eandem rectam verticalem incidat, et centrum gravitatis sectionis aquae. Hinc enim evenit, uti supra vidimus, ut dum oscillationes a nave peraguntur, modo sint minimae, centrum gravitatis in quiete permaneat, quod plurimum invat ad istum motum maxime tranquillum efficiendum. Non solum autem figura parabolica ad hunc effectum producendum est accommodata, sed praeterea omnes parabolae cuiusque ordinis idem praestant, · innumerabilesque aliae curvae, quae ita sunt comparatae ut areae earum *MQM* proportionales sint ipsis ordinatis *MPM* sectionis aquae; siquidem spina corporis aquae innatantis est horizontalis. At si tota spina non est linea recta, sed vel tota curva, vel tantum ad proram puppimque sursum erecta, tum peculiaribus opus est curvis ad idem commodum obtinendum. Quamobrem primo parabolas superiorum

graduum pro casu, quo tota spina est recta horizontalis, evolvam, ac deinde curvas idoneas ad spinas non rectas investigabo.

PROPOSITIO 37

PROBLEMA

379. *Si sectio aquae fuerit curva quaecunque AMBMA praedita diametroAB sub qua in plano verticali existat spina recta horizontalis EF* (Fig. 63), *ad quam terminetur pars corporis aquae immersa parabolis cuiusvis ordinis MQ, vertices in M habentibus, axesque horizontales MM; determinare stabilitatem respectu axis AB.*

SOLUTIO

Positis ut ante $AP = x$, $PM = y$ et $AE = PQ = c$, consideretur sectio transversalis *MQM* seorsim (Fig. 64), in qua sumta abscissa *MX* sit $= t$ et applicata $XY = u$; natura vero huius parabolae exprimatur hac aequatione

$$u = \frac{t^n}{p^{n-1}},$$

existente p parametro. Quia autem facto

$$t = MP = y, \text{ sit } u = PQ = c$$

erit

$$c = \frac{y^n}{p^{n-1}} \text{ atque } p^{n-1} = \frac{y^n}{c}.$$

Area autem *MXY* erit

$$= \frac{t^{n+1}}{(n+1)p^{n-1}} = \frac{ct^{n+1}}{(n+1)y^n}$$

unde totius sectionis *MQM* prodibit area $= \frac{2cy}{(n+1)}$. Deinde huius sectionis centrum gravitatis situm erit in o ut sit

$$Po = \frac{\int uudt}{2\int udt}$$

posito post integrationem $t = y$. At est

$$\int u^2 dt = \frac{\int t^{2n} dt}{p^{2n-2}} = \frac{t^{2n+1}}{(2n+1)p^{2n-2}} = \frac{c^2 t^{2n+1}}{(2n+1)y^{2n}} = \frac{c^2 y}{2n+1}$$

posito $t = y$. Quare cum sit

$$2\int udt = \frac{2cy}{n+1}, \text{ erit } Po = \frac{(n+1)c}{2(2n+1)}.$$

Cum igitur omnium sectionum transversalium centra gravitatis in eandem rectam horizontalem cadant, partis submersae centrum magnitudinis situm erit in *O,* ut sit

$$IO = \frac{(n+1)c}{2(2n+1)}.$$

Capacitas autem partis submersae erit

$$= \int \frac{2cydx}{n+1}$$

$= \dfrac{c}{n+1}$ in aream *AMBMA* (Fig. 63), si ergo superficies sectionis aquae dicatur = *E,* erit volumen partis submersae $= \dfrac{cE}{n+1}$. Sit denique totius corporis centrum gravitatis situm in *G,* ut sit $IG = h$, atque pondus totius corporis $= M$, erit

$$GO = h - \frac{(n+1)c}{2(2n+1)},$$

atque in propositione generali (§ 298) fiet

$$\int (y^3 + z^3)dx = 2\int y^3 dx,$$

ob

$$z = y, \text{ et } V = \frac{cE}{n+1}.$$

Hinc igitur orietur stabilitas huius aequilibrii situs respectu axis

$$AB = M\left(h - \frac{(n+1)c}{2(2n+1)} + \frac{2(n+1)\int y^3 dx}{3Ec} \right).$$

Q.E.I.

COROLLARIUM 1

380. Cum quaevis sectio transversalis *MQM* proportionalis sit ordinatae sectionis aquae *MM,* perspicuum est centrum gravitatis sectionis aquae *1* et centrum magnitudinis partis submersae *O* in eandem rectam verticalem *1H*

incidere.

COROLLARIUM 2

381. Dato igitur in eiusmodi corpore centro gravitatis I sectionis aquae, simullocus centri magnitudinis O innotescit; atque in rectam verticalem IOH etiam centrum gravitatis totius corporis G positum sit necesse est.

COROLLARIUM 3

382. Si fiat $n=1$, sectiones transversales fient triangula, ac lineae MQ rectae. Hoc igitur casu erit volumen partis submersae $V=\frac{cE}{2}$ atque stabilitas prodibit

$$=M\left(h-\frac{c}{3}+\frac{4\int y^3dx}{3Ec}\right).$$

COROLLARIUM 4

383. Sin autem sit $n<1$, attamen $n>0$, curvarum $M\,Q$ tangentes in M erunt verticales, atque pars submersa figuram habebit gibbam seu convexam. At si $n>1$ figura fiet concava.

COROLLARIUM 5

384. Si stabilitas respectu cuiuscunque alius axis horizontalis per I transeuntis desideretur in formula, nil erit mutandum, nisi expressio $4\int y^3dx$, quae ad illum axem accommodari debebit. Cetera enim omnia non pendent a positione axis assumti AB.

SCHOLION

385. Eadem proprietas, quam habent tum conicae parabolae tum omnes reliquae cuiusque ordinis, competit in innumerabiles alias curvas, quae id circo eodem successu sectionibus transversalibus MQ tribui poterunt. Omnes enim curvae eodem modo satisfaciunt, quae ita sunt comparatae, ut earum areae MQM quae aequalibus abscissis respondent, ipsis ordinatis MM sint proportionales, quippe ex quo fit, ut partis submersae centrum magnitudinis O verticaliter infra centrum gravitatis 1 sectionis aquae cadat. Pro his igitur curvis aequatio inter u et t ita debet esse comparata, ut primo fiat $u=0$, facto $t=0$, atque ut deinde fiat $u=c$ posito $t=y$. Tertio vero area $\int udt$, si ponatur $t=y$, talem formam induere debebit $\frac{mcy}{n}$. Haec autem requisita sequenti modo impetrabuntur: in genere sit T functio quaecunque nullius dimensionis ipsarum $t=y$, seu

functio quaecunque ipsius $\frac{t}{y}$ quae evanescat facto $t=0$. Haec ergo functio T posito $t=y$ abibit in numerum constantem, qui sit n, quo facto exhibebit ista aequatio $u=\frac{cT}{n}$ curvam quaesito satisfacientem. Namque facto $t=0$, erit $u=0$, atque posito $t=y$ fit $u=c$.
Denique erit

$$\int udt = \int \frac{cTdt}{n} = \frac{cy}{n}\int \frac{Tdt}{y}.$$

At $\int \frac{Tdt}{y}$ dabit functionem ipsius $\frac{t}{y}$, quae ideo abibit in numerum constantem puta m facto $t=y$; unde area sectionis transversalis MQM orietur $=\frac{2ncy}{n}$.
Praeterea vero etiam intervallum Po, quo centrum gravitatis o sectionis transversalis cuiusvis sub horizontem cadit erit constans. Cum enim sit

$$Po = \frac{\int uudt}{2\int udt}$$

posito post integrationem $t=y$, erit

$$\int uudt = \frac{cc}{nn}\int T^2dt = \frac{c^2y}{nn}\int \frac{T^2dt}{y}.$$

Sed $\int \frac{T^2dt}{y}$, dabit functionem ipsius $\frac{t}{y}$, quae facto $t=y$ abibit in numerum constantem, qui sit K, ita ut sit

$$\int uudt = \frac{Kc^2y}{nn},$$

quae expressio divisa per

$$2\int udt = \frac{2mcy}{n}, \text{ dabit } Po = \frac{Kc}{2mn},$$

cui expressioni consequenter aequale quoque est intervallum IO.

PROPOSITIO 38

PROBLEMA

386. *Sit sectio aquae curva quaecunque AMBMA* (Fig. 65) *praedita diametro AB, sub qua in plano verticali pars submersa terminetur ad spinam EHF utcunque curvilineam, invenire figuram idoneam pro sectionibus transversalibus, ut centrum magnitudinis partis submersae O verticaliter infra centrum gravitatis sectionis aquae I cadat.*

SOLUTIO

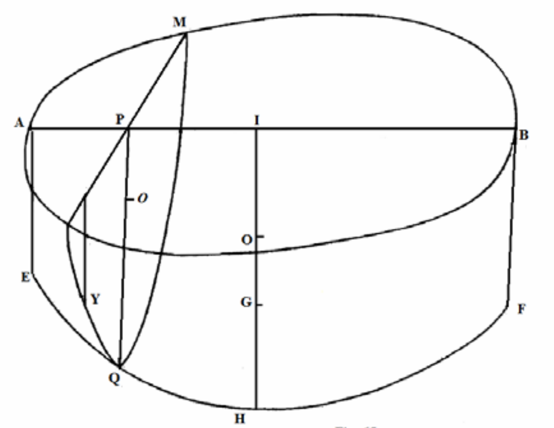


Fig. 65

Positis $AP = x$, $PM = PM = y$, et $PQ = z$ dabitur ob sectionem aquae datam y per x, et ob figuram spinae *EHF* pariter datam etiam z per x. Quaesito autem commodissime satisfiet, si singulis sectionibus transversalibus *MQM* eiusmodi figura tribuatur, ut earum areae fiant proportionales ordinatis *MM* seu ipsis y. Ad hoc efficiendum ducta in sectione transversali applicata quacunque XY, sit, $MX = t$ et $XY = u$ atque assumatur ad naturam curvae *MQ* exprimendam indefinita ista aequatio

$$u = \frac{At^{n-1}}{y^{n-1}} + \frac{Bt^{m-1}}{y^{m-1}} + \frac{Ct^{k-1}}{y^{k-1}},$$

in qua n, m, et k sint numeri unitate maiores, quo facto $t = 0$ fiat $u = 0$. Nunc quia facto $t = y$, fieri debet $u = z$, erit $z = A + B + C$. Porro quaeratur area $\int udt$, quae erit

$$= \frac{At^n}{ny^{n-1}} + \frac{Bt^m}{my^{m-1}} + \frac{Ct^k}{ky^{k-1}};$$

quae cum posito $t = y$, fieri debeat ipsi y proportionalis, ponatur $= cy$, habebiturque

$$c = \frac{A}{n} + \frac{B}{m} + \frac{C}{k}.$$

Ex his conditionibus consequitur

$$B = \frac{kmnc - mnz - m(k-n)A}{n(k-m)}, \text{ atque } C = \frac{knc - lmnz - m(k-n)A}{n(k-m)};$$

quamobrem pro curva quaesita sequens habebitur aequatio:

$$u=\frac{At^{n-1}}{y^{n-1}}+\frac{\left(kmnc-mnz-m(k-n)A\right)t^{m-1}}{n(k-m)y^{m-1}}-\frac{\left(kmnc-knz-k\left(m-n\right)A\right)t^{k-1}}{n\left(k-m\right)y^{k-1}}:$$

in qua praeter exponentes *k, m, n* quantitatem *A* pro arbitrio assumere licet. In quantitate autem *A* eligenda, ad hoc praecipue attendi oportebit, ut applicata *u* continuo crescat, ab *M* ad *Q* progrediendo, atque ut inter puncta *M* et *Q* curva sit ubique convexa, seu ut $\frac{du}{dt}$ continuo decrescat, prius autem assequemur si $\frac{du}{dt}$ ab *M* usque ad *Q* affirmativum valorem retineat; atque adeo in *Q* sit affirmativum. In *Q* vero erit

$$\frac{du}{dt}=\frac{1}{y}\left(\frac{\left(n-1\right)A+(m-1)}{n(k-m)}\left(kmnc-mnz-m(k-n)A\right)-\frac{\left(k-1\right)}{n(k-m)}(kmnc-knz-\ k(m-n)A)\right)$$
$$=\frac{1}{y}\left(\frac{\left(m-n\right)\left(k-n\right)}{n}A-kmc+(k+m-1)z\right).$$

Quare esse debebit

$$A>\frac{kmnc}{\left(m-n\right)(k-n)}$$

quo $\frac{du}{dt}$ maneat affirmativum, etiamsi *z* fiat minimum. At si aliae circumstantiae non admittant, ut singulis sectionibus transversalibus eiusmodi figura inducatur, tum quanto ex una parte puncti *I* sectiones transversales iusto vel maiores vel minores fuerint, tanto quoque vel maiores vel minores ex altera parte fieri debebunt, ut nihilominus centrum magnitudinis partis submersae in rectam *IH* incidat. Q. E. I.

COROLLARIUM 1

387. Si numerorum *n, m,* et *k* ponatur *n* minimus, *m* medius et *k* maximus, ex numero *n* cognoscetur positio tangentis sectionum transversalium in *M.* Nam si $n-1$ fuerit >1 tum tangens erit horizontalis, sin $n-1<1$ verticalis, at si $n=2$ tum angulus erit obliquus.

COROLLARIUM 2

388. Si ergo ponatur $n=\frac{3}{2}$, tangens in *M* non solum fiet verticalis, sed etiam radius osculi in *M* erit finitus. Quare si porro ponatur $m=\frac{5}{2}$ et $k=\frac{7}{2}$

habebitur pro curva haec aequatio

$$u = \frac{A\sqrt{t}}{\sqrt{y}} + \frac{(105c-30z-40A)}{12}\frac{t\sqrt{t}}{y\sqrt{y}} - \frac{(105c-42z-28A)t^2\sqrt{t}}{12y^2\sqrt{y}}.$$

COROLLARIUM 3

389. Quia esse debet

$$A > \frac{kmnc}{(m-n)(k-n)} - \frac{n(k+m-1)x}{(m-n)(k-n)}.$$

Videamus an salva hac conditione tertius terminus O possit evanescere; hinc autem fit

$$A = \frac{mnc - nz}{m-n}.$$

Debebit ergo esse

$$(m+n-1)z > mnc.$$

COROLLARIUM 4

390. Quando ergo spina ita est comparata ut z ad 0 usque decrescat, tum non poterit esse ubique $(m+n-1)z > mnc$, et hancobrem his casibus trinomia functione ipsius t ad u designandum uti oportebit.

SCHOLION

391. Perspicuum autem est eiusmodi occurrere posse casus, quibus z tam diversorum capax sit valorum, ut area sectionum transversalium ipsi y soli omnino non proportionalis reddi queat; siquidem figurae non admodum dissimiles desiderentur earum, quae in navibus adhiberi solent. Nam vel ubi altitudo z maior existit, ibi transversalis nimium coarctata esse deberet, vel ubi z vehementer fit diminuta, ibi area sectionis tanta esse deberet, ut limitibus praescriptis contineri non posset. Eiusmodi igitur casibus eam medelam affere conveniret, cuius in solutione mentionem feci, ut sectiones transversales, quae per regulam nimis deformes prodirent, vel augeantur vel minuantur ex utraque parte aequaliter, quo locus centri gravitatis communia conservetur. At ne tali scientiae minus conveniente correctione sit opus, praestabit tum figuram sectionis aquae, tum spinae ad formam sectionum transversalium idoneam accomodare. Quem in finem pono sectionum transversalium areas tenere rationem compositam amplitudinum in sectione aquae et profunditatum, seu esse ubique ut $yz;$ namque hoc posito quae figura uno casu erit apta ad praxin, eadem locum habebit in omnibus reliquis. Eiusmodi autem curvae hac proprietate praeditae pro sectionibus transversalibus innumerabiles exhiberi possunt, quae omnes sequenti aequatione generali continentur, sit T functio quaecunque

ipsius $\frac{t}{y}$ evanescens posito $t = 0$, quae facto $t = y$ abeat in numerum constantem *n.* tum fiat $u = \frac{yT}{n}$. Ex hac enim aequatione fit $u = 0$, si $t = 0$ et $u = z$ si $t = y$; ac denique erit

$$\int udt = \frac{\int zTdy}{n} = \frac{Zy}{n}\int\frac{Tdy}{y}\ ;\ \text{at} \int\frac{Tdy}{y}$$

facto $t = y$ abit in numerum constatem *m* ita ut area tota fiat $= \frac{2m^2yz}{n}$.

PROPOSITIO 39

PROBLEMA

392. *Si areae sectionum transversalium MQM* (Fig. 65) *fuerint in ratione composita basium MM et profunditatum PQ, invenire tum pro sectione aquae AMBM tum pro spina EHF figuras idoneas, ut centra gravitatis sectionis aquae I et voluminis partis submersae O in eandem rectam verticalem IH incidant.*

SOLUTIO

Sit longitudo diametri sectionis aquae $AB = a$; et quoniam partes sectionis aquae utrinque circa *AB* similes et aequales esse debent, atque curvam *AMBM* ubique concavam versus *AB* esse convenit, sumatur pro ea ista aequatio $y = (A + Bx)\sqrt{(ax - xx)}$. At vero pro spina accipiatur haec aequatio

$$z = \frac{(\alpha + \beta x)(ax - xx)}{A + Bx},$$

quo ea tam in *A* quam in *B* sectioni aquae occurat, idque sub obliquis angulis, prout in navibus fieri solet. Hic scilicet positum est ut ante $AP = x$, $PM = y$ et $PQ = z$. Quo nunc puncta *I* et *O* in eandem rectam verticalem incidant, debet post integrationem peractam facto $x = a$ fieri

$$\frac{\int yxdx}{\int ydx} = \frac{\int yzxdx}{\int yzdx}.$$

Ad haec integralia autem capienda saltem pro casu $x = a$ inserviet hoc theorema vi cuius est

$$\int\left(I+Kx+Lx^{2}+Mx^{3}+\text{etc.}\right)dx\left(ax-xx\right)^{n}$$
$$=\left(I+\frac{(n+1)Ka}{(2n+2)}+\frac{(n+1)(n+2)}{(2n+2)(2n+3)}La^{2}+\frac{(n+1)(n+2)}{(2n+2)(2n+3)(2n+4)}Ma^{3}+\text{etc.}\right)\int dx\left(ax-xx\right)^{n}.$$

Hinc igitur erit

$$\frac{\int yxdx}{\int ydx}=\frac{\int\left(Ax\ +\ Bx^{2}\right)dx\sqrt{(ax-xx)}}{\int\left(A\ +\ Bx\right)dx\sqrt{(ax-xx)}}=\frac{\frac{1}{2}Aa+\frac{5}{16}Ba}{A+\frac{1}{2}Ba}=AI.$$

At cum sit

$$yz=\left(\alpha+\beta x\right)(ax-xx)^{\frac{3}{2}},\text{ erit }\frac{\int yzxdx}{\int yzdx}=\frac{\frac{1}{2}\alpha a+\frac{7}{24}\beta a^{2}}{\alpha+\frac{1}{2}\beta a}.$$

Qui valores inter se aequati dant

$$\frac{A+\frac{5}{8}Ba}{A+\frac{1}{2}Ba}=\frac{\alpha+\frac{7}{12}\beta a}{\alpha+\frac{1}{2}\beta a},$$

unde fit

$$B\beta a=4A\beta-6B\alpha,\ \text{seu}\ \beta=\frac{6B\alpha}{4A-Ba}.$$

Quamobrem si pro sectione aquae assumatur haec aequatio

$$y=\left(m+\frac{nx}{a}\right)\sqrt{(ax-xx)},$$

tum pro spina assumenda erit aequatio haec:

$$z=\frac{\alpha\left((4m-n)a+6nx)(ax-xx)\right)}{(4m-n)(ma+nx)}.$$

Q.E.I.

COROLLARIUM 1

393. Cum sit

$$AI=\frac{\frac{1}{2}Aa+\frac{5}{16}Ba^{2}}{A+\frac{1}{2}Ba},\ \text{ob}\ A=m\ \text{ et }\ B=\frac{n}{a}$$

erit

$$AI=\frac{a\left(\frac{1}{2}m+\frac{5}{16}n\right)}{m+\frac{1}{2}n}.$$

Quamobrem habebitur

$$AI = \tfrac{1}{2}a + \frac{na}{16m+8n}.$$

Quoties igitur n est numerus affirmativus seu m et n numeri eiusdem signi, erit $AI > \tfrac{1}{2}AB$.

COROLLARIUM 2

394. Si $n = 0$, fiet aequatio pro sectione aquae $y = m\sqrt{(ax-xx)}$, quo ergo casu sectio aquae erit ellipsis, cui figura spinae respondet

$$z = \frac{\alpha}{m}(ax-xx),$$

quae ideo erit parabola. Hoc autem casu punctum I in medium rectae AB incidit.

COROLLARIUM 3

395. Si ponatur $m = 0$, quo sectio aquae hac aequatione exprimatur

$$y = \frac{nx}{a}\sqrt{(ax-xx)}$$

erit figura spinae

$$z = \frac{\alpha(a-x)(a-6x)}{nx},$$

quae autem figura est inepta, ob $z = 0$, si est $x = \tfrac{1}{6}a$.

COROLLARIUM 4

396. Ne igitur alicubi inter A et B fiat $z = 0$ necesse est ut sit

$$a + \frac{6nx}{4m-n} > 0,$$

si quidem x inter limites 0 et a continetur. Fit autem

$$a + \frac{6nx}{4m-n} = 0,$$

si est $x = \dfrac{a(n-4m)}{6n}$; quare $\dfrac{n-4m}{6n}$ vel minus esse debet quam 0, vel maius quam I.

COROLLARIUM 5

397. Fiat $n = 4m$, qui casus id habet singulare, quod cum sit pro sectione aquae

$$y = \left(m + \frac{4mx}{a} \right)\sqrt{(ax - xx)}\ ,,$$

fiat pro spina $z = \dfrac{\alpha xx(a-x)}{a(a+4x)}$, cuius igitur tangens in AI erit horizontalis. At intervallum prodit $= \frac{1}{2}a + \frac{1}{12}a = \frac{7}{12}a.$

SCHOLION 1

398. Propositio haec latissime patet, atque omnes fere figuras, quae vulgo in constructione navium adhiberi solent, in se complectitur. Est enim ad infinitas figuras sectionum transversalium accomodata, prout ex§ 391 videre licet, atque insuper innumerabiles in se continet figuras sectionum aquae ab usu non abhorentes; ita ut ex ea tam de constructione navium iudicari, quam novae navium formae idoneae inveniri queant, quae quidem hactenus expositis principiis sint consentaneae. Latiore quidem sensu, si opus fuisset, solutionem adornare potuissemus, si pro sectione aquae eiusmodi aequationem

$$y = \left(A + Bx + Cx^2 + Da^3 + \text{etc.}\right)\sqrt{(ax - xx)},$$

pro spina vero hanc aequationem

$$z = \frac{\left(\alpha + \beta x + \gamma x^2 + \delta a^3 + \text{etc.}\right)(ax - xx)}{\left(A + Bx + Cx^2 + Da^3 + \text{etc.}\right)}$$

assumsissemus; tum enim in figura spinae plures litterae indeterminatae relictae fuissent, quarum determinatione multo plures figurae produci potuissent. Prodiisset autem haec aequatio figuram spinae ad datam sectionem aquae accommodans

$$\frac{\frac{1}{2}A + \frac{5}{16}Ba^2 + \frac{7}{32}Ca^3 + \frac{21}{128}Da^4 + \text{etc.}}{A + \frac{1}{2}Ba + \frac{5}{16}Ca^2 + \frac{7}{32}Da^3 + \text{etc.}} = \frac{\frac{1}{2}\alpha a + \frac{1}{24}\beta a^2 + \frac{3}{16}\gamma a^3 + \frac{3}{256}\delta a^4 + \text{etc.}}{\alpha + \frac{1}{2}\beta a + \frac{7}{24}\gamma a^2 + \frac{3}{16}\delta a^3 + \text{etc.}} = AI.$$

Ex qua innumeris modis relatio coefficientium A, B, C, D etc. atque α, β, γ, δ, etc. definiri potest.

SCHOLION 2

399. Satis iam fuse in isto capite omnia, quae ad stabilitatem corporum aquae in quopiam aequilibrii situ insidentium, cognoscendam et diiudicandam pertinent, explanasse mihi videor, neque quicquam deesse videtur, quod in hac doctrina amplius desiderari possit. Quamobrem huic capiti finem imponam eo progressurus quo ea, quae hic tractata sunt, magnam afferent utilitatem. In sequente enim capite in effectum propius inquiram, quem vires quaecunque navem seu corpus aquae innatans quodcunque sollicitantes producant quo intelligatur, quid tam vis venti et remorum, quam

gubernaculum et allisio aquae ipsius in nave efficiant. Cum autem motus progressivus ipse sine calculo resistentiae cognosci nequeat, quem in quinto denique capite plenius exponere constitui, hic tantum sollicitationem ad istum motum et accelerationem momentaneam considerasse contentus ero. In hoc vero praecipue incumbam, ut quantum quaevis potentiae navi ex situ aequilibrii deturbent, accurate definiam, atque in hunc finem ad tres supra memoratos axes, quos in quaque navi concipere licet, imprimis respiciam, circa quos omnis inclinatio et declinatio fieri censenda est.