CHAPTER SIX

THE RESISTANCE WHICH ANY BODIES MAY EXPERIENCE MOVING DIRECTLY THROUGH WATER, ACCORDING TO BODY FORM.

PROPOSITION 61

PROBLEMA

612. *ATDEb* (Fig. 93) *shall be the figure of the foremost part of a ship immersed in water, and divided into two equal and similar parts by the vertical plane ACD ; and this figure shall be progressing through water along the direction CAL: to determine the resistance, which this shape will experience in its motion.*

SOLUTION

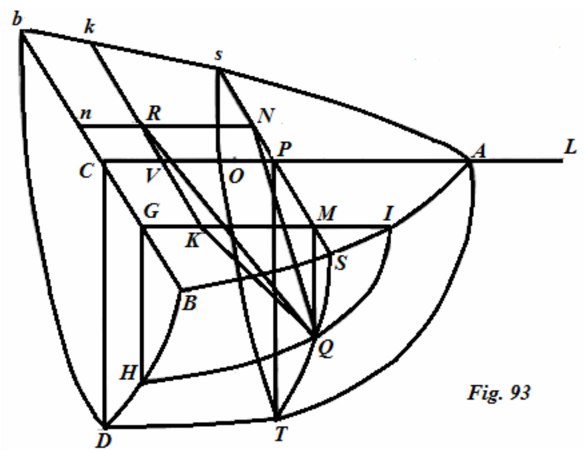


Fig. 93

In this figure of the anterior part or prow of a ship, or of any other similar body floating in water, that part is represented which is immersed in the water, and the surface of which is subject to the water resistance along the direction of the course. Therefore so that the horizontal section is the plane of the water *ABb*, thus the vertical plane *ACD* will divide that same part into two similar and equal parts *ACDB* and *ACDb,* so that all the horizontal right lines drawn in the plane *ACD* shall be just as many as the diameters of the horizontal section, or in the plane *ABb* of the proposed parallel volume. Therefore since the motion of this body in water may be made along the direction of the horizontal *CAL,* it is evident the mean direction of the resistance must lie in the diametric plane *ACD* itself ; so that part of the resistive force will retard the motion, part will raise the body from the water, if indeed the mean direction were not horizontal, but pointing upwards. Therefore for this the twofold effect of the resistance will be required to be defined, in the first place there shall be the height corresponding to the speed, by which the body is progressing in the direction *CAL* will be due to the height *v.* Then with the right line *AO* taken for the axis, on that the will be taken $AP = x$, and the vertical section *STs* may be considered made through the point *P* normal to the diametric plane *ACD*, on the base *Ss* of which some part may be put $PM = y$, and with the vertical $MQ = z$. Therefore in this manner the proposed point *Q* may be defined by an equation between the three variables x, y et z. Moreover this equation may be reduced to this differential equation $dz = Pdx + Qdy$, in which P and Q shall be certain functions of x and y themselves, not involving z, and this equation on account of the similar and equal parts situated on each side $ACDB$, $ACDb$ will express the nature of the diametric plane ACD. Now so that it shall be apparent an element of the surface taken at Q shall strike the water under some angle Q, either the surface tangent plane at Q or some right line QR normal to the surface must be defined at the point Q. Therefore we will investigate the position of this normal line QR, which in the end at first we will consider only the section STs, the nature of which on account of x being constant will be expressed by this equation $dz = Qdy$, from which thus the position of the normal QN will be defined for the arc SQT, so that the subnormal shall become

$$MN = -\frac{zdz}{dy} = -Qz \text{ from which there will become } PN = -y - Qz.$$

Whereby if MN may be drawn perpendicular to NR in the plane ABn, all the right lines from Q drawn to the right line NR will be normal to the curve SQT at the point Q ; of which likewise shall be normal to the same surface at the point Q, which will be found in this manner. The plane vertical section $IMGH$ may be considered through the points M and Q parallel to the diameter ACD, and the nature of the curve IQH on account of constant y is expressed by this equation $dz = Pdx$. Now the right line QK shall be normal to the curve IQH at the point Q, under the normal there will become

$$MK = \frac{zdz}{dx} = Pz.$$

If therefore in the plane ABb, KVR may be drawn normal to the right line MK, also all the right lines drawn from Q to the line KR will be normal to the curve IQH at Q. And thus since the right lines NR and KR will intersect each other at the point R, with there being

$$AV = x + Pz, \text{ and } VR = PN = -y - Qz,$$

of which the one VR is perpendicular to the other AV, the right line QR will be normal both to the curve SQT as well as to the curve IQH at the point Q ; and on this account this right line QR will be normal to the surface at the point Q. Therefore the angle at which the element of the surface strikes the water at Q, will be the right complement of this angle which the normal QR makes with the direction of the course CAL or which the right line RN makes parallel to this, which angle is QRN. But on account of
[Note the two distinct uses of the letter Q here; presumably Euler used this technique to indicate where the derivative was to be evaluated.]

$$MN = -Qz, \text{ there will be } QN = z\sqrt{1 + PP + QQ}$$

and on account of

$$NR = MK = Pz \text{ there will be } QR = z\sqrt{1 + PP + QQ}$$

from which the sine of the angle QRN will become

$$=\frac{\sqrt{1+QQ}}{\sqrt{1+PP+QQ}}, \text{truly the cosine } =\frac{P}{\sqrt{1+P^2+Q^2}},$$

which cosine as well as the sine will be that of the angle under which the element of the surface at Q will be forced against the water. Whereby if the element of the surface may be put $=dS$, the force of the resistance which it will experience $=\frac{P^2vdS}{1+P^2+Q^2}$, and the direction of this force shall be placed along the normal to the surface QR. Moreover it will be required to express the element of the surface dS by the differentials of the coordinates x, y and z, from which the whole resistance may be able to be deduced by integration. Therefore the abscissa x may be considered to be increased by the element dx, and the applied line y by the element dy; and the infinitely small rectangle $dxdy$ will arise at P placed in the plane ABb, the inclination of which from its angles drawn vertically downwards the element dS will correspond on the surface, the inclination of which to the plane ABb, which will provide for the equal angle MQR :

$$dS=dxdy\sqrt{1+P^2+Q^2}.$$

Therefore the resistance which the element dS will experience hence will be $=\frac{P^2vdxdy}{\sqrt{1+P^2+Q^2}}$, and its direction will be incident along the normal QR. Now this force of resistance may be resolved into three parts normal to each other the directions of which shall be parallel to the three coordinates AP, PM, and MQ. Therefore since these three forces may be able to be considered to be applied at the point R , the figure at R will be driven vertically upwards by the force $=\frac{P^2vdxdy}{1+P^2+Q^2}$; then it will be forced to move in the direction Rn parallel to the axis AC by the force $=\frac{P^3vdxdy}{1+P^2+Q^2}$; and finally it will be forced to move in the direction Rk of the right line Bs by the parallel force $=\frac{-P^2Qvdxdy}{1+P^2+Q^2}$. Now if the resistance of the element in the other half $ACDb$ may be deduced in a similar manner, and that may be joined with that found, the forces in the directions parallel to Ss will cancel each other out mutually; but at V the body will be forced vertically upwards by the force $=\frac{2P^2vdxdy}{1+P^2+Q^2}$; and likewise it will be forced backwards in the direction of the axis VC by the force $=\frac{2P^3vdxdy}{1+P^2+Q^2}$. Therefore from the resistance, which is apparent, the portion of the surface from the two sections STs and the other parallel to this, and with the interval dx of the abscissas put in place the figure will be urged backwards in the direction AC by the force

$$= 2vdx\int \frac{P^3 dy}{1+P^2+Q^2},$$

which with the integral in which x is put constant thus will be resolved so that it may vanish on putting $y=0$, and then there may be put $y=PS$. Again truly it will be acted on by the force

$$= 2vdx\int \frac{P^2 dy}{1+P^2+Q^2},$$

of which the moment of the force with respect of the point A will be equal to

$$= 2vdx\int \frac{P^2\left(x+Pz\right)dy}{1+P^2+Q^2};$$

which integrals are required to be taken in the same manner as before. Therefore so that the whole resistance experienced from the water for the whole surface, is reduced to two forces of which the one will be acting backwards in the direction AC by the force

$$= 2v\int dx\int \frac{P^3 dy}{1+P^2+Q^2},$$

where it is to be observed for the integral $=\int \frac{P^3 dy}{1+P^2+Q^2}$ taken in the prescribed manner to be a function of x only ; from which the latter integral

$$\int dx\int \frac{P^3 dy}{1+P^2+Q^2},$$

must be taken thus so that it shall vanish on putting $x=0$, and with this done there must be put in place $x=AC$, so that the resistance of the whole body proposed may be obtained. Truly likewise the figure will be forced vertically upwards by the force

$$= 2v\int dx\int \frac{P^2 dy}{1+P^2+Q^2},$$

of which, since the moment of the force shall be

$$= 2\int vdx\int \frac{P^2\left(x+Pz\right)dy}{1+P^2+Q^2};$$

that is agreed to be applied at the point O of the axis AC, thus so that there shall become

$$AO=\frac{\int dx\int\frac{P^2\left(x+Pz\right)dy}{1+P^2+Q^2}}{\int dx\int\frac{P^2dy}{1+P^2+Q^2}},$$

with the integrals there to be read, which it is understood to be taken. Therefore from both these equivalent resistive forces, the mean direction of the total resistance, which will pass through the point *O* in the plane *ACD*, and since the angle *AC* will be set up the tangent of which will be place

$$=\frac{\int dx\int\frac{P^2\left(x+Pz\right)dy}{1+P^2+Q^2}}{\int dx\int\frac{P^3dy}{1+P^2+Q^2}}$$

under which angle the mean direction of the resistance from *O* will incline upwards towards the prow. Q.E.I.

COROLLARY 1

613. Therefore the direction of the motion of the ship progressing along the direction *AL* will be retarded by the resistance of the force

$$=2v\int dx\int\frac{P^3dy}{1+P^2+Q^2},$$

which expression indicates the volume of water the weight of which is equal to the strength of the resistance itself.
[Recall that the density of water is taken as 1; no distinction is made between mass and weight .]

COROLLARY 2

614. But since in addition the ship may be forced upwards by the force

$$=2v\int dx\int\frac{P^2dy}{1+P^2+Q^2},$$

the ship is considered to be made lighter by so great a force, and that will be raised from the water, by a force equivalent also the weight of the water, the volume of which is indicated by that same expression.

COROLLARY 3

615. Truly besides, unless the mean direction of the resistance shall pass through the centre of gravity, the ship will be turned by the resistance about the latitudinal axis of the ship, and the prow will be either raised or lowered, just as the direction of the resistance shall be acting either directly above or below the centre of gravity.

COROLLARY 4

616. Finally, it is evident from the expressions found, the effect of all the resistances which act both in retarding the ship as well as raising or lowering the inclination, are agreed to follow on account of doubling the speed by which the ship shall be moved forwards.

COROLLARY 5

616. Thus from the given formulas the calculation will lead to the whole surface of this body.
Since an element of the surface dS shall be

$$= dxdy\sqrt{1+P^2+Q^2},$$

at first

$$dy\sqrt{1+P^2+Q^2},$$

will be integrated, with x put constant, thus so that the integral shall vanish on putting $y=0$ and then there shall be put $y=PS$, with which done the integral will become some function of x, thus so that

$$\int dx\int dy\sqrt{1+P^2+Q^2}$$

may be able to be assigned, so that with the integral taken twice on putting $x=AC$, it will be provided for the whole surface.

COROLLARY 6

617. But for finding the volume of the whole figure $ABDb$, shall be $PT=t$ and $PS=s$, and t and s will be functions of x itself assignable from the equation

$$dz = Pdx + Qdy.$$

Then truly the area will be

$$PTS = \int zdy = -\int ydz$$

since $z=0$, when there shall become

$$y = s = -\int Qydy.$$

Thus the integral $\int Qydy$ thus may be taken with x put constant, thus so that the integral may vanish on putting $y = 0$ and then there may be put $y = s$. With which done, $2\int -dx\int Qydy$ will give the volume of the whole figure on putting $x = AC$ after the integration.

COROLLARY 7

618. Since *ABDb* shall be the whole and only surface to experience the resistance, if indeed the ship may be progressing in the direction *AL*, it is necessary that the plane *BDb* shall be the widest transverse section of the ship, and in addition so that all the tangent planes of these parts *ABDb* shall be inclined towards the prow.

COROLLARY 8

619. Hence it is deduced also, if the figure *ABDb* were half of the same denser than water body, and this body may be moved in the direction *AL* either deeper, or to be completely submerged in the water, then so great a resistance will be going to be allowed along the direction *AC*, which will become

$$= 4v\int dx\int \frac{P^3 dy}{1 + P^2 + Q^2}.$$

SCHOLIUM

620. From the differential equation $dz = Pdx + Qdy$, of which indeed we take the known integral, by which we express the nature of the known surface *ATDB*, that surface itself is known perfectly. Indeed in the first place the section of the water *ABb* will become known if there may be made $z = 0$, in which case if there may be put $PS = S$, there will become $y = s$ and the equation $Pdx + Qds = 0$ will show the nature of the water section or the relation between $AP = x$ and $PS = s$. In a similar manner any other horizontal section will become known, on putting $z =$ constant or $dz = 0$, from the equation $Pdx + Qdy = 0$, in which x will denote the abscissa taken parallel to the axis *AC* itself and y will denote the applied line. But moreover the same equation will be produced from all these sections

$$Pdx + Qdy = 0,$$

yet hence all these sections shall not be considered equal to each other, since the equation

$$Pdx + Qdy = 0$$

shall be the differential, and in the integration innumerable constants shall be able to be received in the integration. Moreover for any horizontal section the integral of the formula $Pdx + Qdy$ must be put equal to the value of *z,* or to the interval, by which some section may be distant from the section of the water *ABb*. Truly the formula of the differential $Pdx + Qdy$ will allow an integration always, since generally there is $dz = Pdx + Qdy$ and both P and Q are considered not to depend on z, thus so that $Pdx + Qdy$ shall be the differential of these functions of x et y, to which z may be made equal. On account of which P and Q will be functions of x and y themselves, so that, if there were $dP = Rdx + Bdy$ and $dQ = Tdx + Udy$, there is going to become $S = T$, from which the general connection between P and Q is considered. But if P and Q were functions, in which everywhere x and y may represent a number of the same dimension n ; for example , there will become $Px + Qy = (n+1)z$, from which at once the value of Q itself is found from the given value of P . Then also the nature of the vertical diametrical plane *ACD* will be expressed by putting $y = 0$, in which case there becomes $z = PT = t$, thus so that this equation $dt = Pdx$ may be had between $AP = x$ and $PT = t$, put in place at *P,* which generally is a function of both x and y, $y = 0$. Finally the nature of the widest transverse section of the ship *BDb* will be had from the known equation $dz = Pdx + Qdy$ on putting $x = AC = a$; then indeed on account of $CG = y$ and $GH = z$, there will become $dz = Qdy$. Moreover, just as $dz = Pdx + Qdy$ from the canonical equation, the nature of the whole surface *ATDB* is known, thus in turn the nature of the canonical equation will be elicited from the given nature of the surface. Indeed if the equations may be given both for the section of the water *ACB,* as well as for the diametric plane *ATD,* and also for the individual cross sections *SPT,* it will be allowed to determine the length $MQ = z$, which is sent from whatever point of the section as far as to the surface of the water M ; and in this manner z is expressed by a quantity composed from x, and y composed from constants, which differential value will give the canonical equation $dz = Pdx + Qdy$ expressing the value of the surface. We will set out particular kinds of surfaces in the following problems, and we will define the resistance, which each experiences in advancing directly forwards in the water; after which we will have reduced the particular kinds to a canonical equation of this form

$$dz = Pdx + Qdy.$$

PROPOSITION 62

PROBLEM

621. *The part of a body floating in water, which shall be turning in the water, shall be in the shape of the cone ABDb* (Fig. 94) *having the given base BDb and the vertex at A, thus so that its surface will be terminated by the right line HA draw from the individual base points BDb drawn to the vertex A , and this figure may be moved along the direction of the axis CAL, to determine the resistance which it will experience.*

SOLUTION

Therefore in this body the section of the water *BAb* will be an isosceles triangle, the diametric plane *ACD* truly will be a right angled triangle. Then any transverse section of the base *STs* will be parallel to the widest parallel section *BDb,* or the base *BDb* itself.

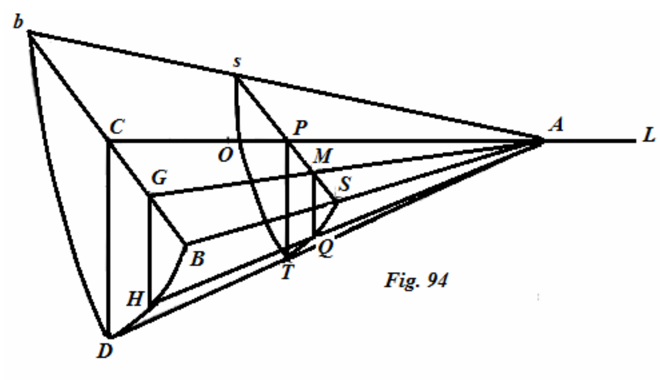


Fig. 94

Therefore half the base *CBD,* certainly to which the other half *CbD* is similar and equal, thus any given curve so that its nature may become known by an equation between the coordinates *CG* and *GH.* Therefore on putting

$$CG = r \text{ and } GH = u\,,$$

u will be some function of r. Now with the right lines *GA* and *HA* drawn, and on putting $AC = a$, there will become on account of the similar triangles :

$$AC(a) : Ap(x) = CG(r) : PM(y) = GH(u) : MQ(z)$$

from which there becomes

$$y = \frac{rx}{a}, \text{ et } z = \frac{uy}{r} = \frac{ux}{a}.$$

There shall become $du = pdr$, with p being some function of r, on account of which there shall be

$$r = \frac{ay}{x},\ dr = \frac{axdy - aydx}{xx} \text{ and } du = \frac{apxdy - apydx}{xx};$$

from which there becomes

$$dz = \frac{udx}{a} + pdy - \frac{pydx}{x},$$

which equation compared with the general canonical equation $dz = Pdx + Qdy$ gives

$$P=\frac{n}{a}-\frac{py}{x}=\frac{u-pr}{a}\quad\text{on account of }\; y=\frac{rx}{a}$$

and $Q=p$*;* from which there becomes

$$1+P^2+Q^2=1+p^2+\frac{(u-pr)^2}{aa}.$$

Truly for the resistance requiring to be defined as before it will be required that all the following be found

$$\int\frac{P^2dy}{1+P^2+Q^2},\quad \int\frac{P^3dy}{1+P^2+Q^2},\quad \int\frac{P^2\left(x+Pz\right)dy}{1+P^2+Q^2}$$

with x made constant, and with the integrals thus taken so that they may vanish on putting $y=0$, to put $y=PS$ or $z=0$. But with x constant there is $dy=\frac{xdr}{a}$, from which there becomes

$$\int\frac{P^2dy}{1+P^2+Q^2}=\frac{x}{a}\int\frac{\left(u-pr^2\right)dr}{a^2+a^2p^2+\left(u-pr\right)^2},$$

$$\int\frac{P^3dy}{1+P^2+Q^2}=\frac{x}{a^2}\int\frac{\left(u-pr^2\right)^3dr}{a^2+a^2p^2+\left(u-pr\right)^2},$$

and

$$\int\frac{P^2\left(x+Pz\right)dy}{1+P^2+Q^2}=\frac{xx}{a^3}\int\frac{\left(u-pr^2\right)\left(a^2+u^2-pru\right)dr}{a^2+a^2p^2+\left(u-pr\right)^2},$$

which since thus they were taken so that they vanish on putting $y=0$ or $r=0$, there must be put $z=0$ or $u=0$. Truly since the integrals found in this manner will not depend on x, the total resistance by which the figure will be forced backwards along the direction AC will become

$$=2v\int\frac{xdx}{a^2}\int\frac{\left(u-pr\right)^3dr}{a^2+a^2p^2+\left(u-pr\right)^2}=v\int\frac{\left(u-pr\right)^3dr}{a^2+a^2p^2+\left(u-pr\right)^2},$$

with the integral taken in the same manner as before. Likewise truly by this resistance this conical body will be raised upwards by the force

$$=2v\int\frac{xdx}{a^2}\int\frac{\left(u-pr\right)^2dr}{a^2+a^2p^2+\left(u-pr\right)^2}=av\int\frac{\left(u-pr\right)^2dr}{a^2+a^2p^2+\left(u-pr\right)^2},$$

of which the direction of the vertical force will pass through the point *O,* so that there shall become

$$AO=\frac{\int\frac{xxdx}{a^2}\int\frac{(u-pr)^2\left(a^2+u^2-pru\right)dr}{a^2+a^2p^2+(u-pr)^2}}{\int\frac{xdx}{a}\int\frac{(u-pr)^2\,dr}{a^2+a^2p^2+(u-pr)^2}},$$

or

$$AO=\frac{2\int\frac{(u-pr)^2\left(a^2+u^2-pru\right)dr}{a^2+a^2p^2+(u-pr)^2}}{3a\int\frac{(u-pr)^2\,dr}{a^2+a^2p^2+(u-pr)^2}},$$

And from these two forces, together with the known point *O,* the total force of the resistance becomes known.
Q. E. I.

COROLLARY I

622. First it is understood from the formulas found so that the further the vertex A may stand apart from the base *BDb*, there the smaller to become the force of the resistance, which the figure experiences, indeed the resistance not to hold any assignable ratio for the variation of the length of the axis $AC=a$.

COROLLARY 2

623. But if the length *AC* were exceedingly great so that besides *a* the remaining quantities pertaining to the base *BDb* shall be able to be ignored, then the strength of the resistance in the direction *AC* will be

$$=\frac{v}{a^2}\int\frac{(u-pr)^3\,dr}{1+pp};$$

but the force by which it is pushed upwards

$$=\frac{v}{a}\int\frac{(u-pr)^2\,dr}{1+pp},$$

the direction of which will pass through the point *O,* with there being $AC=\frac{2}{3}a.$

COROLLARY 3

624. Therefore in this case the strength of the resistance acting backwards on the body in the direction itself will be had as the square of the length of the cone AC. But the force acting upwards will maintain the inverse ratio of the length of the cone: evidently if the length of the cone were exceedingly great.

COROLLARY 4

625. Since the area of the base BDb shall be $= 2\int udr$ with $r = CB$ or $u = 0$ put in place after the integration, the resistance, which is experienced by the base $= 2v\int udr$, if it shall be moving along CA with the same speed in the water, and its direction shall be normal to the base, and passing through its centre of gravity.

COROLLARY 5

626. Truly likewise the case, where only the base may be moving, will be obtained if there may be put $a = 0$. But then the resistance of the force pushing upwards vanishes, moreover the retarding force will be $= v\int (u - pr)dr = v\int udr - v\int rdu$. But if after the integration thus performed so that zero may be produced, if there may be put $r = 0$, there may put $u = 0$, then there becomes $\int rdu = -\int udr$, from which the retarding resistance produced $= 2v\int udr$.

COROLLARY 6

627. The whole surface of this body is

$$= 2\int dx \int dy\sqrt{1 + P^2 + Q^2}$$

(§ 610). Truly there shall be

$$\int dy\sqrt{1 + P^2 + Q^2} = \int \frac{dy}{a}\sqrt{a^2 + a^2p^2 + (u - pr)^2};$$

which, since x may be put constant, will become

$$\frac{x}{a^2}\int dr\sqrt{a^2 + a^2p^2 + (u - pr)^2},$$

from which the total surface produced

$$=\int dr\sqrt{a^2+a^2p^2+(u-pr)^2}$$

with $x=a$ put in place after the latter integration.

COROLLARY 7

628. Finally, since the volume shall be

$$=2\int -dx\int Qydy \ (\S 617) \text{ on account of } Q=p \text{ and } y=\frac{rx}{a},$$

that will become

$$=2\int -dx\int \frac{x^2 prdr}{aa}=2\int -\frac{xxdx}{aa}\int rdu=\tfrac{2}{3}a\int udr$$

with $\int udr$ denoting the area *BCD;* that which indeed is apparent from the elements of geometry.

SCHOLIUM 1

629. Therefore in this first proposition we have subjected the most easy kind of bodies to examination, which within itself includes conical bodies of all kinds: for not only the right cone which has a circular base shall be contained in that, but also oblique cones, certainly which can be reduced to right cones with some conic section taken for the base, then also generally here they pertain to all bodies, which are generated from some given base to a certain high point by right lines drawn, hence they pertain also to pyramids besides cones drawn in some direction having curvilinear bases. But here we consider only conical bodies of this kind according to our principles, which have two equal and similar parts situated on each side of the diametric plane, so that the whole treatment shall be adapted especially for ships. Truly since yet generally the integral formulas remain to be considered, concerning the integration of which may not be evident, it will help to be set out certain special cases, for which a given determined figure is accepted for the base *BDb*.

EXAMPLE 1

630. The submerged part (Fig. 95) of the triangular prism *ABDb* shall be the part which experiences the resistance of the water, of which the base or the widest section *BDb* is an isosceles triangle, in which there shall be $CB = Cb = b$ and $CD = c$. Therefore on putting $CG = r$ and $GH = u$, there will become $c : u = b : b - r$, and hence $u = c - \frac{cr}{b}$ and $du = -\frac{cdr}{b}$, from which there becomes $p = -\frac{c}{b}$.

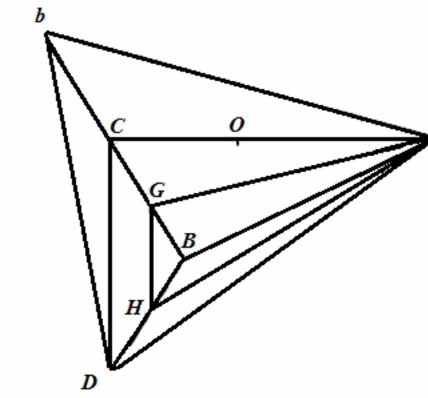


Fig. 95

Now if this pyramid shall be progressing along the direction *AL* with the speed due to the height *v*, and the length *AC* shall be put $= a$, the resistive force will be found on account of

$$ur - pr = c \quad \text{and} \quad aa + aapp = \frac{aa(bb+cc)}{bb},$$

retarding in the direction *AC*

$$= v\int \frac{(u - pr)^3 dr}{a^2 + a^2p^2 + (u - pr)^2} - v\int \frac{b^2c^3dr}{aa(bb+cc) + bbcc}$$

from which after integration on putting $r = b$, this force of the resistance contrary to the direction of the motion produced

$$= \frac{b^3c^3v}{a^2b^2 + a^2c^2 + b^2c^2}.$$

Then since there shall be

$$\int \frac{(u-pr)^2 dr}{a^2 + a^2p^2 + (u - pr)^2} = \int \frac{b^2c^2dr}{a^2b^2 + a^2c^2 + b^2c^2},$$

the force of the resistance acting vertically will be

$$= \frac{a^2b^3c^2v}{a^2b^2 + a^2c^2 + b^2c^2},$$

the direction of which will pass straight through the point *O*, with there being

$$AO = \frac{2\int (aa+cu)dr}{3ab} = \frac{2aa+cc}{3a}.$$

Truly the volume of this whole pyramid *ABDb* will be

$$=\frac{2a}{3}\int udr=\frac{abc}{3};$$

truly the surface intruding into the water, or the two triangles *ABD* and *AbD*

$$=\int dr(aa+aapp+(u-pr)^2)=\sqrt{aabb+aacc+bbcc}.$$

COROLLARY 1

631. Therefore since the base *BDb* shall be $=bc$, and the surface striking against the water

$$=\sqrt{a^2b^2+a^2c^2+b^2c^2}\ .$$

the resistance retarding the motion is equal to the height due to the speed multiplied by the cube of the base and divided by the square of the surface.

COROLLARY 2

632. Therefore with the base *BDb* remaining, the same resistance there will be smaller, where the surface of the body were greater, which is experienced by the water resistance; for the resistance of the motion to vary inversely proportional as the square of the surface.

COROLLARY 3

633. The base *BDb* shall be made constant or $bc=ff$, so that there shall become $c=\frac{ff}{b}$, and the resistance retarding the motion will be

$$=\frac{bbf^4v}{a^2b^4+a^2f^4+bbf^4}$$

from which it is understood the resistance to become a minimum, if either b or c will be had a maximum amount, moreover the resistance will be a maximum if there were $b=c$.

COROLLARY 4

634. Since in this case both *ff* as well as the position *a* shall be constant, and $\frac{1}{3}aff$ will denote the volume of the figure, it is apparent among all the triangular prisms which have equal bases and heights experience the same maximum resistance, of which the base shall be an isosceles triangle and *D* a rectangle.

COROLLARY 5

635. Therefore so that the angle *BDb* differs more from a right angle, there the pyramid will experience a smaller resistance in its motion; with all else being equal. Evidently with both the base as well as the length remaining of the same magnitude.

COROLLARY 6

636. If the base *BDb* standing alone were struck directly by the water with a speed corresponding to the height *v*, the resistance experienced will be $= bcv$. From which the resistance of the pyramid itself will be had to the resistance of the base as b^2c^2 to

$$a^2b^2 + a^2c^2 + b^2c^2,$$

from which it is understood the resistance of the base there to be greater to the resistance of the pyramid, where the greater shall be its height *a*.

COROLLARY 7

637. But with the width of the base *Bb* remaining and with the volume of the pyramid of the same magnitude, the resistance thus will be smaller, where the depth $CD = c$ were smaller, or where the length *AC* of the pyramid were taken longer.

COROLLARY 8

638. Finally it is to be observed the force of the resistance by which the body itself is forced upwards and may be raised from the water to be had itself to the force of the resistance opposing the motion itself to be had as *a* to *c*, that is as *AC* to *CD*. From which the pyramid thus will be forced upwards more, where its axis *AC* shall be longer, or where the cusp at *A* were sharper.

EXAMPLE 2

639. Our conical body may be changed into a right semi-cone, thus so that the base *BDb,* as well as all the sections parallel to *STs,* shall become semicircles (Fig. 94). Moreover the height of this cone may be put to become $AC = a$, which likewise is the direction along which this cone with a speed corresponding to the height *v*. Therefore on putting the radius of the base *BDb*

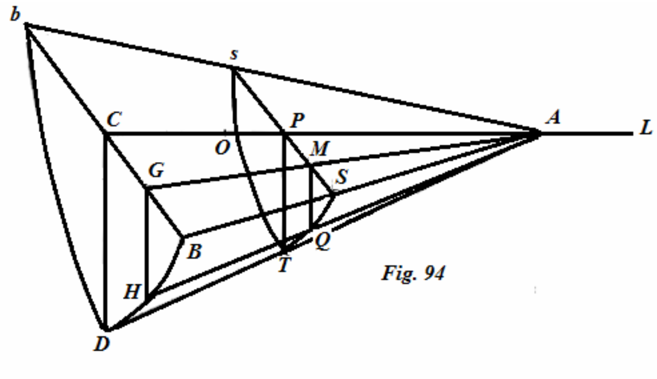


Fig. 94

$$BC = CD = b,$$

there will be on account of $CG = r$ and $GH = u$, from the nature of the circle $u = \surd(bb - rr)$; from which there becomes :

$$p = \frac{-r}{\surd(bb-rr)}, \text{ and } 1 + pp = \frac{bb}{bb-rr}$$

and

$$u - pr = \frac{1}{\surd(bb-rr)}$$

From these there becomes

$$\int \frac{(u-pr)^3 dr}{a^2(1+pp)+(u-pr)^2} = \int \frac{b^4 dr}{\left(a^2+b^2\right)\surd\left(bb-rr\right)} = \frac{\pi b^4}{\left(2a^2+2b^2\right)}$$

on putting $r = b$ after the integration, and with $\pi:1$ denoting the ratio of the periphery to the diameter of the circle. On account of which the resistive force, which acts along the horizontal in the direction *AC* will $= \dfrac{\pi b^4 v}{2\left(a^2+b^2\right)}$. Again since there shall be

$$\int \frac{(u-pr)^2 dr}{a^2\left(1+p^2\right)+(u-pr)^2} = \int \frac{b^2 dr}{\left(a^2+b^2\right)} = \frac{\pi b^3}{\left(a^2+b^2\right)}$$

and

$$\int \frac{(u-pr)^2\left(a^2+u^2-pru\right)dr}{a^2\left(1+p^2\right)+(u-pr)^2} = \int bbdr = b^3$$

will be the resistance force forcing the body upwards $=\frac{ab^3v}{a^2+b^2}$, and the direction of this force will pass through the point *O,* thus so that there shall become

$$AO=\frac{2aa+2bb}{3a}.$$

Moreover the volume of this body will be

$$=\frac{2a}{3}\int dr\sqrt{(bb-rr)}=\frac{\pi abb}{6},$$

and the conic surface, which experiences the resistance will be produced

$$=\int\frac{bdr\sqrt{(aa+bb)}}{\sqrt{(bb-rr)}}=\frac{\pi b}{2}\sqrt{(a^2+b^2)},$$

which indeed are deduced most easily from the known properties of the cone.

COROLLARY 1

640. Since the base of the semi-cone or of the semi-circle *BDb* shall be $=\frac{\pi bb}{2}$, if that may be moved in the same direction *CA* in water its resistance will become $=\frac{\pi bbv}{2}$. From which the resistance of the cone itself will be had to the resistance of the base as b^2 to a^2+b^2, that is as CD^2 to AD^2.

COROLLARIUM 2

641. The semi-circle *BDb* may be changed into an equally large isosceles triangle, and the cone will be changed into a pyramid of which the length *a* shall be the same. Moreover with half the width of the base of this pyramid put in place, $CB=\beta$, and with the height

$$CD=\gamma \text{ there will become } \beta\gamma=\frac{\pi b^2}{2},$$

and the resistance of this pyramid will be $\frac{\beta^3\gamma^3v}{a^2\beta^2+a^2\gamma^2+\beta^2\gamma^2}$.

COROLLARY 3

642. Therefore since there shall become $bb = \frac{2\beta\gamma}{\pi}$, the resistance of the cone equally high and of equal size $= \frac{2\beta^2\gamma^2 v}{\pi a^2 + 2\beta\gamma}$, from which the resistance of the cone itself will be had to the resistance of the pyramid of equal height and base as

$$2a^2\beta^2 + 2a^2\gamma^2 + 2\beta^2\gamma^2 \text{ to } \pi a^2\beta\gamma + 2\beta^2\gamma^2.$$

COROLLARY 4

643. Therefore the resistance of the cone will be equal to the resistance of the pyramid of the same base and of the same height, if there were

$$\beta^2 + \gamma^2 = \frac{\pi\beta\gamma}{2} \text{ or } \frac{\beta}{\gamma} = \frac{\pi}{4} \pm \sqrt{\left(\frac{\pi^2}{16} - 1\right)},$$

that is never. Whereby the resistance of the cone is greater always than the resistance of the pyramid.

EXAMPLE 3

644. Now the base to the cone BDb (Fig. 94) shall be the semi-ellipse described with centre C, in which case the figure will be changed into a scalene cone. But there shall be put

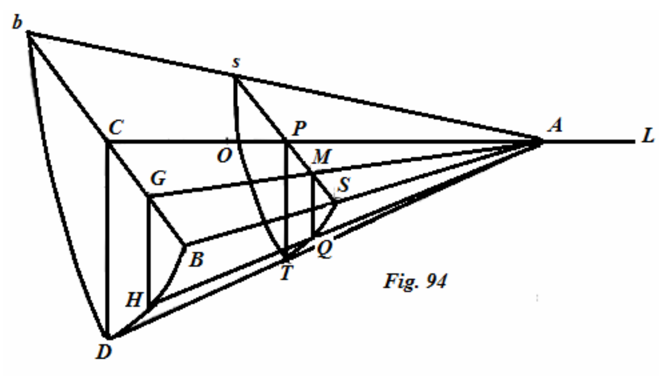


Fig. 94

$$CB = Cb = b, \text{ and } CD = c,$$

from the nature of the ellipse there will become $u = \frac{c}{b}\sqrt{(bb - rr)}$, from which there becomes

$$p = \frac{-cr}{b\sqrt{(bb - rr)}} \text{ and } 1 + pp = \frac{b^4 + (cc - bb)rr}{b^2(b^2 - rr)} -$$

and

$$u-pr=\frac{bc}{\surd(bb-rr)},$$

and hence

$$a^2(1+pp)+(u-pr)^2=\frac{a^2b^4+b^4c^2+a^2(cc-bb)rr}{b^2(b^2-rr)}.$$

From these there is found:

$$\int\frac{(u-pr)^3\,dr}{a^2(1+pp)+(u-pr)^2}=\int\frac{b^5c^3dr}{\left(a^2b^4+b^4c^2+a^2(cc-bb)rr\right)\surd\left(b^2-r^2\right)}$$

of which the integral on putting

$$r=b \text{ is } =\frac{\pi b^2c^2}{2\surd(aa+bb)(aa+cc)};$$

from which the strength of the resistance, which retards the motion and acts in the direction *AC* is

$$=\frac{\pi b^2c^2v}{2\surd(aa+bb)(aa+cc)}.$$

Then there is

$$\int\frac{(u-pr)^2\,dr}{a^2+a^2p^2+(u-pr)^2}=\int\frac{b^4c^2dr}{b^4\left(a^2+c^2\right)+a^2(cc-bb)r^2}$$

of which the integral will depend on the quadrature of the circle if $c>b$, but if $c<b$, the integral will depend on logarithms. But since it shall not pertain much to our principles, how great the body may be forced to rise from the resistance, and in which direction, we will not need to linger over this investigation; but it shall suffice to have determined the true resistance, by which the motion may be retarded.

COROLLARY 1

645. Since in the expression of the resistance of the resistance found

$$\frac{\pi b^2c^2v}{2\surd\left(a^2+b^2\right)\left(a^2+c^2\right)},$$

the semi axes of the conjugate bases *b* et *c* are present equally, these can be interchanged with each other with the resistance remaining the same. That is provided either the semi-axis *b* or the semi-axis *c* of the ellipse *BDb* will produce the same resistance.

COROLLARY 2

646. If the area of the base *BDb* which is $\frac{\pi bc}{2}$ may be called $= A$, on account of

$$\frac{b}{\surd(a^2+b^2)} = \text{sin. ang. } CAB \text{ and } \frac{c}{\surd(a^2+c^2)} = \text{sin. ang.} CAD,$$

the resistance $= Av\sin CAB\sin CAD$; where noting Av to express the resistance of the base *BDb* if that alone may be moved forwards in the direction *CA*.

COROLLARY 3

647. If the circle of the same area as the ellipse *BDb* may be substituted, its radius will be $= \surd bc$, and the resistance which the cone will experience here will be $= \frac{\pi b^2c^2v}{2\left(a^2+bc\right)}$.

Therefore the resistance of the circular cone will be had to the resistance of the elliptical cone of equal bases and heights will be had as

$$\surd(a^2+b^2)(a^2+c^2) \text{ to } a^2+bc.$$

COROLLARY 4

648. Therefore unless there shall be $b = c$, the resistance of a circular cone always will be greater than the resistance of an elliptical cone. Indeed with the squares taken there is seen to become

$$a^4+a^2b^2+a^2c^2+b^2c^2 > a^4+2a^2bc+bbcc,$$

since there is always $bb+cc > 2bc$, unless there shall be $b = c$.

COROLLARY 5

649. Therefore with the elliptic base area *BDb* and with the same height of the cone *AG*, the resistance will be a maxima, if the base shall be changed into a semicircle. Thus the resistance will be smaller, where a greater inequality will intervene between the height and the width of the base.

SCHOLIUM 2

650. Therefore it is clear enough from these in the end the conical body cannot be assigned which may experience the minimum resistance. For if the height of the cone *a* may remain constant, the resistance thus may emerge smaller , where the base *BDb* may be taken smaller with all else being equal. But if in addition a given area may be attributed to the base, the resistance always can be diminished more on putting the inequality between its height *CD* and width *CB* greater. On this account we cannot solve this problem, where either between all the cones completely, or only between those of equal volumes that one is desired, which may experience the minimum resistance. Therefore we shall move on to other kinds of bodies and we shall examine how the resistance may be had in these themselves. Truly we will consider the shapes of bodies of this kind, in which besides a single curve may remain undetermined, just as happens in these conical bodies in which the base alone remained indeterminate.

PROPOSITION 63

PROBLEM

651. *The anterior part submerged of the ship shall be in motion directly forwards experiencing the resistance from a wedge-shaped cone* [here to be called *cono-cuneis* in the original Latin;], *taken in the widest sense AEDHBbhD* (Fig. 96) *from the given curve as far as to the base BDb and thus generated from the vertical line AFE, so that its surface will be terminated by the horizontal lines HF, hF drawn from the singular point of the base of the perimeter BDb and drawn from the right line AE; and this figure shall be progressing directly in the water along the direction of the axis CAL: to determine the resistance which it may experience.*

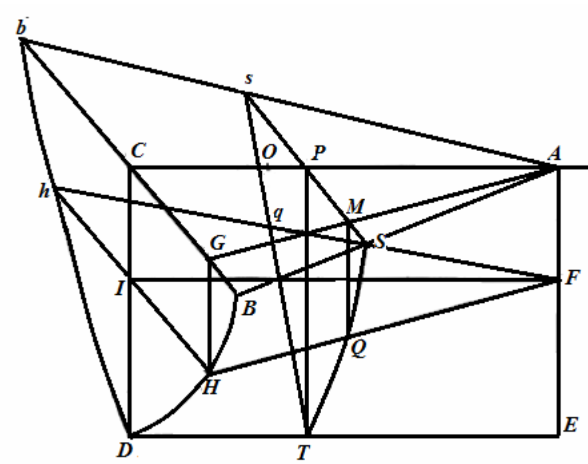


Fig. 96

SOLUTION

Therefore in this figure the vertical plane diametric figure *ACDE* will be a rectangular parallelogram, and the section of the water the isosceles triangle *ABb* ; and in a similar manner all the horizontal sections *FHh* will be triangles with equal legs. Again it is evident from the construction, all the vertical sections made through the line *AE*, of which kind *AGHF* is to be a rectangular parallelogram of this kind. Therefore the whole figure in the prow is defined by the sharpness in the vertical right line *AFE;* moreover the widest section of the normal vertical axis *AC* will be the base of this conical wedge *BDb,* on which the nature of the whole figure depends. Therefore with the length put $AC = a$, the abscissa $CG = r$ may be taken in the base and the applied line $GH = u$, and on account of the base an equation will be given between *u* and *r,* or *u* by *r.* Moreover, there shall be $du = pdr$, and the magnitude *p* will be known by *r.* Now the vertical section *STs* may be considered parallel to the base, for which there shall be $AP = x$, and another section *AGHF* shall be made through *GH* and *AE,* which will be the rectangle, and the side *HF* of that shall be placed in the surface of the figure. Therefore on putting $PM = y$ and $MQ = z$ there will be $z = GH = u$, and $x : y = a : r$, from which there becomes $y = \frac{rx}{a}$. From these there is found :

$$dr = \frac{axdy - aydx}{xx}, \text{ and } dz = du = \frac{apxdy - apydx}{xx}.$$

Therefore this equation will be had for the surface of this wedge-shaped cone:

$$dz = \frac{-apydx}{xx} + \frac{apdy}{x},$$

which compared with the canonical equation $dz = Pdx + Qdy$ gives :

$$P = \frac{-apy}{xx} = \frac{-pr}{x}, \text{ on account of } y = \frac{rx}{u}, \text{ and } Q = \frac{ap}{x}.$$

Hence there arises

$$1 + P^2 + Q^2 = \frac{x^2 + p^2\left(a^2 + r^2\right)}{x^2},$$

and the integral formulas of proposition 61 in which the position *x* is constant are changed in the following, on account of $dy = \frac{xdr}{a}$, since *x* is constant: evidently there shall become

$$\int \frac{P^3dy}{1 + P^2 + Q^2} = -\int \frac{p^3r^3dr}{axx + ap^2\left(a^2 + r^2\right)};$$

and

$$\int \frac{P^2dy}{1 + P^2 + Q^2} = \int \frac{p^2r^2xdr}{axx + ap^2\left(a^2 + r^2\right)},$$

and since there shall be

$$x + Pz = x - \frac{pru}{x}$$

there will become

$$\int \frac{P^2\left(x+Pz\right)dy}{1+P^2+Q^2} = \int \frac{p^2r^2\left(xx-pru\right)dr}{axx+ap^2\left(a^2+r^2\right)}$$

which integrals thus are required to be taken on putting x constant, so that they shall vanish on putting $r=0$, then truly there must be put $r=CB$ or $u=0$. Then for the resistance itself requiring to be found this integral must be taken

$$\int dx \int \frac{P^3dy}{1+P^2+Q^2} = -\int \frac{dx}{a} \int \frac{p^3r^3dr}{xx+p^2\left(a^2+r^2\right)}.$$

But since after the integration of the latter formulae, r and p will not depend on x, the question is reduced to this, so that

$$\frac{-p^3r^3drdx}{ax^2+ap^2\left(a^2+r^2\right)}$$

may be integrated twice by putting x into the one integration and the constants r and p into the other ; likewise it is the case for whichever integration to be made from the beginning. Whereby we may put initially p and r to be constants and there will become for the integral

$$\frac{-p^2r^3dr}{a\sqrt{\left(a^2+r^2\right)}}\text{A tang.}\frac{a}{p\sqrt{\left(a^2+r^2\right)}};$$

after the integration it will be required to use $x=a$. Therefore with the other integration put in place first, and after putting $r=CB$ or $u=0$, there will be produced:

$$\int dx \int \frac{P^3dy}{1+P^2+Q^2} = \frac{-p^2r^3dr}{a\sqrt{\left(a^2+r^2\right)}}\text{A tang.}\frac{a}{p\sqrt{\left(a^2+r^2\right)}}.$$

On this account if the angular-wedge cone may be moved along the axis in the direction CAL with the speed due to the altitude v, the strength of the resistance, by which it is repelled along the direction AC

$$= \frac{-2v}{a} \int \frac{p^2r^3dr}{\sqrt{\left(a^2+r^2\right)}}\text{A tang.}\frac{a}{p\sqrt{\left(a^2+r^2\right)}}.$$

with the integrations being resolved in a similar manner, there will become

$$\int dx\int\frac{P^2dy}{1+P^2+Q^2}=\iint\frac{p^2r^2xdxdr}{axx+ap^2\left(a^2+r^2\right)},$$

which it will be required to integrate twice, in turn either x or truly r and p will be required to be constants; then on putting at first r constant, there will become

$$\int dx\int\frac{P^2dy}{1+P^2+Q^2}=\int\frac{p^2r^2dr}{a}l\frac{\sqrt{\left(a^2+p^2\left(a^2+r^2\right)\right)}}{p\sqrt{\left(a^2+r^2\right)}}=\int\frac{p^2r^2dr}{2a}l\frac{a^2+a^2p^2+p^2r^2}{a^2p^2+p^2r^2}.$$

Therefore on making $r=CB$ or $u=0$ after the integration, the strength of the resistance will be produced , by which the body will be urged to rise vertically upwards

$$=\frac{v}{a}\int p^2r^2dr\,l\frac{a^2+p^2\left(a^2+r^2\right)}{p^2\left(a^2+r^2\right)}.$$

Finally for the point of the application of this force, which shall be required to be found at *O,* this differential formula must be integrated twice

$$\frac{p^2r^2\left(x^2-pru\right)dxdr}{axx+app\left(aa+rr\right)}.$$

In the first place only x may be made variable, and after the integration on putting $x=a$ there will be had for the other integration :

$$\int p^2r^2dr\left(1-\frac{\left(p\left(a^2+r^2\right)+ru\right)}{a\sqrt{\left(a^2+r^2\right)}}\text{A tang.}\frac{a}{p\sqrt{\left(a^2+r^2\right)}}\right);$$

so that the integral, since there were put $u=0$, divided by the integral found before :

$$\int\frac{p^2r^2dr}{2a}\cdot l\frac{a^2+p^2\left(a^2+r^2\right)}{p^2\left(a^2+r^2\right)}$$

will give the distance AO of the point *O,* through which the force of the resistance must pass from the prow *A.*

Q. E. I.

COROLLARY 1

652. Therefore whatever curve may be taken for the base *BDb*, the determination of the resistance contrary to the motion, which is

$$=\frac{-2v}{a}\int\frac{p^2c^3dr}{\surd\left(a^2+r^2\right)}\cdot\text{Atang.}\frac{a}{p\surd\left(a^2+r^2\right)},$$

which is the square of the circle required. But the strength of the contrary resistance, which acts upwards, will depend on logarithms.

COROLLARIUM 2

653. Also from these formulas it is observed each strength of the resistance thus to become smaller when the length shall greater; for each will vanish if there may be put $a=\infty.$ Truly while a becomes greater, the strength of the horizontal resistance will decrease, as well as the vertical resistance.

COROLLARY 3

654. If the length $AC=a$ were so very great with respect of the base *BDb,* so that p and r may vanish before a, the resistance of the horizontal force

$$=\frac{-2v}{aa}\int p^2r^3dr\cdot\text{Atang.}\frac{1}{p};$$

truly the resistance of the vertical force will become

$$=\frac{v}{a}\int p^2r^2dr\,l\,\frac{1+pp}{pp}.$$

COROLLARY 4

655. But if the length $AC=a$ may vanish, so that the whole figure may be changed into the base *BDb* only, then the horizontal resistance will become

$$=\frac{-2v}{a}\int p^2r^2dr\cdot\text{Atang.}\frac{a}{pr}=-2v\int prdr=2v\int udr,$$

just as it will be evident of course, that the vertical resistance will vanish.

COROLLARY 5

656. Certainly the whole volume of this wedge-shaped cone will be found from § 617 , which is

$$2\int -dx\int Qydy = 2\int -dx\int \frac{xprdr}{a}.$$

Which, since x shall be constant in the first integration, will become

$$-2\int \frac{xdx}{a}\int prdr = \int \frac{2xdx}{a}\int udr,$$

and $\int udr$ denotes the area *CBD*. From which the whole volume $= a\int udr$, which indeed is apparent at once.

COROLLARY 6

657.Moreover the surface of this wedge-shaped cone meeting the water is, from § 616

$$= 2\int dx\int dy\sqrt{(1+P^2+ Q^2)} = 2\int dx\int \frac{dr}{a}\sqrt{\left(x^2+p^2\left(a^2+ r^2\right)\right)}.$$

From which, since this differential formula must be integrated twice,

$$\frac{2dxdr}{a}\sqrt{\left(x^2+p^2\left(a^2+ r^2\right)\right)},$$

either x or r will be required to be made constant in turn. Moreover, if at first r may be made constant, there will become for the integral

$$\frac{xdr}{a}\sqrt{\left(x^2+p^2\left(a^2+r^2\right)\right)}+\frac{p^2dr\left(a^2+r^2\right)}{a}\cdot l\frac{x+\sqrt{\left(x^2+p^2\left(a^2+r^2\right)\right)}}{p\sqrt{\left(a^2+r^2\right)}}$$

Therefore on putting $x = a$, the wedge-shaped cone sought will be

$$\int dr\sqrt{\left(a^2+p^2\left(a^2+r^2\right)\right)}+\int\frac{p^2dr\left(a^2+r^2\right)}{a}\cdot l\frac{a+\sqrt{\left(a^2+p^2\left(a^2+r^2\right)\right)}}{p\sqrt{\left(aa+rr\right)}}$$

COROLLARY 7

658. Therefore with the surface of any kind of wedge-shaped cone required to be found it will depend either on logarithms or the quadrature of the hyperbola, and in addition on the quadratures of these others, unless these other differential formulas may be allowed to be integrated.

SCHOLIUM

659. Whatever the shapes of this kind, which we have called here by the name conical wedge, thus began to be considered recently, yet has been seen to be advanced here as the following kind of body, since they have a great affinity with the conical bodies, which have been considered by us as the first kind. Though indeed, if we may consider the simplicity of the construction, in the first place cylindrical and prismatic bodies deserve to be put in the first place, yet these we will not discuss here, since the resistance which they experience and may become known there from the preceding, which have been brought forwards concerned with plane figures, thereupon now may be indicated.
For if all the horizontal sections were similar and equal amongst themselves to the parallel diametric plane, then the resistance will be obtained from the resistance of the single sections, that are required to be introduced into the height of the figure. But if all the sections amongst themselves were equal and similar to the parallel diametrical plane, then equally the resistance of the individual sections will be obtained on being multiplying by the width, just as will be apparent at once on being attended to. But here we accept the wedge-shaped cone in a wider sense than Wallis considered, for we may consider any curve in place of the base *BDb*, since Wallis had assumed only the circle. But generally the nature of all these angular-wedge cone curves will become known from the canonical equation found:

$$dz = -\frac{apydx}{xx} + \frac{apdy}{x},$$

in which since p will be some function of r and $r = \frac{ay}{x}$, p will become some function of x and y of zero dimensions. Whereby for the angular-wedge cone there will become

$$dz = -\frac{ap\left(ydx - xdy\right)}{xx},$$

and since there shall become

$$\frac{xdy - ydx}{xx} = d \cdot \frac{y}{x}$$

will be equal to the function z of x and y of zero dimensions. From which it will be able to be seen for each equation offered for some surface the figure shall be an angular-wedge cone or otherwise. Similarly the nature of a cone-shaped body will become known from the canonical equation found above :

$$dz = \frac{udx}{a} - \frac{pydx}{x} + pdy,$$

which since there shall be $u = \frac{az}{x}$ will be changed into this :

$$\frac{dz}{x} - \frac{zdx}{xx} = \frac{pdy}{x} - \frac{pydx}{xx}.$$

Truly since on account of $r = \frac{ay}{x}$, p is some function of zero dimensions of x and y, $z =$ to the product from x into some function of zero dimensions of x and y themselves. Therefore just as $\frac{z}{x}$ is equal to a function of zero dimensions of x and y the whole equation will be for a conical surface. Therefore every equation between x, y and z, in which these three variables will constitute a number everywhere of the same dimension, will express the nature of this same conic. But every equation between x, y and z will be prepared thus so that only the two variables x and y will implement a number of the same dimensions, will show the surface of this same angular-wedge cone.

EXAMPLE 1

660. *BDb* may be changed into the base of an isosceles triangle, in which case the body *ABDb* will be a mixture from a pyramid and from a cone. The half width of this base $OB = Ob = b$, and the height

$CD = c$, will be $u = c - \frac{cr}{b}$, and $p = -\frac{c}{b}$.

Therefore since the resistance, which this body must experience moving along the direction *CA* with a speed corresponding to the height *v*, the force found pushing back in the direction *AC* shall be

$$= \frac{-2v}{a}\int \frac{p^2 r^3 dr}{\sqrt{\left(a^2 + r^2\right)}} \text{ Atang.} \frac{a}{p\sqrt{\left(a^2 + r^2\right)}}$$

will become in this case

$$= \frac{2c^2 v}{ab^2}\int \frac{r^3 dr}{\sqrt{\left(a^2 + r^2\right)}} \text{ Atang.} \frac{ab}{c\sqrt{\left(a^2 + r^2\right)}}.$$

Moreover since there will become

$$\int \frac{r^3 dr}{\sqrt{\left(a^2+r^2\right)}} = \frac{2a^3}{3} + \frac{\left(r^2-2a^2\right)\sqrt{\left(a^2+r^2\right)}}{3}$$

there will become

$$\int \frac{r^3 dr}{\sqrt{\left(a^2+r^2\right)}} \text{ Atang.} \frac{ab}{c\sqrt{\left(a^2+r^2\right)}}$$

$$= \frac{\left(r^2-2a^2\right)r^3\sqrt{\left(a^2+r^2\right)}}{3} \text{Atang.} \frac{ab}{c\sqrt{\left(a^2+r^2\right)}} + \frac{abc}{3}\int \frac{rdr\left(r^2-2a^2\right)}{a^2c^2+a^2b^2+c^2r^2}$$

$$= \frac{\left(r^2-2a^2\right)r^3\sqrt{\left(a^2+r^2\right)}}{3} \text{Atang.} \frac{ab}{c\sqrt{\left(a^2+r^2\right)}} + \frac{abr^2}{6c} - \frac{a^3b\left(bb+3cc\right)}{6c^2} l \frac{\left(a^2b^2+a^2c^2+c^2r^2\right)}{a^2\left(b^2+c^2\right)}$$

$$-\frac{a^2v\left(bb+3cc\right)}{3bc} l \frac{\left(a^2c^2+a^2b^2+c^2b^2\right)}{a^2\left(b^2+c^2\right)}.$$

[C.Truesdell's corrections have been adopted here from the *OO* edition.]

with such a constant added, so that zero may be produced on putting $r=0$. Now there may be put $r=b$, and the whole resistance which the figure will experience in the direction AC will become

$$= \frac{2ccv\left(bb-2aa\right)r^3\sqrt{\left(a^2+b^2\right)}}{3ab^2} \text{Atang.} \frac{ab}{c\sqrt{\left(a^2+b^2\right)}} + \frac{bcv}{3} - \frac{2a^2v\left(bb+3cc\right)}{3bc} l \frac{bc+\sqrt{\left(a^2b^2+a^2c^2+b^2c^2\right)}}{\alpha\sqrt{\left(b^2+c^2\right)}}$$

$$-\frac{a^2v\left(bb+3cc\right)}{3bc} l \frac{\left(a^2c^2+a^2b^2+c^2b^2\right)}{a^2\left(b^2+c^2\right)}$$

Then the resistive force which acts upwards will be

$$= \frac{v}{a}\int p^2r^2drl \frac{a^2+p^2\left(a^2+r^2\right)}{p^2\left(a^2+r^2\right)} = \frac{ccv}{abb}\int r^2drl \frac{a^2b^2+a^2c^2+c^2r^2}{cc\left(a^2+r^2\right)},$$

which expression cannot be shown more conveniently, on account of which the resistance shall suffice, by which the motion may be retarded, certainly to which we will attend mainly, to be determined by finite quantities.

COROLLARY 1

661. If the length $AC=a$ were much greater than b and c, the resistance may be extracted more conveniently from the differential formula, which will be changed into this :

$$\frac{2ccv}{a^2b^2}\int r^3 dr \text{Atang.}\frac{b}{c},$$

of which the integral, on putting

$$r = b \text{ is } = \frac{b^2c^2v}{2a^2}\text{Atang.}\frac{b}{c},$$

which is the retarding resistance.

COROLLARY 2

662. Therefore if the area of the base *BDb* may be given, which is *bc,* and the length *AC* were extremely great, the resistance thus will become smaller, where the fraction $\frac{b}{c}$ will have been made smaller, that is, where the angle *BDb* was more acute. Therefore the resistance will be a maximum, if the ratio $b:c$ may be taken infinitely great, yet in which case the resistance will be finite, on account of $\text{Atang.}\infty = \frac{\pi}{2}$.

COROLLARY 3

663. Also the resistance can be expressed generally by a series conveniently for any length *a.* Indeed since there shall be

$$\text{A tang.}\frac{ab}{c\sqrt{\left(a^2+r^2\right)}} = \frac{ab}{\sqrt{\left(a^2+r^2\right)}} - \frac{a^3b^3}{3c^3\left(a^2+r^2\right)^{\frac{3}{2}}} + \frac{a^5b^5}{5c\ \left(a^2+r^2\right)^{\frac{5}{2}}} - \text{etc.}$$

the resistance will be, on putting $r = b$ after the integration:

$$\frac{2c^2v}{abb}\int\frac{r^3dr}{\sqrt{\left(a^2+r^2\right)}}\text{Atang.}\frac{ab}{c\sqrt{\left(a^2+r^2\right)}}$$

$$= v\left(\begin{array}{l} bc - \dfrac{2a^2\left(b^2+3c^2\right)}{3bc}l\sqrt{\dfrac{a^2+b^2}{a^2}} - \dfrac{a^4b\left(3bb+5cc\right)}{1\cdot3\cdot5\cdot c^3\left(a^2+b^2\right)} \\ + \dfrac{a^6b^3\left(5bb+7cc\right)}{2\cdot5\cdot7c^5\left(a^2+b^2\right)^2} - \dfrac{a^8b^5\left(7bb+9cc\right)}{3\cdot7\cdot9c^7\left(a^2+b^2\right)^3} + \dfrac{a^{10}b^7\left(9bb+11cc\right)}{4\cdot9\cdot11c^9\left(a^2+b^2\right)^4} - \text{etc.} \end{array}\right)$$

which converges strongly if a were very small.

COROLLARY 4

664. But if a series may be desired, which shall converge strongly, if a shall be a very large quantity, the resistance retarding the motion

$$=\frac{4a^2c^2v}{3bb}\text{Atang.}\frac{b}{c}-\frac{2ccv\left(2a^2-b^2\right)\surd\left(a^2+b^2\right)}{3ab^2}\text{Atang.}\frac{ab}{c\surd\left(a^2+b^2\right)}-\frac{2bc^2v}{3\left(bb+cc\right)}$$
$$+\frac{v\left(b^2+3c^2\right)}{3bc}\left(\frac{b^4c^4}{2a^2\left(b^2+c^2\right)^2}-\frac{b^6c^6}{3a^4\left(b^2+c^2\right)^5}+\frac{b^8c^8}{4a^6\left(b^2+c^2\right)^4}-\text{etc.}\right).$$

COROLLARY 5

665. Truly the volume of this body is found $=\frac{abc}{2}$, moreover the surface with its base and the water section removed will be

$$=\int\frac{dr}{b}\surd\left(a^2b^2+a^2c^2+c^2r^2\right)+\frac{cc}{abb}\int dr\left(a^2+r^2\right)l\frac{ab+\surd\left(a^2b^2+a^2c^2+c^2r^2\right)}{c\surd\left(a^2+r^2\right)}.$$

Of which the integral on putting

$$\frac{bb+cc}{cc}=m,$$

and by making $r=b$, is found

$$=\frac{c}{2}\surd\left(ma^2+b^2\right)+\frac{ma^2c}{2b}l\frac{b+\surd\left(ma^2+b^2\right)}{a\surd m}+\frac{cc\left(3aa+bb\right)a^2c}{ab}l\frac{ab+c\surd\left(ma^2+b^2\right)}{c\surd\left(a^2+b^2\right)}$$
$$+\frac{cc}{3bb}\int\frac{\left(3a^2+r^2\right)\left(\left(m-1\right)ac+b\surd\left(ma^2+r^2\right)\right)r^2dr}{\left(a^2+r^2\right)\left(ab+a\surd\left(ma^2+r^2\right)\right)\surd\left(ma^2+r^2\right)}$$

thus so that the integration of this formula remains.

COROLLARY 6

666. The case where $m=2$ or $b=c$ becomes a little more simple, for the surface will be produced

$$=\frac{c}{2}\sqrt{\left(2a^2+c^2\right)}+a^2l\frac{c+\sqrt{\left(2a^2+c^2\right)}}{a\sqrt{2}}+\frac{c\left(3a^2+c^2\right)}{3a}l\frac{a+\sqrt{\left(2a^2+c^2\right)}}{\sqrt{\left(a^2+c^2\right)}}$$

$$+\frac{1}{3}\int\frac{\left(3a^2+r^2\right)r^2dr}{\left(a^2+r^2\right)\sqrt{\left(2a^2+r^2\right)}}=\frac{c}{2}\sqrt{\left(2a^2+c^2+a^2\right)}l\frac{c+\sqrt{\left(2a^2+c^2\right)}}{a\sqrt{2}}+\frac{c\left(3a^2+c^2\right)}{3a}l\frac{a+\sqrt{\left(2a^2+c^2\right)}}{\sqrt{\left(a^2+c^2\right)}}$$

$$+\frac{c}{6}\sqrt{\left(2a^2+r^2\right)}+\frac{aa}{3}l\frac{c+\sqrt{\left(3a^2+c^2\right)}}{a\sqrt{2}}-\frac{2a^2}{3}\text{Atang.}\frac{c}{\sqrt{\left(2a^2+c^2\right)}}.$$

Therefore the surface sought will be

$$=\frac{2c}{3}\sqrt{\left(2a^2+c^2\right)}+\frac{4a^2}{3}l\frac{c+\sqrt{\left(2a^2+c^2\right)}}{a\sqrt{2}}+\frac{c\left(3a^2+c^2\right)}{3a}l\frac{a+\sqrt{\left(2a^2+c^2\right)}}{\sqrt{\left(a^2+c^2\right)}}-\frac{2a^2}{3}\text{Atang.}\frac{c}{\sqrt{\left(2a^2+c^2\right)}}.$$

COROLLARY 7

667. If in addition there shall be $c=a$, thus so that there shall become $AC=CB=CD$, the surface will be

$$=\frac{2aa}{\sqrt{3}}+\frac{4aa}{3}l\left(2+\sqrt{3}\right)-\frac{\pi a^2}{9}\ ;$$

the approximate value of this expression is $a^2\cdot 2{,}56156,$ or the surface itself will be had to the base approximately as $2\frac{1}{2}$ to 1.

EXEMPLUM 2

668. Now if our Wallis angular-wedge cone, or the base *BDb,* may be changed into a semicircle, the diameter of which shall be $CB=CD=b$. Therefore there will become $u=\sqrt{\left(b^2-r^2\right)}$, and thus

$$p=\frac{-r}{\sqrt{\left(bb-rr\right)}},$$

therefore with this value substituted, the force resisting the motion will become

$$=\frac{2v}{a}\int\frac{r^5dr}{\surd\left(b^2-r^2\right)\surd\left(a^2+r^2\right)}\ \text{Atang.}\frac{a\surd\left(bb-rr\right)}{r\surd\left(a^2+r^2\right)}.$$

But although there shall become

$$\int\frac{r^5dr}{\surd\left(b^2-r^2\right)\surd\left(a^2+r^2\right)}=\frac{\left(a^2+r^2\right)^{\frac{3}{2}}}{3}+\left(a^2-b^2\right)\surd\left(a^2+r^2\right)+\frac{b^4}{2\surd\left(a^2+b^2\right)}l\frac{\surd\left(a^2+b^2\right)+\surd\left(a^2+r^2\right)}{\surd\left(a^2+b^2\right)-\surd\left(a^2+r^2\right)}$$

yet hence it is not much help for the whole integration. For if the tangent of which the arc is

$$\frac{a\surd\left(b^2-r^2\right)}{r\surd\left(a^2+r^2\right)}$$

may be resolved in series, indeed the integration of the individual terms in

$$\frac{r^5dr}{\left(b^2-r^2\surd a^2+r^2\right)}$$

may avoid being expanded out more easily, but an infinite constant must be added, from which zero may be produced on putting $r=0$. This inconvenience may be avoided in a certain way if in place of this arc, the equivalent expression may be inserted

$$\frac{\pi}{2}-\text{Atang.}\frac{r\surd\left(a^2+r^2\right)}{a\surd\left(bb-rr\right)},$$

but in whatever manner the calculation may be put in place, nothing is derived worthy of the effort, on account of which we shall abandon the angular-wedge cone, evidently to be progressing now to the treatment of another common kind of bodies, evidently to rounded bodies.

PROPOSITION 64

PROBLEM

669. *ABb shall be the section of the water with some curve agreeing with the two equal and similar parts ACB, ACb* (Fig. 97), *and all the vertical sections of the semicircle STs normal to the diametric plane ACD, or which return to the same height, shall generate the body ABDb by the rotation of the curve ACB about the axis AC; and this body shall be moved in the water straight in the direction CAL; to determine the resistance which it may experience.*

SOLUTION

It is understood from the construction of this body, not only the diametric plane *ATD* but also all the sections passing through the axis *AC* to be similar and equal curves of the semi-section of the water *ASBC*. Therefore since the given curve *ASB* may be put in place, on calling $AP = x$, and $PS = PT = s$, an equation will be given between x and s, or s will be a certain function of x, thus so that if there may be put $ds = pdx$, p equally is going to be a function of x. Now with both the remaining coordinates taken

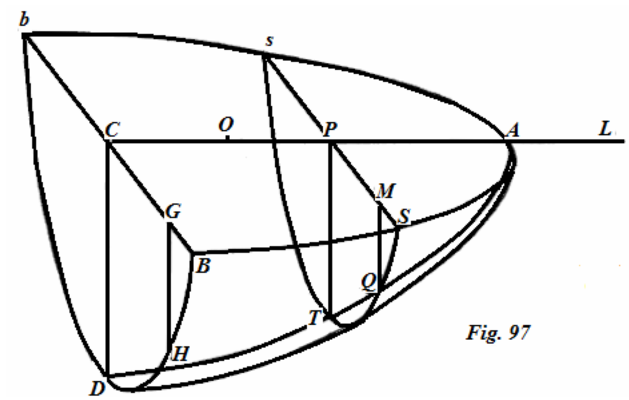


Fig. 97

$PM = y$ and $MQ = z$, since the section *SQTs* is the semicircle with centre *P* described, the radius of which is $PS = PT = s$, there will become $z^2 + y^2 = s^2$ and $z = \surd(s^2 - y^2)$;; from which there becomes

$$dz = \frac{sds - ydy}{\surd(s^2 - y^2)} = \frac{psdx - ydy}{\surd(s^2 - y^2)},$$

from which equation the nature of the surface of this body is expressed. Therefore this equation, if compared with the canonical equation assumed above: $dz = Pdx + Qdy$, will become

$$P = \frac{ps}{\surd(s^2 - y^2)} \text{ and } Q = \frac{-y}{\surd(s^2 - y^2)}.$$

Now we may put the section *BDb* to be the widest of all with there being $AC = a$, or the width *Bb* to be the maximum; and the whole surface *ABDb* will be experiencing the resistance; and the speed with which this body is moving forwards in the water along the direction *AL* must correspond to the height v. From these premises from prop. 61, the resistance will be defined in the following manner: since there shall become

$$1 + P^2 + Q^2 = \frac{(1 + p^2)s^2}{ss - y^2},$$

there will become:

$$\frac{P^3dy}{1 + P^2 + Q^2} = \frac{p^3sdy}{(1 + pp)\surd(ss - y^2)} \text{ and } \frac{P^2dy}{1 + P^2 + Q^2} = \frac{p^2dy}{1 + p^2}$$

and

$$\frac{P^2(x+Pz)dy}{1+P^2+Q^2}=\frac{p^2(x+ps)dy}{1+p^2},$$

which differentials on putting x and hence the dependent quantities p and s to be constants, are to be taken thus in order that they may vanish on putting $y=0$, from which done there must be put

$$y=PS=s.$$

Moreover in this manner there will be found

$$\int\frac{P^3dy}{1+P^2+Q^2}=\frac{\pi p^3 s}{2\left(1+p^2\right)}$$

with π denoting the periphery of the circle, of which the diameter is 1; and

$$\int\frac{P^2dy}{1+P^2+Q^2}=\frac{p^2 s}{1+p^2},$$

as well as

$$\int\frac{p^2(x+Pzdy)}{1+P^2+Q^2}=\frac{p^2 s(x+ps)}{1+pp}.$$

Now on putting x, p and as s to be had as variables the horizontal force, by which the body is repelled in the direction AC

$$=\pi v\int\frac{p^3 sdx}{1+p^2};$$

in which the integral, since it will have been taken thus, so that it shall vanish on putting $x=0$, there must become also $x=a$. Then the force of the resistance, by which the body will be acted on upwards, is

$$=2v\int\frac{p^2 sdx}{1+pp},$$

and this force will be acting through the point of the axis O with there being

$$AO=\frac{\int\frac{p^2 sdx(x+ps)}{1+pp}}{\int\frac{p^2 sdx}{1+pp}},$$

with the individual integrals taken thus so that they may vanish on putting $x=0$, and then on making $x=a$. Q.E.I.

COROLLARY 1

670. If the section of the water *ABb* shall have the tangent at *B* normal to *Bb* or parallel to the axis *AC*, then all the tangents of the plane surface at the points *H* of the section *BDb* will be normal to this section itself.

COROLLARY 2

671. In the similar manner as the tangent angle of the water-section is established with the axis *PA* at *S,* all the tangential planes will establish the same angle at the individual points *Q* of the section *STs*, and the individual elements *Q* of the section *STs* will experience the same resistance as the element situated at *S* experiences.

COROLLARY 3

672. In order to know the volume of this whole body from § 617, the first requiring to be integrated is the differential :

$$-Qydy = \frac{y^2 dy}{\surd(ss-yy)},$$

the integral of which, on putting $y = s$ after the integration is $= \frac{\pi ss}{4}$. From which the whole volume becomes $= \frac{\pi}{2}\int ssdx$, on putting $x = a$ after the integration.

COROLLARY 4

673. Then since the surface *ABDb* in general shall become

$$= 2\int dx \int dy \surd(1+P^2+Q^2),$$

the surface of our rounded solid

$$= 2\int dx \int \frac{sdy\surd(1+pp)}{\surd(ss-yy)} = \pi\int sdx\surd(1+pp),$$

in which with the integral thus taken, so that it may vanish on putting $x = 0$, there must be taken $x = a$.

COROLLARY 5

674. If the whole round solid, which is generated while the figure *ACB* is turned around the axis *AC,* may move completely in the water along the direction of the axis *CAL*, then it will experience twice as great a resistance to the motion in the opposite direction, and that resistance thus will become $=2\pi v\int\frac{p^3 sdx}{1+pp}$.

SCHOLIUM

675. Round bodies of this kind to be almost the only ones considered by those, who have investigated the resistance I calculate, but they have sought the resistance of these bodies by another tedious manner, appropriate for a body of this kind. Indeed they have derived the resistance from that consideration, which we have indicated in the second corollary, though by a way which is much easier, than that which we have followed here, yet since it may not be apparent for other kinds of bodies, we have preferred the general method to be used. Hence moreover the nature of all rounded bodies generally becomes known by the general equation found from those $z^2+y^2=s^2$ evidently with the abscissa x on the axis AC, z^2+y^2 is always equal to a certain function of x, and as often as such an equation occurs, that will be applied just as often to the volume of the rounded shape. But so that the resistance of bodies of this kind may become known more fully, it will help to establish some particular cases, for which the curve determined for the section of the water *ACB* is accepted.

EXAMPLE 1

676. The isosceles triangle *ABb* shall be the first section of the water (Fig. 94), or the body *ABDb* of half the right circular cone, which case, though now it has been treated before, yet that has been seen to be brought forwards here, where it may be considered more conveniently, and thus that same proposition will be made clear. And thus with the radius of the base put to become $BC=CD=b$ there will become $a:b=x:s$, and thus

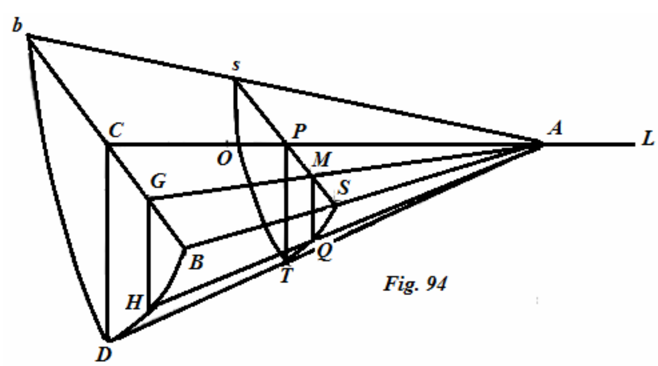


Fig. 94

$$s=\frac{bx}{a},\ \text{and}\ p=\frac{b}{a}.$$

From which the resistance of the horizontal force will become

$$=\pi v\int\frac{p^3sdx}{1+p^2}=\frac{\pi b^4}{aa}v\int\frac{xdx}{a^2+b^2}=\frac{\pi b^4v}{2\left(a^2+b^2\right)};$$

moreover the vertical force arising from the resistance, by which the body will be raised from the water, will become

$$=2v\int\frac{p^2sdx}{1+pp}=\frac{2b^3v}{a}\int\frac{xdx}{aa+bb}=\frac{ab^3v}{a^2+b^2}.$$

Finally the point O at which this force will be applied, will be defined thus: since there shall be

$$AO=\frac{\int p^2sdx\left(x+ps\right):\left(1+pp\right)}{\int p^2sdx\left(1+pp\right)}$$

for our case there will become

$$AO=\frac{\left(aa+bb\right)\int xxdx}{aa\int xdx}=\frac{2\left(aa+bb\right)}{3a},$$

which all agree precisely with § 639 found above.

EXEMPLUM 2

677. *ABb* shall be the semicircular water section described with centre C (Fig. 97), of which therefore the radius $AC=CB=CD$ will be $=a$, therefore in this case our body will be changed into the fourth part of the sphere described with centre C with radius $AC=a$. Therefore from the nature of the circle there will be $s=\surd(2ax-xx)$ and

$$p=\frac{a-x}{\surd(2ax-xx)},\text{ and } 1+pp=\frac{aa}{2ax-xx}.$$

With these substituted there will be produced

$$\frac{p^3sdx}{1+pp}=\frac{\left(a-x\right)^3dx}{aa}$$

of which the integral is

$$\frac{a^2}{4}-\frac{(a-x)^4}{4a^2},$$

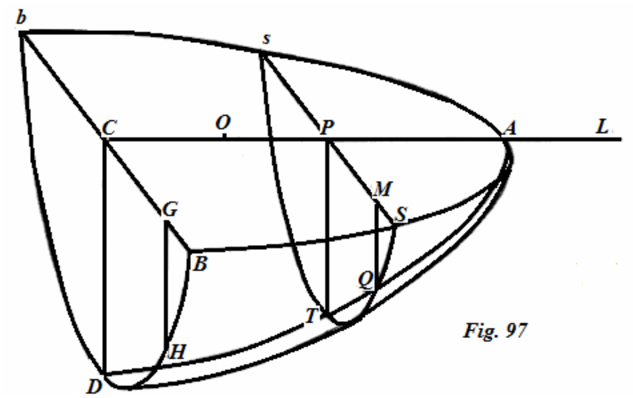


Fig. 97

which on putting $x=a$ becomes $=\frac{a^2}{4}$. Therefore the horizontal resistance, which this part of the sphere experiences in its motion, will be $=\frac{\pi a^2 v}{4}$. Then since

$$\frac{p^2 sdx}{1+pp}=\frac{(a-x)^2\,dx}{aa}\surd\left(2ax-xx\right),$$

its integral will be on putting $x=a$ after the integration $=\frac{\pi a^2}{16}$, from which this body will be forced upwards by the force of the resistance $=\frac{\pi a^2 v}{8}$. Finally since there shall be $x+ps=a$, there will become

$$\int\frac{p^2 sdx\left(x+ps\right)}{\left(1+pp\right)}=\int\frac{(a-x)^2\,dx}{a}\surd\left(2ax-xx\right)=\frac{\pi a^3}{16},$$

from which point O through which the vertical force of the resistance will pass, incident on the centre of the sphere C. Truly the volume of this spherical quadrant will be

$$=\frac{\pi}{2}\int ssdx=\int\left(2ax-xx\right)dx=\frac{\pi a^3}{3},$$

and its surface

$$=\pi\int sdx\surd(1+pp)=\pi\int adx=\pi a^2;$$

which indeed follow at once from the nature of the sphere.

COROLLARY 1

678. Therefore the vertical resistive force, which is $=\frac{\pi a^2 v}{8}$, is half as great as its horizontal force, by which the motion is retarded. Therefore the mean direction of the resistance will pass through O and the angle situated in the vertical diametrical plane ACD will constitute an angle with AC of which the tangent will be $=\frac{1}{2}$.

COROLLARY 2

679. Since the area of the base *BDb* shall be $=\frac{\pi a^2}{2}$ if the base alone may be moving in the water along *CA* with the same speed, its resistance would become $=\frac{\pi a^2 v}{2}$; thus so that the resistance of the horizontal figure *ABDb* shall be half as great as the resistance of the base.

COROLLARY 3

680. Also it is understood how great a resistance a whole globe may experience moving in water ; indeed since it shall be opposed by half of its resistance, that resistance will become $=\frac{\pi a^2 v}{4}$, if its radius may be put $=a$. And thus a globe moving in water will experience half as much resistance as its maximum circle will experience.

COROLLARY 4

681. Hence the resistances, which diverse globes moving in water may experience, will be in the ratio composed from double the diameters and double the speeds, with which they are progressing.

EXAMPLE 3

682. The figure floating on water shall be part of an elliptic spheroid *ABDb* of this kind (Fig. 97), so that the water section *ABb* shall be a semi ellipse having the centre at *C* and the conjugate semi axes shall be $AC=a$ and $BC=b$, from the nature of the ellipse there will become:

$$s=\frac{b}{a}\surd\left(2ax-xx\right) \text{ and hence } p=\frac{b(a-x)}{a\surd\left(2ax-xx\right)}$$

and

$$1+pp=\frac{a^2b^2+2a(a^2-b^2)x-\left(a^2-b^2\right)xx}{a^2(2ax-xx)}.$$

Therefore for the resistance requiring to be known the following integral formulas are required to be known, the first of which is $\int\frac{p^3sdx}{1+pp}$, which will be changed into

$$\frac{b^4}{a^2}\int\frac{(a-x)^3\,dx}{a^2b^2+2a(a^2-b^2)x-\left(a^2-b^2\right)xx},$$

the integral of which is

$$\frac{-b^4}{2\left(a^2-b^2\right)}+\frac{a^2b^4}{\left(a^2-b^2\right)^2}l\frac{a}{b}.$$

From this the force of the resistance of which its contrary direction is AC will be

$$=\pi b^2v\left(\frac{a^2b^2}{\left(a^2-b^2\right)^2}l\frac{a}{b}-\frac{b^2}{2\left(a^2-b^2\right)}\right),$$

or the same force will be expressed by the series

$$=\frac{\pi b^4v}{2a^2}\left(\frac{1}{2}+\frac{a^2-b^2}{3a^2}+\frac{\left(a^2-b^2\right)^2}{4a^4}+\frac{\left(a^2-b^2\right)^3}{5a^6}+\frac{\left(a^2-b^2\right)^4}{6a^8}+\text{etc.}\right)$$

which thus will converge more, where the difference between a and b were smaller. Then since there shall be

$$\int\frac{p^2sdx}{1+pp}=\frac{b^3}{a}\int\frac{(a-x)^2\,dx\sqrt{\left(2ax-xx\right)}}{a^2b^2+2a(a^2-b^2)x-\left(a^2-b^2\right)x^2},$$

the integral of this, on putting $x=a$, will become the following quantity $\frac{\pi ab^3}{4\left(a+b\right)^2}$; from which the vertical force of the resistance is

$$=\frac{\pi ab^3v}{4\left(a+b\right)^2};$$

but we cannot determine the direction itself of this same force or the place of application on account of the prolixity of the calculation.

COROLLARY 1

683. If the ellipse *ABb* may be changed into a circle thus so that there shall be $a = b$; then the horizontal resistance will be freed from logarithms, and the given series will become $= \frac{\pi a^2 v}{4}$. Truly the force by which it is pushed upwards will become $= \frac{\pi a^2 v}{8}$, as now found before.

COROLLARY 2

684. If the ellipse *ABb* may differ minimally from a circle thus so that there shall be $b = a + \alpha$, with α denoting an extremely small quantity, the force of the resistance from the series along $AC = \frac{\pi a^2 v}{4} + \frac{2\pi a\alpha v}{3} = \frac{\pi b^2 v}{4} + \frac{\pi b\alpha v}{6}$, on account of $a = b - \alpha$.

COROLLARY 3

685. Therefore with the axis *ACa* remaining, the resistance thus will emerge greater, where $BC = b$ increases more. But if b may remain the same, the resistance will decrease with the increase of the axis $AC = a$. And from that expression for the resistance

$$\pi b^2 v\left(\frac{a^2b^2}{\left(a^2-b^2\right)^2} l\frac{a}{b} - \frac{b^2}{2\left(a^2-b^2\right)}\right)$$

it is understood if a may become infinitely great, then the resistance vanishes completely.

COROLLARY 4

686. Therefore the resistance retarding the motion will be diminished by augmenting the length of the elliptic spheroid AC and by diminishing the width $BC = b$. From which so that where the axes of the ellipse were more unequal between themselves, thus a smaller resistance will result.

COROLLARY 5

687. Since the volume in general shall be $= \frac{\pi}{2}\int ssdx$ for our case the volume of the elliptic spheroid

$$ABDb = \frac{\pi b^2}{2aa}\int\left(2ax - xx\right)dx = \frac{\pi ab^2}{3},$$

on putting $x = a$ after the integration.

COROLLARY 6

688. Finally the surface of this spheroid, which in general is

$$\pi\int sdx\surd(1+pp),\text{will become } = \frac{\pi b}{a^2}\int dx\surd\left(a^2b^2+2a\left(a^2-b^2\right)x-\left(a^2-b^2\right)x^2\right),$$

which expression on putting $a - x = u$ will be changed into this

$$-\frac{\pi b}{a^2}\int du\surd(a^4-(a^2-b^2)u^2) = \frac{-\pi a^2 b}{2\surd(a^2-b^2)}\left(\text{Asin}\frac{u\surd\left(aa-bb\right)}{aa}+\frac{u\surd\left(aa-bb\right)}{a^4}\surd(a^4-(aa-bb)uu)\right)$$
$$+\frac{\pi a^2 b}{2\surd(a^2-b^2)}\left(\text{Asin}\frac{\surd\left(aa-bb\right)}{a}+\frac{b\surd\left(aa-bb\right)}{a^2}\right).$$

Therefore on putting $x = a$ or $u = 0$ the total surface will be produced

$$= \frac{-\pi a^2 b}{2\surd(a^2-b^2)}\left(\text{Asin}\frac{\surd\left(a^2-b^2\right)}{a}+\frac{b\surd\left(a^2-b^2\right)}{aa}\right) = \frac{\pi bb}{2}+\frac{\pi a^2 b}{2\surd(a^2-b^2)}\text{Asin}\frac{\surd\left(a^2-b^2\right)}{a}.$$

COROLLARY 7

689. Whereby if a and b may not differ from each other in turn, on account of

$$\text{Asin}\frac{\surd\left(a^2-b^2\right)}{a} = \text{Atang}\frac{\surd\left(a^2-b^2\right)}{a} = \frac{\surd(a^2-b^2)}{b}-\frac{\left(a^2-b^2\right)^{\frac{3}{2}}}{3b^3}+\frac{\left(a^2-b^2\right)^{\frac{5}{2}}}{5b^5}-\text{etc.}$$

this expression will suffice for finding the surface:

$$\frac{\pi}{2}\left(bb+aa-\frac{a^2\left(aa-bb\right)}{3b^2}+\frac{aa\left(a^2-b^2\right)^2}{5b^4}-\frac{aa\left(a^2-b^2\right)^3}{7b^6}+\text{etc.}\right)$$

which is strongly convergent.

PROPOSITION 65

PROBLEM

690. *All the vertical sections STs shall remain normal to the axis AC of the semicircle as before* (Fig. 97), *and the nature of the curve ASBC of the section of the water may be sought which shall form the volume of this kind ABDb, so that along this direction CAL it may experience the minimum resistance, and likewise truly it shall be especially large.*

SOLUTION

With the abscissa $AP = x$ put in place as before in the water section, and with the applied line $PS = s$, and $ds = pdx$, the resistance, which this rounded volume will experience corresponding to this water section, will be as $\int \frac{p^3 sdx}{1+pp}$, which formula therefore must be a minimum. This term $\frac{p^3 s}{1+pp}$ will be differentiated on the boundary, and its differential will become

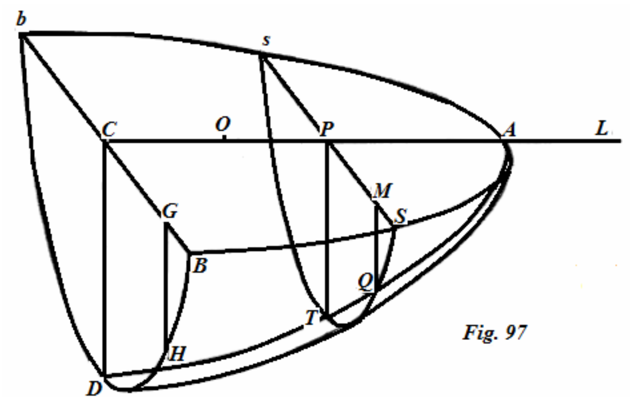


Fig. 97

$$\frac{p^3 ds}{1+pp} + \frac{(3p^2+p^4)sdp}{(1+pp)^2},$$

from which the following rule given above in § 523 will arrive at this value:

$$\frac{p^3}{1+pp} - \frac{1}{dx} d.\frac{(3p^2+p^4)s}{(1+pp)^2},$$

which must be put $=0$, if the volume may be desired, which will be experiencing the absolute minimum resistance. But since in addition the volume must be a maximum, truly the volume will be as $\int ssdx$, and to this formula there shall correspond this same value $2s$, some multiple of which is equal to the value required to be put in place. Hence therefore this same equation will be obtained

$$\frac{2s}{c} = \frac{p^3}{(1+pp)^2} - \frac{1}{dx} d \cdot \frac{(3pp+p^4)s}{(1+pp)^2};$$

it may be multiplied by *ds or pdx,* there will be had

$$\frac{2sds}{c}=\frac{p^3ds}{1+pp}-pd\cdot\frac{\left(3pp+p^4\right)s}{\left(1+pp\right)^2}=d\cdot\frac{p^3ds}{1+pp}-d\cdot\frac{\left(3pp+p^4\right)ps}{\left(1+pp\right)^2}$$

from which the integral will become

$$\frac{ss}{c}-f=\frac{p^3s}{1+pp}-\frac{\left(3pp+p^4\right)ps}{\left(1+pp\right)^2}=\frac{-2p^3s}{\left(1+pp\right)^2}$$

or

$$ss=cf-\frac{2cp^3s}{\left(1+pp\right)^2};$$

and from the equation it is understood it is not possible for $s=0$, yet which condition is required of the question, unless there shall be $f=0$. Therefore there may be put $f=0$, and c negative, so that there will become

$$s=\frac{2cp^3}{\left(1+pp\right)^2}.$$

But since there shall be $ds=pdx$, there will become

$$x=\frac{s}{p}+\int\frac{sdp}{pp}=\frac{2cpp}{\left(1+pp\right)^2}+2c\int\frac{pdp}{\left(1+pp\right)^2}=\frac{2cpp}{\left(1+pp\right)^2}-\frac{c}{1+pp}+\text{Const.}$$

from which there will arise

$$x=\text{ Const.}+\frac{-c+cpp}{\left(1+pp\right)^2}.$$

Truly since x must vanish in the same case as s, but s may vanish in two cases, of which the one is if $p=0$, and the other if $p=\infty$, the constant must be determined from that. Therefore at the point *A,* there shall become $p=0$, or the tangent of the curve AC at A shall lie on the same right line *AL,* and the $\text{Const.}=c$, from which there will become

$$x=\frac{3cpp+cp^4}{\left(1+pp\right)^2},$$

and

$$s=\frac{2cp^3}{\left(1+pp\right)^2},$$

and this curve will generate the volume, which will experience the minimum resistance on account of the acute cusp at *A*, truly in the other case, where at *A* there becomes $p=\infty$, certainly a body will be produced of the maximum resistance, which case equally lies hidden in the question. On account of which the curve sought thus will be prepared thus so that for the abscissa

$$x=\frac{3cpp+cp^4}{(1+pp)^2},$$

will correspond the applied line

$$s=\frac{2cp^3}{(1+pp)^2}$$

from which it is understood the section of the water sought *ASB* required to satisfy the question to become an algebraic curve; which thus among all the other equal volumes generated that lead to such a volume, so that in the direction of the axis *AL,* the motion will experience the minimum resistance.
Q.E.I.

COROLLARY 1

691. Since the curve *ASB,* which produces the solid figure of the maximum resistance, will result from the same equation by increasing the abscissa x by a constant quantity, it is understood that each curve, evidently both that solid of the minimum resistance, as well as that which will produce a solid of the maximum resistance, to be part of the same curve continued.

COROLLARY 2

692. Therefore since s will vanish for two cases, or the curve *ASB* crosses the axis *AC* at two points, without doubt the first if $p=0$ in which case also x becomes $=0$ and then if $p=\infty$, in which case there becomes $x=c$, the first concurrence will give the curve producing the minimum resistance, the latter truly the curve, to which the solid of the maximum volume will correspond.

COROLLARY 3

693. Because the equation found

$$ss=cf-\frac{2cp^3s}{(1+pp)^2}$$

on putting $f=0$, is divisible by *s*, it is apparent the equation $s=0$ also to contain the case contained in the question. Moreover it is evident this case to provide that curve which produces the volume of the smallest capacity.

COROLLARY 4

694. Since there shall be

$$x=\frac{3cp^2+cp^4}{(1+pp)^2} \text{ and } s=\frac{2cp^3}{(1+pp)^2}$$

it is understood by continually attributing greater values of *p* than the initial value made from $p=0$, both *x* as well as *s* to increase to a certain terminal value, then truly to decrease again. But *x* and *s* will be made maxima if there shall be put $p=\sqrt{3}$, in that place where the tangent of the curve constitutes an angle of 60 degrees with the axis *AC*. Moreover in this case there will be

$$x=\frac{9c}{8} \text{ and } s=\frac{3c\sqrt{3}}{8}.$$

COROLLARY 5

695. But if this equation may be compared with §532, this curve may be taken to be compared with that curve found above, which amongst all the others containing the same area it shall be allowed to have the minimum resistance. Therefore the curve found here will be that triangular curve *AMBCDNA* [Fig. 81, Ch. 5].

COROLLARY 6

696. Therefore the portion *AMB* of this curve rotated about the axis *AC* will produce the solid, which likewise will have the maximum capacity, and which will experience the minimum resistance moved along the direction of the axis *CA*. Truly the other part *BCD* rotated around the same axis *CE* will give the solid experiencing the maximum resistance.

COROLLARY 7

697. Therefore in accordance with this curve, which is itself similar and equal on both sides of the axis *ACE*, *CA* will be the tangent at *A;* from which it may ascend and descend as far as to *B* and *D,* with there being

$$AE=\frac{9c}{8} \text{ and } BE=DE=\frac{3c\sqrt{3}}{8}.$$

Then from the cusps *B* and *D* united with the axis at *C* with there being $AC=c$: truly the three parts of this *AMB, BCD* and *AND* will be equal and similar amongst themselves.

SCHOLIUM

698. This same problem differs from the others, which will have been treated by this argument, with that condition being omitted, by which likewise it has been accustomed to propose a solid of the greatest capacity to be required, thus so that between all the curves entirely that shall be tried to be determined, which shall produce a solid from being rotated about the axis, so that in the direction of the axis the minimum resistance may be experienced. But in this manner no suitable curve is found satisfying what is sought, indeed this same case will be resolved from our solution by putting $c=\infty$, from which there becomes

$$s=\frac{f\left(1+pp\right)^{2}}{2p^{3}}$$

from which at not time can there become $s=0$, and thus the desired curve will never concur with the axis, that which is contrary to the intended condition. On account of which it has been viewed to omit this same question completely, and in its place I shall endeavour to propose a case, from which besides the minimum resistance the maximum capacity is required. Indeed this question thus has been adapted more to our principles, since for ships not only the minimum resistance is desired, but likewise it may be required for ships to have the greatest capacities. But it is seen easily the figure found to differ exceedingly from the customary shapes of ships, and other circumstances to prohibit, why of such a shape or the way, why such a shape or at least a resemblance may not be attributed to ships. Moreover it is noteworthy to observe that the curve found shall be algebraic; of which truly the ordinate thus shall be found, with p being eliminated. Since there shall be

$$x=\frac{3cp^{2}+cp^{4}}{\left(1+pp\right)^{2}} \text{ and } s=\frac{2cp^{3}}{\left(1+pp\right)^{2}},$$

there will become

$$\sqrt{\left(xx-3ss\right)}=\frac{3cpp-cp^{4}}{\left(1+pp\right)^{2}}$$

and thence

$$\frac{x}{\sqrt{\left(xx-3ss\right)}}=\frac{3+pp}{3-pp};$$

from which there becomes

$$pp=\frac{3x-3\sqrt{\left(xx-ss\right)}}{x+\sqrt{\left(xx-ss\right)}}=\frac{\left(x-\sqrt{\left(xx-ss\right)}\right)^{2}}{ss}$$

and

$$p=\frac{x-\sqrt{(xx-ss)}}{s}.$$

Again there is

$$pp+1=\frac{2xx-2ss-2x\sqrt{(xx-3ss)}}{ss},$$

and

$$pp+3=\frac{2xx-2x\sqrt{(xx-3ss)}}{ss}.$$

Moreover, with these values substituted into the equation $(1+pp)^2x=cpp(3+pp)$ and with the irrationality removed this same equation will emerge

$$4s^4+8xxss-36cxss+27ccss-4cx^3+4x^4=0.$$

Moreover, by putting $c=2a$ this equation will arise:

$$s^4+2xxss-18axss+27a^2ss-2ax^3+x^4=0,$$

thus so that curves satisfying the equation shall pertain to lines of the fourth order so that the satisfying curve found shall pertain to lines of the fourth order. Therefore is it elicited from this equation:

$$ss=-xx+9ax-\frac{27}{2}a^2\pm\frac{(9a-4x)\sqrt{a(9a-4x)}}{2},$$

from which the construction of the curve shall not become difficult. Truly the nature of the parts here representing *AMB* will become known more conveniently from this series:

$$ss=xx\left(\frac{1}{6}\cdot\frac{4x}{9a}+\frac{1\cdot3}{6\cdot8}\cdot\frac{4^2\cdot x^2}{9^2\cdot a^2}+\frac{1\cdot3\cdot5}{6\cdot8\cdot10}\cdot\frac{4^3\cdot x^3}{9^3\cdot a^3}+\frac{1\cdot3\cdot5\cdot7}{6\cdot8\cdot10\cdot12}\cdot\frac{4^4\cdot x^4}{9^4\cdot a^4}+\text{etc.}\right)$$

or, on putting

$$\frac{9a}{4}=b,\ \text{so that there shall be}\ b=\frac{9c}{8}=AE\,,$$

there will become:

$$ss=xx\left(\frac{1}{6}\cdot\frac{x}{b}+\frac{1\cdot3x^2}{6\cdot8b^2}+\frac{1\cdot3\cdot5x^3}{6\cdot8\cdot10b^3}+\frac{1\cdot3\cdot5\cdot7x^4}{6\cdot8\cdot10\cdot12b^4}+\text{etc.}\right),$$

from which equation it is easily understood the tangent at *A* to fall on the axis *AC*, which is seen from the upper equation with more difficulty. But now we may progress to other kinds of bodies less able to be determined than those treated up to this stage, in which evidently two arbitrary curves shall remain.

PROPOSITION 66

PROBLEM

699. *Not only shall the section of the water be ABb but also the greatest section BDb shall be some given curve* (Fig. 97), *and the solid ABDb shall have this property, so that all the vertical sections STs normal to the axis AC shall be similar sections BDb and this body shall be move in the water along the direction CAL; it will be required to determine resistance which it may experience.*

SOLUTION

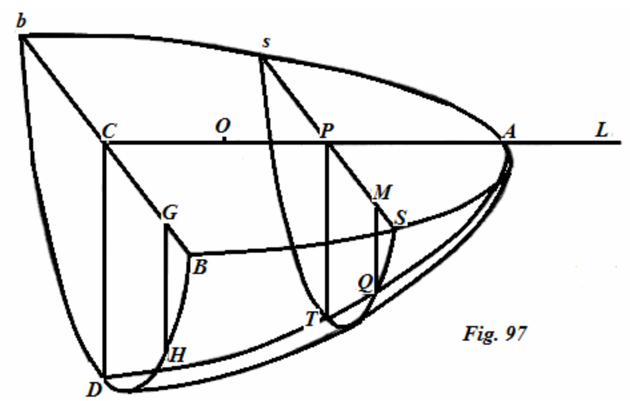


Fig. 97

In the first place since the section *ABb* or rather half of this *ACB* shall be some given curve; with the abscissa $AP = x$ taken in that, and with the applied line put in place $PS = s$, s will be some given function of *x*. Then since also the curve *BDb* or rather with its half given, *BDC* shall be put in place and that with the coordinates $CG = r$ and $GH = u$, the equation will be given between *u* and *r,* and *u* will be equal to a certain function of *r.* Now since the section *STP* shall be similar to the section *BDC,* the lines in these will maintain a ratio similar to *PS* to *CB*. Therefore on putting $CB = b$, and for the section *SPT* with the coordinates taken $PM = y$, and $MQ = z$ with *r* and *u* similar to these, there will become

$$y = \frac{rs}{b} \text{ and } z = \frac{us}{b}.$$

Now since *s* shall be a function of *x,* there may be put $ds = pdx$, so that *p* shall be a function of *x,* and similarly on account of *u* being a function of *r* there may be put $du = qdr$, so that *q* shall be a function of *r.* Therefore from these in place there will become

$$dy = \frac{rpdx}{b} + \frac{sdr}{b}, \text{ and } dz = \frac{updx}{b} + \frac{sqdr}{b};$$

from which on account of

$$\frac{sdr}{b} = dy - \frac{rpdx}{b}$$

the following equation will emerge between the three coordinates *x, y,* and *z* by which the nature of the proposed surface may be held:

$$dz = \frac{(u-qr)\,pdx}{b} + qdy;$$

which will provide a comparison with the general equation assumed in proposition 61 :

$$dz = Pdx + Qdy$$

$$P = \frac{(u-qr)\,p}{b}$$

and $Q = q$, where the magnitudes *s* and *p* requiring to be observed depend on *x* only, *u* and *q* on *r,* while *r* and *x* do not depend on each other. Now for the resistance requiring to be found opposed to the motion, first it will be required first to find the integral of this formula $\frac{P^3dy}{1+P^2+Q^2}$ with *x* constant, and after the integration to make $y = s$. Therefore since *x* is constant, there will become

$$dy = \frac{sdr}{b},$$

and on account of

$$1+P^2+Q^2 = \frac{b^2+\left(u^2-qr\right)^2+b^2q^2}{b^2}$$

there will become

$$\frac{P^3dy}{1+P^2+Q^2} = \frac{(u-qr)^3\,p^3sdr}{b^4(1+qq)+b^2p^2(u-qr)^2}$$

in the integration of which *p* and *s* must be considered as constant quantities. Therefore with the integral

$$\frac{p^3s}{b^2}\int\frac{(u-qr)^3\,dr}{b^2(1+qq)+p^2(u-qr)^2}$$

found, thus so that it may vanish on putting $r = 0$, and then by making $r = b$, for this integral is required to be multiplied by *dx,* and again with the integral taken, indeed with a single variable *x* present within, and with the integration being performed there must be put $x = AC = a$. Or , so that this formula returns the same result :

$$\frac{(u-qr)^3 p^3 sdrdx}{b^4(1+qq)+b^2p^2(u-qr)^2}$$

is required to be integrated, on integrating by *x, p* in turn, and with *s* being put constant, moreover in the alternate integration with *r, q* and *u* constant; likewise indeed by which the first integration may be put in place. Moreover with the magnitude, which is produced by the twofold integration, following which $r=b$ and $x=a$ is put in place, is designated by this form

$$\iint \frac{(u-qr)^3 p^3 sdrdx}{b^4(1+qq)+b^2p^2(u-qr)^2};$$

the strength of the resistance, which repels the body backwards along the direction *AC* will become

$$=\frac{2v}{bb}\iint \frac{(u-qr)^3 p^3 sdrdx}{b^2(1+qq)+p^2(u-qr)^2}.$$

Moreover the business of finding the vertical force acting upwards on the body is carried out in a similar manner :

$$\frac{2v}{b}\iint \frac{(u-qr)^2 p^2 sdrdx}{b^2(1+qq)+p^2(u-qr)^2}.$$

Finally if the value may be sought in the same manner

$$\iint \frac{(u-qr)^2\left(bbx+pus(u-qr)\right)p^2 sdrdx}{b^4(1+qq)+b^2p^2(u-qr)^2}$$

and this may be divided by

$$\iint \frac{(u-qr)^2 p^3 sdrdx}{b^2(1+qq)+pp(u-qr)^2},$$

the distance *AO* will be produced, and from that the position of the point *O* through which the vertical force of the resistance passes. Q. E. I.

COROLLARY 1

700. Since there shall be

$$\frac{(u-qr)^3 dr}{b^2(1+qq)+p^2(u-qr)^2}=dr\left(\frac{(u-qr)^3}{b^2(1+qq)}-\frac{p^2(u-qr)^5}{b^4(1+qq)^2}+\frac{p^4(u-qr)^7}{b^6(1+qq)^3}-\text{etc.}\right)$$

there will become

$$\iint\frac{(u-qr)^3 p^3 sdrdx}{b^2(1+qq)+p^2(u-qr)^2}=\frac{1}{bb}\int p^3 sdx\cdot\int\frac{(u-qr)^3 dr}{(1+qq)}$$
$$-\frac{1}{b^4}\int p^5 sdx\cdot\int\frac{(u-qr)^5 dr}{(1+qq)^2}+\frac{1}{b^6}\int p^7 sdx\cdot\int\frac{(u-qr)^7 dr}{(1+qq)^3}-\text{etc.}$$

in which integrations the variables *r* and *x* have been separated from each other in a straight forwards manner.

COROLLARY 2

701. Therefore if the individual differential formulas, in which only *r* is present and thence the depending magnitudes *u* and *q* thus may be integrated so that they may vanish on putting $r=b$, and in a similar manner the other integral formulas in which only *x, s* and *p* present may be integrated, and then there may be put $x=a$, the desired value of the formula will be obtained

$$\iint\frac{(u-qr)^3 p^3 sdrdx}{b^2(1+qq)+p^2(u-qr)^2}.$$

COROLLARY 3

702. Therefore in a similar manner the remaining differential formulae, which require a twofold integration, thus will be able to be expressed by series, so that the two variables *x* and *r* in short will be separated from each other in turn; with which done the individual formulas themselves will be had able to be integrated without being with respect to the rest.

COROLLARY 4

703. Since the volume of the body in general shall be $=-2\int dx\int Qydy,$ where in the integral $\int Qydy\quad x$ is put constant, for our case there will be, on account of $y=\frac{rs}{b}$ and $dy=\frac{sdr}{b}$ and $Q=q$, the formula

$$\int Qydy=\int\frac{qrs^2dr}{bb}=\frac{s^2}{b^2}\int rdu=-\frac{s^2}{b^2}\int udr,$$

where $\int udr$ denotes the area BCD from where the whole volume will be $=2\int\frac{ssdx}{bb}\int udr.$

COROLLARY 5

704. Truly the surface of the solid figure $ABDb$ will be found from the general formula

$$2\int dx\int dy\sqrt{(1+P^2+Q^2)}\ ,$$

which on account of x being constant, will be changed into

$$2\iint\frac{sdrdx}{bb}\sqrt{\left(bb(1+qq)+pp(u-qr)^2\right)}$$

where there is need of a double integral, the one in which $r,$ the other in which $x,$ is put constant.

PROPOSITION 67

PROBLEM

705. *If the vertical section BDb were given normal to the axis AC* (Fig. 97), *to which all the remaining parallel sections STs themselves shall be similes; to determine the curve ASB, from which the solid ABDb arises for its capacity, which may experience the minimum resistance, if indeed it may be moving in water along the direction of the axis CAL.*

SOLUTION

With $BC = b,\ CG = r,$ remaining as before, and $GH = u$, and on putting $du = qdr$, thus so that *u* and *q* shall be going to become given functions of *r* itself, there shall become $AP = x,\ PS = s$ and on putting $ds = pdx$, from which the resistance will become as

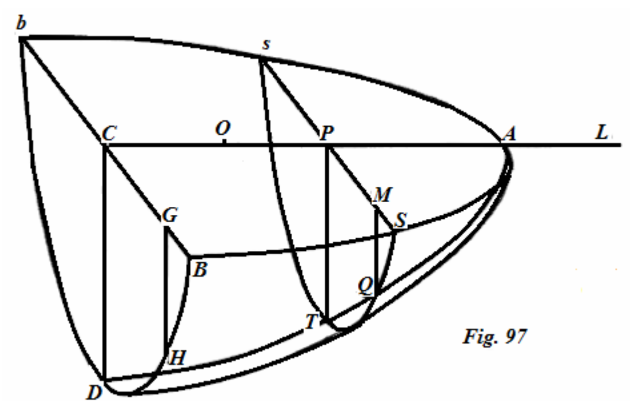


Fig. 97

$$\iint \frac{(u-qr)^3\ p^3 sdrdx}{b^2(s+qq)+pp(u-qr)^2},$$

which minimum quantity thus must be integrated twice. Moreover there the first integration may be considered in which *r* may be put in place, thence with the dependents *s* and *q* put constant, and after the integration to become $x = AC = a$; it is clear in the other integration that the nature of the curve *ASB* not to be held more fully. On account of which it is required so that the quantity, which is produced by the first integration, may be rendered a minimum. Moreover here *dx* is multiplied by

$$\int \frac{(u-qr)^3\ p^3 sdr}{b^2(1+qq)+pp(u-qr)^2},$$

in which only *p* and *s* are variable quantities. For the sake of brevity there is put:

$$u - qr = t \text{ and } 1+qq = w^2,$$

and this formula will be had

$$\int \frac{t^3 p^3 sdr}{bbw^2 + ttp^2},$$

which differentiated with *r, t* and *w* always made constant gives

$$p^3 ds\int\frac{t^3 dr}{bbw^2+ttp^2}+ppdp\int\frac{\left(3b^2w^2+ttp^2\right)t^3 sdr}{\left(bbw^2+ttp^2\right)^2}$$

from which this value itself arises for the determination of the minimum required :

$$p^3\int\frac{t^3 dr}{bbw^2+ttp^2}-\frac{1}{dx}d\cdot pp\int\frac{\left(3b^2w^2+ttp^2\right)t^3 sdr}{\left(b^2w^2+ttp^2\right)^2},$$

which put in place must become $=0$ unless an account of the capacity may be required to be had. Truly the capacity is as $\int ssdx\int udr$, in which the multiplication of the integral is dx by $ss\int udr$, of which the differential is $2sds\int udr$, from which the value for the maximum requiring to be determined is $2s\int udr$. Therefore from these values taken together this equation will emerge

$$\frac{2s\int udr}{c}=p^3\int\frac{t^3 dr}{bbw^2+ttp^2}-\frac{1}{dx}d\cdot pp\int\frac{\left(3b^2w^2+t^2p^2\right)t^3 sdr}{\left(b^2w^2+t^2p^2\right)^2},$$

which multiplied by $pdx=ds$, and integrated gives

$$\frac{ss\int udr}{c}-f^3=\int\frac{t^3p^3 sdr}{bbw^2+ttp^2}-\int\frac{\left(3b^2w^2+t^2p^2\right)t^3p^3 sdr}{\left(b^2w^2+t^2p^2\right)^2}=-\int\frac{2b^2w^2t^3p^3 sdr}{\left(b^2w^2+t^2p^2\right)^2}.$$

So that there may be able to become $s=0$, it is necessary that there shall become $f=0$, thus so that with c made negative this equation shall be had for the curve sought :

$$s=\frac{2b^2cp^3}{\int udr}\int\frac{w^2t^3 dr}{\left(b^2w^2+t^2p^2\right)^2},$$

to which the following value of x will correspond

$$x=\int\frac{ds}{p}=\frac{s}{p}+\int\frac{sdp}{pp}$$
$$=\text{Const.}+\frac{2b^2cp^2}{\int udr}\int\frac{w^2t^3dr}{\left(b^2w^2+t^2p^2\right)^2}-\frac{b^2c}{\int udr}\int\frac{w^2tdr}{b^2w^2+t^2p^2}$$
$$=\text{Const.}-\frac{b^2c}{\int udr}\int\frac{\left(b^2w^2-t^2p^2\right)w^2tdr}{\left(b^2w^2+t^2p^2\right)^2}.$$

So that *x* likewise shall vanish, if there may become $p=0$, certainly so that in which case likewise there becomes $s=0$, there will become

$$\text{Const.}=\frac{b^2c}{\int udr}\int\frac{tdr}{b^2};$$

so that there shall become

$$x=\frac{cpp}{\int udr}\int\frac{\left(3b^2w^2+t^2p^2\right)t^3dr}{\left(b^2w^2+t^2p^2\right)^2}.$$

Moreover since $\int udr$ has a constant value on account of our variables *x, s* and *p,* that may be taken into the constant *c*, and with the former values restored for *w* et *t,* this construction will be had:

$$x=\frac{cpp}{bb}\int\frac{\left(3b^2\left(1+qq\right)+pp(u-qr)^2\right)(u-qr)^3dx}{\left(b^2\left(1+q^2\right)+p^2\left(u-qr\right)^2\right)^2}$$

and

$$s=2cp^3\int\frac{\left(1+qq\right)(u-qr)^3dr}{\left(b^2\left(1+q^2\right)+p^2\left(u-qr\right)^2\right)^2}.$$

Which integral formulas disturb the construction least, in these *p* may be taken constant, and thus the given integration actually may be able to be resolved from the equation between *r* and *u* ; thus moreover the integration must be resolved so that it shall produce 0 on putting $r=0$, with which done there becomes $r=b$. Q.E.I.

COROLLARIUM 1

706. Therefore this curve will have the tangent at *A* equally incident on the axis *AL*, since initially as both *x* and *s* vanish there shall be $p = 0$. In addition truly the curve may fall on the axis *AC* at another location, which will eventuate if $p = \infty$, for in this case there becomes

$$s = 0 \text{ and } x = \frac{c}{bb}\int (u - qr)dr = \frac{2c}{bb}\int udr;$$

or *x* will be equal to the area of the base *BDb* by $\frac{c}{bb}$, or there will become

$$x = \frac{2c \cdot BCD}{BC^2}.$$

COROLLARY 2

707. At that other point, where the curve again cuts the axis *AC*, the tangent will be normal to the axis *AC*, from which this same solid part of curve will generate the maximum resistance being experienced.

COROLLARY 3

708. Since in addition the axis *AC* shall be the diameter of the curve found, since from which it agrees, since with *p* made negative *x* remains, truly *s* will be changed into its negative, and the curve will not be very different from that which we have found before, since the section *BDb* shall be a semicircle.

COROLLARY 4

709. But from the beginning where there becomes $p = 0$, with *p* increasing both the abscissa *x* as well as the applied line *s* increase as far as to a to a certain limit, which term is found by differentiating

$$\int \frac{p^3(1+qq)(u-qr)^3 dr}{\left(b^2(1+q^2)+p^2(u-qr)^2\right)^2}$$

on putting only *p* to be variable, and by making the differential $= 0$.

COROLLARY 5

710. Moreover, the following equation will be found from this absolute differentiation, from which the value of p itself will be determined :

$$0=\int \frac{p^2\left(3b^2\left(1+q^2\right)-p^2\left(u-qr\right)^2\right)\left(1+q^2\right)(u-qr)^3\,dr}{\left(b^2\left(1+q^2\right)+p^2\left(u-qr\right)^2\right)^3}$$

which integration must be completed in the prescribed manner, and at last by putting $r=b$.

COROLLARY 6

711. If there were $\left(u-qr\right)^2=ff(1+qq)$ which happens, if the curve BDb were a semicircle, then the magnitude p will be able to be eliminated from the integral formulas. Certainly in this case

$$x=\frac{cf^3p^2(3bb+ffpp)}{bb(bb+ffpp)^2}\int dr\sqrt{(1+qq)}$$

and

$$S=\frac{2cf^3p^3}{(bb+ffpp)^2}\int dr\sqrt{(1+qq)}.$$

COROLLARY 7

712. Therefore if $\int dr\sqrt{(1+qq)}$ or the arc BD may be considered as a constant magnitude in c, there will become

$$x=\frac{c^5p^2(3bb+ffpp)}{bb(bb+ffpp)^2}\ \text{ and }\ S=\frac{2c^5p^3}{(bb+ffpp)^2}.$$

SCHOLIUM

713. This property requiring to be noted remains, which is

$$\left(u-qr\right)^2=ff(1+qq)\quad\text{or}\quad -u+qr=f\sqrt{(1+qq)},$$

to belong to no other curve except the circle. For with the differentials taken, on account of $du=qdr$ there will become

$$rdq = \frac{fqdq}{\surd(1+qq)} \quad \text{and thus} \quad r = \frac{fq}{\surd(1+qq)}$$

or also, on account of division, $dq = 0$, which is the case for the first right line mentioned. Then since there shall be $u = qr - f\surd(1+qq)$, there will become

$$u = \frac{fqq}{\surd(1+qq)} - f\surd(1+qq) = -\frac{f}{\surd(1+qq)}.$$

Therefore there will become

$$\frac{r}{u} = -q$$

from which there becomes

$$r = -\frac{fr}{\surd\left(r^2+u^2\right)} \text{ or } f = -\surd\left(r^2+u^2\right).$$

But since on making $u = 0$ there must become $r = b$ there will become $f = -b$, and thence

$$b^2 = r^2 + uu.$$

And thus the case mentioned where there becomes $(u-qr)^2 = ff(1+qq)$ does not occur, unless the section *BDb* were a semicircle or isosceles triangle. Finally that also generally occurs here, so that, whatever kind of curve *BDb* were, the satisfying curve sought *AB* always shall emerge algebraic, since the integral formulas shall not affect the algebraic construction.

PROPOSITION 68

PROBLEM

714. *If both the widest section BDb, as well as the shape of the spine ASD or the diametric section ACD, of the body ABDb shall be given* (Fig. 98), *and the shape of the solid shall be prepared thus, so that all the vertical parallel sections shall be similar to the same mean section ACd: to determine the resistance, which this body will perceive, if it shall be moved forwards in the direction CAL in water.*

SOLUTION

Since at first an equation shall be given of the diametrical vertical section *ACD* between its abscissa $AR = r$ and the applied line $RS = s$, so that s will be equal to a function of r and there shall be going to become $ds = pdr$ with p an even function of r. Then the interval shall be $AC = a$, where the vertex *A* shall be the most distant from the section *BDb*, and for this section *BDC* the abscissa may be put $CG = y$, certainly which will arise equal to the second variable $PM = y$, of these three x, y and z, which will enter into the local equation of the whole equation of the surface, and the applied line $GH = u$, and on this account this curve will be a certain known function u of y, thus so that on putting $du = qdy$, also q will be a function of y; truly on putting $y = 0$, $GH = u$ will be changed into *CD*, which shall be $= c$ thus so that c may become a value of u on putting $y = 0$ as well as a function of s on putting $r = a$. Now since the section *FGH*, parallel to the section *ACD*, shall be similar to the same, there will become $CD : AC = GH : FG$, from which there becomes $FG = \frac{au}{c}$. Now with the point *M* taken in the section *FGH*, with the homologous point *R* in the section *ACD*, there will become

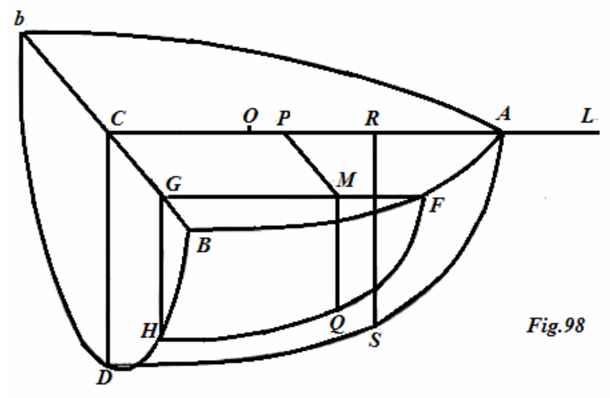


Fig.98

$$FM = \frac{ru}{c}, \text{ and } MQ = z = \frac{su}{c}.$$

Again the normal $MP = y$ may be drawn from *M* to the axis *AC*, certainly which is equal to *CG* itself, and on putting

$$AP = x \text{ there will become } CP = a - x = GM = \frac{au}{c} - \frac{ru}{c},$$

from which there shall become

$$x = a - \frac{(a-r)u}{c}.$$

Whereby since from the curves *ACD* and *BCD* for the following given variables x, y and z we will have the values

$$x = a - \frac{(a-r)u}{c}, \quad y = y \text{ and } z = \frac{su}{c},$$

there will become

$$dx = \frac{-aqdy + rqdy + udr}{c} \text{ and } dz = \frac{sqdy + updr}{c},$$

where since there shall be

$$\frac{udr}{c} = dx + \frac{aqdy - rqdy}{c},$$

there will become

$$dz = pdx + \frac{(ap - rp + s)qdy}{c},$$

which equation compared with the canonical form $dz = Pdx + Qdy$ provides

$$P = p \text{ and } Q = \frac{(ap - rp + s)q}{c},$$

thus so that there shall become

$$1 + P^2 + Q^2 = \frac{c^2 + c^2 p^2 + (ap - rp + s)^2 q^2}{c^2}:$$

which expressions include two independent variables, evidently *y,* and through *y* the given *u* and *q,* and from *r* the two given *s* and *p.* Hence there will become

$$\frac{P^3 dy}{1 + P^2 + Q^2} = \frac{c^2 p^3 dy}{c^2 \left(1 + p^2\right) + q^2 (ap - rp + s)^2}$$

only *y* is variable in the integration of this differential, and *r, s* and *p* may be considered as if constants. But thus for the complete integration so that 0 may be produced on putting $y = 0$, there must become $y = BC$ or $u = 0$; with which done a function of *r* only will be produced which must be multiplied by *dx* in order to be integrated again. But since *dx* with *y* constant may become $= \frac{udr}{c}$, and thus may depend on *y*, this same twofold integration is required to be put in place in the reverse manner, by putting *y* constant initially. For since the general formula for defining the resistance is

$$\iint \frac{P^3 dydx}{1 + P^2 + Q^2},$$

which requires a double integration the one with *x* made constant, the other with *y* put constant, since there $dx = \frac{udr}{c}$, for our case will be changed into this

$$\iint \frac{cp^3 udrdy}{c^2 \left(1 + p^2\right) + q^2 (ap - rp + s)^2}$$

of which the value equally is required to be evaluated by a twofold integration, in the first of which *y* is required to be constant along with *u* and *q,* in the other truly *r* must be put constant with *p* and *s*. And thus in this manner likewise the matter is required to be resolved beyond the first integration. But each integration thus must be completed, so that the integrals may be extended through all the values of the variables *r* and *y*. Therefore with this reminder, the horizontal force of the resistance repelling in the direction *AC* will be produced

$$=2cv\iint\frac{p^3udrdy}{c^2\left(1+p^2\right)+q^2(ap-rp+s)^2}.$$

Truly the force of the resistance forcing the body upwards will be

$$=2cv\iint\frac{p^2udrdy}{c^2\left(1+p^2\right)+q^2(ap-rp+s)^2}.$$

For the place of this application or the point *O* requiring to be found on account of

$$x+Pz=\frac{ac-(a-r)u+psu}{c}$$

this same magnitude

$$\iint\frac{\left(ac-(a-r)u+psu\right)p^2udrdy}{c^2\left(1+p^2\right)+q^2(ap-rp+s)^2}$$

must be divided by

$$\iint\frac{cp^2udrdy}{c^2\left(1+p^2\right)+q^2(ap-rp+s)^2},$$

and the quotient will indicate the distance *AC.* Q. E. I.

COROLLARY 1

715. Since the applied line *GF* in the water section *BAb* shall have a constant ratio to the applied line *GH* of the largest section *BDb*, the curve *CBA* will be related to the curve *CBD,* so that if the curve *CBA* for the water section may be given, then the curve *CBD* may become known most easily.

COROLLARY 2

716. There the solution of a similar problem will remain, if in place of the curve *BCD* the water section *ACB* may be given *;* on account of which provided that all the vertical sections *FGH* may be similar to each other, likewise the solution will be found, whether the curve *ACB* may be given or the alternative *BCD.*

COROLLARY 3

717. Again since the volume of the whole solid *ABDb* generally is $=-2\iint Qydydx$ in our case on account of

$$dx=\frac{udr}{c} \text{ and } Q=\frac{(ap-pr+s)q}{c},$$

the volume

$$=\frac{-2}{cc}\iint(ap-pr+s)quydrdy.$$

COROLLARY 4

718. Of these two integrations the first may be put in place, in which *y,* and likewise *u* and *q* may be put as constants, there shall become

$$\int(ap-pr+s)dr=\int sdr+\int\left(a-r\right)ds=2 \text{ area } ACD,$$

if after the integration there may be put $r=a$. Therefore this area *ACD* may be called $=ff$, the area of the solid may be called

$$=\frac{-4ff}{cc}\int quydy.$$

COROLLARY 5

719. Then since there shall be $\int quydy=\int uydu$ there will become

$$\int quydy=\frac{u^2y}{2}-\tfrac{1}{2}\int u^2dy=-\tfrac{1}{2}\int u^2dy$$

on putting $u=0$. Whereby the whole volume will be produced

$$=\frac{2ff}{cc}\int u^2dy,$$

from which the same expression arises from the nature of the construction.

SCHOLIUM

720. Since the figure of the water section *ACB* is determined only from the widest section *BCD* and does not depend on the shape of the diametric section *ACD*, likewise also that question is resolved, where the resistance of the solid is sought, which thus is generated from the given curves *ACB* and *ACD* so that all the *FGH* parallel to the diametric plane *ACD* shall be similar to each other; thus so that there is no need to treat this question separately. In a similar manner to the preceding case (Fig. 97), where the water section *ACB* and the widest section *BCD* were given, moreover for this all the parallel sections *SPT* are similar to each other, the curve *ATD* is determined everywhere only from the curve *ASB,* and indeed everywhere has that same ratio *PT* to *PS* as *CD* has to *CB,* thus so that the curve *ATD* shall be related to the curve *ASB* : I say moreover affine curves, which have a common abscissa, and of which the applied lines with equal corresponding abscissas, maintain a given ratio between each other ; thus so that all ellipses having a common axis according to this definition, are affine curves ; but soon we will establish this definition for more. On account of this same relationship, which intercedes between the sections *ACB* et *ACD* there is also another question we have not touched on, by which it may be possible to inquire into the resistance of solids of this kind, which thus may be generated from the given curves *BCD* and *ACD* so that all the sections *SPT* parallel to the section *ACD* likewise will be similar to each other. Hence in the following also, where it will be sufficient to assume for all the given horizontal sections of the curves *BCD* and *ACD* to be put in place to be similar to each other, since in an equivalent way, the one shall be similar to the other. Therefore with this agreed on the number of problems going to be treated, if indeed we may wish to make a perfect enumeration, to be diminished according to this central tenet.

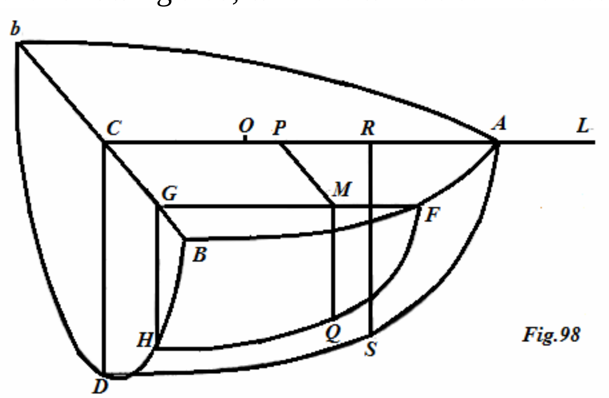


Fig.98

EXAMPLE 1

721. We may put all the vertical sections *FGH* parallel to the diametric section *ACD* to be quadrants of the circles described with centres *G*, or the volume *ABDb* generated by the rotation of the figure *BDb* around the fixed axis *Bb* (Fig. 98). Therefore *ACB* shall be a quadrant of the circle, and thus $c = a$, and $s = \surd(2ar - rr)$, from which there becomes

$$p = \frac{(a-r)}{\surd(2ar-rr)},$$

and

$$1+pp=\frac{a^2}{2ar-rr},$$

and also

$$ap-rp+s=\frac{as}{\sqrt{(2ar-rr)}}.$$

With these substituted the force of the horizontal resistance produced

$$=\frac{2v}{a^2}\iint\frac{(a-r)^3udrdy}{(1+qq)\sqrt{(2ar-rr)}}.$$

At first there may be put u constant along with y and q, and the integral

$$\int\frac{(a-r)^3dr}{\sqrt{(2ar-rr)}}$$

on putting $r=a$ after the integration, will become $=\frac{2}{3}a^3$ whereby a single integration remains, and thus the resistance sought

$$=\frac{4v}{3}\int\frac{udy}{1+qq},$$

which integral thus is required to be accepted, so that it shall vanish on putting $y=0$, and then there may be put $u=0$. But the resistance of the vertical force, by which the body will be urged to move upwards will be

$$=\frac{2v}{a^3}\iint\frac{(a-r)^2udrdy}{(1+qq)}$$

truly the first integration on putting y constant, by making $r=a$ gives

$$\int(a-r)^2dr=\frac{a^3}{3}.$$

Hence therefore the vertical force of the resistance becomes

$$=\frac{2v}{3}\int\frac{udy}{1+qq}.$$

Finally since there shall be

$$\frac{ac-(a-r)u+psu}{a}=a,$$

the interval $AO=a$, or the point *O,* at which that vertical force is applied, will lie at the same point *C.*

COROLLARY 1

722. Therefore in bodies of this kind, which are round with respect to the axis *Bb*, have a constant ratio of the resistance of the horizontal force to the vertical force; evidently themselves will have the vertical force to the horizontal force as 1 to 2, thus so that the vertical force shall half as large as the horizontal force.

COROLLARY *2*

723. If the widest section *BDb* were a semicircle also, thus so that the body may become the fourth part of a sphere, on account of $CB=CD=a$, there will be

$$u=\surd\left(a^2-y^2\right) \text{ and } q=-\frac{y}{\surd\left(a^2-y^2\right)};$$

whereby there will become

$$\int\frac{udy}{1+qq}=\int\frac{dy\left(a^2-y^2\right)^{\frac{3}{2}}}{a^2}=\frac{3\pi a^2}{16},$$

thus so that a horizontal resistance shall be produced $=\frac{\pi a^2}{4}$ and the vertical force $=\frac{\pi a^2}{8}$.

COROLLARY 3

724. If the widest section *BDb* may become an isosceles triangle, thus so that there shall be

$$BC=Cb=b$$

there will become

$$u=a-\frac{ay}{b}, \text{ and } q=\frac{-a}{b}.$$

From these there will become

$$\int\frac{udy}{1+qq}=\frac{ab}{aa+bb}\int\left(b-y\right)dy=\frac{ab^2}{2\left(a^2+b^2\right)},$$

whereby the horizontal resistance will be $=\frac{2ab^2}{3(aa+bb)}$ and the vertical resistance $=\frac{ab^2}{2\left(a^2+b^2\right)}$.

COROLLARY 4

725. It is understood from this case the resistance from the other parts there to be smaller, where the difference between the width *BC* and the height *CD* were greater. Indeed with *b* remaining in these formulas, the resistance becomes a maximum, if there may be put $a=b$.

EXAMPLE 2

726. Now all the vertical sections *FGH*, which are parallel to the diametric section shall be the quadrants of ellipses similar to each other; and the diametric section *ACD* equally will be the elliptic quadrant of which the one semi axis $AC=a$, the other $CD=c$, from which there will become

$$s=\frac{c}{a}\sqrt{(2ar-rr)}$$

and

$$p=\frac{c(a-r)}{a\sqrt{(2ar-rr)}}.$$

For the sake of brevity there shall be $a-r=t$, there will become

$$s=\frac{c}{a}\sqrt{\left(a^2-t^2\right)} \text{ and } p=\frac{ct}{a\sqrt{\left(a^2-t^2\right)}}$$

and

$$1+pp=\frac{a^4-\left(a^2-c^2\right)t^2}{a^2\left(a^2-t^2\right)}$$

and again

$$(a-r)p+s=\frac{ac}{\sqrt{\left(a^2-t^2\right)}};$$

from which there becomes

$$\frac{p^3udrdy}{c^2\left(1+p^2\right)+q^2(ap-rp+s)^2}=\frac{-ct^3udtdy}{\left(a^5(1+qq)-a\left(a^2-c^2\right)t^2\right)\sqrt{\left(a^2-t^2\right)}}.$$

Initially this formula is integrated by putting y and u as well as q constants, thus so that for the integral it shall vanish on putting $t = a$, with which done there shall become fiat $t = 0$; and there will arise

$$\frac{cudy}{a^2 - c^2}\left(-1 + \frac{a^2(1+qq)}{\sqrt{(a^2-c^2)(a^2q^2+c^2)}}\text{Atang.}\frac{\sqrt{(a^2-c^2)}}{\sqrt{(a^2q^2+c^2)}}\right),$$

or by the series

$$\frac{cudy}{a^2q^2+c^2}\left(1 - \frac{a^2(1+q^2)}{3(a^2q^2+c^2)} + \frac{a^2(1+q^2)(a^2-c^2)}{5(a^2q^2+c^2)^2} - \frac{a^2(1+q^2)(a^2-c^2)^2}{7(a^2q^2+c^2)^3} + \text{etc.}\right),$$

which provides a more convenient use than that expression, certainly which if $c > a$ may cease to depend on the quadrature of the circle, but is reduced to logarithms. And thus hence the force of the horizontal resistance, which this body may experience, will be

$$= 2c^2v\int\frac{udy}{a^2q^2+c^2}\left(1 - \frac{a^2(1+q^2)}{3(a^2q^2+c^2)} + \frac{a^2(1+q^2)(a^2-c^2)}{5(a^2q^2+c^2)^2} - \frac{a^2(1+q^2)(a^2-c^2)^2}{7(a^2q^2+c^2)^3} + \text{etc.}\right),$$

with the integration thus absolute so that with the integral $= 0$ if there may be put $y = c$, and then there must be put $y = CB$ or $u = 0$.

EXAMPLE 3

727. Now both the curve ACD as well as BCD shall be the quadrant of an ellipse, thus so that the semi axes of the elliptic quadrant ACD shall be $AC = a$ and $CD = c$; truly of the other BCD the semi axes shall be $BC = b$ and $CD = c$; therefore in the first place as before there will be

$$s = \frac{c}{a}\sqrt{(2ar - rr)}$$

and

$$p = \frac{c(a-r)}{a\sqrt{(2ar-rr)}}$$

or on putting $a - r = t$ there will be

$$s = \frac{c}{a}\sqrt{(a^2 - tt)},\ \ p = \frac{ct}{a\sqrt{(a^2-t^2)}},\ \ 1 + p^2 = \frac{a^4 - (a^2-c^2)t^2}{a(a^2-t^2)};$$

and the formula being used for finding the horizontal resistance

$$=\frac{p^3udrdy}{c^2\left(1+p^2\right)+q^2\left(ap-rp+s\right)^2}$$

will become

$$=\frac{-ct^3udtdy}{a\left(a^4\left(1+qq\right)-\left(a^2-c^2\right)t^2\right)\sqrt{\left(a^2-t^2\right)}}.$$

Now again since there shall be

$$u=\frac{c}{b}\sqrt{\left(b^2-y^2\right)}$$

there shall become

$$q=\frac{-cy}{b\sqrt{\left(b^2-y^2\right)}}\ \text{ and }\ 1+qq=\frac{b^4-\left(b^2-c^2\right)y^2}{b^2\left(bb-yy\right)},$$

and that differential formula will be transformed into this :

$$\frac{-bc^2t^3dtdy\left(b^2-y^2\right)^{\frac{3}{2}}}{a\left(a^4b^4-a^4\left(b^2-c^2\right)y^2-b^4\left(a^2-c^2\right)t^2+b^2\left(a^2-c^2\right)t^2y^2\right)\sqrt{\left(a^2-t^2\right)}}$$

of which the integral on putting t constant will be found

$$=\frac{\pi b^2c^2t^3dt}{4a\sqrt{\left(a^2-t^2\right)}}\left(\frac{2a^6c^3-b\left(3a^4c^2-a^4b^2+b^2\left(a^2-c^2\right)t^2\right)\sqrt{\left(a^4-\left(a^2-c^2\right)t^2\right)}}{\left(a^4\left(bb-cc\right)-b^2\left(a^2-c^2\right)t^2\right)^2\sqrt{\left(a^4-\left(a^2-c^2\right)t^2\right)}}\right)$$

which formula integrated again and on putting $t=a$ after the integration, if it may be multiplied by $2cv$ will give the horizontal force of the resistance by which the motion will be retarded. But hence it shall be concluded, with little more to be added to the usefulness by which we may define the nature of the curve BCD according to the method of maxima and minima, to which the minimum resistance may correspond.

PROPOSITION 69

PROBLEM

728. If a diametric section ACD (Fig. 98) may be given for which all the parallel sections are similar, to determine the nature of the curve BCD, which the solid figure will generate, moving in the direction CAL, for which its capacity may experience the minimum resistance.

SOLUTION

With $AR = r$, and $RS = s$ remaining as before, on account of the given curve *ACD, s* and also *p* will be given on putting $ds = pdr$ by means of *r*. Moreover, for the curve requiring to be found there shall be $CG = y$ and $GH = u$, and $du = qdy$, with which in place this expression must become the minimum

$$\iint \frac{p^3 udrdy}{c^2\left(1+p^2\right)+q^2\left(ap-rp+s\right)^2}$$

or

$$\int udy \int \frac{p^3 dr}{c^2\left(1+p^2\right)+q^2\left(ap-rp+s\right)^2}.$$

For the sake of brevity there may be put

$$\frac{1+p^2}{p^3} = w^2 \text{ and } \frac{\left(ap-rp+s\right)^2}{p^3} = t^2,$$

thus so that the quantities *t* and *w* may not depend on *y* ; and the formula requiring to return this formula will be

$$\int udy \int \frac{dr}{c^2w^2+t^2q^2},$$

in which since *dy* shall be multiplied by

$$u\int \frac{dr}{c^2w^2+t^2q^2}$$

and its differential may be taken by putting *r* with *w* and *t,* always to be constants, which will become

$$du\int\frac{dr}{c^2w^2+t^2q^2}-\int\frac{2ut^2qdqdr}{\left(c^2w^2+t^2q^2\right)^2}$$

where the summation signs is only with regard to the magnitudes *r, w,* and *t*, truly *u* and *q* are put constant. Hence therefore for the minimum value serving to be found, there will be

$$\int\frac{dr}{c^2w^2+t^2q^2}+\frac{1}{dy}d\cdot 2uq\int\frac{t^2dr}{\left(c^2w^2+t^2q^2\right)^2},$$

which must be put $=0$ unless likewise a capacity may be introduced into the computation which must be a maximum. But the capacity is as $\int u^2dy,$ from which the same value for the maximum requiring to be found provides the value $2u$. Therefore from these values, the following equation may be assembled presenting the nature of the curve sought

$$\frac{2u}{cf}=\int\frac{dr}{c^2w^2+t^2q^2}+\frac{1}{dy}d\cdot 2uq\int\frac{t^2dr}{\left(c^2w^2+t^2q^2\right)^2}.$$

Both sides may be multiplied by $du=qdy$, and there will be produced

$$\frac{2udu}{cf}=du\int\frac{dr}{c^2w^2+t^2q^2}+qd\cdot 2uq\int\frac{t^2dr}{\left(c^2w^2+t^2q^2\right)^2}$$
$$=d\cdot u\int\frac{dr}{c^2w^2+t^2q^2}+\int\frac{2ut^2qdqdr}{\left(c^2w^2+t^2q^2\right)^2}+qd\cdot 2uq\int\frac{t^2dr}{\left(c^2w^2+t^2q^2\right)^2}$$

of which the integral is

$$\frac{u^2}{cf}=\int\frac{udr}{c^2w^2+t^2q^2}+\int\frac{2ut^2q^2dr}{\left(c^2w^2+t^2q^2\right)^2}+\text{Const.}=\int\frac{u\left(c^2w^2+3t^2q^2\right)dr}{\left(c^2w^2+t^2q^2\right)^2}+\text{Const.}$$

Truly since at some point there must become $u=0$, but this can never happen unless there shall be the $\text{Const.}=0$, and there will become

$$u = cf\int \frac{\left(c^2w^2+3t^2q^2\right)dr}{\left(c^2w^2+t^2q^2\right)^2}.$$

Truly since there shall be $du = qdy$, there will become:

$$y = \frac{u}{q} + \int \frac{udq}{qq};$$

but there is

$$\int \frac{udq}{qq} = cf\iint \frac{\left(c^2w^2+3t^2q^2\right)drdq}{qq\left(c^2w^2+t^2q^2\right)^2} = h - cf\int \frac{dr}{q\left(c^2w^2+t^2q^2\right)},$$

from which there becomes

$$y = h + 2cf\int \frac{t^2qdr}{\left(c^2w^2+t^2q^2\right)^2}.$$

On account of which with w^2 and t^2 restored in place with the assumed values, this construction of the curve sought will emerge :

$$y = h + 2cf\int \frac{p^3\left(ap-rp+s\right)^2 qdr}{\left(c^2\left(1+p^2\right)+q^2\left(ap-rp+s\right)^2\right)^2}$$

and

$$u = cf\int \frac{\left(c^2\left(1+p^2\right)+3q^2\left(ap-rp+s\right)^2\right)^2 p^3dr}{\left(c^2\left(1+p^2\right)+q^2\left(ap-rp+s\right)^2\right)^2},$$

which integrations do not impede construction, since in these q may be put in place to be constant, and shall not be an impediment, unless the curve sought shall not be algebraic. Q. E. I.

COROLLARY 1

729. Because u vanished if there shall be $q = \infty$, it is understood the tangent at B to the curve BD to be normal to the right line CB, or vertical moreover in this case to produce $y = h$: whereby if there may be called $CB = b$, there will become $h = b$.

COROLLARY 2

730. Since the curve approaches towards *CB* by progressing from *D* towards *B*, q will have a negative value everywhere. From which there will become $y = 0$ if there were

$$b = -2\, cf \int \frac{p^3 \left(ap - rp + s\right)^2 qdr}{\left(c^2\left(1 + p^2\right) + q^2\left(ap - rp + s\right)^2\right)^2}.$$

COROLLARY 3

731. But u will obtain the maximum value if q itself may be attributed the value so that there may become

$$0 = \int \frac{p^3\left(c^2\left(1 + p^2\right) - 3q^2\left(ap - rp + s\right)^2\right)\left(ap - rp + s\right)^2 dr}{\left(c^2\left(1 + p^2\right) + q^2\left(ap - rp + s\right)^2\right)^3}$$

with the integration to be resolved in the absolute manner; clearly so that it may vanish on making $r = 0$, and then there may be put $r = a$.

EXAMPLE

732. The diametric section *ACD* shall be a triangle right angled at C, or *ASD* a right line, there will become $s = \frac{cr}{a}$, and $p = \frac{c}{a}$, and also

$$1 + pp = \frac{aa + cc}{aa},$$

and likewise $ap - rp + s = c$; with these substituted there will become

$$\int \frac{p^3\left(ap - rp + s\right)^2 dr}{\left(cc\left(1 + pp\right) + q^2\left(ap - rp + s\right)^2\right)^2} = \int \frac{acqdr}{\left(a^2 + c^2 + a^2q^2\right)^2} = \frac{a^2cq}{\left(a^2 + c^2 + a^2q^2\right)^2};$$

and

$$\int \frac{p^3\left(c^2\left(1 + p^2\right) + 3q^2\left(ap - rp + s\right)^2\right)dr}{\left(c^2\left(1 + pp\right) + qq\left(ap - rp + s\right)^2\right)^2} = \int \frac{c\left(a^2 + c^2 + a^2q^2\right)dr}{a\left(a^2 + c^2 + a^2q^2\right)^2} = \frac{c\left(a^2 + c^2 + 3a^2q^2\right)}{\left(a^2 + c^2 + a^2q^2\right)^2}.$$

On account of which for the curve *BCD* which for the maximum volume will experience the minimum resistance that same equation will be obtained :

$$y = b + \frac{2a^2c^2 fq}{\left(a^2 + c^2 + a^2q^2\right)^2}$$

to which there corresponds

$$u = \frac{ccf\left(a^2 + c^2 + 3a^2q^2\right)}{\left(a^2 + c^2 + a^2q^2\right)^2}.$$

Therefore u will have the maximum value if there may be taken

$$q = \pm \frac{\sqrt{\left(a^2 + c^2\right)}}{a\sqrt{3}}.$$

Therefore if for the maximum value of u there may be put $CD = c$, there will become

$$f = \frac{s\left(a^2 + c^2\right)}{9c};$$

then since in this case y must vanish, there will become

$$b = \frac{-ac}{\sqrt{\left(a^2 + c^2\right)}};$$

from which the nature and figure of the desired curve is readily recognised. Likewise moreover this curve is understood to be algebraic.

PROPOSITION 70

PROBLEM

733. If both the section with the greatest width *BDC* as well as the water section *ACB* (Fig. 99) were given, and all the horizontal sections *FIH* shall be similar to this water section, to determine the resistance, which this body will experience moving in the water along the direction *CAL*.

SOLUTION

Since the curve AVB is given, the abscissa for that may be put $CT = t$ and the applied line $TV = u$, and the equation between u and t will be given, and on putting $du = qdt$, q will be some function of t. Again for the curve DHB there may be put the abscissa $CG = r$, and the applied line $GH = z$, since this applied line GH will be equal to the third of the three variable x, y, z, which enter into the equation for the surface ; moreover there shall be $dz = pdr$, thus so that p shall be going to become a function of r. Now if the constant quantities may be called $AC = a$, $CB = Cb = b$ and $CD = c$, the homologous sides of the similar figures ACB and FIH will be called CB, b and $HI = r$; whereby if there may be taken $b : r = t : IK$ so that there shall become

$$IK = \frac{rt}{b}, \text{ there will become } KQ = \frac{ru}{b}.$$

Truly on calling $AP = x$, $PM = y$, and $MQ = z$, there will become

$$x = a - \frac{rt}{b}, \quad y = \frac{ru}{b}, \quad \text{and} \quad z = z,$$

and of which the first equations give

$$dx = \frac{-rdt - tdr}{b} \quad \text{and} \quad dy = \frac{rqdt + udr}{b},$$

from which there becomes

$$dr = \frac{bdy + bqdx}{u - tq} \quad \text{and} \quad dt = \frac{-budx - btdy}{r(u - q)}.$$

Now since there shall be $dz = pdr$, there will become

$$dz = \frac{bpqdx + bpdy}{u - tq},$$

which equation expresses the nature of the surface proposed. Therefore this equation compared with the general equation to be assumed $dz = Pdx + Qdy$, gives

$$P = \frac{bpq}{u - tq} \quad \text{and} \quad Q = \frac{bp}{u - tq},$$

from which there becomes

$$1+P^2+Q^2=\frac{b^2p^2\left(1+q^2\right)+\left(u-tq\right)^2}{\left(u-tq\right)^2}.$$

Now for the value of

$$\frac{P^3dxdy}{1+P^2+Q^2}$$

itself requiring to be found, it must be observed, while *dy* may be considered, *dx* must be treated as constant; with which done, moreover with $dx=0$, there becomes

$$dr=\frac{-rdt}{t},$$

and thus

$$dy=\frac{-rudt}{bt}+\frac{rqdt}{b}=\frac{-rdt\left(u-tq\right)}{bt};$$

and while *dx* shall be considered, *dy* is required to be put as constant, or

$$dt=\frac{-udr}{rq}$$

from which there becomes

$$dx=\frac{+udr}{bq}-\frac{tdr}{b}=\frac{dr\left(u-tq\right)}{bq}.$$

But since hence it shall not be clear how the variables *r* and *t* may be able to be distinguished from each other, in place of either of the elements *dx* and *dy* it will be required to introduce the third element *dz*, since that itself may be present in the assumed variable quantities. Moreover there is

$$dxdy=\frac{dzdx}{Q}=\frac{dzdy}{P};$$

for while *x* may be considered as constant, in place of *dy* there can be written $\frac{dz}{Q}$, and while *y* may be assumed constant in place of *dx* there can be written $\frac{dz}{P}$ from which we come upon this formula $\frac{p^2dzdy}{1+P^2+Q^2}$, which must be integrated twice, the one by integrating on putting *z* constant, the other by putting *y* constant. But there is $dz=pdr$, and if *z* may be put constant,

$$dy=\frac{rqdt}{b};$$

on account of which the general formula $\frac{p^2dzdy}{1+P^2+Q^2}$, for our case becomes

$$\frac{bp^3q^3rdrdt}{b^2p^2\left(1+q^2\right)+\left(u-tq\right)^2},$$

which twice integrated on the one hand by taking r constant, on the other by taking t constant. And indeed in the first place each integral is required to be put in place thus so that it shall vanish, provided the initial value taken for the variable r or $t=0$, and the integral is taken again on making either $r=b$ or $t=a$. Therefore from these according to the forewarned manner if the height $=v$ may be put to correspond to the speed with which the body is progressing, the resistive force repelling the body along the direction AC

$$=2bv\iint\frac{p^3q^3rdrdt}{b^2p^2\left(1+q^2\right)+\left(u-tq\right)^2}.$$

Thereon the vertical force arising from the resistance, which is

$$\iint\frac{P^2dxdy}{1+P^2+Q^2}\ =2v\iint\frac{Pdxdy}{1+P^2+Q^2}$$

will become for our case

$$=2v\iint\frac{p^2q^2rdrdt\left(u-tq\right)}{b^2p^2\left(1+q^2\right)+\left(u-tq^2\right)^2}$$

which applied line will be at the point O of the axis AC, the distance of which from the point A will be found if

$$\iint\frac{\left(ab\left(u-tq\right)-rt\left(u-tq\right)+b^2pqz\right)p^2q^2rdrdt}{b^2p^2\left(1+q^2\right)+\left(u-tq\right)^2}$$

is divided in the prescribed manner established by

$$\iint\frac{\left(u-tq\right)p^2q^2rdrdt}{b^2p^2\left(1+q^2\right)+\left(u-tq\right)^2}.$$

And with these known the effect of the whole resistance will become known. Q. E. I.

COROLLARY 1

734. Hence the figure of the diametric section *AFD* is defined most easily from the curve CBD. For since there is $BC : HI = AC : FI$ the applied lines *FI* and *HI* maintain the same given ratio between themselves corresponding to the same abscissa CI; from which the curve *AFD* will be related to the curve *BHD*.

COROLLARY 2

735. Hence on account of this problem, from which the position of the curve *BHD* will have given the curve *AFD*, truly all the horizontal sections shall be similar to each other, so that in the present question, it will be resolved in the same manner, and thus the solution will not differ from this, unless by writing *a* in place of *b* if indeed r and *z* shall denote the coordinates of the curve *DFH*.

COROLLARIUM 3

736. Since the volume in general shall be $= -2\iint Qydxdy = -2\iint Qdxdz$ on putting $\frac{dxdz}{Q}$ in place of *dxdy*, for our case the volume will become

$$= \frac{2}{bb}\iint pr^2 udrdt = \frac{2}{bb}\int pr^2 dr \int udt.$$

Therefore since $\int udt$ may express the area ACB, that may be said $= ff$, the volume

$$= \frac{2ff}{bb}\int pr^2 dr = \frac{2ff}{bb}\int r^2 dz$$

on putting $r = b$ after integration thus so that the absolute 0 may be produced, if there may become $r = 0$.

COROLLARY 4

737. Moreover the surface *ABDb* traveling in the water generally is

$$2\iint dxdy\,\surd\left(1+P^2+Q^2\right) = 2\iint \frac{dxdz}{Q}\surd\left(1+P^2+Q^2\right).$$

On account of which, in our case this surface is expressed by this formula :

$$2\iint \frac{rdrdt}{bb}\sqrt{\left(b^2p^2\left(1+q^2\right)+\left(u-tq\right)^2\right)}.$$

COROLLARY 5

738. Also it is possible to deduce, however many of the curves *CBD* and *CAD* were related to each other, just as many horizontal sections of the body to be similar to each other to be found. Therefore since the vertical sections shall be similar to the parallel section *CBD*, if the curves *CBA* and *CDA* were related to each other, it is understood in turn if the three curves *CBD, CAD* and *CAB* were related to each other, then all the sections parallel to one of these sections will be parallel to each other in turn.

SCHOLIUM

739. From which it shall be evident, in whatever manner the differential formulas given above in which *dxdy* is present, they shall be able to be reduced to others in which either *dxdz* or *dydz* shall be present, thus it is required to observe *dxdy* to be entered into these formulas, which were present in the element of the surface $dxdy\sqrt{\left(1+P^2+Q^2\right)}$.
Moreover, since this element has arisen from the canonical equation $dz = Pdx + Qdy$, in a similar manner from this same canonical equation ,

$$dy=\frac{dz}{Q}-\frac{Pdx}{Q}$$

this element of the surface shall arise:

$$\frac{dxdz}{Q}\sqrt{\left(1+P^2+Q^2\right)},$$

and from this equation:

$$dx=\frac{dz}{P}-\frac{Qdy}{P},$$

this same element of the surface

$$\frac{dydz}{P}\sqrt{\left(1+P^2+Q^2\right)}.$$

Therefore since these three elements integrated twice shall give rise to the whole surface, it is evident these can be substituted for each other mutually. On account of which the formulae for the resistance found above can be reduced to other equivalent forms, which it will be allowed to use in place of these. Thus the force of the horizontal resistance which was found above :

$$=2v\iint\frac{P^3dydx}{1+P^2+Q^2}$$

also can be expressed also in this manner:

$$2v\iint\frac{P^2dydx}{1+P^2+Q^2}$$

or in this manner:

$$2v\iint\frac{P^3dxdz}{Q\left(1+P^2+Q^2\right)}.$$

In a similar manner the vertical force of the resistance can be expressed in these three different ways; evidently there shall be either

$$2v\iint\frac{P^2dxdy}{1+P^2+Q^2}$$

or

$$2v\iint\frac{Pdydz}{1+P^2+Q^2}$$

or

$$2v\iint\frac{P^2dxdz}{Q\left(1+P^2+Q^2\right)};$$

from which formulas any case presented from these can be agreed to be used, which have been prepared thus for the calculation of the variable quantity, so that one or other of the variables contained in the formula may depend on the assumed variables. Thus in this case there was a need for formulas of this kind to be use in which *dz* might be present, since it may be assumed *z* was found between these assumed variables.

EXAMPLE

740. All the horizontal sections shall be of the semicircle *HFh*, or the volume arising from the rotation of the figure *CBD* about the axis *CD*, and the figure *CBA* the quarter of a circle and therefore $b=a[=CA]$, and from the nature of the circle $[CT=t;TV=u]$; $u=\surd\left(a^2-t^2\right)$ and

$$q\left[=\frac{du}{dt}\right]=\frac{-t}{\surd\left(a^2-t^2\right)}$$

and

$$u-tq=\frac{a^2}{\surd\left(a^2-t^2\right)},\ \text{ and }\ 1+qq=\frac{a^2}{a^2-t^2}.$$

With these in place the force of the horizontal resistance retarding the motion in the horizontal direction *AC*

$$=\frac{2v}{a^3}\iint\frac{p^2t^3rdrdt}{(1+pp)\sqrt{(a^2-t^2)}}=\frac{2v}{a^3}\int\frac{t^3dt}{\sqrt{(a^2-t^2)}}\int\frac{p^3rdr}{1+pp}$$

where the variables t and r are separated from each other in turn. Indeed this formula will have a negative sign, but in place of this with care a + may be substituted since it shall depend on the transformation of the general formula with the square root sign, in which each sign agrees equally. But there becomes

$$\int\frac{t^3dt}{\sqrt{(a^2-t^2)}}=\frac{2}{3}a^3$$

on putting $t=a$ after the integration, from which the force of the horizontal resistance is

$$=\frac{4}{3}v\int\frac{p^3rdr}{1+pp}.$$

In a similar manner the force of the horizontal resistance

$$=\frac{2v}{a^2}\iint\frac{p^2t^2rdrdt}{(1+pp)\sqrt{(a^2-t^2)}}=\frac{2v}{a^2}\int\frac{t^2dt}{\sqrt{(a^2-t^2)}}\int\frac{p^2rdr}{1+pp}=\frac{\pi v}{2}\int\frac{p^2rdr}{1+pp},$$

which is

$$\int\frac{ttdt}{\sqrt{(a^2-t^2)}}=\frac{\pi}{4}.$$

But concerning the position of the applied line *O*, since the formula becomes less simple, we will not be concerned with this.

COROLLARY 1

741. If this same solid shape may be inverted so that *BDb* shall become the water section and *BAb* the widest section, and this shape must be moved in the direction *CD* with a speed corresponding to the height v then the resistance retarding the motion will be

$$=\pi v\int\frac{rdr}{1+p^2}.$$

For in this case all the vertical sections normal to the axis *CD* will be semicircles.

COROLLARY 2

742. Therefore the resistance of this body, if it shall be moving along the direction *CA* itself will be had to the resistance of the same body moved in the direction *CD* as

$$\frac{4}{3}\int\frac{p^3rdr}{1+pp} \text{ to } \pi\int\frac{rdr}{1+pp}.$$

COROLLARY 3

743. Therefore if the figure *BDb* may be changed into an isosceles triangle, or the body into a semi cone with the right axis CD and there may be put $CD = c$ with there being $BC = AC = a$, there will become

$$z = c - \frac{cr}{a} \text{ and } p = -\frac{c}{a}.$$

Therefore the resistance which this cone will be experiencing moving in the direction *CA* will become

$$= -\frac{4v}{3a}\int\frac{c^3rdr}{a^2+c^2} = \frac{2ac^3v}{3\left(a^2+c^2\right)},$$

with the sign ignored as now noted above.

COROLLARY 4

744. But the resistance, which the same semi cone will suffer moving in the direction of the axis *CD*, will become

$$= \pi v\int\frac{a^2rdr}{a^2+c^2} = \frac{\pi a^4v}{2\left(a^2+c^2\right)}.$$

Whereby this resistance itself is had as $\frac{\pi a^3}{2}$ to $\frac{\pi c^3}{3}$. From which these two resistances will be equal to each other if there were

$$c^3 = \frac{3\pi a^3}{4}, \text{ or } \frac{c}{a} = \sqrt[3]{\frac{3\pi}{4}},$$

or if there shall be

$$CD : CB = \sqrt[3]{6\pi} : 2 = 2,661341 : 2,$$

from which approximately there becomes $CD : CB = 4:3$.

COROLLARIUM 5

745. If the section *BDb* also may be put semicircular, thus so that the body shall become a spherical quadrant, both resistances must become the same. Moreover the resistance for the motion along *CA* becomes, on account of

$$z=\sqrt{\left(a^2-r^2\right)} \text{ and } p=\frac{-r}{\sqrt{\left(a^2-r^2\right)}},$$

$$=\frac{4}{3}v\int\frac{r^4dr}{a^2\sqrt{\left(a^2-r^2\right)}}=\frac{2\pi a^2v}{3}\frac{1}{2}\cdot\frac{3}{4}=\frac{\pi a^2v}{4}.$$

But for the motion along the direction *CD,* the resistance will be

$$=\pi v\int\frac{rdr\left(a^2-r^2\right)}{a^2}=\frac{\pi a^2v}{4}.$$

SCHOLIUM

746. Therefore we have resolved all the cases for these propositions for which the sections of the body shall be similar to each other, both for the vertical as well as the horizontal sections to be parallel to each other, as well as the cases of the diametric and the widest sections. And there has been a need for the determination of the three principal sections for bodies of this kind: clearly only two of the water section, the widest section, and diametric section need be given, since from this condition the third section can be determined at once. But besides these kinds of bodies, the natures of which have certain similar sections parallel to each other, innumerable other kinds of bodies, for the establishment of which neither the space nor the time will be at hand. Truly of these other kinds, it will help the first to be subjected to examination, to be applied next to the figures of ships. Evidently we will consider shapes of this kind, in which the sections amongst themselves shall be related to each other, whether they be horizontal or vertical sections, and of which we accept here so-called affine shapes in a much wider sense than may be called similar. Indeed we call figures affine, in which with the abscissas taken in a given ratio, the corresponding applied lines also may hold a constant ratio, from which definition it is understood similar figures to be contained as if under a special kind of affinity, indeed affine figures may avoid being similar, if the applied lines may hold the same ratio as the abscissas; moreover affine and non similar figures may produce figures, if the ratios of the abscissas and of the applied lines were unequal. Thus all ellipses are affine figures amongst themselves, since with the abscissas assumed in the ratio of the transverse axis, the corresponding applied lines are held in the ratio of the conjugate axis, if indeed the abscissas may be taken on the transverse axes. Also in a similar manner all the right angled triangular figures are affine amongst themselves. Ttherefore, for any given curve having a given base and height, it will be easy to describe another curve affine to that, which may have any prescribed base and height. For if the base of the given curve shall be $=a$ and the height $=b$, and if some abscissa called x may be taken

on the base, and if the corresponding applied line parallel to the height shall be y, in the manner of the above base, another base A for another height B for an affine curve may be constructed, on the base the abscissa A may be taken $=\frac{Ax}{a}$, and the corresponding applied line $=\frac{By}{b}$, and the curve described in this manner will be affine to the former curve. Therefore with these noted it will not be difficult to approach the following problems.

PROPOSITION 71

PROBLEM

747. *All three principal sections shall be given, namely the water section ACB, the widest section BCD and the diametric section ACD (Fig. 100); truly the volume shall be prepared, so that all the sections STP parallel to the widest section BDC shall be affine to the same, and this body shall be moving in the water along the direction CAL: to determine the resistance which it may experience.*

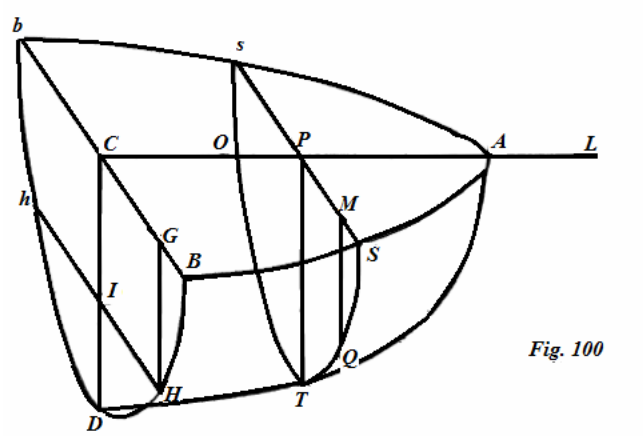


Fig. 100

SOLUTION

Since at first the diametrical section ATD shall be given, for that there may be put the abscissa $CP = r$, and the applied line $PT = s$, and the relation between r and s shall be such that $ds = pdr$. In the second place on account of the curve CBA or the given water section, there may be put for the $CP = t$, and the applied line $PS = u$, and there shall become $du = qdt$. In the third place, for the widest section CBD the abscissa shall be $CG = \tau$ and the applied line $GH = \gamma$ and $d\gamma = \rho d\tau$. Some section SPT from these may be considered parallel to the given section BCD for the given curve in place, which from the nature of the affine solid will be of the section BCD itself; and for the nature of the solid requiring to be expressed these three variables $AP = x$, $PM = y$ and $MQ = z$ may be taken and from the former notations applied to this case $t = r$, and $x = a - r$ with the length $AC = a$. Now since the base of the section SPT shall be $PS = u$ and the height $PT = s$; truly the base of the section BCD may be put $BC = b$ and the height $CD = c$;

hence on account of the relationship if there shall be $PM = y = \frac{u\tau}{b}$, there will be $MQ = z = \frac{s\gamma}{c}$. Now on account of $x = a - r$ there will be $dr = -dx$; and

$$dy = \frac{ud\tau + \tau q dr}{b}$$

on account of $t = r$ and

$$dz = \frac{s\rho d\tau + \gamma p dr}{c}.$$

Therefore since there shall be

$$d\tau = \frac{bdy}{u} + \frac{\tau q dx}{u}$$

on account of $dr = -dx$ there will become

$$dz = \frac{(s\tau q\rho - u\gamma p)dx}{cu} + \frac{bs\rho dy}{cu},$$

which equation compared with the general assumed above $dz = Pdx + Qdy$ gives

$$P = \frac{s\tau q\rho - u\gamma p}{cu} \text{ and } Q = \frac{bs\rho}{cu},$$

from which there becomes

$$1 + P^2 + Q^2 = \frac{c^2u^2 + b^2s^2\rho^2 + (s\tau q\rho - u\gamma p)^2}{c^2u^2}.$$

We may proceed now to the formulas

$$\frac{P^3dxdy}{1+P^2+Q^2}, \quad \frac{P^2dxdy}{1+P^2+Q^2} \text{ and } \frac{P^2(x+Pz)dzdy}{1+P^2+Q^2},$$

for the resistance and direction requiring to be found for the forces determined, which require a twofold integration, the one in which x and the other in which y may be put constant.

Therefore since there shall be $dx = -dr$, and on putting x constant there shall become $dy = \frac{ud\tau}{b}$; these values may be substituted in place of dx and dy, so that there may become

$$dxdy = -\frac{udrd\tau}{b};$$

and if in these formulas the integration may be put in place with *r* constant, likewise there will be constant quantities depending on *r* such as *s, t, u, p, q*, truly in the other integration in which *r* is put constant, in addition γ and ρ will be constants. But the integration in which *r* is put constant thus is resolved so that the integral shall vanish on putting $\tau = 0$, and then there may be put $\tau = b$ or $\gamma = 0$; in a similar manner the other integration in which τ is put constant is required to be resolved, so that the integral shall vanish on putting $r = 0$, and with this done there may be put $r = a$. Likewise moreover from which the integration may be started, since the variables *r* and τ, and the remaining, which are given by these two, in turn shall not depend on each other. Therefore from these premises the force of the horizontal resistance opposing the motion and acting along the direction *AC*

$$= \frac{-2v}{bc}\iint \frac{(s\tau q\rho - u\gamma p)^3\, drd\tau}{c^2u^2 + b^2s^2\rho^2 + (s\tau q\rho - u\gamma p)^2};$$

truly the force of the vertical resistance, by which the body is acted on upwards will be

$$= \frac{-2v}{b}\iint \frac{(s\tau q\rho - u\gamma p)^2\, udrd\tau}{c^2u^2 + b^2s^2\rho^2 + (s\tau q\rho - u\gamma p)^2}.$$

But the point *O*, in which this applied force may be considered to be found on dividing this expression

$$\iint \frac{P^2(x + Pz)udrd\tau}{1 + P^2 + Q^2}$$

by this one

$$\iint \frac{P^2 udrd\tau}{1 + P^2 + Q^2},$$

indeed the quotient will give the interval *AO*. Q. E. I.

COROLLARY 1

748. Since in general the volume shall be $= -2\iint Qydxdy$, moreover for our case there shall be

$$-dxdy = \frac{udrd\tau}{b},\quad y = \frac{u\tau}{b}\quad \text{and}\quad Q = \frac{bs\rho}{cu},$$

for our case the volume of the solid will be

$$=\frac{2}{bc}\iint us\tau\rho drd\tau=\frac{2}{bc}\int usdr\int\tau\rho d\tau.$$

Truly there is

$$\int\tau\rho d\tau=\int\tau d\gamma=-\text{area } BCD;$$

therefore if this area may be called *ff,* the volume of the solid will be

$$=\frac{-2ff}{bc}\int usdr.$$

COROLLARY 2

749. If the diametric section *ACD* shall be affine to the water section, then all the sections parallel to *BCD* itself likewise will be similar. Moreover then there will become $s:u=c:b$ as well as

$$u=\frac{bs}{c} \text{ and } q=\frac{bp}{c},$$

with which values substituted the above expressions found are found for the similar sections.

COROLLARY 3

750. If the line *DTA* may be changed into a horizontal right line there will become $s=c$ and $p=0,$ hence the force of the horizontal resistance will become

$$=\frac{-2v}{b}\iint\frac{\tau^3q^3\rho^3drd\tau}{u^2+b^2\rho^2+\tau^2q^2\rho^2}.$$

And if the area *ACB* may be put $=gg$, with the area $BCD=ff$, the volume of this body $=\frac{2ffgg}{b}.$

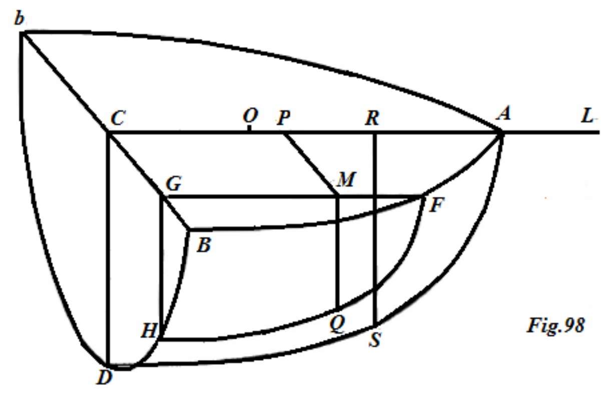
PROPOSITION 72

PROBLEM

751. *Again the three principal sections shall be ACB, ACD and BCD (Fig. 98), and all the sections FGH parallel to the diametric section ACD shall be affine to the same section: and this body may be moved in water along the direction CAL; to determine the resistance which it may experience.*

SOLUTION

Again as before so that for the diametrical section ACD the abscissa $CR = r$ and the applied line $RS = s$ and $ds = pdr$.



Then for the water section CBA the abscissa itself shall be taken on AC and for that $GF = t$, and the applied line corresponding to that, which shall be equal u equal to CG, and $du = qdt$. For the third and final section BCD the abscissa shall be $CG = \tau$ and with the applied line $GH = \gamma$ with there being $d\gamma = \rho d\tau$. If now the vertical section may be considered FGH parallel to the diametric section ACD there will become $u = \tau$, and $d\tau = qdt$, from which $\tau,\ q,\ u,\ \gamma$ and ρ will be functions of t and on which they will depend, and there will become $d\gamma = q\rho dt$. Therefore on putting $AC = a,\ BC = b$ and $CD = c,$ since the figure FGH is affine to the figure ACD the abscissa may be taken in that

$$GM = \frac{tr}{a},$$

and the applied line will be

$$MQ = \frac{\gamma s}{c}.$$

On account of which if there may be called $AP = x,\ PM = y$ but $MQ = z,$ there will be

$$x = a - \frac{tr}{a}\ ,\ y = \tau = u \ \text{ and } \ z = \frac{\gamma s}{c}.$$

Therefore since there shall be

$$dy = qdt,\ \text{or}\ dt = \frac{dy}{q},$$

there will be

$$dx = \frac{-ry}{aq} - \frac{tdr}{a} \ \text{ and } \ dz = \frac{\gamma pdr}{c} + \frac{sq\rho dt}{c} = \frac{\gamma pdr}{c} + \frac{s\rho dt}{c},$$

from which there becomes

$$dz = \frac{-a\gamma pdx}{ct} + \frac{(tsq\rho - r\gamma p)dy}{ctq};$$

which compared with the general equation $dz = Pdx + Qdy$ gives

$$p = \frac{-a\gamma p}{ct} \text{ and } Q = tsq\rho - \frac{r\gamma p}{ctq}$$

and

$$1 + P^2 + Q^2 = \frac{c^2t^2q^2 + a^2\gamma^2 p^2 q^2 + \left(tsq\rho - r\gamma p\right)^2}{c^2t^2q^2}.$$

But so that it may pertain for the differential formulas put in place to depend on *dxdy*, and not to depend on *x* and *y* individually; since there shall be $dy = qdt$, and thus *y* will depend on *t* only, there will become

$$dx = \frac{-tdr}{a}; \text{ on account of } dy = 0,$$

when the equation concerned with *dx* is sought. Therefore there will become

$$dxdy = \frac{-tqdrdt}{a},$$

and the force of the horizontal resistance acting in the direction *AC* will become

$$= \frac{2a^2v}{c} \iint \frac{\gamma^3 p^3 q^3 drdt}{c^2t^2q^2 + a^2\gamma^2 p^2 q^2 + \left(tsq\rho - r\gamma p\right)^2}$$

where there is need for a twofold integration, the one in which *t* is made constant, and with that u, γ, q and ρ, in the other *r* is put constant with its functions *s* and *p*. Truly in a similar manner the force of the vertical resistance will be

$$= -2av \iint \frac{\gamma^2 p^2 q^3 tdrdt}{c^2t^2q^2 + a^2\gamma^2 p^2 q^2 + \left(tsq\rho - r\gamma p\right)^2}$$

the point of application of which will be *O*, and its interval *AO* will be the quotient which results from the division of this quantity

$$\iint \frac{P^2\left(x + Pz\right)tqdrdt}{1 + P^2 + Q^2}$$

by this

$$\iint \frac{P^2 tqdrdt}{1 + P^2 + Q^2}.$$

Q.E.I.

COROLLARY 1

752. The volume of this body will be found from the general formula

$$-2\iint Qydxdy,$$

which for our case will be changed into this

$$\frac{2}{ac}\iint\left(tsuq\rho drdt - ru\gamma pdrdt\right)$$

which for the first integration with t put constant, gives

$$\frac{2ff}{ac}\int\left(tuq\rho + u\gamma\right)dt = \frac{2ff}{ac}\int t\gamma du$$

on account of $q\rho dt = d\gamma,$ with ff denoting the area ACD.

COROLLARY 2

753. If the curves CBA and CBD were affine, that is, $GF : GH = a : c$,
thus so that there shall become

$$\gamma = \frac{ct}{a} \quad \text{and} \quad q\rho = \frac{c}{a},$$

all the sections FGH will become similar to each other, and the horizontal resistance of the body will be

$$= 2av\iint \frac{tp^3q^3drdt}{a^2q^2\left(1+p^2\right)+\left(s-rp\right)^2},$$

as now agrees with the above.

COROLLARY 3

754. If the curve BD may be changed into a right line parallel to BC itself, thus so that the widest section BDb may become a rectangle, there will become $\gamma = c$ and $\rho = 0$; therefore the horizontal resistance of this solid or retarding the motion is

$$= 2a^2v\iint \frac{p^3q^3drdt}{a^2p^2q^2 + r^2p^2 + t^2q^2}.$$

COROLLARY 4

755. Since in this expression p and q, and likewise r and t are present equally, it is understood the sections ACB and ACD with the same resistance maintained can be interchanged with each other, if indeed the widest section were a rectangular parallelogram.

COROLLARY 5

756. If in addition the sections ACB and ACD may become right angled triangles, in which case the volume will be changed into a curvilinear pyramid of which the base will be a rectangle, truly with the vertex A. Therefore since in this case there shall be

$$u = b - \frac{bt}{a}$$

and hence

$$q = -\frac{b}{a}, \quad \text{and} \quad s = c - \frac{cr}{a}$$

and hence

$$p = -\frac{c}{a},$$

the resistance of this body will be

$$= \frac{2b^3c^3v}{a^2}\iint \frac{drdt}{b^2c^2 + b^2t^2 + c^2r^2} = \frac{2b^3c^3v}{a^2}\int \frac{dr}{\sqrt{\left(b^2 + r^2\right)}}\text{Atang. } \frac{ab}{c\sqrt{\left(a^2 + r^2\right)}}.$$

PROPOSITION 73

PROBLEM

757. *Again the three principal sections shall be ACB, ACD and BCD* (Fig. 99)*, and the body ABDb shall be prepared thus, so that all the horizontal sections FHI shall be affine amongst themselves: and this body shall be moved along the direction AC in water: to determine the resistance which it will experience.*

SOLUTION

In the first place the abscissa for the diametric section ACD shall be assumed to be the axis CA, and for that abscissa IF itself shall be taken $= r$ and for the corresponding applied line which will be $= CI = GH = s$, and there shall become $ds = pdr$. Then for the water section CBA the abscissa $CT = t$ and the applied line $TV = u$ and there shall be $du = qdt$. In the third place for the widest section the abscissa shall be $CG = \tau$, and the applied line $GH = \gamma$ with there being $d\gamma = \rho d\tau$. If now some horizontal section FIH may be considered, the above denominations for that will produce the applied lines $\gamma = s$ and hence $pdr = \rho d\tau$. But since the section FIH is affine to the section ACB, and if there shall be put $AC = a$, $BC = b$ and $CD = c$, and there may be taken $IK = \frac{rt}{a}$, the corresponding applied line will be $KQ = \frac{\tau u}{b}$. If now there may be put $AP = x$, $PM = y$ and $MQ = z$, there will be

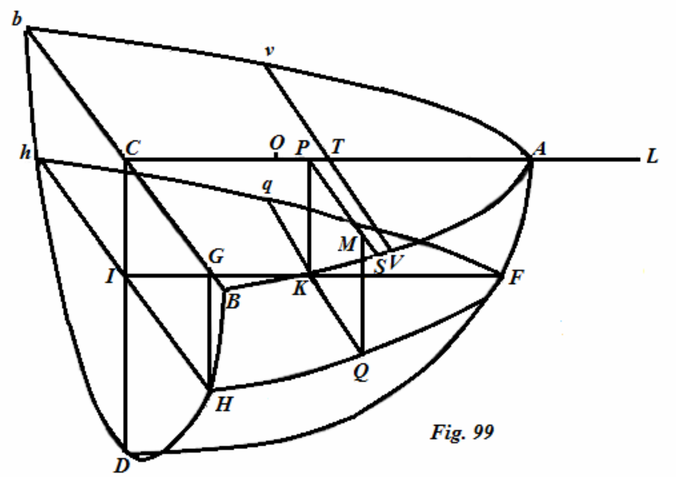


Fig. 99

$$x = a - \frac{tr}{a}, \quad y = \frac{\tau u}{b} \quad \text{and} \quad z = \gamma = s;$$

from which there becomes

$$dz = pdr, \quad dy = \frac{\tau q dt}{b} + \frac{updr}{b\rho},$$

on account of

$$d\tau = \frac{pdr}{\rho}, \text{ and } dx = \frac{-rdt}{a} - \frac{tdr}{a};$$

from which the following equation between x, y and z may be established:

$$dz = \frac{bpr\rho dy + a\tau pq\rho dx}{urp - t\tau q\rho}$$

which since compared with the general equation assumed above gives

$$P = \frac{a\tau pq\rho dx}{urp - t\tau q\rho} \quad \text{and} \quad Q = \frac{bpr\rho dy}{urp - t\tau q\rho}$$

thus so that there shall be

$$1+P^2+Q^2=\frac{p^2\rho^2\left(a^2\tau^2q^2+b^2r^2\right)+\left(urp-t\tau q\rho\right)^2}{\left(urp-t\tau q\rho\right)^2}$$

Now since z may be determined by a single variable of the investigation, it will be agreed to assume formulas for determining the resistance in which there shall be dz.
Indeed since there shall be $dz = pdr$, and on putting z or r constant there shall become

$$dy=\frac{\tau qdt}{b},$$

there will become

$$dzdy=\frac{\tau pqdrdt}{b},$$

in which two variables independent from each other are present, with the one r and the magnitudes given by that $s,\ p,\ \gamma,\ \tau,$ and ρ, and the other truly by t, with u and q, which in the integration are required to be separated properly from each other, thus while the one group of variables are put in place, the other group are to be treated as constants.
Now since the force of the horizontal resistance or the force acting along the direction AC shall be

$$\iint\frac{P^3dzdy}{1+P^2+Q^2}$$

for our case this resistance will become

$$=\frac{2a^2v}{b}\iint\frac{\tau^3p^3q^3\rho^3drdt}{p^2\rho^2\left(a^2\tau^2q^2+b^2r^2\right)+\left(urp-t\tau q\rho\right)^2}$$

which now is taken first more often, must be integrated from these in the due manner. But the vertical force of the resistance shall become

$$=\frac{2av}{b}\iint\frac{\tau^2p^2q^2\rho drdt\left(urp-t\tau q\rho\right)}{p^2\rho^2\left(a^2\tau^2q^2+b^2r^2\right)+\left(urp-t\tau q\rho\right)^2}$$

[C. Truesdell has given corrected versions of these two integrals in his edited version of the corresponding volume in the *O.O*. edition.]

moreover the position or the point O where this force is considered to be applied, will be found in that manner, as we have given generally, evidently how great an interval of AC may result if

$$\iint\frac{P\left(x+Pz\right)\tau pqdrdt}{1+P^2+Q^2}$$

may be divided by

$$\iint \frac{P\tau pqdrdt}{1+P^2+Q^2},$$

with the integrations of each made in the true absolute mode. Q. E. I.

COROLLARY 1

758. For the volume of this solid found it will be required to consider this expression $2\iint ydxdz$; for which since it is required to put in place $dz = pdr$ and with z constant

$$dx = \frac{-rdt}{a} \text{ and } y = \frac{\tau u}{b}$$

the volume will become

$$= \frac{2}{ab}\iint \tau urpdrdt = \frac{2}{ab}\int udt \cdot \int \tau rpdt.$$

COROLLARY 2

759. Truly since the integral $\int udt$ gives the area ACB, which if it may be called $= ff$, the volume will become

$$\frac{2ff}{ab}\int \tau rpdt = \frac{2ff}{ab}\int \tau rds$$

on account of $pdr = ds$, or

$$= \frac{2ff}{ab}\int \tau rd\gamma \text{ on account of } d\gamma = ds,$$

which integration will depend on the nature of each of the curves CDA and CDB.

COROLLARY 3

760. If the right line AFD may become vertical there will be $r = a$ and $p = \infty$, from which the horizontal resistance, after ρdr being put in the formula found in place of pdr, gives

$$= \frac{2v}{b}\iint \frac{\tau^3 q^3 \rho dtd\tau}{b^2+u^2+\tau^2 q^2}.$$

SCHOLIUM

761. We have described in detail for these the resistance set out at length, which bodies experience with the given diametric plane moving forwards in water; indeed scarcely the figure, which certainly must be considered suitable for ships, which shall not be present in the kinds of bodies treated. Therefore an arrangement will be required so that also, so that we may progress to the consideration of plane figures to be used for oblique motion, but since with plane figures this treatment may stand out to be so difficult, with much more difficulty, when the question is concerned with moving bodies, this investigation will become impossible, and besides whatever the direction the resistive force would have to be elicited by the most outstanding calculation, yet thence of little use for the perfection of ships we might follow. On account of which from these hindering causes we will try to impose in this final chapter, that which we may be able to make in the following without notable inconvenience, since these are brought forwards, which are concerned with plane figures, if they may be moved in an oblique motion, it may be able to estimate well enough the direction of the resistance and the centre of the resistance.

CAPUT SEXTUM

DE RESISTENTIA, QUAM CORPORA QUAECUNQUE IN AQUA MOTU DIRECTO LATA PATIUNTUR

PROPOSITIO 61

PROBLEMA

612. *Sit AT DEb* (Fig. 93) *figura navis anterior aquae immersa et plano diametrali verticali ACD in duas portiones aequales et similes diremta; haecque figura in aqua cursu directo progrediatur secundum directionem CAL: determinare resistentiam, quam haec figura in motu suo patietur.*

SOLUTIO

Repraesentatur in hac figura partis anterioris seu prorae navigii aliusve corporis similis aquae innatantis ea portio quae aquae est immersa, cuiusque superficies in cursu directo ab aqua resistentiam patitur. In ea igitur est planum horizontale *ABb* sectio aquae, planum verticale *ACD* dirimit istam

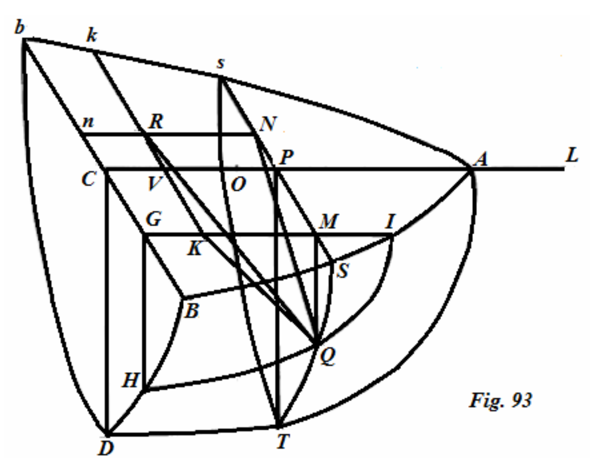


Fig. 93

portionem ita in duas partes similes et aequales *ACDB* et *ACDb,* ut omnes rectae horizontales in plano *ACD* ductae sint totidem diametri sectionum horizontalium seu plano *ABb* parallelarum solidi propositi. Cum igitur motus huius corporis in aqua fiat secundum directionem horizontalem *CAL,* manifestum est mediam resistentiae directionem incidere debere in ipsum planum diametrale *ACD;* unde vis resistentiae partim motum retardabit, partim corpus ex aqua elevabit, si quidem media directio non fuerit horizontalis, sed sursum vergens. Ad hunc ergo resistentiae duplicem effectum definiendum, sit primo altitudo celeritati, qua corpus in directione *CAL* progreditur debita altitudini v. Deinde sumta recta *AO* pro axe sit in ea abscissa $AP = x$, atque per punctum *P* facta concipiatur sectio verticalis *STs* ad planum diametrale *ACD* normalis, in cuius basi *Ss* ponatur portio quaecunque $PM = y$, et verticalis $MQ = z$. Definietur igitur hoc modo in superficie corporis propositi punctum *Q* per aequationem inter tres variabiles x, y et z. Sit autem ista aequatio reducta ad hanc aequationem differentialem $dz = Pdx + Qdy$, in qua *P* et *Q* sint functiones quaepiam ipsarum x et y, non involventes z; haecque aequatio ob partes utrinque circa diametrale planum *ACD* sitas similes et aequales utriusque medietatis *AC DB, ACDb* naturam exprimet. Iam quo pateat sub quonam angulo elementum superficiei in *Q* sumtum in aquam impingat, vel planum tangens superficiem in *Q* vel recta normalis *QR* ad superficiem in puncto *Q* definiri debebit. Investigemus ergo positionem normalis huius *QR,* quem in finem primo solum sectionem *STs* considerabimus, cuius natura ob x
constans hac exprimetur aequatione $dz = Qdy$, ex qua ita definietur positio normalis *QN* ad arcum *SQT,* ut sit subnormalis

$$MN = -\frac{zdz}{dy} = -Qz \text{ unde fit } PN = -y - Qz.$$

Quare si in plano *ABn* ad *MN* ducatur perpendicularis *NR,* omnes rectae ex *Q* ad hanc rectam *NR* ductae ad curvam *SQT* in puncto *Q* erunt normales; quarum quae simul ad ipsam superficiem in puncto *Q* sit normalis, reperietur hoc modo. Per puncta *M* et *Q* concipiatur sectio verticalis *IMGH* plano diametrali *ACD* parallela, ac curvae *IQH* ob y constans natura exprimetur hac aequatione $dz = Pdx$. Sit nunc recta *QK* normalis ad curvam *IQH* in puncto *Q,* erit sub normalis

$$MK = \frac{zdz}{dx} = Pz.$$

Si ergo in plano *ABb* ad rectam *MK* ducatur normalis *KVR,* omnes quoque rectae ex *Q* ad lineam *KR* ductae normales erunt in *Q* ad curvam *IQH.* Cum itaque rectae *NR* et *KR* sese intersecent in puncto *R,* existente

$$AV = x + Pz, \text{ et } VR = PN = -y - Qz,$$

quarum haec *VR* ad alteram *AV* est perpendicularis, erit recta *QR* in puncto *Q* tam ad curvam *SQT* quam *IQH* normalis; et hancobrem haec recta *QR* normalis erit ad superficiem ipsam in puncto *Q.* Angulus ergo quo superficiei elementum in *Q* in aquam

impingit, complementum erit ad rectum eius anguli quem normalis *QR* cum directione cursus *CAL* seu cum recta *RN* huic parallela
constituit, qui angulus est *QRN.* At ob

$$MN = -Qz, \text{ erit } QN = z\sqrt{1+PP+QQ}$$

et ob

$$NR = MK = Pz \text{ erit } QR = z\sqrt{1+PP+QQ}$$

unde anguli *QRN* sinus erit

$$= \frac{\sqrt{1+QQ}}{\sqrt{1+PP+QQ}}, \text{cosinus vero} = \frac{P}{\sqrt{1+P^2+Q^2}},$$

qui cosinus simul sinus erit anguli sub quo superficiei elementum in *Q* situm in aquam impingit. Quare si elementum superficiei ponatur $= dS$, erit vis resistentiae quam patietur $= \dfrac{P^2 vdS}{1+P^2+Q^2}$, huiusque vis directio sita erit in ipsa normali *QR* ad superficiem. Oportet autem elementum superficiei *dS* per differentialia cordinatarum *x, y* et *z* exprimi, quo per integrationem totalis resistentia colligi queat. Concipiatur igitur abscissa *x* crescere elemento *dx,* et applicata *y* elemento *dy;* orieturque in *P* rectangulum infinite parvum *dxdy* in plano *ABb* positum, cui ex angulis eius deorsum ductis verticalibus in superficie respondebit elementum *dS,* cuius inclinatio ad planum *ABb,* quae aequalis angulo *MQR* praebebit

$$dS = dxdy\sqrt{1+P^2+Q^2}.$$

Hinc ergo resistentia quam elementum *dS* patietur erit $= \dfrac{P^2 vdxdy}{\sqrt{1+P^2+Q^2}}$, eiusque directio incidet in normalem *QR.* Resolvatur nunc haec resistentiae vis in ternas inter se normales quarum directiones sint parallelae coordinatis tribus *AP, PM,* et *MQ.* Cum igitur hae tres vires concipi queant in puncto *R* applicatae, figura in *R* verticaliter sursum pelletur vi $= \dfrac{P^2 vdxdy}{1+P^2+Q^2}$; tum urgebitur in directione *Rn* axi *AC* parallela vi $= \dfrac{P^3 vdxdy}{1+P^2+Q^2}$;
denique urgebitur in directione *Rk* rectae *Bs* parallela vi $= \dfrac{-P^2 Qvdxdy}{1+P^2+Q^2}$. . Si nunc resistentia elementi in altera medietate *ACDb* analogi simili modo colligatur, eaque cum inventa coniungatur, vires in directionibus ipsi *Ss* parallelis se mutuo destruent;
at in *V* corpus verticaliter sursum pelletur vi $= \dfrac{2P^2 vdxdy}{1+P^2+Q^2}$; simulque in directione axis *VC* directe retrorsum urgebitur vi $= \dfrac{2P^3 vdxdy}{1+P^2+Q^2}$. A resistentia igitur, quam patitur portio

superficiei a duabus sectionibus *STs* et altera huic parallela et intervallo *dx* dissita abscissae figura retrorsum urgebitur in directione *AC* vi

$$= 2vdx\int \frac{P^3 dy}{1+P^2+Q^2},$$

quae integratio in qua ponitur *x* constans ita absolvatur ut evanescat posito $y = 0$, tumque ponatur $y = PS$. Sursum vero urgebitur vi

$$= 2vdx\int \frac{P^2 dy}{1+P^2+Q^2},$$

cuius vis momentum respectu puncti *A* erit

$$= 2vdx\int \frac{P^2\left(x+Pz\right)dy}{1+P^2+Q^2};$$

quae integralia eodem modo quo ante sunt accipienda. Totalis ergo resistentia quam integra superficies ab aqua patietur, reducitur ad duas vires quarum altera retrorsum urgebitur in directione *AC* vi

$$= 2v\int dx\int \frac{P^3 dy}{1+P^2+Q^2},$$

ubi notandum integrale $= \int \frac{P^3 dy}{1+P^2+Q^2}$ praescripto modo sumtum fore functionem ipsius *x* tantum; ex quo posterius integrale

$$\int dx\int \frac{P^3 dy}{1+P^2+Q^2},$$

ita sumi debet ut evanescat posito $x = 0$, hocque facto poni debet $x = AC$, quo resistentia totius corporis propositi obtineatur. Simul vero figura sursum verticaliter urgebitur vi

$$= 2v\int dx\int \frac{P^2 dy}{1+P^2+Q^2},$$

cuius vis momentum cum sit

$$= 2\int vdx\int \frac{P^2\left(x+Pz\right)dy}{1+P^2+Q^2}$$

ea censenda est applicata in puncta *O* axis *AC,* ita ut sit

$$AO = \frac{\int dx \int \frac{P^2(x+Pz)dy}{1+P^2+Q^2}}{\int dx \int \frac{P^2 dy}{1+P^2+Q^2}}$$

integralibus ea lege, qua est praeceptum sumtis. Ex his ergo ambabus viribus resistentiae aequivalentibus reperietur media totius resistentiae directio, quae per punctum *O* in plano *ACD* transibit, atque cum *AC* angulum constituet cuius tangens erit sub quo angulo media directio resistentiae ex *O* versus puppim sursum verget
Q.E.I.

COROLLARIUM 1

613. Navis igitur cursus directo secundum directionem *AL* progrediens a resistentia retardabitur vi

$$= 2v \int dx \int \frac{P^3 dy}{1+P^2+Q^2},$$

quae expressio volumen aquae indicat cuius pondus ipsi vi resistentiae est aequale.

COROLLARIUM 2

614. Cum autem navis insuper sursum urgeatur vi

$$= 2v \int dx \int \frac{P^2 dy}{1+P^2+Q^2},$$

tanta vi navis quasi levior facta est censenda, eaque ex aqua attolletur, aequivalet vero etiam ponderi aquae, cuius volumen ista expressione indicatur.

COROLLARIUM 3

615. Praeterea vero, nisi media directio resistentiae per ipsum gravitatis centrum transeat, navis a resistentia circa axem latitudinalem convertetur, eiusque prora vel elevabitur vel deprimetur, prout directio resistentiae vel supra vel infra centrum gravitatis dirigatur.

COROLLARIUM 4

616. Denique ex inventis expressionibus manifestum est, omnes resistentiae effectus, qui tum in retardanda tum allevanda tum inclinanda navi consistunt rationem sequi duplicatam celeritatum, quibus navis promovetur.

COROLLARIUM 5

616. Superficies tota huius corporis ex datis formulis ita calculo subducetur. Cum elementum superficiei dS sit

$$= dxdy\sqrt{1+P^2+Q^2},$$

integretur prima

$$dy\sqrt{1+P^2+Q^2},$$

posito x constante ita ut integrale evanescat posito $y = 0$ tumque ponatur $y = PS$, quo facto integrale abibit in functionem quandam ipsius x, ita ut

$$\int dx \int dy\sqrt{1+P^2+Q^2}$$

assignari queat, quod integrale posito $x = AC$ bis sumtum, totam superficiem praebebit.

COROLLARIUM 6

617. Ad soliditatem autem totius figurae $ABDb$ inveniendam, sit $PT = t$ et $PS = s$ erunt t et s functiones ipsius x ex aequatione

$$dz = Pdx + Qdy$$

assignabiles. Tum vero erit area

$$PTS = \int zdy = -\int ydz$$

ob $z = 0$ quando fit

$$y = s = -\int Qydy.$$

Integrale $\int Qydy$ ita sumatur posito x constante, ut evanescat posito $y = 0$ tumque ponatur $y = s$. Quo facto $2\int -dx \int Qydy$ posito post integrationem $x = AC$ dabit soliditatem totius figurae.

COROLLARIUM 7

618. Cum superficies $ABDb$ ponatur tota atque sola resistentiam pati, si quidem navis in directione AL progrediatur, necesse est ut planum BDb sit amplissima navis sectio transversalis, atque insuper ut omnia totius huius portionis $ABDb$ plana tangentia versus proram inclinent.

COROLLARIUM 8

619. Hinc etiam colligitur, si figura *ABDb* fuerit semissis corporis cuiusdam aqua gravioris, hocque corpus in aqua vel descendat vel totum aquae submersum moveatur in directione *AL,* tum resistentiam esse passurum secundum directionem *AC* tantum, quae erit

$$= 4v \int dx \int \frac{P^3 dy}{1+P^2+Q^2}.$$

SCHOLION

620. Ex aequatione differentiali $dz = Pdx + Qdy$, cuius quidem integrale notum esse assumimus, qua naturam superficiei *ATDB* expressimus, tota ista superficies perfecte cognoscitur. Sectio enim aquae *ABb* primo cognoscetur si fiat $z = 0$, quo casu si ponatur $PS = S$, fit $y = s$ atque aequatio $Pdx + Qds = 0$ naturam sectionis aquae seu relationem inter $AP = x$ et $PS = s$ exhibebit. Simili modo quaevis alia sectio horizontalis innotescet ponendo $z =$ constanti seu $dz = 0$, ex aequatione $Pdx + Qdy = 0$, in qua x abscissam in axe ipsi *AC* parallelo sumtam et y applicatam denotabit. Quamvis autem pro his omnibus sectionibus eadem prodeat aequatio

$$Pdx + Qdy = 0,$$

tamen hinc omnes inter se aequales non sint censendae, cum aequatio

$$Pdx + Qdy = 0$$

sit differentialis et in integratione innumerabiles constantes recipere queat. Pro qualibet autem sectione horizontali integrale formulae $Pdx + Qdy$ aequale poni debet valori ipsius z, seu intervallo, quo quaeque sectio a sectione aquae *ABb* distat. Semper vero formula differentialis $Pdx + Qdy$ integrationem admittet, quia generaliter est $dz = Pdx + Qdy$ atque P et Q a z non pendere ponuntur, ita ut $Pdx + Qdy$ sit differentiale eius functionis ipsarum x et y, cui z aequatur. Hancobrem P et Q eiusmodi erunt functiones ipsarum x et y, ut si fuerit
$dP = Rdx + Bdy$ et $dQ = Tdx + Udy$, futurum sit
$S = T$, unde generatim nexus inter P et Q inspicitur. Sin autem P et Q fuerint functiones, in quibus x et y ubique eundem dimensionum numerum puta n teneant, erit $Px + Qy = (n+1)z$, unde immediate ex dato valore ipsius P valor ipsius Q reperitur. Deinde etiam natura plani diametralis verticalis *ACD* exprimetur ponendo $y = 0$, quo casu fit $z = PT = t$, ita ut habeatur inter

$AP = x$ et $PT = t$ ista aequatio $dt = Pdx$, posito in *P,* quae generaliter est functio ipsarum *x* et y, $y = 0$. Natura denique sectionis navis transversalis amplissimae *BDb* habebitur cognita ex aequatione $dz = Pdx + Qdy$ ponendo $x = AC = a$; tum enim ob $CG = y$ et $GH = z$ erit $dz = Qdy$. Quemadmodum autem ex aequatione canonica $dz = Pdx + Qdy$ natura totius superficiei *ATDB* cognoscitur, ita vicissim ex data superficiei natura aequatio canonica elicietur. Si enim dentur aequationes tum pro sectione aquae *ACB,* tum pro plano diametrali *ATD,* tum etiam pro singulis sectionibus transversalibus *SPT,* definire licebit longitudinem $MQ = z$, quae ex quovis puncto *M* sectionis aquae deorsum usque ad superficiem demittitur; hocque modo *z* exprimetur per quantitatem ex *x,* et *y* ex constantibus compositam, qui valor differentiatus dabit $dz = Pdx + Qdy$ aequationem canonicam naturam superficiei exprimentem. Praecipuas igitur huiusmodi superficierum species in sequentibus problematis evolvemus, atque resistentiam, quam quaeque in aqua directe promota patitur, definiemus; postquam praecipuas species ad aequationem canonicam huius formae

$$dz = Pdx + Qdy$$

reduxerimus.

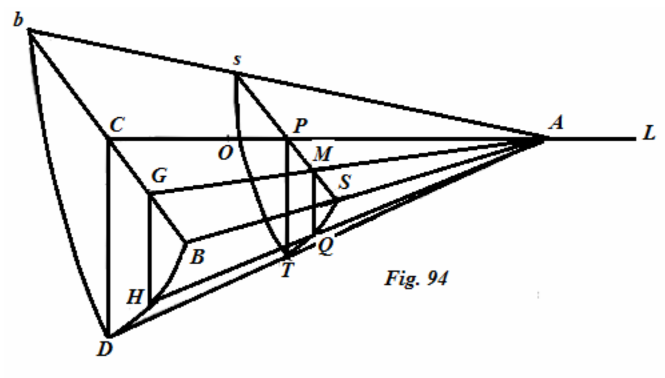


Fig. 94

PROPOSITIO 62

PROBLEMA

621. *Sit pars corporis aquae innatantis, quae in aqua versatur, figura conica ABDb* (Fig. 94) *basin habens datam BDb et verticem in A ita ut eius superficies terminetur lineis rectis HA ex singulis basis BDb punctis ad verticem A ductis, moveaturque haec figura secundum directionem axis CAL, determinare resistentiam quam patietur.*

SOLUTIO

In hoc igitur corpore sectio aquae *BAb* erit triangulum isosceles, planum diametrale *ACD* vero triangulum rectangulum. Deinde quaevis sectio transversalis *STs* basi seu sectioni amplissimae *BDb* parallela erit ipsi basi *BDb*similis. Sit ergo semissis basis *CBD,* quippe cui altera semissis *CbD* similis est et aequalis, curva quaecunque data ita ut eius natura sit cognita per aequationem inter coordinatas *CG* et *GH.* Positis igitur $CG = r$ et $GH = u$, erit *u* functio quaecunque ipsius *r.* Ductis iam rectis *GA* et *HA* positoque $AC = a$ erit ob triangula similia

$$AC(a) : Ap(x) = CG(r) : PM(y) = GH(u) : MQ(z)$$

unde erit $y = \frac{rx}{a}$, et $z = \frac{uy}{r} = \frac{ux}{a}$.

Sit $du = pdr$, existente p functione quadam ipsius r, erit ob

$$r = \frac{ay}{x},\ dr = \frac{axdy - aydx}{xx} \text{ atque } du = \frac{apxdy - apydx}{xx};$$

unde fit

$$dz = \frac{udx}{a} + pdy - \frac{pydx}{x},$$

quae aequatio cum generali canonica comparata $dz = Pdx + Qdy$ dat

$$P = \frac{n}{a} - \frac{py}{x} = \frac{u - pr}{a} \text{ ob } y = \frac{rx}{a}$$

atque $Q = p$; unde obtinetur

$$1 + P^2 + Q^2 = 1 + p^2 + \frac{(u - pr)^2}{aa}.$$

Ad resistentiam vero definiendam oportet ante omnia sequentia invenire integralia

$$\int \frac{P^2 dy}{1 + P^2 + Q^2},\ \int \frac{P^3 dy}{1 + P^2 + Q^2},\ \int \frac{P^2 \left(x + Pz\right) dy}{1 + P^2 + Q^2}$$

posito x constante, integralibusque ita sumtis ut evanescant posito $y = 0$, ponere $y = PS$ vel $z = 0$. At posito x constante est $dy = \dfrac{xdr}{a}$, unde fit

$$\int \frac{P^2 dy}{1 + P^2 + Q^2} = \frac{x}{a} \int \frac{\left(u - pr^2\right) dr}{a^2 + a^2 p^2 + \left(u - pr\right)^2},$$

$$\int \frac{P^3 dy}{1 + P^2 + Q^2} = \frac{x}{a^2} \int \frac{\left(u - pr^2\right)^3 dr}{a^2 + a^2 p^2 + \left(u - pr\right)^2},$$

atque

$$\int \frac{P^2 \left(x + Pz\right) dy}{1 + P^2 + Q^2} = \frac{xx}{a^3} \int \frac{\left(u - pr^2\right)\left(a^2 + u^2 - pru\right) dr}{a^2 + a^2 p^2 + \left(u - pr\right)^2}$$

quae cum ita fuerint accepta ut evanescant posito $y = 0$ seu $r = 0$, poni debet $z = 0$ seu $u = 0$. Quoniam vero ista integralia hoc modo inventa ab x non pendebunt, erit totalis resistentia qua figura secundum directionem AC retropellitur

$$= 2v\int \frac{xdx}{a^2}\int \frac{(u-pr)^3\, dr}{a^2+a^2p^2+(u-pr)^2} = v\int \frac{(u-pr)^3\, dr}{a^2+a^2p^2+(u-pr)^2},$$

integrali hoc eodem modo accepto quo modo est praeceptum. Simul vero a resistentia corpus hoc conicum sursum urgebitur vi

$$= 2v\int \frac{xdx}{a^2}\int \frac{(u-pr)^2\, dr}{a^2+a^2p^2+(u-pr)^2} = av\int \frac{(u-pr)^2\, dr}{a^2+a^2p^2+(u-pr)^2},$$

cuius vis directio verticalis transibit per punctum O ita ut sit

$$AO = \frac{\int \frac{xxdx}{a^2}\int \frac{(u-pr)^2\left(a^2+u^2-pru\right)dr}{a^2+a^2p^2+(u-pr)^2}}{\int \frac{xdx}{a}\int \frac{(u-pr)^2\, dr}{a^2+a^2p^2+(u-pr)^2}},$$

seu

$$AO = \frac{2\int \frac{(u-pr)^2\left(a^2+u^2-pru\right)dr}{a^2+a^2p^2+(u-pr)^2}}{3a\int \frac{(u-pr)^2\, dr}{a^2+a^2p^2+(u-pr)^2}},$$

Ex hisque duabus viribus una cum puncto O cognitis tota resistentiae vis innotescit.
Q. E. I.

COROLLARIUM I

622. Intelligitur ex formulis inventis primum quo longius vertex A a basi BDb distet, eo minorem fore vim resistentiae, quam figura patitur, resistentiam vero non tenere rationem quampiam assignabilem pro varietate longitudinis axis $AC = a$.

COROLLARIUM 2

623. At si longitudo AC fuerit vehementer magna ut prae a reliquae quantitates ad basem BDb pertinentes negligi queant, tum resistentiae vis horizontalis in directione AC erit

$$= \frac{v}{a^2}\int \frac{(u-pr)^3\, dr}{1+pp}$$

vis autem qua sursum pelletur

$$=\frac{v}{a}\int\frac{(u-pr)^2\,dr}{1+pp}$$

cuius directio transibit per punctum O existente $AC=\frac{2}{3}a$.

COROLLARIUM 3

624. Hoc ergo casu vis resistentiae corpus retropellentis in directione AC reciproce se habebit ut quadratum longitudinis coni AC. At vis sursum pellens rationem tenebit reciprocam longitudinis coni: scilicet si longitudo coni fuerit vehementer magna.

COROLLARIUM 4

625. Cum area basis BDb sit $=2\int udr$ posito post integrationem $r=CB$ seu $u=0$, erit resistentia, quam basis pateretur, si directe secundum CA eadem celeritate in aqua moveretur $=2v\int udr$, eiusque directio esset normalis ad basin et per eius centrum gravitatis transiret.

COROLLARIUM 5

626. Idem vero casus, quo sola basis promovetur, obtinetur si fiat $a=0$. Tum autem resistentiae vis sursum urgens evanescit, vis autem retroagens erit $=v\int(u-pr)dr=v\int udr-v\int rdu$. At si post integrationem ita peractam ut prodeat nihil, si ponatur $r=0$, ponatur $u=0$, tum est $\int rdu=-\int udr$, ex quo resistentia retropellens prodit $=2v\int udr$.

COROLLARIUM 6

627. Tota superficies huius corporis est

$$=2\int dx\int dy\sqrt{1+P^2+Q^2}$$

(§ 610). Sit vero

$$=\int dy\sqrt{1+P^2+Q^2}=\int\frac{dy}{a}\sqrt{a^2+a^2p^2+(u-pr)^2};$$

quae cum ponatur x constans abit in

$$\frac{x}{a^2}\int dr\sqrt{a^2+a^2p^2+(u-pr)^2},$$

unde tota superficies prodit

$$=\int dr\sqrt{a^2+a^2p^2+(u-pr)^2}$$

posito post integrationem postremam $x=a$.

COROLLARIUM 7

628. Cum denique soliditas sit

$$=2\int -dx\int Qydy \ (\S 617) \text{ ob } Q=p \text{ et } y=\frac{rx}{a},$$

fiet ea

$$=2\int -dx\int \frac{x^2prdr}{aa}=2\int -\frac{xxdx}{aa}\int rdu=\tfrac{2}{3}a\int udr$$

denotante $\int udr$ aream *BCD;* id quod quidem ultro patet ex elementis Geometriae.

SCHOLION 1

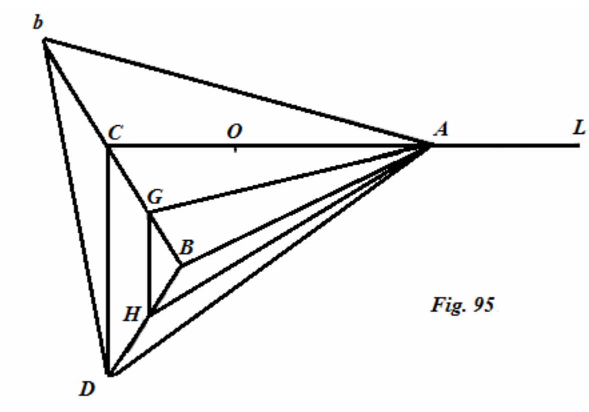


Fig. 95

629. In hac ergo propositione primam atque facillimam corporum speciem examini subiecimus, quae omnis generis corpora conoidica sub se complectitur: non solum enim conus rectus qui basin habet circularem in ea continetur, sed etiam coni obliqui, quippe qui ad rectos reduci possunt sumta quapiam sectione conica pro basi, deinde etiam generaliter huc pertinent omnia corpora, quae ex data quacunque base ad punctum quoddam sublime ductis lineis rectis generantur, quorsum praeter conos bases curvilineas habentes etiam pyramides pertinent. Hic autem secundum nostrum institutum eiusmodi tantum corpora conoidica consideramus, quae duas habent partes similes et aequales ex utraque plani diametralis

parte sitas, quo tota tractatio navibus maxime sit accommodata. Cum vero rem tam generaliter concipiendo formulae supersint integrales, de quarum integratione non constat, iuvabit casus quosdam speciales evolvere, quibus data figura determinata pro basi *BDb* accipitur.

EXEMPLUM 1

630. Sit pars aquae submersa quae resistentiam sentit pyramis triangularis *ABDb* (Fig. 95) cuius basis seu sectio amplissima *BDb* est triangulum isosceles, in quo sit $CB = Cb = b$ et $CD = c.$ Posito ergo $CG = r$ et $GH = u$, erit $c : u = b : b - r$, hincque

$$u = c - \frac{cr}{b} \text{ et } du = -\frac{cdr}{b}, \text{ unde fit } p = -\frac{c}{b}.$$

Si nunc haec pyramis directe progrediatur secundum directionem *AL* celeritate debita altitudini *v,* atque longitudo *AC* ponatur $= a$, reperietur ob

$$ur - pr = c \quad \text{et} \quad aa + aapp = \frac{aa(bb + cc)}{bb}$$

resistentiae vis in directione *AC* retropellens

$$= v\int \frac{(u - pr)^3\, dr}{a^2 + a^2p^2 + (u - pr)^2} - v\int \frac{b^2c^3dr}{aa(bb + cc) + bbcc}$$

unde post integrationem posito $r = b$ prodit ista resistentiae vis motui directe contraria

$$= \frac{b^3c^3v}{a^2b^2 + a^2c^2 + b^2c^2}.$$

Deinde cum sit

$$\int \frac{(u - pr)^2\, dr}{a^2 + a^2p^2 + (u - pr)^2} = \int \frac{b^2c^2dr}{a^2b^2 + a^2c^2 + b^2c^2}$$

erit vis resistentiae verticaliter sursum urgens

$$= \frac{a^2b^2c^2v}{a^2b^2 + a^2c^2 + b^2c^2},$$

cuius directio transibit per punctum *O* existente

$$AO = \frac{2\int (aa + cu)dr}{3ab} = \frac{2aa + cc}{3a}.$$

Soliditas vero totius huius pyramidis *ABDb* erit

$$=\frac{2a}{3}\int udr=\frac{abc}{3};$$

superficies vero in aquam irruens seu duo triangula *ABD* et *AbD*

$$=\int dr(aa+aapp+(u-pr)^2)=\sqrt{aabb+aacc+bbcc}.$$

COROLLARIUM 1

631. Cum igitur basis *BDb* sit $=bc$, et superficies in aquam impingens

$$\sqrt{a^2b^2+a^2c^2+b^2c^2}\ .$$

erit resistentia motum retardans aequalis altitudini celeritati debitae ducta in cubum basis et divisae per quadratum superficiei.

COROLLARIUM 2

632. Manente igitur basi *BDb,* eadem resistentia eo erit minor, quo maior fuerit superficies corporis, quae ab aqua resistentiam patitur; est enim resistentia motui contraria reciproce ut quadratum superficiei.

COROLLARIUM 3

633. Ponatur basis *BDb* constans seu $bc=ff$, ut sit $c=\frac{ff}{b}$, erit resistentia motum retardans

$$=\frac{bbf^4v}{a^2b^4+a^2f^4+bbf^4}$$

unde intelligitur resistentiam fore minimam, si vel b vel c maximam habuerit quantitatem, maxima autem erit resistentia si fuerit $b=c$.

COROLLARIUM 4

634. Cum in hoc casu tam ff quam a positum sit constans, atque $\frac{1}{3}aff$ denotet soliditatem figurae, patet inter omnes pyramides triangulares quae aequales bases et altitudines habent eam maximam pati resistentiam, cuius basis sit triangulum isosceles ad *D* rectangulum.

COROLLARIUM 5

635. Quo magis igitur angulus *BDb* differt a recto, eo minorem pyramis in motu suo sentiet resistentiam; ceteris paribus. Scilicet manentibus tum basi tum longitudine eiusdem quantitatis.

COROLLARIUM 6

636. Si basis *BDb* nuda contra a quam directe impingat eadem celeritate altitudini v debita, resistentiam sentiret $= bcv$. Ex quo resistentia pyramidis se habebit ad resistentiam basis ut b^2c^2 ad $a^2b^2 + a^2c^2 + b^2c^2$ unde intelligitur resistentiam basis eo esse maiorem resistentia pyramidis, quo maior sit eius altitudo a.

COROLLARIUM 7

637. Manente autem latitudine basis *Bb* et soliditate pyramidis eiusdem quantitatis, resistentia eo erit minor, quo minor fuerit profunditas $CD = c$, seu quo longior capiatur pyramidis longitudo *AC*.

COROLLARIUM 8

638. Denique notandum est vim resistentiae qua corpus sursum pellitur et ex aqua elevatur se habere ad vim resistentiae motui contrariam ut se habet a ad c, hoc est ut *AC* ad *CD*. Unde pyramis eo magis sursum pelletur, quo longior sit eius axis *AC*, seu quo fuerit acutior cuspis in *A*.

EXEMPLUM 2

639. Abeat corpus nostrum conoidicum in semiconum rectum, ita ut tam basis *BDb* quam omnes sectiones ipsi parallelae *STs* sint semicirculi (Fig. 94). Ponatur autem huius coni altitudo $AC = a$, quae simul est directio secundum quam hic conus movetur celeritate altitudini v debita. Posito igitur basis *BDb*
semidiametro

$$BC = CD = b,$$

erit ob $CG = r$ et $GH = u$ ex natura circuli $u = \surd(bb - rr)$; unde fit

$$p = \frac{-r}{\surd(bb-rr)}, \text{ et } 1 + pp = \frac{bb}{bb-rr}$$

atque

$$u - pr = \frac{1}{\surd(bb-rr)}$$

Ex his fit

$$\int \frac{(u-pr)^3 dr}{a^2\left(1+pp\right)+(u-pr)^2} = \int \frac{b^4 dr}{\left(a^2+b^2\right)\sqrt{\left(bb-rr\right)}} = \frac{\pi b^4}{\left(2a^2+2b^2\right)}$$

posito post integrationem $r=b$, et $\pi:1$ denotante rationem peripheriae ad diametrum. Quamobrem resistentiae vis, quae urget secundum directionem horizontalem AC erit $= \frac{\pi b^4 v}{\left(2a^2+2b^2\right)}$. Porro cum sit

$$\int \frac{(u-pr)^2 dr}{a^2\left(1+p^2\right)+(u-pr)^2} = \int \frac{b^2 dr}{\left(a^2+b^2\right)} = \frac{\pi b^3}{\left(a^2+b^2\right)}$$

atque

$$\int \frac{(u-pr)^2\left(a^2+u^2-pru\right)dr}{a^2\left(1+p^2\right)+(u-pr)^2} = \int bbdr = b^3$$

erit resistentiae vis corpus verticaliter sursum pellens $= \frac{ab^3 v}{a^2+b^2}$, huiusque vis directio transibit per punctum O, ita ut sit

$$AO = \frac{2aa+2bb}{3a}.$$

Soliditas ceterum huius corporis erit

$$= \frac{2a}{3}\int dr\sqrt{(bb-rr)} = \frac{\pi abb}{6}$$

atque superficies conica, quae resistentiam sentit prodibit

$$= \int \frac{bdr\sqrt{(aa+bb)}}{\sqrt{(bb-rr)}} = \frac{\pi b}{2}\sqrt{(a^2+b^2)},$$

quae quidem facillime ex notis coni proprietatibus deducuntur.

COROLLARIUM 1

640. Cum basis semiconi seu semicirculus BDb sit , si ea $= \frac{\pi bb}{2}$, moveretur in eadem directione CA in aqua foret eius resistentia $= \frac{\pi bbv}{2}$. Unde resistentia ipsius coni se habebit ad resistentiam basis ut b^2 ad a^2+b^2, hoc est ut CD^2 ad AD^2.

COROLLARIUM 2

641. Mutetur semicirculus *BDb* in triangulum isosceles aeque capax, conusque abibit in pyramidem cuius longitudo a sit eadem. Positis autem dimidia latitudine basis huius pyramidis, $CB = \beta$, et altitudine

$$CD = \gamma \text{ erit } \beta\gamma = \frac{\pi b^2}{2},$$

et resistentia pyramidis huius erit $\dfrac{\beta^3\gamma^3 v}{a^2\beta^2 + a^2\gamma^2 + \beta^2\gamma^2}$.

COROLLARIUM 3

642. Cum igitur sit $bb = \dfrac{2\beta\gamma}{\pi}$, erit resistentia coni aeque alti et aeque capacis $\dfrac{2\beta^2\gamma^2 v}{\pi a^2 + 2\beta\gamma}$, unde resistentia coni se habebit ad resistentiam pyramidis aequalis altitudinis et basis ut

$$2a^2\beta^2 + 2a^2\gamma^2 + 2\beta^2\gamma^2 \text{ ad } \pi a^2\beta\gamma + 2\beta^2\gamma^2.$$

COROLLARIUM 4

643. Resistentia ergo coni aequalis erit resistentiae pyramidis eiusdem basis eiusdemque altitudinis, si fuerit

$$\beta^2 + \gamma^2 = \frac{\pi\beta\gamma}{2} \text{ seu } \frac{\beta}{\gamma} = \frac{\pi}{4} \pm \sqrt{\left(\frac{\pi^2}{16} - 1\right)},$$

hoc est nunquam. Quare resistentia coni semper maior est quam resistentia pyramidis.

EXEMPLUM 3

644. Sit nunc basis coni *BDb* (Fig. 94) semiellipsis centro C descripta, quo casu figura abibit in conum scalenum. Sed ponatur

$$CB = Cb = b, \text{ et } CD = c,$$

erit ex natura ellipsis

$$u = \frac{c}{b}\sqrt{(bb - rr)},$$

unde fit

$$p = \frac{-cr}{b\sqrt{(bb - rr)}} \text{ et } 1 + pp = \frac{b^4 + (cc - bb)rr}{b^2(b^2 - rr)}$$

atque

$$u - pr = \frac{bc}{\sqrt{(bb - rr)}},$$

hincque

$$a^2(1 + pp) + (u - pr)^2 = \frac{a^2b^4 + b^4c^2 + a^2(cc - bb)rr}{b^2(b^2 - rr)}.$$

Ex his reperitur

$$\int \frac{(u - pr)^3\,dr}{a^2(1 + pp) + (u - pr)^2} = \int \frac{b^5c^3dr}{\left(a^2b^4 + b^4c^2 + a^2(cc - bb)rr\right)\sqrt{(b^2 - r^2)}}$$

cuius integrale posito

$$r = b \text{ est } = \frac{\pi b^2c^2}{2\sqrt{(aa + bb)(aa + cc)}};$$

unde resistentiae vis, quae motum retardat et in directione *AC* urget est

$$= \frac{\pi b^2c^2v}{2\sqrt{(aa + bb)(aa + cc)}}.$$

Deinde est

$$\int \frac{(u - pr)^3\,dr}{a^2 + a^2p^2 + (u - pr)^2} = \int \frac{b^4c^2dr}{b^4(a^2 + c^2) + a^2(cc - bb)r^2}$$

cuius integrale a quadratura circuli pendebit si $c > b$, at si $c < b$ pendebit a logarithmis. Cum autem ad nostrum institutum non multum pertineat, quantum corpus sursum urgeatur a resistentia, et in quanam directione, huic investigationi operam non impendemus; sed sufficiat veram resistentiam, qua motus retardatur, determinasse.

COROLLARIUM 1

645. Quoniam in expressione resistentiae inventa

$$= \frac{\pi b^2c^2v}{2\sqrt{(a^2 + b^2)(a^2 + c^2)}},$$

semiaxes coniugati basis b et c aequaliter insunt, ii inter se commutari possunt manente eadem resistentia. Hoc est dummodo ellipsis BDb alter semiaxis sit b alter vero c resistentia prodit eadem.

COROLLARIUM 2

646. Si area basis BDb quae est $\frac{\pi bc}{2}$ dicatur $= A$, ob

$$\frac{b}{\sqrt{(a^2+b^2)}} = \text{sinui ang.}CAB \quad \text{et} \quad \frac{c}{\sqrt{(a^2+c^2)}} = \text{sin. ang.}CAD,$$

erit resistentia $= Av\sin CAB\sin CAD$; ubi notandum Av exprimere resistentiam basis BDb si ea nuda in directione CA promoveretur.

COROLLARIUM 3

647. Si loco ellipsis BDb substituatur circulus eiusdem areae, erit eius radius $= \sqrt{bc}$, atque resistantia, quam hic conus patietur erit $= \frac{\pi b^2c^2v}{2\left(a^2+bc\right)}$. Resistentia igitur coni circularis se habebit ad resistentiam coni elliptici aequalis basis aequalisque altitudinis ut

$$\sqrt{(a^2+b^2)(a^2+c^2)} \text{ ad } a^2+bc.$$

COROLLARIUM 4

648. Nisi ergo sit $b=c$, resistentia coni circularis semper erit maior quam resistentia coni elliptici. Sumtis enim quadratis perspicuum est esse
$a^4+a^2b^2+a^2c^2+b^2c^2 > a^4+2a^2bc+bbcc$,
quia semper est $bb+cc > 2bc$, nisi sit $b=c$.

COROLLARIUM 5

649. Manente ergo area basis elliptica BDb et altitudine coni AG eadem, resistentia erit maxima, si basis abeat in semicirculum. Eo minor igitur erit resistentia, quo maior inaequalitas inter altitudinem et latitudinem basis intercedet.

SCHOLION 2

650. Ex his igitur satis perspicuum est corpus conoidicum, quod minimam

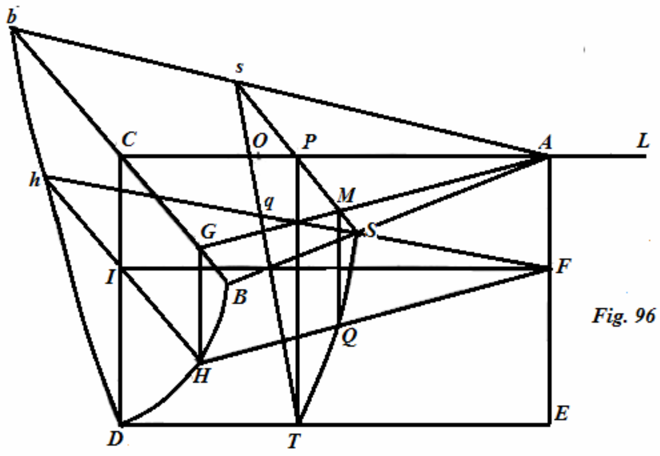


Fig. 96

patiatur resistentiam in finitis assignari non posse. Nam si altitudo coni a maneat constans, resistentia eo b minor evadet, quo minor accipiatur basis *BDb* ceteris paribus. At si insuper basi data area tribuatur, resistentia semper magis diminui potest inaequalitatem inter eius altitudinem *CD* et latitudinem *CB* maiorem ponendo. Hancobrem istud problema non attingemus, quo vel inter omnes conos absolute, vel inter aequicapaces tantum is desideretur qui minimam patiatur resistentiam. Ad alias igitur corporum species progrediamur et quomodo resistentia se in iis habeat, inquiramus. Eius modi vero adhuc contemplabimur corporum figuras, in quibus unica curva maneat indeterminata, quemadmodum evenit in his corporibus conoidicis in quibus sola basis supererat indeterminata.

PROPOSITIO 63

PROBLEMA

651. *Sit partis submersae navis pars anterior in motu directo resistentiam patiens cono cuneus latissimo sensu acceptus AEDHBbhD* (Fig. 96) *ex data curva tanquam basi BDb et recta verticali AFE ita generatus ut eius superficies terminetur rectis horizontalibus HF, hF ex singulis perimetri basis BDb puncti & ad rectam AE ductis; haecque figura cursu directo in aqua progrediatur secundum directionem axis CAL: determinare resistentiam quam patietur.*

SOLUTIO

In hac igitur figura planum verticale diametrale *ACDE* erit parallelogrammum rectangulum, atque sectio aquae *ABb* triangulum isosceles; similique modo omnes sectiones horizontales *FHh* erunt triangula aequicrura. Porro ex constructione apparet omnes sectiones verticales per rectam *AE* factas, cuius modi est *AGHF* esse parallelogramma rectangula. Tota ergo figura in prora definit in aciem rectilineam verticalem *AFE;* amplissima autem sectio verticalis axi *AC* normalis erit basis huius cono cunei *BDb,* a cuius natura totius figurae natura pendet. Posita ergo longitudine $AC = a$, sumatur in basi abscissa $CG = r$ et applicata $GH = u$, atque ob basin datam dabitur aequatio inter *u* et *r*, seu *u* per *r*. Sit autem $du = pdr$, et quantitas *p* erit cognita per *r*. Concipiatur nunc sectio verticalis *STs* basi parallela, pro qua sit $AP = x$, et per *GH* et *AE* alia fiat sectio *AGHF*, quae erit rectangulum, eiusque latus *HF* in superficie figurae erit situm. Positis ergo $PM = y$ et $MQ = z$ erit $z = GH = u$, atque $x : y = a : r$, unde fit $y = \frac{rx}{a}$. Ex his reperitur

$$dr = \frac{axdy - aydx}{xx}, \text{ et } dz = du = \frac{apxdy - apydx}{xx}.$$

Pro superficie igitur huius cono-cunei ista habetur aequatio

$$dz = \frac{-apydx}{xx} + \frac{apdy}{x},$$

qua cum aequatione canonica $dz = Pdx + Qdy$ comparata dat

$$P = \frac{-apy}{xx} = \frac{-pr}{x}, \text{ ob } y = \frac{rx}{u}, \text{ atque } Q = \frac{ap}{x}.$$

Hinc oritur

$$1 + P^2 + Q^2 = \frac{x^2 + p^2\left(a^2 + r^2\right)}{x^2},$$

atque formulae integrales propositionis 61 in quibus positum est *x* constans in sequentes transmutantur, ob $dy = \frac{xdr}{a}$ quia *x* est constans: scilicet fit

$$\int \frac{P^3 dy}{1 + P^2 + Q^2} = -\int \frac{p^3 r^3 dr}{axx + ap^2\left(a^2 + r^2\right)};$$

et

$$\int \frac{P^2 dy}{1 + P^2 + Q^2} = \int \frac{p^2 r^2 dr}{axx + ap^2\left(a^2 + r^2\right)},$$

atque cum sit

$$x + Pz = x - \frac{pru}{x}$$

erit

$$\int \frac{P^2\left(x + Pz\right)dy}{1 + P^2 + Q^2} = \int \frac{p^2 r^2\left(xx - pru\right)dr}{axx + ap^2\left(a^2 + r^2\right)}$$

quae integralia ita sunt accipienda posito x constante, ut evanescant posito $r = 0$, tum vero poni debet $r = CB$ seu $u = 0$. Ad resistentiam deinde ipsam inveniendam sumi debet hoc integrale

$$\int dx \int \frac{P^3 dy}{1 + P^2 + Q^2} = -\int \frac{dx}{a} \int \frac{p^3 r^3 dr}{xx + p^2\left(a^2 + r^2\right)}.$$

At quoniam post integrationem posterioris formulae r et p ab x non pendebunt, quaestio huc est reducta ut

$$\frac{-p^3 r^3 dr dx}{ax^2 + ap^2\left(a^2 + r^2\right)}$$

bis integretur ponendo in altera integratione x in altera vero r et p constantes; perinde autem est ab utra integratione initium fiat. Quare ponamus primo pet r constantes eritque integrale

$$\frac{-p^2 r^3 dr}{a\sqrt{\left(a^2 + r^2\right)}} \text{A tang.} \frac{a}{p\sqrt{\left(a^2 + r^2\right)}}$$

posito post integrationem uti oportet $x = a$. Integratione ergo altera instituta et postea posito $r = CB$ seu $u = 0$, prodibit

$$\int dx \int \frac{P^3 dy}{1 + P^2 + Q^2} = \frac{-p^2 r^3 dr}{a\sqrt{\left(a^2 + r^2\right)}} \text{A tang.} \frac{a}{p\sqrt{\left(a^2 + r^2\right)}}.$$

Hancobrem si cono-cuneus moveatur secundum directionem axis CAL celeritate altitudini v debita, erit resistentiae vis, qua secundum directionem AC repelletur

$$= \frac{-2v}{a} \int \frac{p^2 r^3 dr}{a\sqrt{\left(a^2 + r^2\right)}} \text{A tang.} \frac{a}{p\sqrt{\left(a^2 + r^2\right)}}.$$

simili modo integrationes absolvendo erit

$$\int dx \int \frac{P^2 dy}{1 + P^2 + Q^2} = \iint \frac{p^2 r^2 dr}{axx + ap^2\left(a^2 + r^2\right)},$$

ubi bis integrari oportet, altera vice x altera vero r et p ponendo constantes;
posito igitur primo r constante, erit

$$\int dx \int \frac{P^2 dy}{1+P^2+Q^2} = \int \frac{p^2 r^2 dr}{a} l \frac{\surd\left(a^2+p^2\left(a^2+r^2\right)\right)}{p\surd\left(a^2+r^2\right)} = \int \frac{p^2 r^2 dr}{2a} l \frac{a^2+a^2p^2+p^2r^2}{a^2p^2+p^2r^2}.$$

Facto ergo post integrationem $r = CB$ seu $u = 0$ prodibit vis resistentiae, qua corpus verticaliter sursum urgebitur

$$\frac{v}{a}\int p^2 r^2 dr\, l \frac{a^2+p^2\left(a^2+r^2\right)}{p^2\left(a^2+r^2\right)}.$$

Denique ad locum applicationis huius vis, qui sit in *O* inveniendum his integrari debet haec formula differentialis

$$\frac{p^2 r^2\left(x^2-pru\right)dxdr}{axx+app\left(aa+rr\right)}.$$

Ponatur primo *x* tantum variabile, positoque post integrationem $x = a$ habebitur pro altera integratione

$$\int p^2 r^2 dr\left(1-\frac{\left(p\left(a^2+r^2\right)+ru\right)}{a\surd\left(a^2+r^2\right)}\text{A tang.}\frac{a}{p\surd\left(a^2+r^2\right)}\right);$$

quod integrale, cum positum fuerit $u = 0$, divisum per integrale ante inventum

$$\int \frac{p^2 r^2 dr}{2a}\cdot l \frac{a^2+p^2\left(a^2+r^2\right)}{p^2\left(a^2+r^2\right)}$$

dabit distantiam *AO* puncti *O,* per quod resistentiae vis verticalis transit a prora *A.* Q. E. I.

COROLLARIUM 1

652. Quaecunque ergo curva pro basi *BDb* accipiatur, resistentiae motui contrariae determinatio, quae est

$$= \frac{-2v}{a}\int \frac{p^2 c^3 dr}{\surd\left(a^2+r^2\right)}\cdot \text{A tang.}\frac{a}{p\surd\left(a^2+r^2\right)}$$

quadraturam circuli requirit. At contra resistentiae vis, quae sursum urget pendet a logarithmis.

COROLLARIUM 2

653. Ex his formulis etiam perspicitur utramque resistentiae vim eo fore minorem quo maior sit longitudo; utraque enim evanescit si ponatur $a=\infty$. Magis vero dum crescit *a,* decrescit vis resistentiae horizontalis quam verticalis.

COROLLARIUM 3

654. Si longitudo $AC=a$ fuerit tam magna respectu basis *BDb,* ut *p* et *r* prae *a* evanescant, erit resistentiae vis horizontalis

$$=\frac{-2v}{aa}\int p^2c^3dr\cdot \text{A tang.}\frac{1}{p}$$

resistentiae vero vis verticalis erit

$$=\frac{v}{a}\int p^2r^2dr\, l\,\frac{1+pp}{pp}.$$

COROLLARIUM 4

655. At si longitudo $AC=a$ evanescat, ut tota figura abeat in solam basem *BDb,* tum resistentia horizontalis fiet

$$=\frac{-2v}{a}\int p^2r^2dr\cdot \text{A tang.}\frac{a}{pr}=-2v\int prdr=2v\int udr,$$

prout per se patet, at resistentia verticalis evanescet.

COROLLARIUM 5

656. Soliditas totius huius cono-cunei reperitur ex § 617 quippe quae est

$$2\int -dx\int Qydy=\ 2\int -dx\int \frac{xprdr}{a}.$$

Quae cum x in priore integratione sit constans, abit in

$$-2\int \frac{xdx}{a}\int prdr=\int \frac{2xdx}{a}\int udr,$$

denotatque $\int udr$ aream *CBD.* Unde tota soliditas $=a\int udr$, quae quidem sponte patet.

COROLLARIUM 6

657. Superficies autem huius cono-cunei in aquam incurrentis est ex§ 616

$$= = 2\int dx\int dy\sqrt{(1+P^2+Q^2)} = 2\int dx\int \frac{dr}{a}\sqrt{\left(x^2+p^2\left(a^2+r^2\right)\right)}.$$

Unde his integrari debet haec formula differentialis

$$\frac{2dxdr}{a}\sqrt{\left(x^2+p^2\left(a^2+r^2\right)\right)},$$

altera vice x altera r ponendo constans. Si autem primo r ponatur constans, erit integrale

$$\frac{xdr}{a}\sqrt{\left(x^2+p^2\left(a^2+r^2\right)\right)}+\frac{p^2dr\left(a^2+r^2\right)}{a}\cdot l\frac{x+\sqrt{\left(x^2+p^2\left(a^2+r^2\right)\right)}}{p\sqrt{\left(a^2+r^2\right)}}$$

Posito igitur $x=a$, erit superficies cono-cunei quaesita

$$\int dr\sqrt{\left(a^2+p^2\left(a^2+r^2\right)\right)}+\int\frac{p^2dr\left(a^2+r^2\right)}{a}\cdot l\frac{a+\sqrt{\left(a^2+p^2\left(a^2+r^2\right)\right)}}{p\sqrt{\left(aa+rr\right)}}$$

COROLLARIUM 7

658. Inventio ergo superficierum cono-cunei cuiuscunque pendet a logarithmis seu quadratura hyperbolae, atque insuper ab aliis quadraturis, nisi formulae illae differentiales integrationem admittant.

SCHOLION

659. Quamvis huiusmodi figurae, quas hic cono-cunei nomine appellamus, non ita pridem considerari coeperint, eas tamen hic tanquam secundam corporum speciem proferre visum est, quoniam magnam habent affinitatem cum corporibus conicis, quae nobis primam speciem constituerunt. Quanquam enim, si simplicitatem constructionis spectemus, corpora cylindrica et prismatica primo loco collocari merentur, tamen eas hic prorsus ne quidem attingemus, cum resistentia, quam patiuntur, ex praecedentibus, quae de figuris planis sunt prolata, facillime innotescat, ibique iam indicata sit. Nam si omnes sectiones horizontales sunt inter se similes et aequales, resistentia obtinebitur ex resistentia unicae sectionis, eam ducendo in altitudinem figurae. At si omnes sectiones plano diametrali parallelae fuerint inter se aequales et similes, tum pariter resistentia habebitur resistentiam unicae sectionis hanc per latitudinem multiplicando, quemadmodum attendenti sponte patebit. Hic autem vocabulum

cono-cunei in latiore sensu accipimus, quam WALIISIUS, curvam enim quamcunque basis *BDb* loco contemplamur, cum WALIISIUS circulum tantum assumserit. Generatim autem omnium horum cono-cuneorum natura cognoscetur ex aequatione canonica inventa

$$dz = -\frac{apydx}{xx} + \frac{apdy}{x}$$

in qua cum sit p functio quaecunque ipsius r et $r = \frac{ay}{x}$, fiet p functio quaecunque ipsarum x et y nullius dimensionis. Quare pro cono-cuneis erit

$$dz = -\frac{ap\left(ydx - xdy\right)}{xx},$$

et cum sit

$$\frac{xdy - ydx}{xx} = d \cdot \frac{y}{x}$$

aequabitur z functioni nullius dimensionis ipsarum x et y. Unde ex quaque oblata aequatione pro quapiam superficie perspici poterit utrum figura sit cono-cuneus an secus. Similiter natura corporum conicorum innotescet ex aequatione canonica supra inventa

$$dz = \frac{udx}{a} - \frac{pydx}{x} + pdy,$$

quae cum sit $u = \frac{az}{x}$ abit in hanc

$$\frac{dz}{x} - \frac{zdx}{xx} = \frac{pdy}{x} - \frac{pydx}{xx}.$$

Quoniam vero ob $r = \frac{ay}{x}$ est p functio quaecunque nullius dimensionis ipsarum x et y, erit $z =$ producto ex x in functionem nullius dimensionis ipsarum x et y. Quoties igitur $\frac{z}{x}$ aequatur functioni nullius dimensionis ipsarum x et y toties aequatio erit pro superficie conica. Omnis ergo aequatio inter x et y et z, in qua hae tres variabiles ubique eundem dimensionum numerum constituunt, naturam exprimet coni cuiusdam. At omnis aequatio inter x, y et z ita comparata ut tantum binae variabiles x et y ubique eundem dimensionum numerum adimpleant, superficiem cono-cunei cuiusdam exhibebit.

EXEMPLUM 1

660. Abeat basis *BDb* cono-cunei in triangulum isosceles, quo casu corpus *ABDb* mixtum erit ex pyramide et cuneo. Sit semilatitudo huius basis
OB = *Ob* = *b,* et altiitudo

$CD = c$, erit $u = c - \frac{cr}{b}$, atque $p = -\frac{c}{b}$.

Cum igitur resistentiae, quam hoc corpus celeritate altitudini v debita secundum directionem CA promotum patitur, vis retrourgens in directione AC inventa sit

$$= \frac{-2v}{a} \int \frac{p^2 r^3 dr}{\sqrt{\left(a^2+r^2\right)}} \text{ Atang.} \frac{a}{p\sqrt{\left(a^2+r^2\right)}}$$

fiet ea hoc casu

$$= \frac{2c^2 v}{ab^2} \int \frac{r^3 dr}{\sqrt{\left(a^2+r^2\right)}} \text{ Atang.} \frac{ab}{c\sqrt{\left(a^2+r^2\right)}}.$$

Cum autem sit

$$\int \frac{r^3 dr}{\sqrt{\left(a^2+r^2\right)}} = \frac{2a^3}{3} + \frac{\left(r^2-2a^2\right)\sqrt{\left(a^2+r^2\right)}}{3}$$

erit

$$\int \frac{r^3 dr}{\sqrt{\left(a^2+r^2\right)}} \text{ Atang.} \frac{ab}{c\sqrt{\left(a^2+r^2\right)}}$$

$$= \frac{\left(r^2-2a^2\right) r^3 \sqrt{\left(a^2+r^2\right)}}{3} \text{Atang.} \frac{ab}{c\sqrt{\left(a^2+r^2\right)}} + \frac{abc}{3} \int \frac{rdr\left(r^2-2a^2\right)}{a^2c^2+a^2b^2+c^2r^2}$$

$$= \frac{\left(r^2-2a^2\right) r^3 \sqrt{\left(a^2+r^2\right)}}{3} \text{Atang.} \frac{ab}{c\sqrt{\left(a^2+r^2\right)}} + \frac{abr^2}{6c} - \frac{a^3\left(bb+3cc\right)}{3c^2} l \frac{cr+\sqrt{\left(a^2b^2+a^2c^2+c^2r^2\right)}}{a\sqrt{\left(bb+cc\right)}} + \frac{2a^3}{3} \text{Atang.}$$

tali addita constante, ut prodeat nihil posito $r = 0$. Fiat nunc $r = b$, atque integra resistentia quam figura in directione AC sentiet, erit

$$= \frac{2ccv\left(bb-2aa\right) r^3 \sqrt{\left(a^2+b^2\right)}}{3ab^2} \text{Atang.} \frac{ab}{c\sqrt{\left(a^2+b^2\right)}} + \frac{bcv}{3} - \frac{-2a^2 v\left(bb+3cc\right)}{3bc} l \frac{bc+\sqrt{\left(a^2b^2+a^2c^2+b^2c^2\right)}}{\alpha\sqrt{\left(b^2+c^2\right)}} + \frac{4a^2c^2v}{3bb} \text{Atang.} \frac{b}{c}.$$

Deinde vis resistentiae quae sursum urget est

$$= \frac{v}{a} \int p^2 r^2 drl \frac{a^2+p^2\left(a^2+r^2\right)}{p^2\left(a^2+r^2\right)} = \frac{ccv}{abb} \int r^2 drl \frac{a^2b^2+a^2c^2+c^2r^2}{cc\left(a^2+r^2\right)}$$

quae expressio commodius exhiberi non potest, quamobrem suficiat resistentiam, qua motus retardatur, quippe ad quam potissimum attendemus, determinasse per quantitates finitas.

COROLLARIUM 1

661. Si longitudo $AC = a$ fuerit vehementer magna prae b et c resistentia commodius ex formula differentiali eruetur quae abibit in hanc

$$\frac{2ccv}{a^2b^2}\int r^3 dr \text{Atang.}\frac{b}{c},$$

cuius integrale posito

$$r = b \text{ est } = \frac{b^2c^2v}{2a^2}\text{Atang.}\frac{b}{c},$$

quae est resistentia retardans.

COROLLARIUM 2

662. Si igitur detur area basis *BDb,* quae est *bc,* et longitudo *AC* fuerit perquam magna, resistentia eo erit minor, quo minor fuerit fractio $\frac{b}{c}$, hoc est quo acutior fuerit angulus *BDb.* Maxima vero erit resistentia, si capiatur ratio $b:c$ infinita magna, quo tamen casu resistentia erit finita ob $\text{Atang.}\infty = \frac{\pi}{2}$.

COROLLARIUM 3

663. Per seriem etiam commode resistentia exprimi potest generaliter pro quavis longitudine *a.* Cum enim sit

$$\text{A tang.}\frac{ab}{c\sqrt{\left(a^2+r^2\right)}} = \frac{ab}{\sqrt{\left(a^2+r^2\right)}} - \frac{a^3b^3}{3c^3\left(a^2+r^2\right)^{\frac{3}{2}}} + \frac{a^5b^5}{5c\ \left(a^2+r^2\right)^{\frac{5}{2}}} - \text{etc.}$$

erit, posito post integrationem $r = b$ resistentia

$$\frac{2c^2v}{abb}\int\frac{r^3dr}{\sqrt{\left(a^2+r^2\right)}}\text{ Atang.}\frac{ab}{c\sqrt{\left(a^2+r^2\right)}}$$

$$=v\left(\begin{array}{l} bc-\dfrac{2a^2\left(b^2+3c^2\right)}{3bc}l\sqrt{\dfrac{a^2+b^2}{a^2}}-\dfrac{a^4b\left(3bb+5cc\right)}{1\cdot 3\cdot 5\cdot c^3\left(a^2+b^2\right)} \\ +\dfrac{a^6b^3\left(5bb+7cc\right)}{2\cdot 5\cdot 7c^5\left(a^2+b^2\right)^2}-\dfrac{a^8b^5\left(7bb+9cc\right)}{3\cdot 7\cdot 9c^7\left(a^2+b^2\right)^3}+\dfrac{a^{10}b^7\left(9bb+11cc\right)}{4\cdot 9\cdot 11c^9\left(a^2+b^2\right)^4}-\text{etc.}\end{array}\right)$$

quae vehementer convergit si fuerit a valde parvum.

COROLLARIUM 4

664. Si autem series desideretur, quae vehementer convergat, si sit a quantitas valde magna, reperietur resistentia motum retardans

$$=\frac{4a^2c^2v}{3bb}\text{Atang.}\frac{b}{c}-\frac{2ccv\left(2a^2-b^2\right)\sqrt{\left(a^2+b^2\right)}}{3ab^2}\text{Atang.}\frac{ab}{c\sqrt{\left(a^2+b^2\right)}}-\frac{2bc^2v}{3\left(bb+cc\right)}$$
$$+\frac{v\left(b^2+3c^2\right)}{3bc}\left(\frac{b^4c^4}{2a^2\left(b^2+c^2\right)^2}-\frac{b^6c^6}{3a^4\left(b^2+c^2\right)^5}+\frac{b^8c^8}{4a^6\left(b^2+c^2\right)^4}-\text{etc.}\right).$$

COROLLARIUM 5

665. Soliditas vero huius corporis reperitur $=\dfrac{abc}{2}$, superficies autem eius basi et sectione aquae exceptis erit

$$=\int\frac{dr}{b}\sqrt{\left(a^2b^2+a^2c^2+c^2r^2\right)}+\frac{cc}{abb}\int dr\left(a^2+r^2\right)l\frac{ab+\sqrt{\left(a^2b^2+a^2c^2+c^2r^2\right)}}{c\sqrt{\left(a^2+r^2\right)}}.$$

Cuius integrale posito

$$\frac{bb+cc}{cc}=m,$$

et facto $r=b$, reperitur

$$=\frac{c}{2}\sqrt{\left(ma^2+b^2\right)}+\frac{ma^2c}{2b}l\frac{b+\sqrt{\left(ma^2+b^2\right)}}{a\sqrt{m}}+\frac{cc\left(3aa+bb\right)a^2c}{ab}l\frac{ab+c\sqrt{\left(ma^2+b^2\right)}}{c\sqrt{\left(a^2+b^2\right)}}$$

$$+\frac{cc}{3bb}\int\frac{\left(3a^2+r^2\right)\left(\left(m-1\right)ac+b\sqrt{\left(ma^2+r^2\right)}\right)r^2dr}{\left(a^2+r^2\right)\left(ab+a\sqrt{\left(ma^2+r^2\right)}\right)\sqrt{\left(ma^2+r^2\right)}}$$

adeo ut integratio huius formulae restet.

COROLLARIUM 6

666. Casus quo $m=2$ seu $b=c$ aliquanto fit simplicior, prodit enim superficies

$$=\frac{c}{2}\sqrt{\left(2a^2+c^2\right)}+a^2l\frac{c+\sqrt{\left(2a^2+c^2\right)}}{a\sqrt{2}}+\frac{c\left(3a^2+c^2\right)}{3a}l\frac{a+\sqrt{\left(2a^2+c^2\right)}}{\sqrt{\left(a^2+c^2\right)}}$$

$$+\frac{1}{3}\int\frac{\left(3a^2+r^2\right)r^2dr}{\left(a^2+r^2\right)\sqrt{\left(2a^2+r^2\right)}}=\frac{c}{2}\sqrt{\left(2a^2+c^2+a^2\right)}l\frac{c+\sqrt{\left(2a^2+c^2\right)}}{a\sqrt{2}}+\frac{c\left(3a^2+c^2\right)}{3a}l\frac{a+\sqrt{\left(2a^2+c^2\right)}}{\sqrt{\left(a^2+c^2\right)}}$$

$$+\frac{c}{6}\sqrt{\left(2a^2+r^2\right)}+\frac{aa}{3}l\frac{c+\sqrt{\left(3a^2+c^2\right)}}{a\sqrt{2}}-\frac{2a^2}{3}\text{Atang.}\frac{c}{\sqrt{\left(2a^2+c^2\right)}}.$$

Erit ergo superficies quaesita

$$=\frac{2c}{3}\sqrt{\left(2a^2+c^2\right)}+\frac{4a^2}{3}l\frac{c+\sqrt{\left(2a^2+c^2\right)}}{a\sqrt{2}}+\frac{c\left(3a^2+c^2\right)}{3a}l\frac{a+\sqrt{\left(2a^2+c^2\right)}}{\sqrt{\left(a^2+c^2\right)}}-\frac{2a^2}{3}\text{Atang.}\frac{c}{\sqrt{\left(2a^2+c^2\right)}}.$$

COROLLARIUM 7

667. Si insuper sit $c = a$, ita ut sit $AC = CB = CD$ erit superficies

$$= \frac{2aa}{\surd 3} + \frac{4aa}{3} l\left(2 + \surd 3\right) - \frac{\pi a^2}{9} \;;$$

cuius expressionis valor proximus est $a^2 \cdot 2{,}56156$, seu superficies se habet ad basem proxime ut $2\frac{1}{2}$ ad 1.

EXEMPLUM 2

668. Sit nunc corpus nostrum WALLISII cono-cuneus, seu basis *BDb,* abeat in semicirculum, cuius semidiameter sit $CB = CD = b$. Erit igitur $u = \surd\left(b^2 - r^2\right)$, ideoque

$$p = \frac{-r}{\surd\left(bb - rr\right)},$$

hoc ergo valore substituto, invenietur resistentiae vis motum retardans

$$= \frac{2v}{a} \int \frac{r^5 dr}{\surd\left(b^2 - r^2\right)\surd\left(a^2 + r^2\right)} \text{ Atang.} \frac{a\surd\left(bb - rr\right)}{r\surd\left(a^2 + r^2\right)}.$$

Quamvis autem sit

$$\int \frac{r^5 dr}{\surd\left(b^2 - r^2\right)\surd\left(a^2 + r^2\right)} = \frac{\left(a^2 + r^2\right)^{\frac{3}{2}}}{3} + \left(a^2 - b^2\right)\surd\left(a^2 + r^2\right) + \frac{b^4}{2\surd\left(a^2 + b^2\right)} l \frac{\surd\left(a^2 + b^2\right) + \surd\left(a^2 + r^2\right)}{\surd\left(a^2 + b^2\right) - \surd\left(a^2 + r^2\right)}$$

tamen hinc plenaria integratio non multum iuvatur. Deinde si arcus cuius tangens est

$$\frac{a\surd\left(b^2 - r^2\right)}{r\surd\left(a^2 + r^2\right)}$$

in seriem resolvatur, integratio quidem singulorum terminorum in

$$\frac{r^5 dr}{\left(b^2 - r^2 \surd a^2 + r^2\right)}$$

ductorum facilior evaderet, sed constans infinita esset addenda, quo prodeat nihil posito $r = 0$. Hoc incommodum quodammodo evitatur si loco illius arcus, substituatur aequivalens

$$\frac{\pi}{2} - \text{A tang.}\frac{r\sqrt{\left(a^2+r^2\right)}}{a\sqrt{\left(bb-rr\right)}},$$

sed quomodocunque calculus instituatur nihil, cuius operae foret pretium derivatur, quapropter cono-cuneos relinquamus, ad aliam corporum speciem plurimum iam pertractatam, corporum scilicet rotundorum progressuri.

PROPOSITIO 64

PROBLEMA

669. *Sit sectio aquae ABb curva quaecunque ex duabus partibus aequalibus et similibus ACB, ACb constans* (Fig. 97), *atque omnes sectiones verticales STs ad planum diametrale ACD normales semicirculi seu quod eodem redit, sit corpus ABDb genitum conversione curvae ACB circa axem AC; hocque corpus moveatur in aqua directe in directione CAL; determinare resistentiam quam patietur.*

SOLUTIO

Ex constructione huius corporis intelligitur non solum planum diametrale *ATD* sed omnes sectiones per axem *AC* transeuntes fore curvas similes et aequales semisectioni aquae *ASBC.* Cum igitur curva *ASB* data ponatur, vocatis $AP = x$, et $PS = PT = s$, dabitur aequatio inter x et s, seu s erit functio quaedam ipsius x, ita ut si ponatur $ds = pdx$ futura sit p pariter

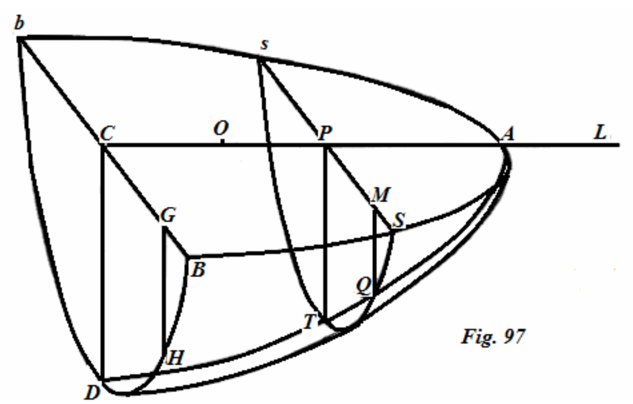


Fig. 97

functio ipsius *x.* Sumtis nunc reliquis ambabus coordinatis $PM = y$ et $MQ = z$ quoniam sectio *SQTs* est semicirculus centro *P* descriptus cuius radius est $PS = PT = s$, erit $z^2 + y^2 = s^2$ et $z = \sqrt{(s^2 - y^2)}$; ; unde fit

$$dz = \frac{sds - ydy}{\sqrt{(s^2 - y^2)}} = \frac{psdx - ydy}{\sqrt{(s^2 - y^2)}},$$

qua aequatione natura superficiei huius corporis exprimitur. Haec ergo aequatio si comparetur cum canonica supra assumta $dz = Pdx + Qdy$, fiet

$$P = \frac{ps}{\sqrt{(s^2 - y^2)}} \text{ et } Q = \frac{-y}{\sqrt{(s^2 - y^2)}}.$$

Ponamus iam sectionem *BDb* omnium sibi parallelarum esse amplissimam existente $AC = a$, seu latitudinem *Bb* esse maximam; ac tota superficies *ABDb* resistentiam patietur; sitque celeritas qua hoc corpus in aqua progreditur secundum directionem *AL* debita altitudini *v.* His praemissis ex prop. 61 resistentia sequenti modo definietur: cum sit

$$1 + P^2 + Q^2 = \frac{\left(1 + p^2\right)s^2}{ss - y^2},$$

erit

$$\frac{P^3 dy}{1 + P^2 + Q^2} = \frac{p^3 sdy}{(1 + pp)\sqrt{\left(ss - y^2\right)}} \text{ et } \frac{P^2 dy}{1 + P^2 + Q^2} = \frac{p^2 dy}{1 + p^2}$$

atque

$$\frac{P^2\left(x + Pz\right)dy}{1 + P^2 + Q^2} = \frac{p^2\left(x + ps\right)dy}{1 + p^2},$$

quae differentialia ponendo *x* et quantitates inde pendentes *p* et *s* constantes ita sunt accipienda ut evanescant posito $y = 0$, quo facto poni debet

$$y = PS = s.$$

Hoc autem modo reperietur

$$\int \frac{P^3 dy}{1 + P^2 + Q^2} = \frac{\pi p^3 s}{2\left(1 + p^2\right)}$$

denotante π peripheriam circuli, cuius diameter est 1; et

$$\int \frac{P^2 dy}{1+P^2+Q^2} = \frac{p^2 s}{1+p^2},$$

atque

$$\int \frac{p^2\left(x+Pzdy\right)}{1+P^2+Q^2} = \frac{p^2 s\left(x+ps\right)}{1+pp}.$$

Nunc positis x et p et s variabilibus habebitur resistentiae vis horizontalis, qua corpus in directione AC repellitur

$$=\pi v\int \frac{p^3 sdx}{1+p^2}$$

in quo integrali, cum ita fuerit acceptum, ut evanescat posito $x=0$, fieri debet $x=a$. Deinde vis resistentiae, qua corpus sursum urgebitur est

$$=2v\int \frac{p^2 sdx}{1+pp},$$

haecque vis transibit per punctum axis O existente

$$AO=\frac{\int \frac{p^2 sdx\left(x+ps\right)}{1+pp}}{\int \frac{p^2 sdx}{1+pp}},$$

singulis his integralibus ita acceptis ut evanescant posito $x=0$, atque tum facto $x=a$. Q.E.I.

COROLLARIUM 1

670. Si sectio aquae *ABb* in *B* habuerit tangentem ad *Bb* normalem seu axi *AC* parallelam, tum omnia plana superficiem tangentia in punctis *H* sectionis *BDb* ad hanc ipsam sectionem erunt normalia.

COROLLARIUM 2

671. Simili modo quem angulum tangens sectionis aquae in *S* constituit cum axe *PA,* eundem angulum plana tangentia omnia in singulis punctis *Q* sectionis *STs* cum axe *PA* constituent, ex quo singula elementa *Q* sectionis *STs* eandem patientur resistentiam, quam patitur aequale elementum in *S* situm.

COROLLARIUM 3

672. Ad soliditatem totius huius corporis cognoscendam ex § 617 primum integrandum est differentiale

$$-Qydy = \frac{y^2dy}{\sqrt{(ss-yy)}},$$

cuius integrale posito $y = s$ post integrationem est $= \frac{\pi ss}{4}$. Unde tota soliditas fit $= \frac{\pi}{2}\int ssdx$ posito post integrationem $x = a$.

COROLLARIUM 4

673. Deinde cum superficies *ABDb* in genere sit

$$= 2\int dx \int dy\sqrt{(1+P^2+Q^2)},$$

erit superficies solidi nostri rotundi

$$= 2\int dx \int \frac{sdy\sqrt{(1+pp)}}{\sqrt{(ss-yy)}} = \pi \int sdx\sqrt{(1+pp)},$$

in quo integrali ita accepto ut evanescat posito $x = 0$, fieri debet $x = a$.

COROLLARIUM 5

674. Si integrum solidum rotundum, quod generatur dum figura *ACB* circa axem *AC* penitus convertitur in aqua secundum directionem axis *CAL* moveatur, tum resistentiam motui directe contrariam patietur duplo maiorem, eaque ideo erit $= 2\pi v\int \frac{p^3sdx}{1+pp}$.

SCHOLION

675. Huiusmodi corpora rotunda fere sola ab iis, qui resistentiam calculo investigarunt, sunt considerata, longe alio autem modo in eorum resistentiam inquisiverunt, huic corporum speciei proprio. Derivaverunt enim resistentiam ex ea consideratione, quam corollario secundo indicavimus, quae via quamquam est multo facilior, quam ea quam hic sumus secuti, tamen quoniam ad alias corporum species non patet, methodo generali uti maluimus. Hinc autem generatim innotescit natura omnium corporum rotundorum per aequationem generalem pro iis inventam $z^2 + y^2 = s^2$ scilicet sumtis abscissis x in axe *AC* est semper $z^2 + y^2$ aequale functioni cuidam ipsius x, et quoties talis aequatio occurrit, toties ea erit ad solidum rotundum. Sed quo resistentia huiusmodi corporum plenius cognoscatur, iuvabit casus nonnullos particulares evolvere, quibus determinata curva pro sectione aquae *ACB* accipitur.

EXEMPLUM 1

676. Sit primo sectio aquae *ABb* triangulum isosceles (Fig. 94), seu corpus *ABDb* semissis coni recti circularis, qui casus, quanquam iam ante est pertractatus, tamen eum hic etiam affere visum est, quo convenientia magis perspiciatur, atque ipsa propositio illustretur. Posita itaque semidiametro basis $BC = CD = b$ erit $a : b = x : s$, ideoque

$$s = \frac{bx}{a}, \text{ et } p = \frac{b}{a}.$$

Unde resistentiae vis horizontalis erit

$$= \pi v \int \frac{p^3 sdx}{1+p^2} = \frac{\pi b^4}{aa} v \int \frac{xdx}{a^2+b^2} = \frac{\pi b^4 v}{2\left(a^2+b^2\right)};$$

vis verticalis autem ex resistentia orta, qua corpus ex aqua elevabitur erit

$$= 2v \int \frac{p^2 sdx}{1+pp} = \frac{2b^3 v}{a} \int \frac{xdx}{aa+bb} = \frac{ab^3 v}{a^2+b^2}.$$

Denique punctum *O* in quo haec vis erit applicata, ita definietur: cum sit

$$AO = \frac{\int p^2 sdx\left(x+ps\right):\left(1+pp\right)}{\int p^2 sdx\left(1+pp\right)}$$

erit pro nostro casu

$$AO = \frac{\left(aa+bb\right)\int xxdx}{aa\int xdx} = \frac{2\left(aa+bb\right)}{3a},$$

quae omnia apprime conveniunt cum supra § 639 inventis.

EXEMPLUM 2

677. Sit sectio aquae *ABb* semicirculus centro *C* descriptus (Fig. 97), cuius propterea radius $AC = CB = CD$ erit $= a$, hoc ergo casu corpus nostrum abibit in quartam partem sphaerae centro *C* radio $AC = a$ descriptae. Ex natura circuli igitur erit
$s = \sqrt{(2ax - xx)}$ atque

$$p = \frac{a-x}{\sqrt{(2ax-xx)}}, \text{ et } 1+pp = \frac{aa}{2ax-xx}.$$

His substitutis prodibit

$$\frac{p^3 sdx}{1+pp}=\frac{(a-x)^3\,dx}{aa}$$

cuius integrale est

$$\frac{a^2}{4}-\frac{(a-x)^4}{4a^2},$$

quod posito $x=a$ fit $=\frac{a^2}{4}.$ Resistentia igitur horizontalis, quam hoc sphaerae frustum in motu suo sentiet, erit $=\frac{\pi a^2 v}{4}.$ Deinde cum sit

$$\frac{p^2 sdx}{1+pp}=\frac{(a-x)^2\,dx}{aa}\surd(2ax-xx),$$

erit eius integrale posito $x=a$ post integrationem $=\frac{\pi a^2}{16},$ unde corpus hoc verticaliter sursum urgebitur a resistentia vi $=\frac{\pi a^2 v}{8}.$ Denique cum sit $x+ps=a,$ erit

$$\int\frac{p^2 sdx(x+ps)}{(1+pp)}=\int\frac{(a-x)^2\,dx}{a}\surd(2ax-xx)=\frac{\pi a^3}{16},$$

ex quo punctum O per quod resistentiae vis verticalis transit, ipsum sphaerae centrum C incidet. Soliditas porro huius sphaerae quadrantis erit

$$\frac{\pi}{2}\int ssdx=\int(2ax-xx)\,dx=\frac{\pi a^3}{3},$$

atque superficies eius

$$=\pi\int sdx\surd(1+pp)=\pi\int adx=\pi a^2;$$

quae quidem ex natura sphaerae sponte fluunt.

COROLLARIUM 1

678. Vis igitur resistentiae verticalis quae est $=\frac{\pi a^2 v}{8}$ duplo minor est quam eius vis horizontalis, qua motus retardatur. Media igitur directio resistentiae transibit per *O* et in plano verticali diametrali *ACD* sita angulum constituet cum *AC* cuius tangens erit $=\frac{1}{2}$.

COROLLARIUM 2

679. Cum basis *BDb* area sit $=\frac{\pi a^2}{2}$ si basis nuda eadem celeritate secundum *CA* moveretur in aqua, foret eius resistentia $=\frac{\pi a^2 v}{2}$; ita ut resistentia horizontalis figurae *ABDb* duplo sit minor, quam resistentia basis.

COROLLARIUM 3

680. Intelligitur etiam quantam resistentiam patiatur globus integer in aqua motus; cum enim eius semissis resistentiae sit opposita, erit resistentia ipsa $=\frac{\pi a^3 v}{4}$, si eius radius ponatur $=a$. Globus itaque in aqua motus duplo minorem patitur resistentiam, quam eius circulus maximus.

COROLLARIUM 4

681. Hinc resistentia, quam diversi globi in aqua moti patiuntur erit in ratione composita ex duplicata diametrorum et duplicata celeritatum, quibus progrediuntur.

EXEMPLUM 3

682. Sit figura aquae innatans *ABDb* sphaeroidis elliptici portio (Fig. 97), eiusmodi ut sectio aquae *ABb* sit semiellipsis centrum habens in *C* cuius semiaxes coniugati sint $AC=a$ et $BC=b$, erit ex natura ellipsis

$$s=\frac{b}{a}\sqrt{(2ax-xx)}\,\text{hincque}\;\; p=\frac{b(a-x)}{a\sqrt{(2ax-xx)}}$$

et

$$1+pp=\frac{a^2b^2+2a(a^2-b^2)x-\left(a^2-b^2\right)xx}{a^2(2ax-xx)}.$$

Ad resistentiam igitur cognoscendam sequentes formulae integrales sunt considerandae, quarum prima est $\int \frac{p^3 sdx}{1+pp}$, quae abit in

$$\int \frac{(a-x)^3 dx}{a^2b^2 + 2a(a^2-b^2)x - \left(a^2-b^2\right)xx},$$

cuius integrale est

$$\frac{-b^4}{2\left(a^2-b^2\right)} + \frac{a^2b^4}{\left(a^2-b^2\right)^2} l \frac{a}{b}.$$

Ex hoc vis resistentiae motui contraria cuius directio est AC erit

$$= \pi b^2 v \left(\frac{a^2b^2}{\left(a^2-b^2\right)^2} l \frac{a}{b} - \frac{b^2}{2\left(a^2-b^2\right)} \right),$$

vel eadem vis per seriem expressa erit

$$= \frac{\pi b^4 v}{2a^2} \left(\frac{1}{2} + \frac{a^2-b^2}{3a^2} + \frac{\left(a^2-b^2\right)^2}{4a^4} + \frac{\left(a^2-b^2\right)^3}{5a^6} + \frac{\left(a^2-b^2\right)^4}{6a^8} + \text{etc.} \right)$$

quae eo magis convergit, quo minor fuerit differentia inter a et $b.$ Deinde cum sit

$$\int \frac{p^2 sdx}{1+pp} = \frac{b^3}{a} \int \frac{(a-x)^2 dx \sqrt{\left(2ax-xx\right)}}{a^2b^2 + 2a(a^2-b^2)x - \left(a^2-b^2\right)x^2},$$

erit eius integrale posito $x = a$, sequens quantitas $\frac{\pi ab^3}{4\left(a+b\right)^2}$; unde vis resistentiae verticalis est

$$= \frac{\pi ab^3 v}{4\left(a+b\right)^2};$$

ipsam autem directionem huius vis seu locum applicationis ob prolixitatem calculi non determinamus.

COROLLARIUM 1

683. Si ellipsis *ABb* abeat in circulum ita ut sit $a = b$; tum resistentia horizontalis a logarithmis liberabitur, fietque per seriem datam $= \frac{\pi a^2 v}{4}$. Vis vero qua sursum pellitur fiet $= \frac{\pi a^2 v}{8}$, uti ante iam est inventum.

COROLLARIUM 2

684. Si ellipsis *ABb* quam minime a circulo discrepet ita ut sit $b = a + \alpha$, denotante α quantitatem valde exiguam, erit ex serie resistentiae vis horizontalis secundum $AC = \frac{\pi a^2 v}{4} + \frac{2\pi a\alpha v}{3} = \frac{\pi b^2 v}{4} + \frac{\pi b\alpha v}{6}$, ob $a = b - \alpha$.

COROLLARIUM 3

685. Manente igitur axe *ACa,* resistentia eo maior evadet, quo magis crescit $BC = b$. At si b maneat eadem, resistentia descrescet crescenta axe $AC = a$. Atque ex ipsa resistentiae expressione

$$\pi b^2 v\left(\frac{a^2b^2}{\left(a^2-b^2\right)^2} l\frac{a}{b} - \frac{b^2}{2\left(a^2-b^2\right)}\right)$$

intelligitur si a fiat infinite magnum, tum resistentiam penitus evanescere.

COROLLARIUM 4

686. Resistentia igitur motum retardans diminuetur augendo longitudinem sphaeroidis elliptici *AC* atque diminuendo latitudinem $BC = b$. Unde quo magis axes ellipsis fuerint inter se inaequales, eo minor evadet resistentia.

COROLLARIUM 5

687. Cum soliditas in genere sit $= \frac{\pi}{2}\int ssdx$ erit pro nostro casu soliditas sphaeroidis elliptici

$$ABDb = \frac{\pi b^2}{2aa}\int\left(2ax - xx\right)dx = \frac{\pi ab^2}{3},$$

posito post integrationem $x = a$.

COROLLARIUM 6

688. Superficies denique huius sphaeroidis, quae in genere est

$$\pi\int sdx\sqrt{(1+pp)}, \text{fiet } =\frac{\pi b}{a^2}\int dx\sqrt{\left(a^2b^2+2a\left(a^2-b^2\right)x-\left(a^2-b^2\right)x^2\right)},$$

quae expressio posito $a-x=u$ transit in hanc

$$-\frac{\pi b}{a^2}\int du\sqrt{(a^4-(a^2-b^2)u^2)}=\frac{-\pi a^2 b}{2\sqrt{(a^2-b^2)}}\left(\text{Asin}\frac{u\sqrt{(aa-bb)}}{aa}+\frac{u\sqrt{(aa-bb)}}{a^4}\sqrt{(a^4-(aa-bb)uu)}\right)$$
$$+\frac{\pi a^2 b}{2\sqrt{(a^2-b^2)}}\left(\text{Asin}\frac{\sqrt{(aa-bb)}}{a}+\frac{b\sqrt{(aa-bb)}}{a^2}\right).$$

Posito ergo $x=a$ seu $u=0$ prodibit tota superficies

$$=\frac{-\pi a^2 b}{2\sqrt{(a^2-b^2)}}\left(\text{Asin}\frac{\sqrt{(a^2-b^2)}}{a}+\frac{b\sqrt{(a^2-b^2)}}{aa}\right)=\frac{\pi bb}{2}+\frac{\pi a^2 b}{2\sqrt{(a^2-b^2)}}\text{Asin}\frac{\sqrt{(a^2-b^2)}}{a}.$$

COROLLARIUM 7

689. Quare si a et b non multum a se invicem discrepent, ob

$$\text{Asin}\frac{\sqrt{(a^2-b^2)}}{a}=\text{Atang}\frac{\sqrt{(a^2-b^2)}}{a}=\frac{\sqrt{(a^2-b^2)}}{b}-\frac{\left(a^2-b^2\right)^{\frac{3}{2}}}{3b^3}+\frac{\left(a^2-b^2\right)^{\frac{5}{2}}}{5b^5}-\text{etc.}$$

superficiei inveniendae inserviet ista expressio

$$\frac{\pi}{2}\left(bb+aa-\frac{a^2(aa-bb)}{3b^2}+\frac{aa\left(a^2-b^2\right)^2}{5b^4}-\frac{aa\left(a^2-b^2\right)^3}{7b^6}+\text{etc.}\right)$$

quae vehementer est convergens.

PROPOSITIO 65

PROBLEMA

690. *Maneant ut ante omnes sectiones verticales STs ad axem AC normales semicirculi* (Fig. 97), *quaeraturque natura curvae ASBC seu sectionis aquae quae formet eiusmodi*

solidum ABDb, quod secundum directionem CAL in aqua motum minimam patiatur resistentiam simul vero maxime sit capax.

SOLUTIO

Positis ut ante in sectione aquae quaesita abscissa $AP = x$, et applicata $PS = s$, atque $ds = pdx$, erit resistentia, quam patietur solidum rotundum huic aquae sectioni respondens, ut $\int \frac{p^3 sdx}{1+pp}$, quae ergo formula debet esse minimum. Hunc in finem differentietur $\frac{p^3 s}{1+pp}$, erit eius differentiale

$$\frac{p^3 ds}{1+pp} + \frac{\left(3p^2+p^4\right) sdp}{\left(1+pp\right)^2},$$

ex quo secundum regulam supra datam § 523 emergit iste valor

$$\frac{p^3}{1+pp} - \frac{1}{dx} d.\frac{\left(3p^2+p^4\right)s}{\left(1+pp\right)^2},$$

qui poni deberet $= 0$, si solidum desideretur, quod absolute minimam pateretur resistentiam. At cum insuper soliditas debeat esse maxima, soliditas vero sit ut $\int ssdx$, huicque formulae respondeat iste valor $2s$, huius multiplum quodcunque illi valori aequale est ponendum. Hinc ergo obtinebitur ista aequatio

$$\frac{2s}{c} = \frac{p^3}{\left(1+pp\right)^2} - \frac{1}{dx} d \cdot \frac{\left(3pp+p^4\right)s}{\left(1+pp\right)^2}$$

multiplicetur per ds seu pdx, habebitur

$$\frac{2sds}{c} = \frac{p^3 ds}{1+pp} - pd \cdot \frac{\left(3pp+p^4\right)s}{\left(1+pp\right)^2} = d \cdot \frac{p^3 ds}{1+pp} - d \cdot \frac{\left(3pp+p^4\right)ps}{\left(1+pp\right)^2}$$

unde integrale erit

$$\frac{ss}{c} - f = \frac{p^3 s}{1+pp} - \frac{\left(3pp+p^4\right)ps}{\left(1+pp\right)^2} = \frac{-p^3 s}{\left(1+pp\right)^2}$$

seu

$$ss = cf - \frac{2cp^3 s}{\left(1+pp\right)^2};$$

ex qua aequatione intelligitur fieri non posse $s=0$, quod tamen conditio quaestionis requirit, nisi sit $f=0$. Ponatur ergo $f=0$, et c negativum erit

$$s=\frac{2cp^3}{(1+pp)^2}.$$

Cum autem sit $ds=pdx$ erit

$$x=\frac{s}{p}+\int\frac{sdp}{pp}=\frac{2cpp}{(1+pp)^2}+2c\int\frac{pdp}{(1+pp)^2}=\frac{2cpp}{(1+pp)^2}-\frac{c}{1+pp}+\text{Const.}$$

unde proveniet

$$x=\text{ Const.}+\frac{-c+cpp}{(1+pp)^2}.$$

Quoniam vero x eodem casu quo s evanescere debet, s autem duobus casibus evanescat, quorum alter est si $p=0$, alter si $p=\infty$, constans ex eo debet determinari. Sit igitur in puncto A, $p=0$, seu tangens curvae AC in A incidat in ipsam rectam $A\,L$, fietque Const. c, ex quo erit

$$x=\frac{3cpp+cp^4}{(1+pp)^2},$$

atque

$$s=\frac{2cp^3}{(1+pp)^2},$$

haecque curva generabit solidum, quod minimam patietur resistentiam ob cuspidem in A acutissimam, contra vero casus, quo in A fit $p=\infty$, producet corpus maximae resistentiae quippe qui casus pariter in quaestione latet. Quamobrem curva quaesita ita erit comparata ut abscissae

$$x=\frac{3cpp+cp^4}{(1+pp)^2},$$

respondeat applicata

$$s=\frac{2cp^3}{(1+pp)^2}$$

unde intelligitur sectionem aquae *ASB* quaestioni satisfacientem fore curvam algebraicam; quae ideo inter omnes alias aequalia solida generantes tale pro ducet solidum, quod in directione axis *AL* motum minimam sufferet resistentiam.
Q. E. I.

COROLLARIUM 1

691. Cum curva *ASB,* quae solidum maximae resistentiae producit, ex eadem aequatione resultet augendo abscissam x quantitate constante, intelligitur utramque curvam tam eam scilicet quae solidum minimae resistentiae, quam eam quae solidum maximae resistentiae producit, portionem esse eiusdem curvae continuae.

COROLLARIUM 2

692. Quoniam igitur s duobus casibus evanescit, seu curva *ASB* in duobus punctis axi *AC* occurrit, primo nimirum si $p=0$ quo casu etiam x fit $=0$ et tum si $p=\infty$, quo casu fit $x=c$, prior concursus dabit curvam producentem minimam resistentiam posterior vero curvam, cui solidum maximae resistentiae respondet.

COROLLARIUM 3

693. Quia aequatio inventa

$$ss = cf - \frac{2cp^3 s}{(1+pp)^2}$$

posito $f=0$, divisibilis est per s, patet aequationem $s=0$ casum quoque continere in quaestione contentum. Perspicuum autem est hunc casum praebere eam curvam quae producit solidum minimae capacitatis.

COROLLARIUM 4

694. Cum sit

$$x = \frac{3cp^2 + cp^4}{(1+pp)^2} \quad \text{et} \quad s = \frac{2cp^3}{(1+pp)^2}$$

intelligitur continuo ipsi p maiorem valorem tribuendo initio facto a $p=0$, tam x quam s usque ad certum terminum crescere, deinde vero iterum decrescere. Maxima autem erunt

x et s si fiat $p = \sqrt{3}$, eu eo loco ubi tangens curvae cum axe AC angulum constituit 60 graduum. Erit autem hoc casu

$$x = \frac{9c}{8} \text{ et } s = \frac{3c\sqrt{3}}{8}.$$

COROLLARIUM 5

695. Si autem haec aequatio cum§ 532 comparetur, deprehendetur haec curva congruere cum ea curva supra inventa, quae inter omnes alias eandem aream continentes patiatur minimam resistentiam. Curva igitur hic inventa erit curva illa triangularis *AM BCDNA* .

COROLLARIUM 6

696. Huius igitur curvae portio *AMB* circa axem *AC* rotata producet solidum, quod simul maximam habebit capacitatem, atque secundum directionem axis *CA* motum minimam patietur resistentiam. Altera vero portio *BCD* circa axem eundem *CE* rotata solidum dabit maximam resistentiam patiens.

COROLLARIUM 7

697. In hac igitur curva, quae ad axem *ACE* utrinque sibi est similis et aequalis ipse axis *CA* erit tangens in *A;* unde ascendet et descendet usque ad *B* et *D,* existente

$$AE = \frac{9c}{8} \text{ et } BE = DE = \frac{3c\sqrt{3}}{8}.$$

Deinde ex cuspidibus *B* et *D* cum axe in *C* unitur existente $AC = c$: eius vero tres portiones *AMB, BCD* et *AND* inter se aequales erunt et similes.

SCHOLION

698. Problema istud ab aliis, qui hoc argumentum pertractaverunt, omissa ea conditione, qua simul solidum capacissimum requiritur, proponi est solitum, ita ut inter omnes omnino curvas eam determinare sint conati, quae circa axem rotata solidum producat quod in directione axis motum minimam pateretur resistentiam. At hoc modo nulla invenitur curva idonea quaesito satisfaciens, resolvetur enim iste casus ex nostra solutione ponendo $c = \infty$, unde fit

$$s = \frac{f\left(1 + pp\right)^2}{2p^3}$$

ex quo nunquam fieri potest $s = 0$, ideoque curva desiderata cum axe nunquam concurreret, id quod est contra conditionem intentam. Hancobrem istam quaestionem hic penitus omittere visum est, eiusque loco praesentem proponere, qua praeter minimam resistentiam maxima capacitas requiritur. Haec enim quaestio eo magis ad institutum nostrum est accommodata, cum in navibus non solum minima resistentia desideretur, sed simul naves maxime capaces esse oporteat. Facile autem perspicitur figuram inventam nimis abhorrere a figuris navium consuetis, aliasque circumstantias prohibere, quominus navibus talis figura vel saltem affinis tribuatur. Ceterum notatu dignum evenit quod curva inventa sit algebraica; cuius vero ordinis sit, eliminando p ita investigabitur. Cum sit

$$x=\frac{3cp^2+cp^4}{(1+pp)^2} \text{ et } s=\frac{2cp^3}{(1+pp)^2}$$

erit

$$\sqrt{(xx-3ss)}=\frac{3cpp-cp^4}{(1+pp)^2}$$

indeque

$$\frac{x}{\sqrt{(xx-3ss)}}=\frac{3+pp}{3-pp};$$

unde fit

$$pp=\frac{3x-3\sqrt{(xx-ss)}}{x+\sqrt{(xx-ss)}}=\frac{(x-\sqrt{(xx-ss)})^2}{ss}$$

et

$$p=\frac{x-\sqrt{(xx-ss)}}{s}.$$

Porro est

$$pp+1=\frac{2xx-2ss-2x\sqrt{(xx-3ss)}}{ss},$$

atque

$$pp+3=\frac{2xx-2x\sqrt{(xx-3ss)}}{ss}.$$

His autem valoribus in aequatione $(1+pp)^2x=cpp(3+pp)$ substitutis atque irrationalitate sublata emerget ista aequatio

$$4s^4+8xxss-36cxss+27ccss-4cx^3+4x^4=0.$$

Posito autem $c=2a$ orietur ista aequatio

$$s^4+2xxss-18axss+27a^2ss-2ax^3+x^4=0,$$

ita ut curva satisfaciens inventa pertineat ad lineas quarti ordinis. Ex hac igitur aequatione elicitur

$$ss = -xx + 9ax - \frac{27}{2}a^2 \pm \frac{(9a-4x)\sqrt{a(9a-4x)}}{2},$$

unde constructio curvae non fit difficilis. Commodius vero partis huc servientis *AMB* natura cognoscetur ex hac serie

$$ss = xx\left(\frac{1}{6}\cdot\frac{4x}{9a} + \frac{1\cdot3}{6\cdot8}\cdot\frac{4^2\cdot x^2}{9^2\cdot a^2} + \frac{1\cdot3\cdot5}{6\cdot8\cdot10}\cdot\frac{4^3\cdot x^3}{9^3\cdot a^3} + \frac{1\cdot3\cdot5\cdot7}{6\cdot8\cdot10\cdot12}\cdot\frac{4^4\cdot x^4}{9^4\cdot a^4} + \text{etc.}\right)$$

vel posito

$$\frac{9a}{4} = b, \text{ ut sit } b = \frac{9c}{8} = AE$$

erit

$$ss = xx\left(\frac{1}{6}\cdot\frac{x}{b} + \frac{1\cdot3x^2}{6\cdot8b^2} + \frac{1\cdot3\cdot5x^3}{6\cdot8\cdot10b^3} + \frac{1\cdot3\cdot5\cdot7x^4}{6\cdot8\cdot10\cdot12b^4} + \text{etc.}\right)$$

ex qua aequatione facile intelligitur tangentem in *A* in axem *AC* incidere, quod ex aequatione superiore difficilius perspicitur. Nunc autem ad alias corporum species progrediamur minus determinatas quam hactenus tractatae, in quibus scilicet duae curvae supersint arbitrariae.

PROPOSITIO 66

PROBLEMA

699. *Sit non solum sectio aquae ABb sed etiam sectio amplissima BDb curva quaecunque data* (Fig. 97), *solidumque ABDb hanc habeat proprietatem, ut omnes sectiones verticales STs ad axem AC normales sint sectioni BDb similes atque moveatur hoc corpus in aqua secundum directionem CAL determinari oportet resistentiam quam patietur.*

SOLUTIO

Primo cum sectio aquae *ABb* seu potius eius semissis *ACB* sit curva quaecunque data; sumta in ea abscissa $AP = x$, et posita applicata $PS = s$, erit *s* functio quaedam ipsius *x* data. Deinde cum etiam curva *BDb* seu potius eius semissis *BDC* data sit positis ad eam coordinatis $CG = r$ et $GH = u$, dabitur aequatio inter *u* et *r,* atque *u* aequabitur functioni cuidam ipsius *r.* Cum nunc sectio *STP* similis sit sectioni *BDC,* lineae in iis homologae tenebunt rationem ut *PS* ad *CB.* Posito igitur $CB = b$, et pro sectione *SPT* sumtis coordinatis $PM = y$, et $MQ = z$ similibus ipsis *r* et *u,* erit

$$y=\frac{rs}{b} \text{ et } z=\frac{us}{b}.$$

Cum nunc s sit functio ipsius x, ponatur $ds = pdx$, ut p sit functio ipsius x, similiterque ob u functionem ipsius r ponatur $du = qdr$, ut q sit functio ipsius r. His igitur factis erit

$$dy=\frac{rpdx}{b}+\frac{sdr}{b}, \text{ et } dz=\frac{updx}{b}+\frac{sqdr}{b};$$

unde ob

$$\frac{sdr}{b}=dy-\frac{rpdx}{b}$$

sequens emergit aequatio inter tres coordinatas x, y, et z qua natura superficiei propositae continetur:

$$dz=\frac{(u-qr)\,pdx}{b}+qdy;$$

quae cum generali aequatione in propositione 61 assumta $dz = Pdx + Qdy$ comparata praebet

$$P=\frac{(u-qr)\,p}{b}$$

et $Q=q$, ubi notandum quantitates s et p a sola x pendere, u vero et q ab r, atque r et x a se mutuo non pendere. Ad resistentiam iam motui contrariam inveniendam oportet primum huius formulae $\frac{P^3dy}{1+P^2+Q^2}$ posito x constante integrale reperire, atque post integrationem facere $y=s$. Quoniam igitur x est constans, erit

$$dy=\frac{sdr}{b},$$

atque ob

$$1+P^2+Q^2=\frac{b^2+\left(u^2-qr\right)^2+b^2q^2}{b^2}$$

fiet

$$\frac{P^3dy}{1+P^2+Q^2}=\frac{(u-qr)^3\,p^3sdr}{b^4(1+qq)+b^2p^2(u-qr)^2}$$

in cuius integrali capiendo p et s tanquam quantitates constantes considerari debent. Invento igitur integrali

$$\frac{p^3 s}{b^2}\int \frac{(u-qr)^3\,dr}{b^2(1+qq)+p^2(u-qr)^2}$$

ita ut evanescat posito $r=0$, tumque facto $r=b$, integrale hoc multiplicandum est per dx, denuoque integrale capiendum, unica enim inerit variabilis x, atque integratione peracta poni debet $x=AC=a$. Vel quod eodem redit ista formula

$$\frac{(u-qr)^3\,p^3 sdrdx}{b^4(1+qq)+b^2p^2(u-qr)^2}$$

his est integranda, in altera integratione x, p, et s ponendo constantia, in altera autem r, q et u; perinde enim est quaenam integratio prius instituatur. Designata autem quantitate, quae per duplicem integrationem, post quam positum est $r=b$ et $x=a$, prodit, per hanc formam

$$\iint \frac{(u-qr)^3\,p^3 sdrdx}{b^4(1+qq)+b^2p^2(u-qr)^2}$$

erit resistentiae vis, quae secundum directionem AC retropellit corpus

$$=\frac{2v}{bb}\iint \frac{(u-qr)^3\,p^3 sdrdx}{b^2(1+qq)+p^2(u-qr)^2}.$$

Simili autem modo rem peragendo reperietur resistentiae vis verticalis corpus sursum sollicitans

$$\frac{2v}{b}\iint \frac{(u-qr)^2\,p^2 sdrdx}{b^2(1+qq)+p^2(u-qr)^2}.$$

Denique si eodem modo quaeratur valor

$$\iint \frac{(u-qr)^2\left(bbx+pus(u-qr)\right)p^2 sdrdx}{b^4(1+qq)+b^2p^2(u-qr)^2}$$

isque dividatur per

$$\iint \frac{(u-qr)^2\,p^3 sdrdx}{b^2(1+qq)+pp(u-qr)^2}$$

prodibit distantia *AO,* ex eaque situs puncti *O* per quod vis resistentiae verticalls transit. Q. E. I.

COROLLARIUM 1

700. Cum sit

$$\frac{(u-qr)^3\,dr}{b^2(1+qq)+p^2(u-qr)^2}=dr\left(\frac{(u-qr)^3}{b^2(1+qq)}-\frac{p^2(u-qr)^5}{b^4(1+qq)^2}+\frac{p^4(u-qr)^7}{b^6(1+qq)^3}-\text{etc.}\right)$$

fiet

$$\iint\frac{(u-qr)^3\,p^3sdrdx}{b^2(1+qq)+p^2(u-qr)^2}=\frac{1}{bb}\int p^3sdx\cdot\int\frac{(u-qr)^3\,dr}{(1+qq)}$$
$$-\frac{1}{b^4}\int p^5sdx\cdot\int\frac{(u-qr)^5\,dr}{(1+qq)^2}+\frac{1}{b^6}\int p^7sdx\cdot\int\frac{(u-qr)^7\,dr}{(1+qq)^3}-\text{etc.}$$

in quibus integrationibus variabiles *r* et *x,* a se invicem prorsus sunt separatae.

COROLLARIUM 2

701. Si igitur singulae formulae differentiales, in quibus tantum inest *r* et quantitates inde pendentes *u* et *q* ita integrentur ut evanescant posito $r=b$, similique modo alterae formulae integrales in quibus tantum insunt *x* et *s* et *p*integrentur, tumque ponatur $x=a$, obtinebitur desideratus valor formulae

$$\iint\frac{(u-qr)^3\,p^3sdrdx}{b^2(1+qq)+p^2(u-qr)^2}$$

COROLLARIUM 3

702. Simili igitur modo reliquae formulae differentiales, quae duplicem integrationem requirunt, per series ita exprimi poterunt, ut binae variabiles *x* et *r* prorsus a se invicem separentur; quo facto singulae sine ullo respectu ad reliquas habito seorsim integrari poterunt.

COROLLARIUM 4

703. Cum soliditas corporis in genere sit $= -2\int dx \int Qydy$, ubi in integratione $\int Qydy$ ponitur *x* constans, erit pro nostro casu ob $y = \frac{rs}{b}$ et $dy = \frac{sdr}{b}$ atque $Q = q$, formula

$$\int Qydy = \int \frac{qrs^2dr}{bb} = \frac{s^2}{b^2}\int rdu = -\frac{s^2}{b^2}\int udr,$$

ubi $\int udr$ denotat aream *BCD* unde tota soliditas erit $= 2\int \frac{ssdx}{bb}\int udr$.

COROLLARIUM 5

704. Superficies vero solidi *ABDb* ex formula generali

$$2\int dx \int dy \sqrt{(1 + P^2 + Q^2)}$$

invenietur, quae ob *x* constans in altera integratione abit in

$$2\iint \frac{sdrdx}{bb}\sqrt{\left(bb(1+qq) + pp(u-qr)^2\right)}$$

ubi duplici integratione est opus, altera in qua *r,* altera in qua *x* ponitur constans.

PROPOSITIO 67

PROBLEMA

705. *Si data fuerit sectio verticalis BDb ad axem AC normalis* (Fig. 97), *cui omnes reliquae sectiones ipsi parallelae STs sint similes; determinare curvam ASB, ex qua natum solidum ABDb pro capacitate sua minimam patiatur resistentiam, si quidem moveatur in aqua secundum directionem axis CAL.*

SOLUTIO

Manentibus ut ante, $BC = b$, $CG = r$, atque $GH = u$, positoque $du = qdr$, ita ut *u* et *q* futurae sint functiones datae ipsius *r,* sit $AP = x$, $PS = s$ ponaturque $ds = pdx$, quibus positis erit resistentia ut

$$\iint \frac{(u-qr)^3\ p^3sdrdx}{b^2(s+qq) + pp(u-qr)^2}$$

quae quantitas ideo bis integrata minimum esse debet. Concipiatur autem integratio ea primum institui in qua r cum inde pendentibus s et q ponitur constans, atque post integrationem fieri $x = AC = a$, manifestum est in altera integratione naturam curvae ASB non amplius contineri. Quo circa requiritur ut quantitas, quae per priorem integrationem prodit, reddatur minima. Multiplicatum autem hic est dx per

$$\int \frac{(u-qr)^3\ p^3 sdr}{b^2(s+qq)+pp(u-qr)^2},$$

in qua p et s tantum sunt quantitates variables. Ponatur brevitatis gratia

$$u-qr=t \text{ et } 1+qq=w^2,$$

habebitur ista formula

$$\int \frac{t^3p^3sdr}{bbw^2+ttp^2},$$

quae differentiata ponendis semper r, t et w constantibus dat

$$p^3ds\int \frac{t^3dr}{bbw^2+ttp^2}+ppdp\int \frac{\left(3b^2w^2+ttp^2\right)t^3sdr}{\left(bbw^2+ttp^2\right)^2}$$

unde oritur iste valor ad determinationem minimi requisitus

$$p^3\int \frac{t^3dr}{bbw^2+ttp^2}-\frac{1}{dx}d\cdot pp\int \frac{\left(3b^2w^2+ttp^2\right)t^3sdr}{\left(b^2w^2+ttp^2\right)^2},$$

qui poni deberet $=0$ nisi capacitatis ratio esset habenda. Capacitas vero est ut $\int ssdx\int udr$, in quo integrali multiplicatum est dx per $ss\int udr$, cuius differentiale est $2sds\int udr$, ex quo valor ad maximum determinandum inserviens est $2s\int udr$. His ergo valoribus coniunctis emerget ista aequatio

$$\frac{2s\int udr}{c}=p^3\int \frac{t^3dr}{bbw^2+ttp^2}-\frac{1}{dx}d\cdot pp\int \frac{\left(3b^2w^2+t^2p^2\right)t^3sdr}{\left(b^2w^2+t^2p^2\right)^2},$$

quae multiplicata per $pdx=ds$, et integrata dat

$$\frac{ss\int udr}{c}-f^3=\int\frac{t^3p^3sdr}{bbw^2+ttp^2}-\int\frac{\left(3b^2w^2+t^2p^2\right)t^3p^3sdr}{\left(b^2w^2+t^2p^2\right)^2}=-\int\frac{2b^2w^2t^3p^3sdr}{\left(b^2w^2+t^2p^2\right)^2}.$$

Quo ergo fieri queat $s=0$, necesse est ut sit $f=0$, ita ut facto c negativo ista habeatur aequatio pro curva quaesita

$$s=\frac{2b^2cp^3}{\int udr}\int\frac{w^2t^3dr}{\left(b^2w^2+t^2p^2\right)^2},$$

cui valor sequens ipsius x valor respondebit

$$\begin{aligned}x&=\int\frac{ds}{p}=\frac{s}{p}+\int\frac{sdp}{pp}\\&=\text{Const.}+\frac{2b^2cp^2}{\int udr}\int\frac{w^2t^3dr}{\left(b^2w^2+t^2p^2\right)^2}-\frac{b^2c}{\int udr}\int\frac{w^2tdr}{b^2w^2+t^2p^2}\\&=\text{Const.}-\frac{b^2c}{\int udr}\int\frac{\left(b^2w^2-t^2p^2\right)w^2tdr}{\left(b^2w^2+t^2p^2\right)^2}.\end{aligned}$$

Quo x simul evanescat, si fit $p=0$, quippe quo casu simul fit $s=0$, fiet

$$\text{Const.}=\frac{b^2c}{\int udr}\int\frac{tdr}{b^2};$$

ita ut fiat

$$x=\frac{cpp}{\int udr}\int\frac{\left(3b^2w^2+t^2p^2\right)t^2dr}{\left(b^2w^2+t^2p^2\right)^2}.$$

Quoniam autem $\int udr$ valorem habet constantem ratione variabilium nostrarum *x, s* et *p,* ea in constanti *c* comprehendatur, atque restitutis pristinis valoribus pro w et *t,* haec habetur constructio:

$$x=\frac{cpp}{bb}\int\frac{\left(3b^2\left(1+qq\right)+pp(u-qr)^2\right)(u-qr)^3dx}{\left(b^2\left(1+q^2\right)+p^2\left(u-qr\right)^2\right)^2}$$

et

$$s = 2cp^3 \int \frac{(1+qq)(u-qr)^3 dr}{\left(b^2(1+q^2)+p^2(u-qr)^2\right)^2}.$$

Quae formulae integrales constructionem minime turbant, cum in iis p constans ponatur, ideoque ex aequatione inter r et u data integratio actu absolvi queat; ita autem integratio absolvi debet ut prodeat 0 posito $r=0$, quo facto faciendum est $r=b$. Q.E.I.

COROLLARIUM 1

706. Haec igitur curva pariter in A tangentem habebit in axem AL incidentem, cum initio quo tam x et s evanescunt sit $p=0$. Insuper vero alio loco curva in axem AC cadet, quod eveniet si $p=\infty$, hoc enim casu fit

$$s=0 \text{ et } x=\frac{c}{bb}\int (u-qr)dr=\frac{2c}{bb}\int udr;$$

seu x aequabitur areae basis BDb ductae in $\frac{c}{bb}$, vel erit

$$x=\frac{2c\cdot BCD}{BC^2}.$$

COROLLARIUM 2

707. In altero hoc puncto, ubi curva iterum in axem AC incidit, tangens erit normalis ad axem AC, ex quo ista curvae portio solidum generabit maximam patiens resistentiam.

COROLLARIUM 3

708. Cum insuper axis AC sit diameter curvae inventae, quod constat ex eo, quia facto p negativo x manet, s vero in sui negativum abit, curva non multum dissimilis erit ei quam ante invenimus, cum sectio BDb sit semicirculus.

COROLLARIUM 4

709. Ab initio autem ubi fit $p=0$, crescente p crescent tum abscissa x quam applicata s usque ad certum terminum, qui terminus reperietur differentiando

$$\int \frac{p^3\left(1+qq\right)(u-qr)^3\,dr}{\left(b^2\left(1+q^2\right)+p^2\left(u-qr\right)^2\right)^2}$$

posito tantum p variabili, faciendoque differentiali $=0$.

COROLLARIUM 5

710. Absoluta autem hac differentiatione reperietur sequens aequatio ex qua valor ipsius p determinabitur

$$0=\int \frac{p^2\left(3b^2\left(1+q^2\right)-p^2\left(u-qr\right)^2\right)\left(1+q^2\right)(u-qr)^3\,dr}{\left(b^2\left(1+q^2\right)+p^2\left(u-qr\right)^2\right)^3}$$

quae integratio modo praescripto perfici debet, posteaque poni $r=b$.

COROLLARIUM 6

711. Si fuerit $\left(u-qr\right)^2=ff(1+qq)$ quod accidit, si curva *BDb* fuerit semicirculus, tum quantitas p ex formulis integralibus eliminari poterit.
Erit nempe hoc casu

$$x=\frac{cf^3p^2(3bb+ffpp)}{bb(bb+ffpp)^2}\int dr\,\surd(1+qq)$$

et

$$S=\frac{2cf^3p^3}{(bb+ffpp)^2}\int dr\,\surd(1+qq).$$

COROLLARIUM 7

712. Si igitur $\int dr\,\surd(1+qq)$ seu arcus *BD* tanquam quantitas constans in c comprehendatur fiet

$$x=\frac{c^5p^2(3bb+ffpp)}{bb(bb+ffpp)^2} \quad \text{et} \quad S=\frac{2c^5p^3}{(bb+ffpp)^2}.$$

SCHOLION

713. Notandum ceterum est hanc proprietatem, qua est

$$\left(u-qr\right)^2=ff(1+qq) \quad \text{seu} \quad -u+qr=f\,\surd(1+qq),$$

in nullam aliam curvam praeter circulum competere. Nam sumtis differentialibus ob $du = qdr$ erit

$$rdq = \frac{fqdq}{\sqrt{(1+qq)}} \quad \text{ideoque} \quad r = \frac{fq}{\sqrt{(1+qq)}}$$

vel etiam propter divisionem $dq = 0$, unde primo linea recta dicta proprietate gaudet. Deinde cum sit $u = qr - f\sqrt{(1+qq)}$ erit

$$u = \frac{fqq}{\sqrt{(1+qq)}} - f\sqrt{(1+qq)} = -\frac{f}{\sqrt{(1+qq)}}.$$

Erit ergo

$$\frac{r}{u} = -q$$

unde fit

$$r = -\frac{fr}{\sqrt{\left(r^2+u^2\right)}} \text{ seu } f = -\sqrt{\left(r^2+u^2\right)}.$$

Quia autem facto $u = 0$ fieri debet $r = b$ erit $f = -b$, indeque

$$b^2 = r^2 + uu.$$

Casus itaque memoratus quo fit $(u-qr)^2 = ff(1+qq)$ locum non habet, nisi sectio *BDb,* fuerit semi circulus vel triangulum isosceles. Denique id etiam hic generaliter locum habet, ut, quaecunque fuerit curva *BDb,* curva *AB* quaestioni satisfaciens semper evadat algebraica, cum formulae integrales constructionem algebraicam non afficiant.

PROPOSITIO 68

PROBLEMA

714. *Si data sit corporis ABDb tum sectio amplissima BDb, tum etiam figura spinae ASD seu sectio diametralis ACD* (Fig. 98), *solidumque ita sit comparatum ut omnes sectiones verticales parallelae sectioni mediae ACd eidem sint similes: determinare resistentiam, quam hoc corpus sentiet, si cursu directo secundum directionem CAL in aqua promoveatur.*

SOLUTIO

Cum primo data sit sectio verticalis diametralis *ACD* dabitur aequatio inter eius abscissam $AR = r$ et applicatam $RS = s$, ita ut *s* aequetur functioni ipsius *r* futurumque sit $ds = pdr$ existente *p* pariter functione ipsius *r.* Deinde sit intervallum $AC = a$, quo vertex *A* a sectione amplissima *BDb* distat, atque pro hac sectione *BDC* ponatur abscissa $CG = y$, quippe quae aequalis evadet secundae variabili $PM = y$, trium illarum *x, y* et *z,* quae in aequationem localem totius superficiei ingredientur, atque applicata $GH = u$, eritque ob hanc curvam cognitam *u* functio quaedam ipsius *y,* ita ut posito

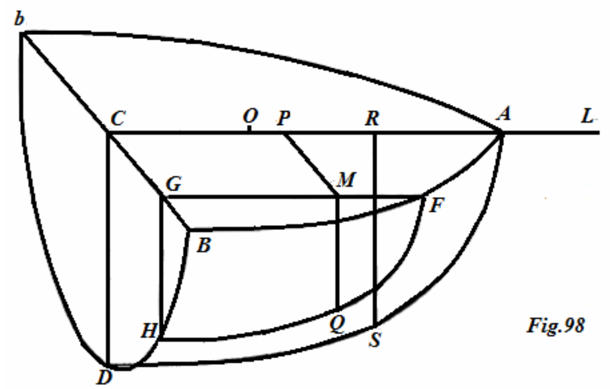


Fig.98

$du = qdy$ futura sit etiam *q* functio ipsius *y;* posito vero $y = 0$, abibit $GH = u$ in *CD,* quae sit $= c$ ita ut *c* tam fiat valor ipsius *u* posito $y = 0$ quam valor ipsius *s* posito $r = a$. Iam cum sectio *FGH,* parallela sectioni *AQD,* eidem sit similis erit $CD : AC = GH : FG$, ex quo fit $FG = \frac{au}{c}$. Sumto nunc in sectione *FGH* puncto *M* homologo puncto *R* in sectione *ACD* erit

$$FM = \frac{ru}{su}, \text{ et } MQ = z = \frac{su}{c}.$$

Porro ex *M* ad axem *AC* ducatur normalis $MP = y$, quippe quae aequalis est ipsi *CG,* et posito

$$AP = x \text{ erit } CP = a - x = GM = \frac{au}{c} - \frac{ru}{c},$$

unde fit

$$x = a - \frac{(a-r)u}{c}.$$

Quare cum ex curvis *ACD* et *BCD* datis sequentes variabilium *x, y* et *z* habeamus valores

$$x = a - \frac{(a-r)u}{c}, \quad y = y \text{ et } z = \frac{su}{c},$$

erit

$$dx = \frac{-aqdy + rqdy + udr}{c} \text{ et } dz = \frac{sqdy + updr}{c},$$

ubi cum sit

$$\frac{udr}{c} = dx + \frac{aqdy - rqdy}{c},$$

fiet

$$dz = pdx + \frac{(ap - rp + s)qdy}{c},$$

quae aequatio cum canonica $dz = Pdx + Qdy$ comparata praebet

$$P = p \text{ et } Q = \frac{(ap - rp + s)qdy}{c},$$

ita ut sit

$$1 + P^2 + Q^2 = \frac{c^2 + c^2p^2 + (ap - rp + s)^2 q^2}{c^2}:$$

quae expressiones duas complectuntur quantitates variabiles a se invicem non pendentes scilicet *y,* et per *y* datas *u* et *q,* atque *r* ex eaque datas *s* et *p.* Hinc erit

$$\frac{P^3dy}{1 + P^2 + Q^2} = \frac{c^2p^3dy}{c^2\left(1 + p^2\right) + q^2(ap - rp + s)^2}$$

in cuius differentialis integratione tantum *y* est variabilis, atque *r, s* et *p* tanquam constantes spectantur. Integratione autem ita absoluta ut prodeat 0, posito $y = 0$ fieri debet $y = BC$ seu $u = 0$; quo facto prodibit functio mero ipsius *r* quae in *dx* ducta denuo integrari debet. Sed cum *dx* posito *y* constanti fiat $= \frac{udr}{c}$, ideoque ab *y* pendeat, duplex ista integratio inverso modo est instituenda, ponendo primo *y* constans. Nam quoniam formula generalis ad resistentiam definiendam est

$$\iint \frac{P^3dydx}{1 + P^2 + Q^2},$$

quae duplicem integrationem requirit alteram posito *x* constante, alteram posito *y* constante, ea ob $dx = \frac{udr}{c}$, pro nostro casu abit in hanc

$$\iint \frac{cp^3udrdy}{c^2\left(1 + p^2\right) + q^2(ap - rp + s)^2}$$

cuius valor pariter duplici integratione est eruendus, in quarum altera *y* cum *u* et *q,* in altera vero *r* cum *p* et *s* poni debet constans. Hocque modo rem absolvendo perinde est utra integratio primum absolvatur. Utraque autem integratio ita perfici debet, ut integralia per omnes valores variabilium *r* et *y* extendantur. Hoc ergo monito prodibit resistentiae vis horizontalis in directione *AC* repellens

$$= 2cv\iint \frac{p^3udrdy}{c^2\left(1+p^2\right)+q^2(ap-rp+s)^2}.$$

Resistentiae vero vis verticalis corpus sursum urgens erit

$$= 2cv\iint \frac{p^2udrdy}{c^2\left(1+p^2\right)+q^2(ap-rp+s)^2}.$$

Ad cuius locum applicationis seu punctum O inveniendum ob

$$x+Pz=\frac{ac-\left(a-r\right)u+psu}{c}$$

ista quantitas

$$\iint \frac{\left(ac-\left(a-r\right)u+psu\right)p^2udrdy}{c^2\left(1+p^2\right)+q^2(ap-rp+s)^2}$$

dividi debet per

$$\iint \frac{cp^2udrdy}{c^2\left(1+p^2\right)+q^2(ap-rp+s)^2},$$

quotusque indicabit intervallum *AC.* Q. E. I.

COROLLARIUM 1

715. Cum in sectione aquae *BAb* applicata *GF* ad applicatam *GH* sectionis amplissimae *BDb* constantem habeat rationem, curva *CBA* affinis erit curvae *CBD,* ut si data sit curva *CBA* sectio aquae *ACB* facillime innotescat.

COROLLARIUM 2

716. Huius igitur problematis solutio similis manebit, si loco curvae *BCD* daretur sectio aquae *ACB;* quamobrem dummodo omnes sectiones verticales *FGH* inter se sint similes, perinde se habebit solutio, sive curva *ACB* detur sive altera *BCD.*

COROLLARIUM 3

717. Quoniam porro tota corporis *ABDb* soliditas generaliter est $=-2\iint Qydydx$ erit pro nostro casu ob

$$dx = \frac{udr}{c} \text{ et } Q = \frac{(ap - pr + s)q}{c}$$

soliditas

$$= \frac{-2}{cc}\iint (ap - pr + s)quydrdy.$$

COROLLARIUM 4

718. Of these two integrations the first may be put in place, in which *y,* and likewise *u* and *q* may be put to be constant, there will become

$$\int (ap - pr + s)dr = \int sdr + \int \left(a - r\right)ds = 2 \text{ area } ACD$$

if there may be put $r = a$ after the integration. Therefore if this area *ACD* may be called $= ff$, the volume of the solid

$$= \frac{-4ff}{cc}\int quydy.$$

COROLLARY 5

719. Then since there shall become $\int quydy = \int uydu$ there will become

$$\int quydy = \frac{u^2y}{2} - \frac{1}{2}\int u^2dy = -\frac{1}{2}\int u^2dy$$

on setting $u = 0$. Whereby the total volume produced

$$= \frac{2ff}{cc}\int u^2dy,$$

which same expression arises from the nature of the construction.

SCHOLIUM

720. Quoniam figura sectionis aquae *ACB* ex sola sectione amplissima *BCD* determinatur neque a figura sectionis diametralis *ACD* pendet, simul etiam ista quaestio est resoluta, qua solidi resistentia quaeritur, quod ex datis curvis *ACB* et *ACD* ita generetur ut omnes sectiones *FGH* plano diametrali *ACD* parallelae sint inter se similes; adeo ut non opus sit hanc quaestionem seorsim tractare. Simili modo in casu praecedente (Fig. 97), quo datae erant sectio aquae *ACB* et sectio amplissima *BCD* huic autem parallelae sectiones omnes *SPT* positae sunt inter se similes, curva *ATD* a sola curva *ASB*

determinatur ubique enim habet *PT* ad *PS* eandem rationem eam scilicet quam habet *CD* ad *CB,* ita ut curva *ATD* affinis sit curvae *ASB* : voco autem curvas affines, quae communem habent abscissam, et quarum applicatae aequalibus abscissis respondentes datam inter se tenent rationem; ita omnes ellipses unum axem communem habentes sunt secundum hanc definitionem curvae affines; sed mox hanc definitionem pluribus evolvemus. Propter istam igitur affinitatem, quae inter sectiones *ACB* et *ACD* intercedit alteram quaestionem etiam non attigimus, qua quaeri posset resistentia eiusmodi solidorum, quae ex datis curvis *BCD* et *ACD* ita generantur ut omnes sectiones *SPT* sectioni *ACD* parallelae ipsi simul sint similes. Hinc etiam in sequentibus, ubi omnes sectiones horizontales inter se similes ponuntur alterutram curvarum *BCD* et *ACD* pro data assumere sufficiet, cum pari modo altera alteri sit affinis. Hoc igitur pacto numerus problematum pertractandorum, si quidem perfectam enumerationem facere volemus, ad sui medietatem diminuitur.

EXEMPLUM 1

721. Ponamus omnes sectiones verticales *FGH* sectioni diametrali *ACD* parallelas esse quadrantes circuli centris *G* descriptos, seu solidum *ABDb* generatum conversione figurae *BDb* circa axem immobilem *Bb* (Fig. 98). Erit ergo *ACB* quadrans circuli, ideoque $c = a,$ et $s = \surd(2ar - rr)$, unde fit

$$p = \frac{(a - r)}{\surd(2ar - rr)},$$

et

$$1 + pp = \frac{a^2}{2ar - rr}$$

atque

$$ap - rp + s = \frac{as}{\surd(2ar - rr)}.$$

His substitutis prodit resistentiae horizontalis vis

$$= \frac{2v}{a^2} \iint \frac{(a - r)^3 udrdy}{(1 + qq)\surd(2ar - rr)}.$$

Ponatur primo *u* cum *y* et *q* constans, atque integrale

$$\int \frac{(a - r)^3 dr}{\surd(2ar - rr)}$$

posito post integrationem $r = a$ fiet $= \frac{2}{3}a^3$ quare unica integratio restat, ideoque erit resistentia quaesita

$$= \frac{4v}{3}\int \frac{udy}{1+qq},$$

quod integrale ita est accipiendum, ut evanescat posito $y = 0$, tumque ponatur $u = 0$. Resistentiae autem vis verticalis, qua corpus sursum urgebitur erit

$$= \frac{2v}{a^3}\iint \frac{(a-r)^2 udrdy}{(1+qq)}$$

prior vero integratio posito y constante, facto $r = a$ dat

$$\int (a-r)^2 dr = \frac{a^3}{3}.$$

Hinc ergo provenit resistentiae vis verticalis

$$= \frac{2v}{3}\int \frac{udy}{1+qq}.$$

Denique cum sit

$$\frac{ac-(a-\ r)u\ \ +psu}{a} = a,$$

erit intervallum $AO = a$, seu punctum O, cui vis illa verticalis est applicata incidet in ipsum punctum C.

COROLLARIUM 1

722. In huiusmodi igitur corporibus, quae respectu axis Bb sunt rotunda, resistentiae vis horizontalis ad verticalem constantem habet rationem; scilicet resistentia verticalis se habebit ad horizontalem ut 1 ad 2, ita ut vis verticalis sit duplo minor, quam horizontalis.

COROLLARIUM *2*

723. Si sectio amplissima BDb quoque fuerit semicirculus ita ut corpus fiat quadrans sphaerae, ob $CB = CD = a$, erit

$$u = \sqrt{\left(a^2 - y^2\right)} \text{ et } q = -\frac{y}{\sqrt{\left(a^2 - y^2\right)}};$$

quare fiet

$$\int \frac{udy}{1+qq} = \int \frac{dy\left(a^2 - y^2\right)^{\frac{3}{2}}}{a^2} = \frac{3\pi a^2}{16},$$

ita ut resistentia horizontalis prodeat $= \dfrac{\pi a^2}{4}$ et verticalis $= \dfrac{\pi a^2}{8}$.

COROLLARIUM 3

724. Si sectio amplissima *BDb* fiat triangulum isosceles, ita ut sit

$$BC = Cb = b$$

erit

$$u = a - \frac{ay}{b}, \text{ et } q = \frac{-a}{b}.$$

Ex his fiet

$$\int \frac{udy}{1+qq} = \frac{ab}{aa+bb} \int (b-y)\, dy = \frac{ab^2}{2\left(a^2+b^2\right)},$$

quare resistentia horizontalis erit $= \dfrac{2ab^2}{3(aa+bb)}$ et verticalis $= \dfrac{ab^2}{2\left(a^2+b^2\right)}$.

COROLLARIUM 4

725. Intelligitur ex hoc casu resistentiam ceteris paribus eo fore minorem, quo maius fuerit discrimen inter latitudinem *BC* et altitudinem *CD*. Manente enim b in his formulis, resistentia fit maxima si ponatur $a = b$.

EXEMPLUM 2

726. Sint nunc omnes sectiones verticales *FGH*, quae sectioni diametrali sunt parallelae quadrantes elliptici inter se similes; eritque sectio diametralis *ACD* pariter quadrans ellipticus cuius alter semiaxis $AC = a$, alter $CD = c$, unde fiet

$$s = \frac{c}{a}\sqrt{(2ar - rr)}$$

atque

$$p = \frac{c(a-r)}{a\sqrt{(2ar-rr)}}.$$

Sit brevitatis gratia $a - r = t$, fiet

$$s=\frac{c}{a}\surd\left(a^2-t^2\right) \quad \text{et} \quad p=\frac{ct}{a\surd\left(a^2-t^2\right)}$$

atque

$$1+pp=\frac{a^4-\left(a^2-c^2\right)t^2}{a^2\left(a^2-t^2\right)}$$

porroque

$$\left(a-r\right)p+s=\frac{ac}{\surd\left(a^2-t^2\right)};$$

ex quibus fit

$$\frac{p^3udrdy}{c^2\left(1+p^2\right)+q^2(ap-rp+s)^2}=\frac{-ct^3udtdy}{\left(a^5\left(1+qq\right)-a\left(a^2-c^2\right)t^2\right)\surd\left(a^2-t^2\right)}.$$

Integretur primo haec formula ponendo y et u et q constantes, ita ut integrale evanescat posito $t=a$, quo facto fiat $t=0$; orieturque

$$\frac{cudy}{a^2-c^2}\left(-1+\frac{a^2\left(1+qq\right)}{\surd\left(a^2-c^2\right)\left(a^2q^2+c^2\right)}\text{Atang.}\frac{\surd\left(a^2-c^2\right)}{\surd\left(a^2q^2+c^2\right)}\right),$$

seu per seriem

$$\frac{cudy}{a^2q^2+c^2}\left(1-\frac{a^2\left(1+q^2\right)}{3\left(a^2q^2+c^2\right)}+\frac{a^2\left(1+q^2\right)\left(a^2-c^2\right)}{5\left(a^2q^2+c^2\right)^2}-\frac{a^2\left(1+q^2\right)\left(a^2-c^2\right)^2}{7\left(a^2q^2+c^2\right)^3}+\text{etc.}\right),$$

quae commodiorem praestat usum quam illa expressio, quippe quae si $c>a$ cessat a quadratura circuli penderet, sed ad logarithmos reducitur. Hinc itaque resistentiae vis horizontalis, quam hoc corpus sentiet, erit

$$=2c^2v\int\frac{udy}{a^2q^2+c^2}\left(1-\frac{a^2\left(1+q^2\right)}{3\left(a^2q^2+c^2\right)}+\frac{a^2\left(1+q^2\right)\left(a^2-c^2\right)}{5\left(a^2q^2+c^2\right)^2}-\frac{a^2\left(1+q^2\right)\left(a^2-c^2\right)^2}{7\left(a^2q^2+c^2\right)^3}+\text{etc.}\right),$$

integratione ita absoluta ut fiat integrale $=0$ si ponanur $y=c$, tumque poni debet $y=CB$ seu $u=0$.

EXEMPLUM 3

727. Sit nunc tam curva *ACD* quam *BCD* quadrans ellipticus, ita ut quadrantis elliptici *ACD* semiaxes sint $AC = a$ et $CD = c$; alterius vero *BCD* semiaxes $BC = b$ et $CD = c$; erit ergo primo ut ante

$$s = \frac{c}{a}\surd\left(2ar - rr\right)$$

et

$$p = \frac{c\left(a - r\right)}{a\surd\left(2ar - rr\right)}$$

seu posito $a - r = t$ erit

$$s = \frac{c}{a}\surd\left(a^2 - tt\right),\ \ p = \frac{ct}{a\surd\left(a^2 - t^2\right)},\ \ 1 + p^2 = \frac{a^4 - \left(a^2 - c^2\right)t^2}{a\left(a^2 - t^2\right)};$$

formulaque resistentiae horizontali inveniendae inserviens

$$= \frac{p^3 udrdy}{c^2\left(1 + p^2\right) + q^2\left(ap - rp + s\right)^2}$$

fiet

$$= \frac{-ct^3 udtdy}{a\left(a^4\left(1 + qq\right) - \left(a^2 - c^2\right)t^2\right)\surd\left(a^2 - t^2\right)}.$$

Cum nunc porro sit

$$u = \frac{c}{b}\surd\left(b^2 - y^2\right)$$

erit

$$q = \frac{-cy}{b\surd\left(b^2 - y^2\right)}\ \ \text{et}\ \ 1 + qq = \frac{b^4 - \left(b^2 - c^2\right)y^2}{b^2\left(bb - yy\right)},$$

atque formula illa differentialis transibit in hanc

$$\frac{-bc^2 t^3 dtdy\left(b^2 - y^2\right)^{\frac{3}{2}}}{a\left(a^4 b^4 - a^4\left(b^2 - c^2\right)y^2 - b^4\left(a^2 - c^2\right)t^2 + b^2\left(a^2 - c^2\right)t^2 y^2\right)\surd\left(a^2 - t^2\right)}$$

cuius integrale posito *t* constanti reperitur

$$=\frac{\pi b^2c^2t^3dt}{4a\sqrt{\left(a^2-t^2\right)}}\left(\frac{2a^6c^3-b\left(3a^4c^2-a^4b^2+b^2\left(a^2-c^2\right)t^2\right)\sqrt{\left(a^4-\left(a^2-c^2\right)t^2\right)}}{\left(a^4\left(bb-cc\right)-b^2\left(a^2-c^2\right)t^2\right)^2\sqrt{\left(a^4-\left(a^2-c^2\right)t^2\right)}}\right)$$

quae formula denuo integrata positoque post integrationem $t=a$, si multiplicetur per $2cv$ dabit vim resistentiae horizontalem qua motus retardabitur. Sed cum parum ad utilitatem hinc concludi queat, per methodum maximorum et minorum naturam curvae BCD definiamus, cui minima resistentia respondeat.

PROPOSITIO 69

PROBLEMA

728. Si data sit sectio diametralis ACD (Fig. 98) cui omnes sectiones parallelae sunt similes, determinare naturam curvae BCD, quae solidum generet quod in directione CAL motum pro sua capacitate patiatur minimam resistentiam.

SOLUTIO

Manentibus ut ante $AR=r$, et $RS=s$ ob curvam ACD datam dabitur s et etiam p posito $ds=pdr$ per r. Pro curva autem invenienda sit $CG=y$ et $GH=u$, et $du=qdy$, quibus positis minimum esse debebit haec expressio

$$\iint\frac{p^3udrdy}{c^2\left(1+p^2\right)+q^2\left(ap-rp+s\right)^2}$$

seu

$$\int udy\int\frac{p^3dr}{c^2\left(1+p^2\right)+q^2\left(ap-rp+s\right)^2}.$$

Ponatur brevitatis gratia

$$\frac{1+p^2}{p^3}=w^2 \quad\text{et}\quad \frac{\left(ap-rp+s\right)^2}{p^3}=t^2,$$

ita ut quantitates t et w ab y non pendeant; eritque formula minima reddenda haec

$$\int udy\int\frac{dr}{c^2w^2+t^2q^2},$$

in qua cum dy multiplicatum sit per

$$u\int\frac{dr}{c^2w^2+t^2q^2}$$

sumatur eius differentiale ponendo semper r et w et t constantes, quod erit

$$du\int\frac{dr}{c^2w^2+t^2q^2}-\int\frac{2ut^2qdqdr}{\left(c^2w^2+t^2q^2\right)^2}$$

ubi signa summatoria tantum ad quantitates $r,\ w,$ et t tanquam variabiles respicit, u vero et q ponit constantes. Hinc igitur valor minimo inveniendo inserviens erit

$$\int\frac{dr}{c^2w^2+t^2q^2}+\frac{1}{dy}d\cdot 2uq\int\frac{t^2dr}{\left(c^2w^2+t^2q^2\right)^2}$$

qui deberet poni $=0$ nisi simul capacitas esset in computum ducenda quae maxima esse debet. At capacitas est ut $\int u^2dy,$ ex qua obtinetur iste valor maximo inveniendo inserviens 2u. Ex his igitur valoribus sequens conficitur aequatio naturam curvae quaesitae praebens

$$\frac{2u}{cf}=\int\frac{dr}{c^2w^2+t^2q^2}+\frac{1}{dy}d\cdot 2uq\int\frac{t^2dr}{\left(c^2w^2+t^2q^2\right)^2}$$

Multiplicetur utrinque per $du=qdy$, prodibit

$$\frac{2udu}{cf}=du\int\frac{dr}{c^2w^2+t^2q^2}+qd\cdot 2uq\int\frac{t^2dr}{\left(c^2w^2+t^2q^2\right)^2}$$
$$=d\cdot u\int\frac{dr}{c^2w^2+t^2q^2}+\int\frac{2ut^2qdqdr}{\left(c^2w^2+t^2q^2\right)^2}+qd\cdot 2uq\int\frac{t^2dr}{\left(c^2w^2+t^2q^2\right)^2}$$

cuius integrale est

$$\frac{u^2}{cf}=\int\frac{udr}{c^2w^2+t^2q^2}+\int\frac{2ut^2q^2dr}{\left(c^2w^2+t^2q^2\right)^2}+\text{Const.}=\int\frac{u\left(c^2w^2+3t^2q^2\right)dr}{\left(c^2w^2+t^2q^2\right)^2}+\text{Const.}$$

Quoniam vero alicubi fieri debet $u=0$, hoc autem nusquam evenire potest nisi sit $\text{Const.}=0$, erit

$$u=cf\int\frac{\left(c^2w^2+3t^2q^2\right)dr}{\left(c^2w^2+t^2q^2\right)^2}.$$

Verum cum sit $du = qdy$, erit

$$y = \frac{u}{q} + \int \frac{udq}{qq};$$

est autem

$$\int \frac{udq}{qq} = cf \iint \frac{\left(c^2w^2 + 3t^2q^2\right)drdq}{qq\left(c^2w^2 + t^2q^2\right)^2} = h - cf \int \frac{dr}{q\left(c^2w^2 + t^2q^2\right)},$$

unde fit

$$y = h + 2cf \int \frac{t^2qdr}{\left(c^2w^2 + t^2q^2\right)^2}.$$

Quamobrem restitutis loco w^2 et t^2 assumtis valoribus ista emerget curvae quaesitae constructio:

$$y = h + 2f \int \frac{p^3\left(ap - rp + s\right)^2 qdr}{\left(c^2\left(1 + p^2\right) + q^2\left(ap - rp + s\right)^2\right)^2}$$

et

$$u = cf \int \frac{\left(c^2\left(1 + p^2\right) + 3q^2\left(ap - rp + s\right)^2\right)^2 p^3dr}{\left(c^2\left(1 + p^2\right) + q^2\left(ap - rp + s\right)^2\right)^2}$$

quae integrationes constructionem non impediunt, cum in iis q ponatur constans, ideoque non impediunt, quominus curva quaesita sit algebraica. Q. E. I.

COROLLARIUM 1

729. Quoniam u evanescit si sit $q = \infty$ intelligitur curvae BD tangentem in B ad rectam CB esse normalem, seu verticalem hoc autem casu prodit $y = h$: quare si dicatur $CB = b$, erit $h = b$.

COROLLARIUM 2

730. Quia curva ex D progrediendo versus B ad CB accedit, habebit q ubique valorem negativum. Ex quo erit $y = 0$ si fuerit

$$b=-2\,cf\int\frac{p^3\left(ap-rp+s\right)^2 qdr}{\left(c^2\left(1+p^2\right)+q^2\left(ap-rp+s\right)^2\right)^2}.$$

COROLLARIUM 3

731. At u obtinebit maximum valorem si ipsi q attribuatur valor ut fiat

$$0=\int\frac{p^3\left(c^2\left(1+p^2\right)-3q^2\left(ap-rp+s\right)^2\right)\left(ap-rp+s\right)^2 dr}{\left(c^2\left(1+p^2\right)+q^2\left(ap-rp+s\right)^2\right)^3}$$

integratione debito modo absoluta; scilicet ut evanescat facto $r=0$, tumque ponatur $r=a$.

EXEMPLUM

732. Sit sectio diametralis ACD triangulum ad O rectangulum, seu ASD linea recta, erit $s=\frac{cr}{a}$, et $p=\frac{c}{a}$, atque

$$1+pp=\frac{aa+cc}{aa},$$

itemque $ap-rp+s=c$; his substitutis erit

$$\int\frac{p^3\left(ap-rp+s\right)^2 dr}{\left(cc\left(1+pp\right)+q^2\left(ap-rp+s\right)^2\right)^2}=\int\frac{acqdr}{\left(a^2+c^2+a^2q^2\right)^2}=\frac{a^2cq}{\left(a^2+c^2+a^2q^2\right)^2};$$

atque

$$\int\frac{p^3\left(c^2\left(1+p^2\right)+3q^2\left(ap-rp+s\right)^2\right)dr}{\left(c^2\left(1+pp\right)+qq\left(ap-rp+s\right)^2\right)^2}=\int\frac{c\left(a^2+c^2+a^2q^2\right)dr}{a\left(a^2+c^2+a^2q^2\right)^2}=\frac{c\left(a^2+c^2+3a^2q^2\right)}{\left(a^2+c^2+a^2q^2\right)^2}.$$

Quocirca pro curva BCD quae solidum producit quod pro maxima capacitate minimam patitur resistentiam ista obtinebitur aequatio

$$y=b+\frac{2a^2c^2fq}{\left(a^2+c^2+a^2q^2\right)^2}$$

cui respondet

$$u=\frac{ccf\left(a^2+c^2+3a^2q^2\right)}{\left(a^2+c^2+a^2q^2\right)^2}.$$

Habebit ergo u maximum valorem si capiatur

$$q=\pm\frac{\sqrt{\left(a^2+c^2\right)}}{a\sqrt{3}}.$$

Si igitur maximus ipsius u valor ponatur $CD=c$, fiet

$$f=\frac{s\left(a^2+c^2\right)}{9c};$$

deinde quia hoc casu y evanescere debet fiet

$$b=\frac{-ac}{\sqrt{\left(a^2+c^2\right)}};$$

ex quibus natura et figura curvae desideratae facile cognoscitur. Simul autem intelligitur hanc curvam fore algebraicam.

PROPOSITIO 70

PROBLEMA

733. Si datae fuerint cum sectio amplissima *BDC* tum sectio aquae *ACB* (Fig. 99), atque huic sectioni aquae omnes sectiones horizontales *FIH* sint similes, determinare resistentiam, quam hoc corpus secundum directionem *CAL* in aqua motum patietur.

SOLUTIO

Quoniam curva AVB est data, ponatur pro ea abscissa $CT = t$ et applicata $TV = u$, dabiturque aequatio inter u et t, atque posito $du = qdt$, erit q functio quaedam ipsius t. Porro pro curva DHB ponatur abscissa $CG = r$, et applicata $GH = z$, quoniam haec applicata GH aequalis erit tertiae variabilium trium x, y, z, quae in aequationem pro superficie ingredientur; sit autem $dz = pdr$, ita ut p futura sit functio ipsius r. Si nunc constantes quantitates

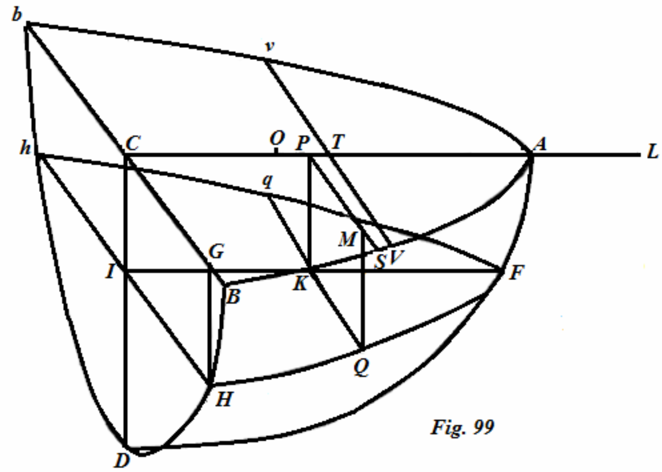


Fig. 99

vocentur $AC = a$, $CB = Cb = b$ et $CD = c$, erunt CB, b et $HI = r$ latera homologa figuram similium ACB et FIH ; quare si capiatur
$b : r = t : IK$ ut sit
Dictis vero $AP = x$, $PM = y$, et $MQ = z$, erit

$$x = a - \frac{rt}{b}, \quad y = \frac{ru}{b}, \quad \text{et } z = z,$$

et quarum priores aequationes dant

$$dx = \frac{-rdt - tdr}{b} \quad \text{et} \quad dy = \frac{rq + udr}{b},$$

ex quibus fit

$$dr = \frac{bdy + bqdx}{u - tq} \quad \text{et} \quad dt = \frac{-budx - btdy}{r(u - q)}.$$

Iam cum sit $dz = pdr$, erit

$$dz = \frac{bpqdx + bpdy}{u - tq},$$

quae aequatio exprimit naturam superficiei propositae. Haec igitur aequatio cum assumta generali $dz = Pdx + Qdy$ comparata dat

$$P = \frac{bpq}{u - tq} \quad \text{et} \quad Q = \frac{bp}{u - tq},$$

unde fit

$$1 + P^2 + Q^2 = \frac{b^2 p^2 \left(1 + q^2\right) + \left(u - tq\right)^2}{\left(u - tq\right)^2}.$$

Ad valorem iam ipsius

$$\frac{P^3 dxdy}{1+P^2+Q^2}$$

inveniendum, notari debet, dum *dy* consideratur, *dx* tanquam constans tractari debere; facto autem $dx=0$ fit

$$dr=\frac{-rdt}{t},$$

adeoque

$$dy=\frac{-rudt}{bt}+\frac{rqdt}{b}=\frac{-rdt(u-tq)}{bt};$$

atque dum *dx* consideratur, *dy* constans est ponendum seu

$$dt=\frac{-udr}{rq}$$

unde fit

$$dx=\frac{+udr}{bq}-\frac{tdr}{b}=\frac{dr(u-\ tq)}{bq}.$$

Sed cum hinc non pateat quomodo variabiles *r* et *t* a se invicem discerni debeant, oportebit loco alterutrius elementorum *dx* et *dy* inducere tertium elementum *dz*, cum id in assumtis quantitatibus variabilibus ipsum contineatur. Est autem

$$dxdy=\frac{dzdx}{Q}=\frac{dzdy}{P};$$

nam dum *x* tanquam constans consideratur loco *dy* scribi potest $\frac{dz}{Q}$, et dum *y* constans assumitur loco *dx* scribere licet $\frac{dz}{P}$ ex quibus hanc nanciscimur formulam $\frac{p^2 dzdy}{1+P^2+Q^2}$, quae bis integrari debet, altera integratione ponendo *z* altera *y* constans. At est $dz = pdr$, et si *z* constans ponitur fit

$$dy=\frac{rqdt}{b};$$

quamobrem fiet formula generalis $\frac{P^2 dxdy}{1+P^2+Q^2}$ pro nostro casu

$$\frac{bp^3q^3rdrdt}{b^2p^2\left(1+q^2\right)+\left(u-tq\right)^2},$$

quae bis integrari debet altera vice ponendo r, altera t constans. Ac primo quidem utraque integratio ita est instituenda ut integrale evanescat, posito vel r vel t, prout vel r vel t pro variabili est sumta $=0$, tumque faciendum est vel $r=b$ vel $t=a$. His igitur de modo integrationum praemonitis si altitudo celeritati, qua corpus progreditur debita ponatur $=v$, erit resistentiae vis horizontalis repellens corpus secundum directionem

$$AC = 2bv\iint\frac{p^3q^3rdrdt}{b^2p^2\left(1+q^2\right)+\left(u-tq\right)^2}.$$

Deinde vis verticalis ex resistentia orta, quae est

$$\iint\frac{P^2dxdy}{1+P^2+Q^2} \ = 2v\iint\frac{Pdxdy}{1+P^2+Q^2}$$

fiet pro nostro casu

$$=2v\iint\frac{p^2q^2rdrdt\left(u-tq\right)}{b^2p^2\left(1+q^2\right)+\left(u-tq^2\right)^2}$$

quae applicata erit in puncto O axis AC, cuius distantia a puncto A reperietur si dividatur

$$\iint\frac{\left(ab\left(u-tq\right)-rt\left(u-tq\right)+b^2pqz\right)p^2q^2rdrdt}{b^2p^2\left(1+q^2\right)+\left(u-tq\right)^2}$$

praescripto modo evolutum per

$$\iint\frac{\left(u-tq\right)p^2q^2rdrdt}{b^2p^2\left(1+q^2\right)+\left(u-tq\right)^2}$$

His que cognitis totius resistentiae effectus cognoscetur. Q. E. I.

COROLLARIUM 1

734. Figura sectionis diametralis *AFD* hinc facillime ex curva CBD definitur. Nam quoniam est $BC:HI = AC:\ FI$ applicatae *FI* et *HI* eidem abscissae *CI* respondentes datam inter se tenent rationem; ex quo curva *AFD* affinis erit curvae *BHD*.

COROLLARIUM 2

735. Hanc obrem problema, quo loco curvae *BHD* data fuisset curva *AFD*, sectiones vero omnes horizontales inter se sint similes, ut in praesente quaestione, simili modo resolvetur, atque adeo solutio ab hac non differet nisi scribendo a loco b siquidem r et z denotent coordinatas curvae *DFH*.

COROLLARIUM 3

736. Cum soliditas in genere sit $=-2\iint Qydxdy=-2\iint Qdxdz$ posito $\frac{dxdz}{Q}$ loco $dxdy$, fiet pro nostro casu soliditas $=\frac{2}{bb}\iint pr^2udrdt=\frac{2}{bb}\int pr^2dr\int udt.$
Cum igitur $\int udt$ exprimat aream *ACB*, dicatur ea $=ff$, erit soliditas $=\frac{2ff}{bb}\int pr^2dr=\frac{2ff}{bb}\int r^2dz$ posito $r=b$ post integrationem ita absolutam ut prodeat 0, si fiat $r=0$.

COROLLARIUM 4

737. Superficies autem *ABDb* in aquam incurrens generaliter est

$$2\iint dxdy\sqrt{\left(1+P^2+Q^3\right)}=2\iint\frac{dxdz}{Q}\sqrt{\left(1+P^2+Q^2\right)}.$$

Quamobrem nostro casu haec superficies exprimetur hac formula

$$2\iint\frac{rdrdt}{bb}\sqrt{\left(b^2p^2\left(1+q^2\right)+\left(u-tq\right)^2\right)}.$$

COROLLARIUM 5

738. Colligere etiam licet, quoties curvae *CBD* et *CAD* fuerint affines, toties corporis omnes sectiones horizontales esse inter se similes. Cum igitur sectiones verticales sectioni *CBD* parallelae similes sint, quando curvae *CBA* et *CDA* fuerint affines, intelligitur si tres curvae *CBD, CAD* et *CAB* fuerint inter se affines, tum omnes sectiones unicuique illarum sectionum parallelas inter se similes fore.

SCHOLION

739. Quo appareat, quomodo formulae differentiales supra datae in quibus *dxdy* inest, ad alias reduci queant in quibus vel *dxdz* vel *dydz* insit, notandum est *dxdy* ideo esse in illas formulas ingressum, quod inerat in elemento superficiei $dxdy\sqrt{\left(1+P^2+Q^2\right)}$. Quoniam autem hoc elementum natum est ex aequatione canonica $dz=Pdx+Qdy$ simili modo ex ista aequatione canonica

$$dy=\frac{dz}{Q}-\frac{Pdx}{Q}$$

nascetur hoc superficiei elementum

$$\frac{dxdz}{Q}\sqrt{\left(1+P^2+Q^2\right)},$$

atque ex hac aequatione

$$dx=\frac{dz}{P}-\frac{Qdy}{P}$$

prodit elementum superficiei istud

$$\frac{dydz}{P}\sqrt{\left(1+P^2+Q^2\right)}.$$

Cum igitur haec tria elementa bis integrata praebeant totam superficiem, manifestum est ea sibi mutuo substitui posse. Hancobrem formulae pro resistentia supra inventae in alias formas aequivalentes reduci possunt, quibus illarum loco uti licebit. Ita resistentiae vis horizontalis quae supra inventa erat

$$=2v\iint\frac{P^3dydx}{1+P^2+Q^2}$$

quoque hoc modo

$$2v\iint\frac{P^2dydx}{1+P^2+Q^2}$$

sive hoc modo

$$2v\iint\frac{P^3dxdz}{Q\left(1+P^2+Q^2\right)}$$

exprimi poterit. Simili modo vis resistentiae verticalis tribus hisce diversis modis exprimi potest; erit scilicet vel

$$2v\iint\frac{P^2dxdy}{1+P^2+Q^2}$$

vel

$$2v\iint\frac{Pdydz}{1+P^2+Q^2}$$

vel

$$2v\iint\frac{P^2dxdz}{Q\left(1+P^2+Q^2\right)};$$

ex quibus formulis quovis casu oblato iis uti conveniet, quae pro ratione quantitatum variabilium ita sunt comparatae, ut alterutra variabilium in formula contentarum ab unica variabilium assumtarum pendeat. Ita in hoc casu opus erat eiusmodi formulis uti in quibus inesset dz, quia z inter ipsas variabiles assumtas reperiebatur.

EXEMPLUM

740. Sint omnes sectiones horizontales HFh semicirculi, seu solidum genitum ex rotatione figurae CBD circa axem CD, erit figura CBA quadrans circuli et propterea $b=a$, atque ex circuli natura $u=\sqrt{\left(a^2-t^2\right)}$ et

$$q=\frac{-t}{\sqrt{\left(a^2-t^2\right)}}$$

atque

$$u-tq=\frac{a^2}{\sqrt{\left(a^2-t^2\right)}},\quad \text{et}\quad 1+qq=\frac{a^2}{a^2-t^2}.$$

His substitutis erit resistantiae vis horizontalis in directione AC motum retardans

$$=\frac{2v}{a^3}\iint\frac{p^2t^3rdrdt}{(1+pp)\sqrt{\left(a^2-t^2\right)}}=\frac{2v}{a^2}\int\frac{t^3dt}{\sqrt{\left(a^2-t^2\right)}}\int\frac{p^3rdr}{1+pp}$$

ubi variabiles t et r a se invicem sunt separatae. Signum quidem haec formula haberet negativum, sed eius loco + tuto substituitur cum transformatio formularum generalis pendeat a signo radicali, in quod utrumque signum aequaliter competit. At est

$$\int\frac{t^3dt}{\sqrt{\left(a^2-t^2\right)}}$$

posito post integrationem $t=a$, unde vis resistentiae horizontalis est

$$=\frac{4}{3}v\int\frac{p^3rdr}{1+pp}.$$

Simili modo erit vis resistentiae verticalis

$$=\frac{2v}{a^2}\iint\frac{p^2t^2rdrdt}{(1+pp)\sqrt{\left(a^2-t^2\right)}}=\frac{2v}{a^3}\int\frac{t^2dt}{\sqrt{\left(a^2-t^2\right)}}\int\frac{p^2rdr}{1+pp}=\frac{\pi v}{2}\int\frac{p^2rdr}{1+pp}.$$

quia est

$$\int \frac{ttdt}{\sqrt{\left(a^2-t^2\right)}}=\frac{\pi}{4}.$$

De loco autem applicationis *O*, quia formula minus fit simplex non erimus solliciti.

COROLLARIUM 1

741. Si idem hoc solidum invertatur ut *BDb* fiat sectio aquae et *BAb* sectio amplissima, atque hoc solidum in directione *CD* celeritate altitudini *v* debita promoveatur tum resistentia motum retardans erit

$$=\pi v \int \frac{rdr}{1+p^2}.$$

Hoc enim casu omnes sectiones verticales axi *CD* normales erunt semi circuli.

COROLLARIUM 2

742. Resistentia ergo huius corporis, si movetur secundum directionem *CA* se habebit ad resistentiam eiusdem corporis moti in directione *CD*

$$\text{ut } \frac{4}{3}\int \frac{p^3 rdr}{1+pp} \text{ ad } \pi \int \frac{rdr}{1+pp}.$$

COROLLARIUM 3

743. Si ergo figura *BDb* abeat in triangulum isosceles, seu corpus in semiconum rectum axis *CD* atque ponatur $CD=c$ existente $BC=AC=a$, erit

$$z=c-\frac{cr}{a} \text{ et } p=-\frac{c}{a}.$$

Resistentia ergo quam hic conus in directione *CA* motus patietur erit

$$\frac{c^3 rdr}{a^2+c^2}=-\frac{4v}{3a}\int \frac{c^3 rdr}{a^2+c^2}=\frac{2ac^3 v}{3\left(a^2+c^2\right)},$$

neglecto signo ut iam notavi.

COROLLARIUM 4

744. Resistentia autem, quam idem semiconus in directione axis *CD* motus sufferet, erit

$$=\pi v\int\frac{a^2rdr}{a^2+c^2}=\frac{\pi a^4v}{2\left(a^2+c^2\right)}.$$

Quare haec resistentia se habet ad priorem ut $\frac{\pi a^3}{2}$ ad $\frac{\pi c^3}{3}$. Unde hae duae resistentiae inter se erunt aequales si fuerit

$$c^3=\frac{3\pi a^3}{4},\text{ seu }\frac{c}{a}=\sqrt[3]{\frac{3\pi}{4}},$$

sive si sit

$$CD:CB=\sqrt[3]{6\pi}:2=2,661341:2,$$

unde fit proxime $CD:CB=4:3$.

COROLLARIUM 5

745. Si ponatur sectio *BDb* etiam semicirculus, ita ut corpus fiat quadrans sphaerae, utraque resistentia debebit esse eadem. Oritur autem ob

$$z=\sqrt{\left(a^2-r^2\right)}\quad\text{et}\quad p=\frac{-r}{\sqrt{\left(a^2-r^2\right)}},$$

resistentia pro motu secundum

$$CA=\frac{4}{3}v\int\frac{r^4dr}{a^2\sqrt{\left(a^2-r^2\right)}}=\frac{2\pi a^2v}{3}\frac{1}{2}\cdot\frac{3}{4}=\frac{\pi a^2v}{4}.$$

Pro motu autem secundum directionem *CD* erit resistentia

$$=\pi v\int\frac{rdr\left(a^2-r^2\right)}{a^2}=\frac{\pi a^2v}{4}.$$

SCHOLION

746. Absolvimus igitur his propositionibus omnes casus quibus corporis sectiones inter se parallelae vel horizontales vel verticales eaeque vel sectioni diametrali vel amplissimae parallelae sunt similes inter se. Atque ad huiusmodi corpora determinanda opus fuit trium sectionum principalium scilicet sectionis aquae, sectionis amplissimae atque sectionis diametralis duas tanquam datas assumere, quia ex hac conditione tertia sectio sponte determinatur. Dantur autem praeter has corporum species, quae sectiones quasdam inter se parallelas similes habent, innumerabiles aliae corporum species, quibus

evolvendis nec locus nec tempus suppeteret. Harum vero primarias aliquas species examini subiicere iuvabit, quae ad navium figuras prope accedant. Eiusmodi scilicet species contemplabimur, in quibus sectiones inter se parallelae vel horizontales vel verticales sint affines, cuius vocis significationem hic in multo latiore sensu accipimus quam vocem similitudinis. Figuras enim affines vocamus, in quibus sumtis abscissis in data ratione, applicatae respondentes quoque constantem teneant rationem, ex qua definitione intelligitur figuras similes sub affinibus tanquam speciem contineri, figurae enim affines evadunt similes, si applicatae eandem rationem teneant, quam abscissae; affines autem et non similes prodeunt figurae, si rationes abscissarum et applicatarum fuerint inaequales. Sic omnes ellipses inter se sunt figurae affines, quoniam abscissis in ratione axium transversorum assumtis respondent applicatae rationem axium coniugatorum tenentes si quidem abscissae in axibus transversis capiantur.
Simili quoque modo omnia triangula rectangula figurae sunt inter se affines. Data igitur curva quacunque datam basin datamque altitudinem habente, facile erit aliam ipsi affinem describere, quae basin quamcunque et altitudinem quamcumque praescriptam habeat. Nam si datae curvae basis sit $=a$ et altitudo $=b$, abscissaque quaecunque in basi accepta vocetur x, eique respondens applicata altitudini parallela sit y, hoc modo super basi alia A ad aliam altitudinem B construetur curva affinis, in basi A sumatur abscissa $=\frac{Ax}{a}$, atque respondens applicata $=\frac{By}{b}$, eritque curva hoc modo descripta priori affinis. His igitur notatis non difficile erit sequentia problemata aggredi.

PROPOSITIO 71

PROBLEMA

747. *Sint omnes tres principales sectiones datae, scilicet sectio aquae ACB, sectio amplissima BCD atque sectio diametralis ACD (Fig. 100); solidum vero ita sit comparatum, ut*

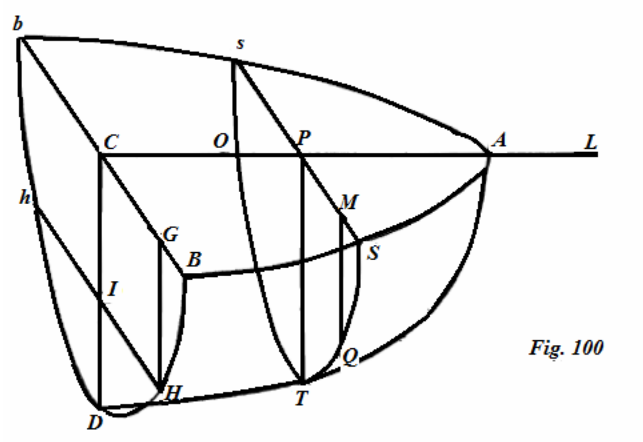


Fig. 100

omnes sectiones STP sectioni amplissimae BDC parallelae eidem sint affines, hocque corpus in aqua moveatur secundum directionem CAL: determinare resistentiam, quam patietur.

SOLUTIO

Cum primum sectio diametralis *ATD* data sit, ponatur pro ea abscissa $CP = r$, et applicata $PT = s$, dabiturque relatio inter r et s sitque $ds = pdr$. Secundo ob curvam *CBA* seu sectionem aquae datam, ponatur pro ea abscissa $CP = t$, et applicata $PS = u$, sitque $du = qdt$. Tertio pro sectione amplissima *CBD* sit abscissa $CG = \tau$ et applicata $GH = \gamma$ atque $d\gamma = \rho d\tau$. His pro curvis datis positis concipiatur sectio quaecunque *SPT* sectioni *BCD* parallela, quae ex natura solidi affinis erit ipsi sectioni *BCD*; atque ad solidi indolem exprimendam accipiantur hae tres variabiles $AP = x$, $PM = y$ et $MQ = z$ eritque prioribus notationibus ad hunc casum accommodatis $t = r$, atque $x = a - r$ posita longitudine $AC = a$. Iam cum sectionis *SPT* basis sit $PS = u$ et altitudo $PT = s$; sectionis vero *BCD* ponatur basis $BC = b$ et altitudo $CD = c$; hinc ob affinitatem si sit $PM = y = \frac{u\tau}{b}$, erit $MQ = z = \frac{s\gamma}{c}$. Nunc ob $x = a - r$ erit $dr = -dx$; atque

$$dy = \frac{ud\tau + \tau qdr}{b}$$

propter $t = r$ et

$$dz = \frac{s\rho d\tau + \gamma pdr}{c}.$$

Cum igitur sit

$$d\tau = \frac{bdy}{u} + \frac{\tau qdx}{u}$$

ob $dr = -dx$ fiet

$$dz = \frac{(s\tau q\rho - u\gamma p)dx}{cu} + \frac{bs\rho dy}{cu},$$

quae aequatio cum generali supra assumta $dz = Pdx + Qdy$ comparata dat

$$P = \frac{s\tau q\rho - u\gamma p}{cu} \quad \text{et} \quad Q = \frac{bs\rho}{cu},$$

unde fit

$$1 + P^2 + Q^2 = \frac{c^2u^2 + b^2s^2\rho^2 + (s\tau q\rho - u\gamma p)^2}{c^2u^2}.$$

Accedamus nunc ad formulas

$$\frac{P^3dxdy}{1+P^2+Q^2},\quad \frac{P^2dxdy}{1+P^2+Q^2}\quad \text{et}\quad \frac{P^2\left(x+Pz\right)dzdy}{1+P^2+Q^2},$$

pro viribus resistentiae et directione determinandis inventas, quae duplicem integrationem requirunt, alteram in qua *x* alteram in qua *y* ponatur constans.

Cum igitur sit $dx=-dr$, atque posito x constante fiat $dy=\frac{ud\tau}{b}$; hi valores substituantur loco *dx* et *dy*, ut fiat

$$dxdy=-\frac{udrd\tau}{b};$$

ac si in illis formulis integratio instituatur posito *r* constante, simul constantes erunt quantitates ab *r* pendentes velut *s, t, u, p, q*, in altera vero integratione in qua *r* ponitur constans, constantes insuper erunt γ et ρ. Integratio autem in qua *r* ponitur constans ita absolvatur ut integrale evanescat posito $\tau=0$, tumque ponatur $\tau=b$ seu $\gamma=0$; simili modo altera integratio in qua τ constans ponitur est absolvenda, ut integrale evanescat posito $\mathrm{r}=0$, hocque facto ponatur $r=a$. Perinde autem est a quanam integratione incipiatur, cum variabiles r et τ, reliquaeque, quae per has duas dantur, a se invicem non pendeant. His igitur praemissis obtinebitur resistentiae vis horizontalis motui contraria et secundum directionem *AC* urgens

$$=\frac{-2v}{bc}\iint\frac{\left(s\tau q\rho-u\gamma p\right)^3 drd\tau}{c^2u^2+b^2s^2\rho^2+\left(s\tau q\rho-u\gamma p\right)^2};$$

vis vero resistentiae verticalis, qua corpus sursum urgetur erit

$$=\frac{-2v}{b}\iint\frac{\left(s\tau q\rho-u\gamma p\right)^2 udrd\tau}{c^2u^2+b^2s^2\rho^2+\left(s\tau q\rho-u\gamma p\right)^2}.$$

Punctum autem *O*, in quo haec vis applicata est concipienda reperietur dividendo hanc expressionem

$$\iint\frac{P^2\left(x+Pz\right)udrd\tau}{1+P^2+Q^2}$$

per istam

$$\iint\frac{P^2udrd\tau}{1+P^2+Q^2},$$

quotus enim dabit intervallum *AO*. Q. E. I.

COROLLARIUM 1

748. Cum soliditas in genere sit $=-2\iint Qydxdy$ pro nostro casu autem sit

$$-dxdy=\frac{udrd\tau}{b},\ \ y=\frac{u\tau}{b}\ \text{ et }\ Q=\frac{bs\rho}{cu},$$

erit nostri solidi capacitas

$$=\frac{2}{bc}\iint us\tau\rho drd\tau=\frac{2}{bc}\int usdr\int\tau\rho d\tau.$$

Est vero

$$\int\tau\rho d\tau=\int\tau d\gamma=-\text{area } BCD;$$

haec ergo area si dicatur ff erit soliditas

$$=\frac{-2ff}{bc}\int usdr.$$

COROLLARIUM 2

749. Si sectio diametralis ACD affinis sit sectioni aquae, tum omnes sectiones ipsi BCD parallelae simul erunt similes. Tum autem fit $s:u=c:b$
atque

$$u=\frac{bs}{c}\text{ et }q=\frac{bp}{c},$$

quibus valoribus substitutis prodeunt supra inventae expressiones pro sectionibus similibus.

COROLLARIUM 3

750. Si linea DTA abeat in rectam horizontalem fiet $s=c$ et $p=0$, hinc resistentiae vis horizontalis fiet

$$=\frac{-2v}{b}\iint\frac{\tau^3q^3\rho^3drd\tau}{u^2+b^2\rho^2+\tau^2q^2\rho^2}.$$

Atque si area ACB ponatur $=gg$, existente area $BCD=ff$ erit soliditas huius corporis $=\dfrac{2ffgg}{b}$.

PROPOSITIO 72

PROBLEMA

751. *Sint iterum datae tres sectiones principales ACB, ACD et BCD (Fig. 98), atque omnes sectiones FGH sectioni diametrali ACD parallelae eidem sectioni sint affines: hocque corpus moveatur in aqua secundum directionem CAL; determinare resistentiam quam patietur.*

SOLUTIO

Sit iterum ut ante pro sectione diametrali *ACD* abscissa $CR = r$ et applicata $RS = s$ atque $ds = pdr$. Deinde pro sectione aquae *CBA* sit abscissa in *AC* sumta ipsique *GF* aequalis $= t$ et ei applicata respondens, quae aequalis erit *CG* sit u atque $du = qdt$. Pro tertia denique sectione *BCD* sit abscissa $CG = -\tau$ et applicata $GH = \gamma$ existente $d\gamma = \rho d\tau$. Si nunc concipiatur sectio verticalis *FGH* parallela sectioni diametrali *ACB* fiet $u = \tau$, et $d\tau = qdt$, unde $\tau,\ q,\ u,\ \gamma$ et ρ functiones erunt ipsius t ab eoque pendebunt, eritque $d\gamma = q\rho dt$. Positis ergo $AC = a,\ BC = b$ et $CD = c,$ quoniam figura *FGH* affinis est figurae *ACD* sumatur in ea abscissa

$$GM = \frac{tr}{a},$$

eritque applicata

$$MQ = \frac{\gamma s}{c}.$$

Quamobrem si vocentur $AP = x,\ PM = y$ at $MQ = z,$ erit

$$x = a - \frac{tr}{a}\ ,\ y = \tau = u \text{ et } z = \frac{\gamma s}{c}.$$

Cum igitur sit

$$dy = qdt,\ \text{seu } dt = \frac{dy}{q},$$

erit

$$dx = \frac{-ry}{aq} - \frac{tdr}{a} \text{ et } dz = \frac{\gamma pdr}{c} + \frac{sq\rho dt}{c} = \frac{\gamma pdr}{c} + \frac{s\rho dt}{c},$$

unde fit

$$dz = \frac{-a\gamma pdx}{ct} + \frac{(tsq\rho - r\gamma p)dy}{ctq};$$

quae cum generali aequatione $dz = Pdx + Qdy$ comparata dat

$$p = \frac{-a\gamma p}{ct} \quad \text{et} \quad Q = tsq\rho - \frac{r\gamma p}{ctq}$$

atque

$$1 + P^2 + Q^2 = \frac{c^2t^2q^2 + a^2\gamma^2p^2q^2 + (tsq\rho - r\gamma p)^2}{c^2t^2q^2}.$$

Quod autem ad formulas differentiales attinet in quibus est *dxdy* atque *x* et *y* a se invicem non pendere, ponuntur; cum sit $dy = qdt$, ideoque *y* a solo *t* pendet erit

$$dx = \frac{-tdr}{a}; \quad \text{ob} \quad dy = 0,$$

quando de *dx* est quaestio. Erit ergo

$$dxdy = \frac{-tqdrdt}{a},$$

atque resistentiae vis horizontalis secundum directionem *AC* sollicitans erit

$$= \frac{2a^2v}{c} \iint \frac{\gamma^3p^3q^3drdt}{c^2t^2q^2 + a^2\gamma^2p^2q^2 + (tsq\rho - r\gamma p)^2}$$

ubi duplici integratione opus est, altera in qua ponitur *t* constans, cum eoque u, γ, q et ρ in altera vero ponitur *r* constans cum suis functionibus *s* et *p*. Simili vero modo erit vis resistantiae verticalis

$$= -2av \iint \frac{\gamma^2p^2q^3tdrdt}{c^2t^2q^2 + a^2\gamma^2p^2q^2 + (tsq\rho - r\gamma p)^2}$$

cuius locus applicationis erit punctum *O*, eiusque intervallum *AO* erit quotus qui resultat ex divisione huius quantitatis

$$\iint \frac{P^2(x + Pz)tqdrdt}{1 + P^2 + Q^2}$$

per hanc

$$\iint \frac{P^2tqdrdt}{1 + P^2 + Q^2}.$$

Q.E.I.

COROLLARIUM 1

752. Soliditas huius corporis reperietur ex formula generali

$$-2\iint Qydxdy,$$

quae pro nostro casu transit in hanc

$$\frac{2}{ac}\iint\left(tsuq\rho drdt - ru\gamma\, pdrdt\right)$$

quae primo integrata posito t constanti, dat

$$\frac{2ff}{ac}\int\left(tuq\rho + u\gamma\right)dt = \frac{2ff}{ac}\int t\gamma du$$

ob $q\rho dt = d\gamma,$ denotante ff aream ACD.

COROLLARIUM 2

753. Si curvae CBA et CBD fuerint affines, hoc est $GF : GH = a : c$,
ita ut sit et

$$\gamma = \frac{ct}{a} \text{ et } q\rho = \frac{c}{a},$$

omnes sectiones FGH fierent inter se similes, atque resistentia corporis horizontalis erit

$$= 2av\iint \frac{tp^3q^3drdt}{a^2q^2\left(1+p^2\right)+\left(s-rp\right)^2}$$

uti iam ex superioribus constat.

COROLLARIUM 3

754. Si curva BD abeat in rectam ipsi BC parallelam, ita ut sectio amplissima BDb fiat rectangulum, erit $\gamma = c$ et $\rho = 0$; huiusmodi solidi ergo resistentia horizontalis seu motum retardans est

$$= 2a^2v\iint \frac{p^3q^3drdt}{a^2p^2q^2 + r^2p^2 + t^2q^2}.$$

COROLLARIUM 4

755. Quoniam in hac expressione *p* et *q*, itemque *r* et *t* aequaliter insunt, intelligitur sectiones *ACB* et *ACD* eadem manente resistentia inter se commutari posse, si quidem sectio amplissima fuerit parallelogrammum rectangulum.

COROLLARIUM 5

756. Si insuper sectiones *ACB* et *ACD* fiant triangula rectangula, quo casu solidum abit in pyramidem curvilineam cuius basis erit rectangulum, vertex vero A. Cum igitur hoc casu sit

$$u = b - \frac{bt}{a}$$

hincque

$$q = -\frac{b}{a}, \quad \text{et} \quad s = c - \frac{cr}{a}$$

hincque

$$p = -\frac{c}{a},$$

erit resistentia huius corporis

$$= \frac{2b^3c^3v}{a^2}\iint \frac{drdt}{b^2c^2 + b^2t^2 + c^2r^2} = \frac{2b^3c^3v}{a^2}\int \frac{dr}{\sqrt{(b^2+r^2)}}\text{Atang.}\ \frac{ab}{c\sqrt{(a^2+r^2)}}.$$

PROPOSITIO 73

PROBLEMA

757. *Datae sint denuo tres sectiones principales ACB, ACD et BCD* (Fig. 99)*, atque corpus ABDb ita sit comparatum, ut omnes sectiones horizontales FHI sint inter se affines: moveaturque hoc corpus secundum directionem AC in aqua: determinare resistentiam quam patietur.*

SOLUTIO

Sit prima pro sectione diametrali *ACD* abscissa in axe *CA* assumta ipsique *IF* aequalis $= r$ eique respondens applicata quae erit $= CI = GH = s$, sitque $ds = pdr$. Tum pro sectione aquae *CBA* sit abscissa $CT = t$ et applicata $TV = u$ sitque $du = qdt$. Tertia pro sectione amplissima sit abscissa $CG = \tau$: et applicata $GH = \gamma$ existente $d\gamma = \rho d\tau$. Si nunc concipiatur sectio horizontalis quaecunque FIH, superiores denominationes ad eam applicatae praebebunt $\gamma = s$ indeque $pdr = \rho d\tau$. Quoniam autem sectio *FIH* affinis est sectioni *ACB*, si ponatur $AC = a$, $BC = b$ et $CD = c$, sumaturque $IK = \frac{rt}{a}$ erit respondens applicata $KQ = \frac{\tau u}{b}$. Si nunc ponatur $AP = x$, $PM = y$ et $MQ = z$ erit

$$x = a - \frac{tr}{a}\ ,\ y = \frac{\tau u}{b} \text{ et } z = \gamma = s;$$

unde fit

$$dz = pdr,\quad dy = \frac{\tau qdt}{b} + \frac{updr}{b\rho},$$

ob

$$d\tau = \frac{pdr}{\rho},\ \text{et } dx = \frac{-rdt}{a} - \frac{tdr}{a};$$

ex quibus sequens aequatio inter *x, y* et *z* conficitur

$$dz = \frac{bpr\rho dy + a\tau pq\rho dx}{urp - t\tau q\rho}$$

quae cum generali aequatione supra assumta comparata dat

$$P = \frac{a\tau pq\rho dx}{urp - t\tau q\rho} \text{ et } Q = \frac{bpr\rho dy}{urp - t\tau q\rho}$$

ita ut sit

$$1 + P^2 + Q^2 = \frac{p^2\rho^2\left(a^2\tau^2q^2 + b^2r^2\right) + \left(urp - t\tau q\rho\right)^2}{\left(urp - t\tau q\rho\right)^2}$$

Iam quoniam *z* per unicam constitutarum variabilium determinatur, eiusmodi formulas ad resistentiam inveniendam assumere convenit in quibus sit *dz*. Cum enim sit $dz = pdr$, et posito *z* seu *r* constante fiat

$$dy = \frac{\tau qdt}{b},$$

erit

$$dzdy = \frac{\tau pqdrdt}{b},$$

in qua duae variabiles a se invicem non pendentes insunt, altera r et quantitates per eam datae $s,\ p,\ \gamma,\ \tau$ et ρ altera vero t, cum u et q, quae in integrationibus probe a se invicem sunt secernendae, ita dum alterae variabiles ponuntur, alterae tanquam constantes tractentur. Cum iam vis resistentiae horizontalis seu secundum directionem AC urgens sit

$$\iint \frac{P^3 dzdy}{1+P^2+Q^2}$$

fiet haec resistentia pro nostro casu

$$=\frac{2a^2v}{b}\iint \frac{\tau^3 p^3 q^3 \rho^3 drdt}{p^2\rho^2\left(a^2\tau^2q^2+b^2r^2\right)+\left(urp-t\tau q\rho\right)^2}$$

quae uti iam saepius est praeceptum, debito modo his debet integrari. At resistentiae vis verticalis fit

$$=\frac{2av}{b}\iint \frac{\tau^2 p^2 q^2 \rho drdt\left(urp-t\tau q\rho\right)}{p^2\rho^2\left(a^2\tau^2q^2+b^2r^2\right)+\left(urp-t\tau q\rho\right)^2}$$

locus autem seu punctum O ubi haec vis applicata est concipienda, reperietur eo modo, quem generaliter dedimus, scilicet intervallum AC est quotus, qui resultat si

$$\iint \frac{P\left(x+Pz\right)\tau pqdrdt}{1+P^2+Q^2}$$

dividatur per

$$\iint \frac{P\tau pqdrdt}{1+P^2+Q^2},$$

integrationibus utrisque legitimo modo absolutis. Q. E. I.

COROLLARIUM 1

758. Ad soliditatem huius solidi inveniendam considerari oportet hanc expressionem $2\iint ydxdz;$ pro qua applicanda quoniam est $dz = pdr$ et posito z constante

$$dx=\frac{-rdt}{a} \text{ atque } y=\frac{\tau u}{b}$$

fiet soliditas

$$=\frac{2}{ab}\iint \tau urpdrdt=\frac{2}{ab}\int udt\cdot\int \tau rpdt.$$

COROLLARIUM 2

759. Quoniam vero $\int udt$ integratum dat aream *ACB*, quae si dicatur $= ff$, erit soliditas

$$\frac{2ff}{ab}\int \tau rpdt = \frac{2ff}{ab}\int \tau rds$$

ob $pdr = ds$ seu est

$$= \frac{2ff}{ab}\int \tau rd\gamma \text{ ob } d\gamma = ds,$$

quae integratio ab utriusque curvae *CDA* et *CDB* natura pendet.

COROLLARIUM 3

760. Si fiat linea *AFD* recta verticalis erit $r = a$ et $p = \infty$, unde resistentia horizontalis, postquam in formula inventa positum erit ρdr loco pdr, prodit

$$= \frac{2v}{b}\iint \frac{\tau^3 q^3 \rho dtd\tau}{b^2 + u^2 + \tau^2 q^2}.$$

SCHOLION

761. Hisce satis prolixe resistentiam, quam corpora plano diametrali praedita in aqua directe promota patiuntur, sumus prosecuti; vix enim figura, quae quidem ad naves esset idonea concipi poterit, quae non in pertractatis corporum speciebus contineatur. Ordo igitur requireret ut etiam, uti in figuris planis fecimus ad motum obliquum considerandum progrederemur, sed cum in figuris planis haec tractatio tam difficilis extitisset, multo maiori difficultati, quando de corporibus quaestio agitatur, haec inquisitio foret obnoxia, et praeterea si quid de directione vis resistentiae per prolixissimos calculos erueretur, tamen parum utilitatis inde ad navium perfectionem consequeremur. Quamobrem his causis impediti isti capiti finem imponere cogimur, id quod sine notabili in sequentibus incommodo facere possumus, cum ea quae de figuris planis si motu obliquo promoveantur, sunt prolata, satis prope media directio resistentiae et centrum resistentiae aestimari queat.