VARIOUS METHODS OF ENQUIRING INTO THE NATURE OF SERIES

Opuscula Analytica 1, 1783, p. 48-63 [E551]

Series of this kind occur often, the origin of which is evident enough, yet the law of these and their nature especially is hidden, and is unable be investigated except by conspicuous consultations with the analytical arts. Indeed it is scarcely permitted to propose artifices of this kind in general, so that the uses of these may be seen more clearly ; but the strength of these methods is shown rather more conveniently in examples, from which likewise both the account and the need of being worked out is understood much more clearly there. Therefore here we will consider the series generally or the progression of the numbers, which is arising, if the powers of the trinomial $1+x+xx$ may be expanded out and from the individual terms only the means, which are affected by the maximum numbers, may be set out in order ; thus indeed a most noteworthy series of numbers arises there, where the law of the progression is less evident. But from that investigation the most beautiful thoughts arise, in which the matter of the great strength of the analytical methods is perceived chiefly. But in the first place this document shows an especially memorable series, to which we must turn by induction with much circumspection, and towards which many investigations of this kind are usually attributed, since here an induction of this kind may occur, which, even though it may seem to be confirmed generally, yet may lead to error.

EXPANSION OF THE POWERS OF THE TRINOMIAL

$$1+x+x^2$$
$$1+2x+3x^2+2x^3+x^4$$
$$1+3x+6x^2+7x^3+6x^4+3x^5+x^6$$
$$1+4x+10x^2+16x^3+19x^4+16x^5+10x^6+4x^7+x^8$$
$$1+5x+15x^2+30x^3+45x^4+51x^5+45x^6+30x^7+15x^8+5x^9+x^{10}$$
$$1+6x+21x^2+50x^3+90x^4+126x^5+141x^6+126x^7+90x^8+50x^9+21x^{10}+6x^{11}+x^{12}$$
etc.

I choose only the middle terms from the individual terms of this kind, which supply this progression

$$x+3x^2+7x^3+19x^4+51x^5+141x^6+\text{etc.},$$

the nature of which I have put to be investigated here, where indeed with the powers of x omitted, the matter is reduced to this numerical progression

1, 3, 7, 19, 51, 141, 393 etc.

CONSIDERATION I

1. This series requiring to be considered soon comes to mind, since any term can be prepared quite consistently from the three preceding terms, since it is clear this series must be continued to infinity by merging together by confounding three geometric progressions from its beginning. Therefore I write below the sum of the three terms for the two terms continued beyond together with the preceding term, truly I note the indices in this manner above:

Indices	0	1	2	3	4	5	6	7	8	9
A	1	1	3	7	19	51	141	393	1107	3139
B		3	3	9	21	57	153	423	1179	3321
C		2	0	2	2	6	12	30	72	182
D		1	0	1	1	3	6	15	36	91
E		1	0	1	1	2	3	5	8	13

where the series *A* has itself been proposed, which from the series *B,* the triplicate of that preceding term, with that taken from the series A, leaves the series *C;* truly that with its terms bisected produces the series D, of which the individual terms are triangular numbers, from which I have written their roots below, from which the series *E* has arisen.

2. In this series *E* the order of the terms will be seen to be prepared thus, so that any one term is seen to be the sum of the two preceding terms, and this conclusion has arisen from inspection, as it may be confirmed by a series of ten terms, thus certainly it will be seen, so that without doubt it may be allowed, why all the terms of the series *D* may not be triangular numbers, nor indeed why the roots of these may not constitute that simple series, by which any term is the sum of the two preceding terms. Certainly we are accustomed often to trust inductions in investigations of this kind, which depend on less firm foundations.

3. If this induction may be agreed to be true, it shall be required to be considered as a discovery of the greatest importance, since thence it shall be possible to designate the general term of the proposed series *A* : clearly the term corresponding to the index *n* will become :

$$\tfrac{1}{10}3^n + \tfrac{1}{10}(-1)^n + \tfrac{1}{5}(\tfrac{3+\sqrt{5}}{2})^n + \tfrac{1}{5}(\tfrac{3-\sqrt{5}}{2})^n + \tfrac{1}{5}(\tfrac{1+\sqrt{5}}{2})^n + \tfrac{1}{5}(\tfrac{1-\sqrt{5}}{2})^n$$

and our progression will be generated by the three recurring series :

										Step of the relation
A	1	1	5	13	41	121	365	1093	3281	$2+3$
B	2	3	7	18	47	123	322	843	2207	$3-1$
C	2	1	3	4	7	11	18	29	47	$1+1$
D	5	5	15	35	95	255	705	1965	5535	and dividing by 5
E	1	1	3	7	19	51	141	393	1107	etc.

Truly the series *D* arises from the recurring series *A, B, C* by the addition of the individual terms, of which the terms divided by 5 produce our progression itself, at least to ten terms.

4. There is no restraint to explain how I may have elicited the general term of this expression, since the above induction, however great the founding force may seem, yet is incompatible with the truth. And indeed at once our progression may be continued further and the operations are established as in § 1, so that there follows :

Indices	5	6	7	8	9	10	11
A	51	141	393	1107	3139	8953	25653
B	57	153	423	1179	3321	9417	26859
C	6	12	30	72	182	464	1206
D	3	6	15	36	91	232	603
E	2	3	5	8	13	−	−

but in series *D* the terms 232 and 603 no longer are triangular numbers and thus none of the other series *E* prevails. Therefore this method of forbidden induction is thus most noteworthy, because I had not yet come across a case of this kind, in which so plausible an induction failed.

CONSIDERATION II

5. Therefore with the nature of our progression rejected from all induction, from that I may undertake to examine its nature carefully. And indeed in the first place it is evident, if in this series

$$x,\ 3x^2,\ 7x^3,\ 19x^4,\ 51x^5,\ 141x^6,\ 393x^7 \text{ etc.}$$

a convenient term of the index n may be put

$$= Nx^n,$$

Nx^n to become the term of this power of x, which arises from the expansion of the formula $(1+x+x^2)^n$. Therefore with the treatment of the trinomial $1+x+xx$ with the two first parts joined together, and there will be

$$(1+x+xx)^n = (1+x)^n + \tfrac{n}{1}xx(1+x)^{n-1} + \tfrac{n(n-1)}{1\cdot 2}x^4(1+x)^{n-2}$$
$$\tfrac{n(n-1)(n-2)}{1\cdot 2\cdot 3}x^6(1+x)^{n-3} + \text{etc.}$$

of which it will be required to elicit the power x^n from the individual members, and thence the sum of everything gathered together will give our term sought Nx^n.

6. But from the first member $(1+x)^n$ or $(x+1)^n$ with the expansion made there arises x^n; but for the following member from the expansion of the formula $(x+1)^{n-1}$ the following term $\frac{n-1}{1}x^{n-2}$ must be taken, which multiplied by $\frac{n}{1}x^2$ gives $\frac{n(n-1)}{1\cdot 1}x^n$. Again for the third member from the formula $(x+1)^{n-2}$ the third term $\frac{(n-2)(n-3)}{1\cdot 2}x^{n-4}$ multiplied by the factor $\frac{n(n-1)}{1\cdot 2}x^4$ produces $\frac{n(n-1)(n-2)(n-3)}{1\cdot 1\cdot 2\cdot 2}x^n$,

and thus with the other members; from which we obtain

$$N = 1 + \tfrac{n(n-1)}{1\cdot 1} + \tfrac{n(n-1)(n-2)(n-3)}{1\cdot 1\cdot 2\cdot 2} + \tfrac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{1\cdot 1\cdot 2\cdot 2\cdot 3\cdot 3} + \text{etc.},$$

of which the number of the parts requiring to be added for any whole number *n* shall be finite; and thus the value of the term N will be able to be assigned easily. The same expression is found more easily, if the power of the trinomial may be expressed thus

$$(x(1+x)+1)^n = x^n(1+x)^n + \tfrac{n}{1}x^{n-1}(1+x)^{n-1}$$
$$+\tfrac{n(n-1)}{1\cdot 2}x^{n-2}(1+x)^{n-2} - \text{etc.},$$

where the coefficient of the power x^n from the first member becomes 1, from the second $\frac{n}{1}\frac{n-1}{1}$, from the third $\frac{n(n-1)}{1\cdot 2}\cdot\frac{(n-2)(n-3)}{1\cdot 2}$ etc., as above.

CONSIDERATION III

7. With the expression found, by which in general the coefficient of the power x^n is defined in our progression, in the first place I observe thus in now way can it be returned any simpler, so that thus it may be reduced to a finite formula. Indeed even if the discovery of a number *N* for the differentio-differential equation shall be able to be recalled, yet that has been prepared thus, so that in no way does it allow a resolution.

Therefore since all the labour in finding a more convenient expression for N may be consumed in a useless manner, I shall press on with this here, so that I may elicit a rule, by which in our progression of terms it may be able to be defined as desired by some preceding terms.

8. To this end I represent our progression thus

$$x,\ 3x^2, 7x^3,\ 19x^4,\ 51x^5,\ \cdot\ \ \cdot\ \ \cdot\ px^{n-2},\ qx^{n-2},\ rx^n,$$

and we are going to investigate, how the number r may be able to be determined by the preceding q and p . But the values p, q, r are obtained from the above series found for N , which, where they may receive analytical operations, I express thus :

$$p = 1 + \frac{(n-2)(n-3)}{1\cdot 1} z^2 + \frac{(n-2)(n-3)(n-4)(n-5)}{1\cdot 1\cdot 2\cdot 2} z^4 + \text{etc.},$$
$$q = 1 + \frac{(n-1)(n-2)}{1\cdot 1} z^2 + \frac{(n-1)(n-2)(n-3)(n-4)}{1\cdot 1\cdot 2\cdot 2} z^4 + \text{etc.},$$
$$r = 1 + \frac{n(n-1)}{1\cdot 1} z^2 + \frac{n(n-1)(n-2)(n-3)}{1\cdot 1\cdot 2\cdot 2} z^4 + \text{etc.},$$

from which we gather, on subtracting any former from the latter

$$\frac{q-p}{2} = \frac{n-2}{1} z^2 + \frac{(n-2)(n-3)(n-4)}{1\cdot 1\cdot 2} z^4 + \frac{(n-2)(n-3)(n-4)(n-5)(n-6)}{1\cdot 1\cdot 2\cdot 2\cdot 3} z^6 + \text{etc.},$$
$$\frac{r-q}{2} = \frac{n-1}{1} z^2 + \frac{(n-1)(n-2)(n-3)}{1\cdot 1\cdot 2} z^4 + \frac{(n-1)(n-2)(n-3)(n-4)(n-5)}{1\cdot 1\cdot 2\cdot 2\cdot 3} z^6 + \text{etc.}$$

9. But with the values q and r differentiated, we obtain :

$$\frac{dq}{2dz} = \frac{(n-1)(n-2)}{1} z + \frac{(n-1)(n-2)(n-3)(n-4)}{1\cdot 1\cdot 2} z^3 + \text{etc.},$$
$$\frac{dr}{2dz} = \frac{n(n-1)}{1} z + \frac{n(n-1)(n-2)(n-3)}{1\cdot 1\cdot 2} z^3 + \text{etc.},$$

which series may be compared easily with the preceding , since evidently there shall be

$$\frac{(n-1)(q-p)}{2} = \frac{zdq}{2dz} \text{ and } \frac{n(r-q)}{2} = \frac{zdr}{2dz},$$

from which we may conclude to become :

$$dq = (n-1)(q-p)\cdot \tfrac{dz}{z}$$

and

$$dr = n(r-q)\cdot \tfrac{dz}{z}$$

10. Then truly the latter forms of the preceding paragraph present on differentiation :

$$\frac{dq-dp}{4dz}=(n-2)z+\frac{(n-2)(n-3)(n-4)}{1\cdot 1}z^3+\frac{(n-2)(n-3)(n-4)(n-5)(n-6)}{1\cdot 1\cdot 2\cdot 2}z^5+\text{etc.},$$

$$\frac{dr-dq}{4dz}=(n-1)z+\frac{(n-1)(n-2)(n-3)}{1\cdot 1}z^3+\frac{(n-1)(n-2)(n-3)(n-4)(n-5)}{1\cdot 1\cdot 2\cdot 2}z^5+\text{etc.},$$

which differ so much from these first ones, because here the coefficients have a single factor extra, but there by differentiation the same factors can be added easily in this manner :

$$\frac{d.pz^{2-n}}{dz}=-(n-2)z^{1-n}-\frac{(n-2)(n-3)(n-4)}{1\cdot 1}z^{3-n}$$
$$-\frac{(n-2)(n-3)(n-4)(n-5)(n-6)}{1\cdot 1\cdot 2\cdot 2}z^{5-n}-\text{etc.}$$

$$\frac{d.qz^{1-n}}{dz}=-(n-1)z^{-n}-\frac{(n-1)(n-2)(n-3)}{1\cdot 1}z^{2-n}$$
$$-\frac{(n-1)(n-2)(n-3)(n-4)(n-5)}{1\cdot 1\cdot 2\cdot 2}z^{4-n}-\text{etc.},$$

from which evidently there becomes :

$$\frac{dq-dp}{4dz}+\frac{z^n d.pz^{2-n}}{dz}=0$$

and

$$\frac{dr-dq}{4dz}+\frac{z^{n+1}d.qz^{1-n}}{dz}=0$$

and with an expansion made:

$$dq-dp+4zzdp-4(n-2)pzdz=0,$$
$$dr-dq+4zzdq-4(n-1)\ qzdz=0.$$

11. Therefore since above we have found the differentials dq and dr expressed by dz, if we may substitute these values into the latter equation, we will obtain

$$\frac{n(r-q)}{z}-\frac{(n-1)(q-p)}{z}+4(n-1)(q-p)z-4(n-1)qz=0,$$

thus so that with the differentials removed a finite relation may be elicited between p, q and r , which is found thus:

$$n(r-q)=(n-1)(q-p)(1-4zz)+4\left(n-1\right)qzz$$

or

$$n(r-q)=(n-1)(q+p(4zz-1)).$$

12. Therefore we have found a relation of this kind between the three values *p, q,* and *r* continued, with the aid of which from two given the third is defined easily, and this is

much more general, than is needed for our case, since this relation shall prevail for any number z . Therefore since in our case there is $z=1$, there will become

$$n(r-q)=(n-1)(q+3p)$$

or

$$r=q+\tfrac{n-1}{n}(q+3p),$$

of which the formula, easily to the benefit of our progression, can be continued in this manner as far as it may please:

A	1	3	7	19	51	141	393	1107	3139
B		3	9	21	57	153	423	1179	
C		6	16	40	108	294	816	2286	
D		3	4	5	6	7	8	9	
E		2	4	8	18	42	102	254	
F		4	12	32	90	252	714	2032.	

Evidently the terms of the series *A*, as far as it were now continued, the same terms tripled may be written below on being moved forwards by a single place, which is the series *B*; while the sum $A+B$ will give the series C, for which divided by the subscribed arithmetical progression D , *C: D* provides the series *E*, from which $C-E$ supplies the series *F*, any term of which added to the term of the top series *A* suggests its following term.

13. Therefore we may continue our progression further in this manner :

A	1107	3139	8953	25653	73789	212941	616227
B	1179	3321	9417	26859	76959	221367	
C	2286	6460	18370	52512	150748	434308	
D	9	10	11	12	13	14	
E	254	646	1670	4376	11596	31022	
F	2032	5814	16700	48136	139152	403286	

from which with the powers of *x* added, since certainly it shall be appropriate for x^0 to be 1, as also it may be clear from the law of the progression found, our progression thus will be had :

$$1,\ 1x,\ 3x^2,\ 7x^3,\ 19x^4,\ 51x^5,\ 141x^6,\ 393x^7,\ 1107x^8,\ 3139x^8,\ 8953x^{10},$$
$$25653x^{11},\ 73789x^{12},\ 212941x^{13},\ 616227x^{14}\ \text{etc.}$$
$$\cdot\ \cdot\ \cdot\ px^{n-2},\ qx^{n-1},\ rx^n$$

and the law of the progression has been prepared thus, to that there shall be:

$$r = q + \tfrac{n-1}{n}(q + 3p) = 2q + 3p - \tfrac{1}{n}(q + 3p).$$

14. But in the first place, it may be observed of the method, by which we have reached that same relation between the three following terms by differentiation, since actually here a ratio of no variable may be had. Now indeed we may observe without difficulty the same relation can be elicited without differentiation, if this multiplication may be used in the three series in § 8 , so that there may become $(A + azz)p + Bq + Cr = 0$. For it will be readily apparent values of this kind can be attribute to the letters *A, a, B* and *C*, so that all the powers of z may become zero, which prevails by putting the above relation itself into effect. Truly by considering the initial matter, this certainly may be seen to be less of an obstacle.

CONSIDERATION IV

15. With this law of the progression found, none the less a curious question occurs, where the sum of this same progression is investigated continued to infinity.
Therefore we may put

$$s = 1 + x + 3x^2 + 7x^3 + \cdot\ \cdot\ \cdot + px^{n-2} + qx^{n-1} + rx^n + \text{etc.},$$

and since we will have found

$$n(r - 2q - 3p) + q + 3p = 0,$$

by differentiating we will introduce this equality in the following manner :

$$\begin{aligned}
\tfrac{ds}{dx} &= 1 + 6x + 21x^2 + \cdot\ \cdot\ \cdot + (n-2)px^{n-3} + (n-1)qx^{n-2} + nrx^{n-1}\\
-\tfrac{2d.xs}{dx} &= -2 - 4x - 18x^2 - \cdot\ \cdot\ \cdot \qquad - 2(n-1)px^{n-2} - 2nqx^{n-1}\\
-\tfrac{3d.x^2s}{dx} &= \quad -6x - 9x^2 - \cdot\ \cdot\ \cdot \qquad - 3npx^{n-1}\\
\hline
s &= 1 + x + 3x^2 + \cdot\ \cdot\ \cdot \qquad + qx^{n-1}\\
3xs &= 3x + 3x^2 + \cdot\ \cdot\ \cdot \qquad + 3px^{n-1},
\end{aligned}$$

from which it follows :

$$\tfrac{ds - 2d.xs - 3d.x^2s}{dx} + s + 3xs = 0$$

or

$$(1 - 2x - 3xx)ds - sdx - 3xsdx = 0.$$

From this equation it is deduced

$$\tfrac{ds}{s} = \tfrac{dx + 3xdx}{1 - 2x - 3xx}$$

and hence on integrating

$$s=\frac{1}{\sqrt{(1-2x-3xx)}}=\frac{1}{\sqrt{(1+x)(1-3x)}}.$$

16. Behold therefore a new origin of our series, certainly which arises from the expansion of this form :

$$\left(1-2x-3xx\right)^{-\frac{1}{2}},$$

with which calculation put in place, this same series is taken to result :

$$1+x+3x^2+7x^3+19x^4+51x^5+141x^6+\text{etc.}$$

Hence truly it is evident likewise, however great the sum of this series is going to become continued to infinity for whatever value of x; where indeed it is required to be observed, if there shall be either $x=-1$ or $x=\frac{1}{3}$, the sum to become infinite ; but if $x>\frac{1}{3}$, the sum is imaginary. Moreover the sum will be finite, if x may be contained between the limits $\frac{1}{3}$ and -1; and beyond these two limits an imaginary sum will be produced always. Thus on taking $x=\frac{1}{4}$ there will be

$$1+\frac{1}{4}+\frac{3}{4^2}+\frac{7}{4^3}+\frac{19}{4^4}+\frac{51}{4^5}+\text{etc.}=\frac{4}{\sqrt{5}}.$$

CONSIDERATION V

17. This investigation for the sum of the middle terms can be extended further from the expansion of the powers of the trinomial taken to be $a+bx+cx^2$. Indeed by putting in general Nx^n for the middle term of the power $(a+bx+cx^2)^n$, the value of the coefficient N thus will be able to be determined : Since there shall be

$$(x(b+cx)+a)^n = x^n(b+cx)^n+\frac{n}{1}ax^{n-1}(b+cx)^{n-1}$$
$$+\frac{n(n-1)}{1\cdot 2}a^2x^{n-2}(b+cx)^{n-2}+\text{etc.},$$

from the individual members the terms may be grouped together for the affected power x^n, and there will be found :

$$N=b^n+\frac{n(n-1)}{1\cdot 1}ab^{n-2}c+\frac{n(n-1)(n-2)(n-3)}{1\cdot 1\cdot 2\cdot 2}a^2b^{n-4}c^2+\text{etc.}$$

or, on putting for the sake of brevity,

$$\frac{ac}{bb}=g$$

there will become :

$$N=b^n\left(1+\frac{n(n-1)}{1\cdot 1}g+\frac{n(n-1)(n-2)(n-3)}{1\cdot 1\cdot 2\cdot 2}g^2+\text{etc.}\right).$$

From which, since by taking $n=0$ there becomes $N=1$, if thus we may represent this progression, these coefficients thus will be found :

$$A=b,$$
$$B=b^2(1+2g),$$
$$C=b^3(1+6g),$$
$$D=b^4(1+12g+6gg),$$
$$E=b^5(1+20g+30gg),$$
$$F=b^6(1+30g+90gg+20g^3).$$

18. In order that we may investigate, how any term may be determined by the two preceding terms, we may expand the series thus:

$$1,\ bx,\ (1+2g)b^2x^2,\ (1+6g)b^3x^3\cdot\ \cdot\ \cdot pb^{n-2}x^{n-2},\ qb^{n-1}x^{n-1},\ rb^nx^n$$

and by writing *zz* for *g* we will have:

$$p=1+\tfrac{(n-2)(n-3)}{1\cdot 1}z^2+\tfrac{(n-2)(n-3)(n-4)(n-5)}{1\cdot 1\cdot 2\cdot 2}z^4+\text{etc.}$$
$$q=1+\tfrac{(n-1)(n-2)}{1\cdot 1}z^2+\tfrac{(n-1)(n-2)(n-3)(n-4)}{1\cdot 1\cdot 2\cdot 2}z^4+\text{etc.}$$
$$r=1+\tfrac{n(n-1)}{1\cdot 1}z^2+\tfrac{n(n-1)(n-2)(n-3)}{1\cdot 1\cdot 2\cdot 2}z^4+\text{etc.},$$

which series are the same, which I have treated now above ; and therefore there will be

$$n(r-q)=(n-1)(q+p(4zz-1)).$$

Therefore in place of *zz* by g being restored in our series, the term *r* in our series thus can be determine by both the preceding, so that there shall become

$$r=q+\tfrac{n-1}{n}(q+(4g-1)p)$$

or

$$r=2q+(4g-1)p-\tfrac{1}{n}(q+(4g-1)p).$$

19. We may put $4g-1=h$, so that there shall become

$$h=\tfrac{4ac-bb}{bb},$$

and since the law of the progression may produce

$$r = 2q + hp - \tfrac{1}{n}(q + hp)$$

and with the powers $b^n x^n$ omitted the initial terms shall both be 1, our progression will be

$$\begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \\ 1, & 1, & \frac{3+h}{2}, & \frac{5+3h}{2}, & \frac{35+30h+3hh}{8}, & \frac{63+70h+15hh}{8}, \end{array}$$

from which on taking $h = 3$ the series treated before will result. But if there may be taken $h = -1$ or $g = 0$, all the terms will become unity, which is clear from the relation

$$n(r - q) = (n-1)(q - p);$$

and indeed once the two contiguous terms p and q are equal, it is necessary all the same may become equal.

CONSIDERATION VI

20. We may put in place the investigation of much more general sums of this progression, and there shall become

$$s = A + Bx + Cx^2 + \cdot \quad \cdot \quad \cdot + px^{n-2} + qx^{n-1} + rx^n + \text{etc.},$$

the law of the progression of this series may be prepared thus, so that there shall be :

$$n(ap + bq + cr) = fp + gq,$$

and by putting in place the calculation as above in § 15 we will have:

$$\begin{aligned} \tfrac{ad.xxs}{dx} &= 2aAx + 3aBxx + \cdots + napx^{n-1} \\ \tfrac{bd.xs}{dx} &= bA + 2bBx + 3bCxx + \cdots + nbqx^{n-1} \\ \tfrac{cds}{dx} &= cB + 2cCx + 3cDxx + \cdots + ncrx^{n-1} \\ \hline gs &= Ag + gBx + gCxx + \cdots + gqx^{n-1} \\ fxs &= \qquad fAx + CBxx + \cdots + fpx^{n-1} \end{aligned}$$

On account of which, it is necessary from the nature of the series, that there becomes

$$\tfrac{ad.xxs + bd.xs + cds}{dx} - (fx + g)s = (b - g)A + cB$$

or

$$\frac{ds(axx+bx+c)}{dx}+s((2a-f)x+(b-g))=(b-g)A+cB.$$

21. Therefore since we may have

$$ds+\frac{sdx((2a-f)x+(b-g))}{(axx+bx+c)}=\frac{(b-g)Adx+cBdx}{(axx+bx+c)},$$

the integration of this equation must be put in place, thus so that on putting $x=0$ there becomes $s=A$, from which this summation presents no difficulty. Therefore we may adapt this to the series found before, which was

$$s=1+x+\tfrac{3+h}{2}x^2+\tfrac{5+h}{2}x^3+\cdot\quad\cdot\quad\cdot+px^{n-2}+qx^{n-1}+rx^n+\text{etc.},$$

for which there is:

$$n(hp+2q-r)=hp+q$$

and

$$\text{A}=1,\ \ \text{B}=1,$$

and, with the application made, there becomes:

$$a=h,\ b=2,\ c=-1,\ f=h,\ g=1,$$

from which the value of the sum s will be required to be defined from this equation :

$$ds+\frac{sdx(hx+1)}{hxx+2x-1}=\frac{Adx-Bdx}{hxx+2x-1}=0$$

and hence there is determined :

$$s\sqrt{(hxx+2x-1)}=\sqrt{-1}$$

or

$$s=\frac{1}{\sqrt{(1-2x-hxx)}}.$$

22. We may restore the value assumed in § 19 :

$$h=4g-1=\frac{4ac-bb}{bb}$$

and in place of x we may write *bx*, so that this series shall be required to be summed :

$$s=1+bx+(bb+2ac)x^2+(b^3+6abc)x^3+(b^4+12abbc+6aacc)x^4+\text{ etc.},$$

and its sum will be

$$s=\frac{1}{\sqrt{(1-2bx+(bb-4ac)xx)}}$$

or

$$s=\frac{1}{\sqrt{((1-bx)^2-4acxx)}}.$$

But the origin of this series is such that its individual terms shall be the middle terms assumed from the powers $(a+bx+cxx)^n$. Then truly the law of progression has been prepared thus, so that from these put in turn into the three terms by the following :

$$px^{n-2},\ \ qx^{n-1},\ \ rx^n$$

the coefficient r may be defined by the two preceding, so that there shall be

$$r=bq+\tfrac{n-1}{n}(bq+(4ac-bb)p)$$

or

$$r=\tfrac{2n-1}{n}bq+\tfrac{n-1}{n}(4ac-bb)p.$$

23. If there may be put $bb=4ac$, thus so that there shall be

$$a+bx+cxx=(\sqrt{a}+x\sqrt{c})^2,$$

any term of our progression thus may be determined by the preceding alone, so that there shall be

$$r=\tfrac{2n-1}{n}\cdot 2q\sqrt{ac}.$$

In this case there may be put

$$a=1,\ c=1\ \text{ and }\ b=2,$$

so that our series may be agreed from the middle terms of the powers

$$(1+2x+xx)^2\ \text{ or }\ (1+x)^{2n},$$

and there will be

$$r=\frac{2(2n-1)}{n}q$$

and with the series itself:

$$s=1+2x+\tfrac{2\cdot 6}{1\cdot 2}x^2+\tfrac{2\cdot 6\cdot 10}{1\cdot 2\cdot 3}x^3+\tfrac{2\cdot 6\cdot 10\cdot 14}{1\cdot 2\cdot 3\cdot 4}x^4+\text{etc.},$$

the sum of which becomes

$$s=\frac{1}{\sqrt{(1-4x)}},$$

as indeed is evident by itself .

CONSIDERATION VII

24. From the preceding series

$$s = 1 + bx + \left(bb + 2ac\right)x^2 + (b^3 + 6abc)x^3 + \text{ etc.}$$

the sum found

$$s = ((1-bx)^2 - 4acxx)^{-\frac{1}{2}},$$

in turn its general term or the coefficient of the power x^n can be elicited. Indeed since with the expansion made in the usually customary manner shall be

$$s = \frac{1}{1-bx} + \frac{1}{2}\cdot\frac{acxx}{(1-bx)^3} + \frac{2\cdot 6}{1\cdot 2}\cdot\frac{a^2c^2x^4}{(1-bx)^5} + \frac{2\cdot 6\cdot 10}{1\cdot 2\cdot 3}\cdot\frac{a^3c^3x^6}{(1-bx)^7} + \text{etc.},$$

the powers x^n may be deduced from the individual members ; indeed from the first there arises

$$b^n x^n,$$

from the second

$$\frac{2}{1}\cdot\frac{n(n-1)}{1\cdot 2}acb^{n-2}x^n,$$

from the third

$$\frac{2\cdot 6}{1\cdot 2}\cdot\frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4}a^2c^2b^{n-4}x^n,$$

which collected into one sum will give

$$b^n x^n (1 + \frac{n(n-1)}{1\cdot 1}\frac{ac}{bb} + \frac{n(n-1)(n-2)(n-3)}{1\cdot 1\cdot 2\cdot 2}\frac{aacc}{b^4} + \text{etc.}),$$

generally as we have elicited above from the start of this series.

VARIA ARTIFICIA IN SERIERUM INDOLEM INQUIRENDI

Commentatio 551 indicis ENESTROEMIANI
Opuscula analytica 1, 1783, p. 48-63

Eiusmodi saepe occurrunt series, quarum origo etsi satis est perspicua, earum tamen lex progressionis et natura maxime est abscondita et nonnisi insignibus adhibitis artificiis analyticis investigari potest. In genere quidem huiusmodi artificia vix ita proponere licet, ut eorum usus luculenter perspiciatur; sed potius eorum vis in exemplis commodissime ostenditur, unde simul ratio ac necessitas ea excogitandi multo clarius intelligitur. Seriem igitur seu numerorum progressionem omnino singularem hic contemplabor, quae oritur, si potestates trinomii $1+x+xx$ evolvantur atque ex singulis termini tantum medii, qui maximis numeris afficiuntur, in ordinem disponantur; ita enim enascitur numerorum series eo magis notatu digna, quo minus lex progressionis perspicitur. Ea autem explorata pulcherrimae affectiones agnoscuntur, in quo negotio maxima vis artificiorum analyticorum potissimum cernitur. Imprimis autem haec series memorabile documentum exhibet, quanta circumspectione in inductione, cui plerumque in huiusmodi investigationibus non parum tribui solet, versari debeamus, cum hic eiusmodi inductio occurrat, quae, etiamsi maxime confirmata videatur, tamen in errorem inducat.

EVOLUTIO POTESTATUM TRINOMII

$$1+x+x^2$$
$$1+2x+3x^2+2x^3+x^4$$
$$1+3x+6x^2+7x^3+6x^4+3x^5+x^6$$
$$1+4x+10x^2+16x^3+19x^4+16x^5+10x^6+4x^7+x^8$$
$$1+5x+15x^2+30x^3+45x^4+51x^5+45x^6+30x^7+15x^8+5x^9+x^{10}$$
$$1+6x+21x^2+50x^3+90x^4+126x^5+141x^6+126x^7+90x^8+50x^9+21x^{10}+6x^{11}+x^{12}$$
etc.

Ex singulis his formis terminos tantum medias excerpo, qui hanc suppeditant progressionem

$$x+3x^2+7x^3+19x^4+51x^5+141x^6+\text{etc.},$$

cuius naturam hic investigare constitui, ubi quidem omissis potestatibus ipsius *x* totum negotium ad hanc progressionem numericam reducitur

1, 3, 7, 19, 51, 141, 393 etc.

CONSIDERATIO I

1. Seriem hanc perpendenti mox in mentem venit quemlibet terminum cum triplo praecedentis haud incongrue comparari posse, quia hanc seriem in infinitum continuatam cum progressione geometrica tripla confundi debere ex eius origine est manifestum. Illi ergo ad duos terminos ultra continuatae terminos praecedentes triplicatos subscribo, indices vero superius noto hoc modo:

Indices	0	1	2	3	4	5	6	7	8	9
A	1	1	3	7	19	51	141	393	1107	3139
B		3	3	9	21	57	153	423	1179	3321
C		2	0	2	2	6	12	30	72	182
D		1	0	1	1	3	6	15	36	91
E		1	0	1	1	2	3	5	8	13

ubi series *A* est ipsa proposita, quae a serie *B,* illius triplicata, ablata relinquit seriem *C;* huius vero terminis bisectis prodit series *D,* cuius singuli termini sunt numeri trigonales, quibus suas subscripsi radices, unde series nata est *E.*

2. In hac serie *E* terminorum ordo ita comparatus videtur, ut quilibet aequetur summae binorum praecedentium, atque haec conclusio ex inspectione nata, quoniam per decem seriei terminos confirmatur, ita certa videtur, ut neque dubitare liceat, quin omnes termini seriei *D* sint numeri trigonales, neque quin eorum radices seriem recurrentem illam simplicem, qua quilibet terminus est aggregatum binorum antecedentium, constituant. Saepe certe in huius generis investigationibus eiusmodi inductionibus confidere solemus, quae minus firmo fundamento innituntur.

3. Si haec inductio veritati esset consentanea, pro invento maximi momenti esset habenda, cum inde adeo terminus generalis seriei propositae *A* assignari posset: terminus scilicet indici n respondens foret

$$\tfrac{1}{10}3^n+\tfrac{1}{10}(-1)^n+\tfrac{1}{5}\left(\tfrac{3+\sqrt{5}}{2}\right)^n+\tfrac{1}{5}\left(\tfrac{3-\sqrt{5}}{2}\right)^n+\tfrac{1}{5}\left(\tfrac{1+\sqrt{5}}{2}\right)^n+\tfrac{1}{5}\left(\tfrac{1-\sqrt{5}}{2}\right)^n$$

et nostra progressio ex sequentibus tribus seriebus recurrentibus nasceretur:

										Scala relationis
A	1	1	5	13	41	121	365	1093	3281	$2+3$
B	2	3	7	18	47	123	322	843	2207	$3-1$
C	2	1	3	4	7	11	18	29	47	$1+1$
D	5	5	15	35	95	255	705	1965	5535	et dividendo per 5
E	1	1	3	7	19	51	141	393	1107	etc.

Ex seriebus nempe recurrentibus *A, B, C* per singulorum terminorum additionem nascitur series *D,* cuius termini per 5 divisi producunt ipsam nostram progressionem, saltem ad decem terminos.

4. Quomodo expressionem huius termini generalis eruerim, haud attinet docere, quandoquidem inductio superior, quantum vis fundata videatur, tamen veritati repugnat. Statim enim ac nostra progressio ulterius continuatur et operationes uti § 1 instituuntur, ut sequitur:

Indices	5	6	7	8	9	10	11
A	51	141	393	1107	3139	8953	25653
B	57	153	423	1179	3321	9417	26859
C	6	12	30	72	182	464	1206
D	3	6	15	36	91	232	603
E	2	3	5	8	13	−	−

in serie *D* termini 232 et 603 non amplius sunt trigonales neque adeo lex seriei *E* alterius valet. Hoc ergo exemplum inductionis illicitae eo magis est notatu dignum, quod mihi quidem eiusmodi casus nondum obtigerit, in quo tam speciosa inductio fefellerit.

CONSIDERATIO II

5. Repudiata ergo omni inductione progressionis nostrae indolem ex ipsa eius natura scrutari aggredior. Ac primo quidem evidens est, si in hac serie

$$x,\ 3x^2,\ 7x^3,\ 19x^4,\ 51x^5,\ 141x^6,\ 393x^7 \text{ etc.}$$

terminus indici *n* conveniens ponatur

$$= Nx^n,$$

fore Nx^n, ipsum terminum huius potestatis ipsius *x*, qui ex evolutione formulae $(1+x+x^2)^n$ nascitur. Trinomium igitur $1+x+xx$ binis prioribus partibus iunctis tanquam binomium tracto eritque

$$(1+x+xx)^n = (1+x)^n + \tfrac{n}{1}xx(1+x)^{n-1} + \tfrac{n(n-1)}{1\cdot 2}x^4(1+x)^{n-2}$$
$$\tfrac{n(n-1)(n-2)}{1\cdot 2\cdot 3}x^6(1+x)^{n-3} + \text{etc.}$$

ex cuius singulis membris potestatem x^n elici oportet, indeque summa omnium collecta dabit nostrum terminum quaesitum Nx^n.

6. Ex primo autem membro $(1+x)^n$ seu $(x+1)^n$ oritur facta evolutione x^n; pro secundo autem membro ex evolutione formulae $(x+1)^{n-1}$ terminus secundus

$$\tfrac{n-1}{1}x^{n-2}$$

capi debet, qui in $\frac{n}{1}x^2$ ductus dat

$$\frac{n(n-1)}{1\cdot 1}x^n.$$

Pro tertio porro membro ex formula $(x+1)^{n-2}$ tertius terminus

$$\frac{(n-2)(n-3)}{1\cdot 2}x^{n-4}$$

in factorem $\frac{n(n-1)}{1\cdot 2}x^4$ ductus praebet

$$\frac{n(n-1)(n-2)(n-3)}{1\cdot 1\cdot 2\cdot 2}x^n$$

sicque de ceteris membris; unde nanciscimur

$$N=1+\frac{n(n-1)}{1\cdot 1}+\frac{n(n-1)(n-2)(n-3)}{1\cdot 1\cdot 2\cdot 2}+\frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{1\cdot 1\cdot 2\cdot 2\cdot 3\cdot 3}+\text{etc.},$$

quarum partium addendarum numerus pro quovis numero integro n fit finitus; sicque valor termini N facile assignari poterit. Facilius eadem expressio reperitur, si potestas trinomii ita evolvatur

$$(x(1+x)+1)^n = x^n(1+x)^n+\frac{n}{1}x^{n-1}(1+x)^{n-1}$$
$$+\frac{n(n-1)}{1\cdot 2}x^{n-2}(1+x)^{n-2}-\text{etc.},$$

ubi potestatis x^n coefficiens ex primo membro fit 1, ex secundo $\frac{n}{1}\frac{n-1}{1}$, tertio $\frac{n(n-1)}{1\cdot 2}\cdot\frac{(n-2)(n-3)}{1\cdot 2}$ etc. ut supra.

CONSIDERATIO III

7. Inventa expressione, qua in genere coefficiens potestatis x^n in nostra progressione definitur, primum observo eam nullo modo ita simpliciorem reddi posse, ut ad formuIam finitam reducatur. Etsi enim inventio numeri N ad aequationem differentio-differentialem revocari possit, ea tamen ita est comparata, ut nullo modo resolutionem admittat. Cum igitur omnis labor in expressione pro N inventa commodius exhibenda inutiliter consumeretur, in id hic incumbam, ut legem eruam, qua in nostra progressione terminus quilibet ex aliquot praecedentibus definiri possit.

8. Hunc in finem progressionem nostram ita repraesento

$$x,\ 3x^2, 7x^3,\ 19x^4,\ 51x^5,\ \cdot\quad\cdot\quad\cdot\ px^{n-2},\ qx^{n-2},\ rx^n,$$

investigaturus, quomodo numerus r per praecedentes q et p determinari possit. Valores autem p, q, r ex superiori serie pro N inventa habentur, quos, quo analyticas operationes recipiant, ita exprimo:

$$p=1+\frac{(n-2)(n-3)}{1\cdot 1}z^2+\frac{(n-2)(n-3)(n-4)(n-5)}{1\cdot 1\cdot 2\cdot 2}z^4+\text{etc.},$$
$$q=1+\frac{(n-1)(n-2)}{1\cdot 1}z^2+\frac{(n-1)(n-2)(n-3)(n-4)}{1\cdot 1\cdot 2\cdot 2}z^4+\text{etc.},$$
$$r=1+\frac{n(n-1)}{1\cdot 1}z^2+\frac{n(n-1)(n-2)(n-3)}{1\cdot 1\cdot 2\cdot 2}z^4+\text{etc.},$$

unde quemlibet a sequente subtrahendo primo colligimus

$$\frac{q-p}{2}=\frac{n-2}{1}z^2+\frac{(n-2)(n-3)(n-4)}{1\cdot 1\cdot 2}z^4+\frac{(n-2)(n-3)(n-4)(n-5)(n-6)}{1\cdot 1\cdot 2\cdot 2\cdot 3}z^6+\text{etc.},$$
$$\frac{r-q}{2}=\frac{n-1}{1}z^2+\frac{(n-1)(n-2)(n-3)}{1\cdot 1\cdot 2}z^4+\frac{(n-1)(n-2)(n-3)(n-4)(n-5)}{1\cdot 1\cdot 2\cdot 2\cdot 3}z^6+\text{etc.}$$

9. Valoribus autem q et r differentiatis nanciscimur

$$\frac{dq}{2dz}=\frac{(n-1)(n-2)}{1}z+\frac{(n-1)(n-2)(n-3)(n-4)}{1\cdot 1\cdot 2}z^3+\text{etc.},$$
$$\frac{dr}{2dz}=\frac{n(n-1)}{1}z+\frac{n(n-1)(n-2)(n-3)}{1\cdot 1\cdot 2}z^3+\text{etc.},$$

quae series cum praecedentibus facile comparantur, cum manifesto sit

$$\frac{(n-1)(q-p)}{2}=\frac{zdq}{2dz}\ \text{et}\ \frac{n(r-q)}{2}=\frac{zdr}{2dz},$$

unde concludimus fore

$$dq=(n-1)(q-p)\cdot\frac{dz}{z}$$

et

$$dr=n(r-q)\cdot\frac{dz}{z}$$

10. Deinde vero formae posteriores paragraphi praecedentis differentiatae praebent:

$$\frac{dq-dp}{4dz}=(n-2)z+\frac{(n-2)(n-3)(n-4)}{1\cdot 1}z^3+\frac{(n-2)(n-3)(n-4)(n-5)(n-6)}{1\cdot 1\cdot 2\cdot 2}z^5+\text{etc.},$$
$$\frac{dr-dq}{4dz}=(n-1)z+\frac{(n-1)(n-2)(n-3)}{1\cdot 1}z^3+\frac{(n-1)(n-2)(n-3)(n-4)(n-5)}{1\cdot 1\cdot 2\cdot 2}z^5+\text{etc.},$$

quae a primis hoc tantum differunt, quod hic coefficientes uno factore abundant, ibi autem per differentiationem iidem factores facile adiici possunt hoc modo:

$$\frac{d\,.\,pz^{2-n}}{dz}=-(n-2)z^{1-n}-\frac{(n-2)(n-3)(n-4)}{1\cdot 1}z^{3-n}$$
$$-\frac{(n-2)(n-3)(n-4)(n-5)(n-6)}{1\cdot 1\cdot 2\cdot 2}z^{5-n}-\text{etc.}$$

$$\frac{d\,.\,qz^{1-n}}{dz}=-(n-1)z^{-n}-\frac{(n-1)(n-2)(n-3)}{1\cdot 1}z^{2-n}$$
$$-\frac{(n-1)(n-2)(n-3)(n-4)(n-5)}{1\cdot 1\cdot 2\cdot 2}z^{4-n}-\text{etc.},$$

unde manifestum est fieri

$$\frac{dq-dp}{4dz}+\frac{z^n d\,.\,pz^{2-n}}{dz}=0$$

et

$$\frac{dr-dq}{4dz}+\frac{z^{n+1}d\,.\,qz^{1-n}}{dz}=0$$

et facta evolutione

$$dq-dp+4zzdp-4(n-2)\,pzdz=0,$$
$$dr-dq+4zzdq-4(n-1)\;\,qzdz=0.$$

11. Cum igitur supra invenerimus differentialia *dq* et *dr* per *dz* expressa, si hos valores in postrema aequatione substituamus, impetrabimus

$$\frac{n(r-q)}{z}-\frac{(n-1)(q-p)}{z}+4(n-1)(q-p)z-4(n-1)qz=0,$$

ita ut differentialibus sublatis hic relatio finita inter *p*, *q* et *r* sit eruta, quae ita se habet

$$n(r-q)=(n-1)(q-p)(1-4zz)+4\left(n\;1\right)qzz$$

seu

$$n(r-q)=(n-1)(q+p(4zz-1)).$$

12. Invenimus ergo inter ternos valores continuos *p, q, r* eiusmodi relationem, cuius ope ex binis datis tertius facile definitur hocque multo generalius, quam pro nostro casu opus est, cum ista relatio pro quocunque numero z valeat. Quoniam igitur nostro casu est $z=1$, erit

$$n(r-q)=(n-1)(q+3p)$$

seu

$$r=q+\tfrac{n-1}{n}(q+3p),$$

cuius formulae beneficio nostra progressio facile, quousque libuerit, continuari potest in hunc modum:

A	1	3	7	19	51	141	393	1107	3139
B		3	9	21	57	153	423	1179	
C		6	16	40	108	294	816	2286	
D		3	4	5	6	7	8	9	
E		2	4	8	18	42	102	254	
F		4	12	32	90	252	714	2032.	

Seriei scilicet *A*, quousque iam fuerit continuata, subscribantur iidem termini triplicati eos uno loco promovendo, quae est series *B*; tum summa $A+B$ dabit seriem C, cui subscripta progressione arithmetica *D* divisio *C: D* praebet seriem *E*, unde $C-E$ suppeditat seriem *F*, cuius quivis terminus ad terminum supremum seriei *A* additus eius sequentem suggerit.

13. Hoc ergo modo nostram progressionem ulterius continuemus:

A	1107	3139	8953	25653	73789	212941	616227
B	1179	3321	9417	26859	76959	221367	
C	2286	6460	18370	52512	150748	434308	
D	9	10	11	12	13	14	
E	254	646	1670	4376	11596	31022	
F	2032	5814	16700	48136	139152	403286	

unde adiunctis potestatibus ipsius *x*, cum terminus ipsi x^0 conveniens certo sit 1, uti etiam ex lege progressionis inventa liquet, nostra progressio ita se habebit:

$$1,\ 1x,\ 3x^2,\ 7x^3,\ 19x^4,\ 51x^5,\ 141x^6,\ 393x^7,\ 1107x^8,\ 3139x^8,\ 8953x^{10},$$
$$25653x^{11},\ 73789x^{12},\ 212941x^{13},\ 616227x^{14}\ \text{etc.}$$
$$\cdot\ \cdot\ \cdot\ px^{n-2},\ qx^{n-1},\ rx^n$$

et lex progressionis ita est comparata, ut sit

$$r=q+\tfrac{n-1}{n}(q+3p)=2q+3p-\tfrac{1}{n}(q+3p).$$

14. Imprimis autem hic notetur artificium, quo per differentialia ad istam relationem inter ternos terminos sequentes pertigimus, cum revera hic nullius variabilitatis ratio habeatur. Iam quidem haud difficulter animadvertimus eandem relationem sine differentiatione erui posse, si in ternis seriebus § 8 haec multiplicatio adhibeatur, ut fiat $(A+azz)p+Bq+Cr=0$. Facile enim patebit litteris *A, a, B* et *C* eiusmodi valores tribui posse, ut omnes ipsius *z* potestates in nihilum abeant, quod efficiendo ipsa superior relatio obtinetur. Verum initio rem consideranti hoc certe minus obvium videbatur.

CONSIDERATIO IV

15. Inventa hac progressionis lege quaestio non minus curiosa occurrit, qua eiusdem progressionis in infinitum continuatae summa investigatur.
Ponamus ergo

$$s=1+x+3x^2+7x^3+\cdot\quad\cdot\quad\cdot+px^{n-2}+\ qx^{n-1}+rx^n+\text{etc.},$$

et cum invenerimus

$$n(r-2q-3p)+q+3p=0\,,$$

hanc aequalitatem differentiando introducemus sequenti modo

$$\begin{array}{rl}
\frac{ds}{dx}=1+6x+21x^2\ +\cdot\ \cdot\ \cdot+(n-2)\,px^{n-3}+ & (n-1)qx^{n-2}+nrx^{n-1}\\
-\frac{2d.xs}{dx}=-2-4x-18x^2-\cdot\ \cdot\ \cdot & -2(n-1)\,px^{n-2}-2nqx^{n-1}\\
-\frac{3d.x^2s}{dx}=\quad-6x-9x^2-\cdot\ \cdot\ \cdot & -3npx^{n-1}\\
\hline
s=1+x+3x^2+\ \cdot\ \cdot\ \cdot & +\ qx^{n-1}\\
3xs=3x+3x^2+\ \cdot\ \cdot\ \cdot & +\ 3px^{n-1},
\end{array}$$

unde consequimur

$$\frac{ds-2d.xs-3d.\,x^2s}{dx}+s+3xs=0$$

seu

$$(1-2x-3xx)ds-sdx-3xsdx=0.$$

Ex hac aequatione colligitur

$$\frac{ds}{s}=\frac{dx+3xdx}{1-2x-3xx}$$

hincque integrando

$$s=\frac{1}{\sqrt{(1-2x-3xx)}}=\frac{1}{\sqrt{(1+x)(1-3x)}}.$$

16. En ergo novam originem nostrae seriei, quippe quae oritur ex evolutione huius formae

$$\left(1-2x-3xx\right)^{-\frac{1}{2}},$$

unde calculo instituto haec ipsa series resultare deprehenditur

$$1+x+3x^2+7x^3+19x^4+51x^5+141x^6+\text{etc.}$$

Simul vero hinc apparet, quanta futura sit summa huius seriei in infinitum continuatae pro quovis valore ipsius *x*; ubi quidem notandum est, si sit vel $x=-1$ vel $x=\frac{1}{3}$, summam fore infinitam; at si $x>\frac{1}{3}$, summa est imaginaria. Finita autem erit summa, si *x* contineatur intra limites $\frac{1}{3}$ et -1; et extra hos limites prodit semper summa imaginaria. Ita sumto $x=\frac{1}{4}$ erit

$$1+\frac{1}{4}+\frac{3}{4^2}+\frac{7}{4^3}+\frac{19}{4^4}+\frac{51}{4^5}+\text{etc.}=\frac{4}{\sqrt{5}}.$$

CONSIDERATIO V

17. Haec investigatio ad seriem terminorum mediorum ex evolutione potestatum trinomii latius accepti $a+bx+cx^2$ extendi potest. Posito enim in genere Nx^n pro termino medio potestatis $(a+bx+cx^2)^n$ valor coefficientis N ita determinari poterit: Cum sit

$$(x(b+cx)+a)^n = x^n(b+cx)^n+\frac{n}{1}ax^{n-1}(b+cx)^{n-1}$$
$$+\frac{n(n-1)}{1\cdot 2}a^2x^{n-2}(b+cx)^{n-2}+\text{etc.},$$

ex singulis membris colligantur termini potestate x^n affecti, ac reperietur

$$N=b^n+\frac{n(n-1)}{1\cdot 1}ab^{n-2}c+\frac{n(n-1)(n-2)(n-3)}{1\cdot 1\cdot 2\cdot 2}a^2b^{n-4}c^2+\text{etc.}$$

seu posito brevitatis gratia

$$\frac{ac}{bb}=g$$

erit

$$N=b^n\left(1+\frac{n(n-1)}{1\cdot 1}g+\frac{n(n-1)(n-2)(n-3)}{1\cdot 1\cdot 2\cdot 2}g^2+\text{etc.}\right).$$

Unde, cum sumto $n=0$ fiat $N=1$, si hanc progressionem ita repraesentemus hi coefficientes ita se habebunt:

$$A=b,$$
$$B=b^2(1+2g),$$
$$C=b^3(1+6g),$$
$$D=b^4(1+12g+6gg),$$
$$E=b^5(1+20g+30gg),$$
$$F=b^6(1+30g+90gg+20g^3).$$

18. Ut investigemus, quomodo quisque terminus per binos praecedentes determinetur, seriem ita exponamus

$$1,\ bx,\ (1+2g)b^2x^2,\ (1+6g)b^3x^3 \cdot\ \cdot\ \cdot pb^{n-2}x^{n-2},\ qb^{n-1}x^{n-1},\ rb^nx^n$$

et pro g scribendo zz habebimus

$$p=1+\tfrac{(n-2)(n-3)}{1\cdot1}z^2+\tfrac{(n-2)(n-3)(n-4)(n-5)}{1\cdot1\cdot2\cdot2}z^4+\text{etc.}$$
$$q=1+\tfrac{(n-1)(n-2)}{1\cdot1}z^2+\tfrac{(n-1)(n-2)(n-3)(n-4)}{1\cdot1\cdot2\cdot2}z^4+\text{etc.}$$
$$r=1+\tfrac{n(n-1)}{1\cdot1}z^2+\tfrac{n(n-1)(n-2)(n-3)}{1\cdot1\cdot2\cdot2}z^4+\text{etc.},$$

quae series sunt eaedem, quas supra iam tractavi; eritque ergo

$$n(r-q)=(n-1)(q+p(4zz-1)).$$

Loco zz ergo restituendo g in serie nostra terminus r ita per ambos praecedentes determinatur, ut sit

$$r=q+\tfrac{n-1}{n}(q+(4g-1)p)$$

seu

$$r=2q+(4g-1)p-\tfrac{1}{n}(q+(4g-1)p).$$

19. Ponamus $4g-1=h$, ut sit

$$h=\tfrac{4ac-bb}{bb},$$

et cum lex progressionis praebeat

$$r=2q+hp-\tfrac{1}{n}(q+hp)$$

et omissis potestatibus b^nx^n bini termini initiales sint 1 et 1, progressio nostra erit

0	1	2	3	4	5
1,	1,	$\frac{3+h}{2}$,	$\frac{5+3h}{2}$,	$\frac{35+30h+3hh}{8}$,	$\frac{63+70h+15hh}{8}$,

unde sumto $h=3$ series ante tractata resultat. Sin autem capiatur $h=-1$ seu $g=0$, omnes termini in unitatem abeunt, id quod ex relatione

$$n(r-q)=(n-1)(q-p)$$

liquet; statim enim ac duo termini contigui p et q sunt aequales, omnes iisdem aequales fiant necesse est.

CONSIDERATIO VI

20. Investigationem summae huius progressionis multo generalius instituamus sitque

$$s = A + Bx + Cx^2 + \cdot \quad \cdot \quad \cdot + px^{n-2} + \; qx^{n-1} + rx^n + \text{etc.},$$

cuius seriei lex progressionis ita comparata concipiatur, ut sit

$$n(ap + bq + cr) = \; fp + gq,$$

et calculum ut supra § 15 instituendo habebimus

$$\begin{aligned}
\tfrac{ad.xxs}{dx} &= 2aAx + 3aBxx + \;\cdots\; + \; napx^{n-1}\\
\tfrac{bd.xs}{dx} &= \; bA + 2bBx + 3bCxx + \;\cdots\; + \; nbqx^{n-1}\\
\tfrac{cds}{dx} &= \; cB + 2cCx + 3cDxx + \;\cdots\; + \; ncrx^{n-1}\\
\hline
gs &= Ag + gBx + gCxx + \;\cdots\; + gqx^{n-1}\\
fxs &= \qquad fAx + \; CBxx + \;\cdots\; + \; fpx^{n-1}
\end{aligned}$$

Quocirca ex indole seriei fiat necesse est

$$\tfrac{ad.xxs+bd.xs+cds}{dx} - \left(fx + g \right) s = \left(b - g \right) A + cB$$

seu

$$\tfrac{ds(axx+bx+c)}{dx} + s((2a - f\,)x \; + (b - g)) = (b - g)A + cB.$$

21. Cum igitur habeamus

$$ds + \tfrac{sdx((2a-f\,)x \; +(b-g))}{(axx+bx+c)} = \tfrac{(b-g)Adx+cBdx}{(axx+bx+c)},$$

aequationis huius integratio ita institui debet, ut posito $x = 0$ fiat $s = A$, ex quo haec summatio nullam habet difficultatem. Accommodemus ergo haec ad seriem ante inventam, quae erat

$$s = 1 + x + \tfrac{3+h}{2}x^2 + \tfrac{5+h}{2}x^3 + \cdot \quad \cdot \quad \cdot + px^{n-2} + \; qx^{n-1} + rx^n + \text{etc.},$$

pro qua est

$$n(hp + 2q - r) = hp + q$$

et

$$\mathrm{A} = 1, \;\; \mathrm{B} = 1,$$

et facta applicatione fiet

$$a = h, \; b = 2, \; c = -1, \; f = h, \; g = 1,$$

unde valorem summae *s* ex hac aequatione definiri oportet

$$ds+\frac{sdx(hx+1)}{hxx+2x-1}=\frac{Adx-Bdx}{hxx+2x-1}=0$$

hincque colligitur

$$s\sqrt{(hxx+2x-1)}=\sqrt{-1}$$

seu

$$s=\frac{1}{\sqrt{(1-2x-hxx)}}.$$

22. Restituamus valorem § 19 assumtum

$$h=4g-1=\frac{4ac-bb}{bb}$$

et loco *x* scribamus *bx*, ut haec series sit summanda

$$s=1+bx+\left(bb+2ac\right)x^2+(b^3+6abc)x^3+(b^4+12abbc+6aacc)x^4+\text{ etc.},$$

eritque eius summa

$$s=\frac{1}{\sqrt{(1-2bx+(bb-4ac)xx)}}$$

seu

$$s=\frac{1}{\sqrt{((1-bx)^2-4acxx)}}.$$

Ipsius autem seriei origo est, ut singuli eius termini sint medii ex potestatibus $(a+bx+cxx)^n$ excerpti. Tum vero lex progressionis ita est comparata, ut positis in ea ternis terminis se invicem sequentibus

$$px^{n-2},\quad qx^{n-1},\quad rx^n$$

coefficiens *r* ita per binos praecedentes definiatur, ut sit

$$r=bq+\frac{n-1}{n}(bq+(4ac-bb)p)$$

seu

$$r=\frac{2n-1}{n}bq+\frac{n-1}{n}(4ac-bb)p.$$

23. Si ponatur $bb=4ac$, ita ut sit

$$a+bx+cxx=(\sqrt{a}+x\sqrt{c})^2,$$

quilibet terminus nostrae progressionis per solum praecedentem ita determinatur, ut sit

$$r=\tfrac{2n-1}{n}\cdot 2q\sqrt{ac}.$$

Ponatur hoc casu

$$a=1,\ c=1\ \text{ et }\ b=2,$$

ut series nostra constet ex terminis mediis potestatum

$$(1+2x+xx)^2\ \text{ seu }\ (1+x)^{2n},$$

eritque

$$r=\tfrac{2(2n-1)}{n}q$$

et ipsa series

$$s=1+2x+\tfrac{2\cdot6}{1\cdot2}x^2+\tfrac{2\cdot6\cdot10}{1\cdot2\cdot3}x^3+\tfrac{2\cdot6\cdot10\cdot14}{1\cdot2\cdot3\cdot4}x^4+\text{etc.}$$

cuius summa fit

$$s=\tfrac{1}{\sqrt{(1-4x)}},$$

uti quidem per se est manifestum.

CONSIDERATIO VII

24. Ex seriei praecedentis

$$s=1+bx+\left(bb+2ac\right)x^2+(b^3+6abc)x^3+\ \text{etc.}$$

summa inventa

$$s=((1-bx)^2-4acxx)^{-\frac{1}{2}}$$

Vicissim eius terminus generalis seu coefficiens potestatis x^n erui potest. Cum enim evolutione more solito facta sit

$$s=\tfrac{1}{1-bx}+\tfrac{1}{2}\cdot\tfrac{acxx}{(1-bx)^3}+\tfrac{2\cdot6}{1\cdot2}\cdot\tfrac{a^2c^2x^4}{(1-bx)^5}+\tfrac{2\cdot6\cdot10}{1\cdot2\cdot3}\cdot\tfrac{a^3c^3x^6}{(1-bx)^7}+\text{etc.},$$

colligantur ex singulis membris potestates x^n; ex primo quidem oritur

$$b^nx^n,$$

ex secundo

$$\tfrac{2}{1}\cdot\tfrac{n(n-1)}{1\cdot2}acb^{n-2}x^n,$$

ex tertio

$$\tfrac{2\cdot6}{1\cdot2}\cdot\tfrac{n(n-1)(n-2)(n-3)}{1\cdot2\cdot3\cdot4}a^2c^2b^{n-4}x^n,$$

qui in unam summam collecti dabunt

$$b^n x^n (1 + \frac{n(n-1)}{1\cdot 1}\frac{ac}{bb} + \frac{n(n-1)(n-2)(n-3)}{1\cdot 1\cdot 2\cdot 2}\frac{aacc}{b^4} + \text{etc.}),$$

omnino uti supra ex huius seriei origine elicuimus.