Concerning Products arising from an Infinite Numbers of Factors

[E122]

Commentaries of the Petersburg Academy of Sciences 11 (1739), 1750, p. 3-31

1. When in analysis there arises quantities of this kind, which cannot be expressed either by rational or irrational numbers, infinite expressions are accustomed to be used expressing these quantities ; which therefore are agreed to be more suitable, so that with the aid of these an understanding and an estimation of the magnitudes of the expressions there may be arrived at more quickly. Therefore the greatest and fullest use of expressions of this kind is for the values of transcending quantities, logarithms are of this kind, circular arcs, and other quantities determined by the quadrature of curves , and the benefit of these representations so very exactly not only with logarithms but also the arcs of circles, and all the other known transcendental quantities we pertain to know. Indeed also infinite expressions of this kind bring a conspicuous use for irrational quantities, and for the algebraic roots of equations truly requiring to be defined by rational numbers in the proximity; which, if the use may be considered, are to be preferred generally by far with the true expressions.

2. But generally between several of the infinite expressions of this kind the most diverse are required to be determined, the first of which includes all the infinite series, with the infinite terms agreeing with the adjoining $+$ or $-$ signs ; which principles have now indeed been improved to such an extent, that several methods may be had whereby not only algebraic but also transcending infinite series of this kind may be expressed, but also with a proposed infinite series requiring to be investigated, a quantity of which kind may be indicated there. For the infinite expressions of each kind will be required to be treated in a twofold manner, of which the one consists in the conversion of algebraic or transcending quantities into infinite expressions, while the other, which the proposed infinite expression may designate, in turn may be inverted in the investigation of that quantity.

3. It is appropriate to refer these to the former kind of infinite expression, which consist of innumerable factors; now although several expressions of this kind have been found and understood, yet neither the way of coming upon these nor the way the values of these can be discerned has been developed anywhere. But equally worthwhile, infinite expressions of this kind are considered, which may be developed, and the first being agreed from an infinite number of terms, nor perhaps a little of the convenience of analysis may be brought for the careful analysis of these. Besides indeed, expressions of this kind which refer to the nature of the quantity can be seen clearly enough, and through which approximate values requiring to be found usually are provided, which present an excellent use for forming the logarithms of these quantities, which brings most often an outstanding usefulness into the calculation. Thus, if some quantity X were transformed into an expression of this kind :

$$\frac{a}{\alpha}\cdot\frac{b}{\beta}\cdot\frac{c}{\gamma}\cdot\frac{d}{\delta}\cdot\frac{e}{\varepsilon}\cdot\text{etc.},$$

the logarithm of the quantity X will be found at once :

$$l\frac{a}{\alpha}+l\frac{b}{\beta}+l\frac{c}{\gamma}+l\frac{d}{\delta}+l\frac{e}{\varepsilon}+\text{etc.},$$

which series therefore converges more, where the factors of that incline more towards unity. For this reason, in this dissertation I have established a theorem of this kind for infinite expressions, indeed so many of my observations began to support the need for this aid, by which that may be brought about more easily by others at some future time.

4. In the first place *Wallis* advanced an expression of this kind containing infinite factors in the *Arithmetica infinitorum,* where it was shown, if the diameter of a circle shall be $=1$, the area of the circle becomes

$$\frac{2\cdot4\cdot4\cdot6\cdot6\cdot8\cdot8\cdot10\cdot10\cdot12}{3\cdot3\cdot5\cdot5\cdot7\cdot7\cdot9\cdot\ 9\cdot11\cdot11}\text{ etc.},$$

[*Vieta* had done something similar earlier in his *Angular Sections*, to be found on this website. Euler apparently was unaware of this earlier development.]
which expression he deduced from the interpolation of the series

$$\frac{2}{3}+\frac{2\cdot4}{3\cdot5}+\frac{2\cdot4\cdot6}{3\cdot5\cdot7}+\text{etc.},$$

the intermediate terms of which he had shown to depend on the quadrature of the circle. Therefore since these expressions must owe their origin to the interpolation of series, it should not seem to be incongruous that this treatment begin with interpolations agreed from infinite products of factors. Indeed since in Book Five of our *Commentaries* [E19], I have treated a method performing interpolations by the quadrature of curves, likewise it will be agreed they may indicate, how a transcending quantity of this kind may arise here on account of an infinite product.

5. Therefore I consider the following progression :

$$\overset{1}{(f+g)}+\overset{2}{(f+g)(f+2g)}+\overset{3}{(f+g)(f+2g)(f+3g)}+\overset{4}{(f+g)(f+2g)(f+3g)(f+4g)}+\text{etc.},$$

any term of which, the index of which is *n,* is found from the preceding by multiplying this by $f+ng$; but I have shown in the dissertation cited [E19], the term of this series, of which the index is *n,* to be :

$$\frac{g^{n+1}\int dx(-lx)^n}{(f+(n+1)g)\int x^{f:g}dx(1-x)^n}$$

with each integration performed thus, so that the integrals may vanish on putting $x=0$, and moreover by making $x=1$.

On account of which this same expression likewise will indicate, how each quadrature of the single intermediate terms may depend. Though indeed, if n shall be a fractional number, thus it is not easily agreed, how such a quadrature $\int dx(-lx)^n$ may be put in place, yet in the same place I have shown on putting $\frac{p}{q}$ in place of n the formula $\int dx(-lx)^{\frac{p}{q}}$ to agree with

$$\sqrt[q]{1\cdot 2\cdot 3\cdot p\left(\tfrac{2p}{q}+1\right)\left(\tfrac{3p}{q}+1\right)\left(\tfrac{4p}{q}+1\right)\cdots\left(\tfrac{qp}{q}+1\right)}$$
$$\times\int dx(x-xx)^{\frac{p}{q}}\cdot\int dx\left(x^2-x^3\right)^{\frac{p}{q}}\cdot\int dx\left(x^3-x^4\right)^{\frac{p}{q}}$$
$$\cdot\int dx\left(x^4-x^5\right)^{\frac{p}{q}}\cdots\int dx\left(x^{q-1}-x^q\right)^{\frac{p}{q}}.$$

with the aid of which reductions the value of $\int dx(-lx)^{\frac{p}{q}}$ can be expressed by the quadratures of the algebraic curves.

6. If now the term z may be put into the assumed series, the index of which is $=\frac{1}{2}$, from the law of the series, the terms of which the indices are $\frac{1}{2}$, $\frac{3}{2}$, $\frac{5}{2}$, etc., will themselves be found in the following manner:

$$z+z(f+\tfrac{3}{2}g)+z(f+\tfrac{3}{2}g)(f+\tfrac{5}{2}g)+z(f+\tfrac{3}{2}g)(f+\tfrac{5}{2}g)(f+\tfrac{7}{2}g)+\text{etc.}$$

But since the assumed progression finally may be found geometrically, these interpolated terms finally emerge as the mean proportionals between neighbouring terms of the series. Whereby if the individual interpolated terms now may be considered from the beginning as mean proportionals, the following approximations will be produced for the term z, of which the index is $\frac{1}{2}$:

$$\text{I. } z=\sqrt{(f+g)},$$
$$\text{II. } z=\sqrt{\frac{(f+g)(f+g)(f+2g)}{1\ \left(f+\frac{3}{2}g\right)\left(f+\frac{3}{2}g\right)}},$$
$$\text{III. } z=\sqrt{\frac{(f+g)(f+g)(f+2g)(f+2g)(f+3g)}{1\ \left(f+\frac{3}{2}g\right)\left(f+\frac{3}{2}g\right)\left(f+\frac{5}{2}g\right)\left(f+\frac{5}{2}g\right)}}$$

etc.,

according to which law of progression, the [general] term of the index $\frac{1}{2}$ is understood truly to be [by induction] :

$$=\left(f+g\right)^{\frac{1}{2}}\sqrt{\frac{(f+g)(f+2g)(f+2g)(f+3g)(f+3g)(f+4g)(f+4g)(f+5g)(f+5g)}{\left(f+\frac{3}{2}g\right)\left(f+\frac{3}{2}g\right)\left(f+\frac{5}{2}g\right)\left(f+\frac{5}{2}g\right)\left(f+\frac{7}{2}g\right)\left(f+\frac{7}{2}g\right)\left(f+\frac{9}{2}g\right)\left(f+\frac{9}{2}g\right)\left(f+\frac{11}{2}g\right)\left(f+\frac{11}{2}g\right)}}\text{etc.}$$

7. Now therefore not only is it certain to be shown with this infinite expression of the terms of the assumed series :

$$\overset{1}{(f+g)}+\overset{2}{(f+g)(f+2g)}+\overset{3}{(f+g)(f+2g)(f+3g)}+\text{etc.},$$

of which the index is $=\frac{1}{2}$, but also the same expression found is returned for the quadratures of curves . Indeed, on putting $n=\frac{1}{2}$ on account of $p=1$ and $q=2$ there becomes [§. 5.]

$$\int dx\left(-lx\right)^{\frac{1}{2}}=\sqrt{1\cdot 2}\int dx\sqrt{\left(x-xx\right)};$$

which expression integrated in the due manner gives the square root from the area of the circle, of which the diameter is $=1$; or on putting $1:\pi$ in the ratio of the diameter to the periphery there will be

$$\int dx\left(-lx\right)^{\frac{1}{2}}=\frac{\sqrt{\pi}}{2}.$$

Hence therefore the same term, of which the index $=\frac{1}{2}$, which we have put to be *z*, is found [*i.e.* putting $x=y^g$]

$$=\frac{g\sqrt{\pi g}}{(2f+3g)\int x^{f:g}dx\sqrt{(1-x)}}=\frac{\sqrt{\pi g}}{(2f+3g)\int y^{f+g-1}dy\sqrt{\left(1-y^{g}\right)}}$$

with the integral treated in the same manner, where before was prescribed in the ratio of the variable x . But by the reduction of formulas of integrals of this kind there is :

$$\int y^{f+g-1}dy\sqrt{\left(1-y^{g}\right)}=\frac{2fg}{(2f+g)(2f+3g)}\int\frac{y^{f-1}dy}{\sqrt{\left(1-y^{g}\right)}}=\frac{2f}{2f+3g}\int y^{f-1}dy\sqrt{\left(1-y^{g}\right)}.$$

With these substituted, there is found :

$$\frac{(2f+g)(2f+3g)(2f+3g)(2f+5g)(2f+5g)(2f+7g)(2f+7g)}{(2f+2g)(2f+2g)(2f+4g)(2f+4g)(2f+6g)(2f+6g)}\text{etc.}$$

$$=\frac{2ff(2f+g)}{\pi g}\left(\int y^{f-1}dy\sqrt{\left(1-y^{g}\right)}\right)^{2}=\frac{2ffg}{\pi(2f+g)}\left(\int\frac{y^{f-1}dy}{\sqrt{\left(1-y^{g}\right)}}\right)^{2}.$$

Therefore by this equation innumerable quadratures are able to be transformed into infinite products and in turn the values of infinite products of this kind can be transformed into the quadratures of curves.

8. So that we may illustrate this equality with examples, let $g=1$ and there will become

$$\int y^{f-1}dy\sqrt{(1-y)} = \frac{2\cdot 2\cdot 4\cdot 6\cdot 8\cdot 10\cdots(2f-2)}{3\cdot 5\cdot 7\cdot 9\cdot 11\cdot 13\cdots(2f+1)} \text{ etc.}$$

From which there becomes

$$\frac{2ff(2f+1)2\cdot 2\cdot 2\cdot 2\cdot 4\cdot 4\cdots(2f-2)(2f-2)}{\pi\cdot 3\cdot 3\cdot 5\cdot 5\cdot 7\cdot 7\cdots(2f+1)(2f+1)} \text{ etc.} = \frac{(2f+1)(2f+3)(2f+3)}{(2f+2)(2f+2)(2f+4)} \text{ etc.,}$$

which expression ordered or reduced to a continued series gives

$$\pi = 4\cdot\frac{2\cdot 4\cdot 4\cdot 6\cdot 6\cdot 8\cdot 8\cdot 10\cdot 10}{3\cdot 3\cdot 5\cdot 5\cdot 7\cdot 7\cdot 9\cdot\ 9\cdot 11} \text{ etc.,}$$

and which is that same formula WALLIS produced, whatever positive number may be substituted in place of f. But this same expression also arises, if there may be put $g=2$ and $f=$ some odd whole number.

9. Therefore, since there shall be

$$\frac{fg}{\pi}\left(\int\frac{y^{f-1}dy}{\sqrt{(1-y^g)}}\right)^2 = \frac{(2f+g)(2f+g)(2f+3g)(2f+3g)(2f+5g)(2f+5g)}{2f(2f+2g)(2f+2g)(2f+4g)(2f+4g)(2f+6g)} \text{ etc.,}$$

likewise there will be

$$\frac{hk}{\pi}\left(\int\frac{y^{h-1}dy}{\sqrt{(1-y^k)}}\right)^2 = \frac{(2h+k)(2h+k)(2h+3k)(2h+3k)(2h+5k)(2h+5k)}{2h(2h+2k)(2h+2k)(2h+4k)(2h+4k)(2h+6k)} \text{ etc.}$$

Whereby with this expression divided by that one the following equation will be obtained, free from the periphery of the circle π

$$\frac{fg\left(\int y^{f-1}dy:\sqrt{(1-y^g)}\right)^2}{hk\left(\int y^{h-1}dy:\sqrt{(1-y^k)}\right)^2} = \frac{2h(2f+g)^2(2h+2k)^2(2f+3g)^2(2h+4k)^2(2f+5g)^2}{2f(2h+k)^2(2f+2g)^2(2h+3k)^2(2f+4g)^2(2h+5k)^2} \text{ etc.,}$$

which, with the square root extracted, provides this equation :

$$\frac{\int y^{f-1}dy:\sqrt{\left(1-y^{g}\right)}}{\int y^{h-1}dy:\sqrt{\left(1-y^{k}\right)}}\cdot\sqrt{\frac{g}{k}}=\frac{2h(2f+g)(2h+2k)(2f+3g)(2h+4k)(2f+5g)}{2f(2h+k)(2f+2g)(2h+3k)(2f+4g)(2h+5k)}\text{ etc.}$$

10. But this infinite expression does not have a constant value; for, even if it may be continued to infinity, yet it will have another value, if the number of factors may be taken even, another, if the number may be taken odd. On account of which, unless there shall be $k=g$, in which case likewise, where the multiplication may be terminated, both factors are required to be accepted, with which done both equations will be obtained, just as the number of factors may be even or odd. But in the first place with this expression set out accurately there will be obtained :

$$\frac{g\int y^{f-1}dy:\sqrt{\left(1-y^{g}\right)}}{k\int y^{h-1}dy:\sqrt{\left(1-y^{k}\right)}}=\frac{2h(2f+g)}{2f(2h+k)}\cdot\frac{(2h+2k)(2f+3g)}{(2f+2g)(2h+3k)}\cdot\frac{(2h+4k)(2f+5g)}{(2f+4g)(2h+5k)}\cdot\frac{(2h+6k)(2f+7g)}{(2f+6g)(2h+7k)}\cdot\text{ etc.}$$

But with the other terms taken equal there will be :

$$\frac{f\int y^{f-1}dy:\sqrt{\left(1-y^{g}\right)}}{h\int y^{h-1}dy:\sqrt{\left(1-y^{k}\right)}}$$
$$=\frac{(2f+g)(2h+2k)}{(2h+k)(2f+2g)}\cdot\frac{(2f+3g)(2h+4k)}{(2h+3k)(2f+4g)}\cdot\frac{(2f+5g)(2h+6k)}{(2h+5k)(2f+6g)}\cdot\frac{(2f+7g)(2h+8k)}{(2h+7k)(2f+8g)}\cdot\text{etc.,}$$

in which expressions the places, where the operation may be allowed to terminate, are at distinct points.

11. Moreover, we will consider more carefully the case, in which there is $k=g$, certainly where the infinite expression can be considered to be consistent with simple factors, and there will become :

$$\frac{\int y^{f-1}dy:\sqrt{\left(1-y^{g}\right)}}{\int y^{h-1}dy:\sqrt{\left(1-y^{g}\right)}}=\frac{2h(2f+g)(2h+2g)(2f+3g)(2h+4g)}{2f(2h+g)(2f+2g)(2h+3g)(2f+4g)}\text{ etc.;}$$

so that which expression may be less than the preceding on account of the same letters mixed together, here we may put $2f=a,\ \ 2h=b$ and $y=x^{2}$, with which substituted there will be produced :

$$\frac{\int x^{a-1}dx:\sqrt{\left(1-x^{2g}\right)}}{\int x^{b-1}dx:\sqrt{\left(1-x^{2g}\right)}}=\frac{b(a+g)(b+2g)(a+3g)(b+4g)(a+5g)}{a(b+g)(a+2g)(b+3g)(a+4g)(b+5g)}\text{ etc.,}$$

which expression with the above § 9 given, which equally by making $y = x^2$ changes into this :

$$\frac{4fg}{\pi}\left(\int \frac{x^{2f-1}dx}{\sqrt{\left(1-x^{2g}\right)}}\right)^2 = \frac{(2f+g)(2f+g)(2f+3g)(2f+3g)(2f+5g)(2f+5g)}{2f(2f+2g)(2f+2g)(2f+4g)(2f+4g)(2f+6g)} \text{ etc.}$$

significant properties will be evident coupled together, the truth of which scarcely will be able to be shown otherwise.

12. Indeed it will be apparent at once, if there may be put $a = 2f$, $b = 2f + g$, that infinite expression is going to be transformed into this above; whereby also with these expressions equal, containing the quadratures of the curves, in this case they will become equal, from which the following equality emerges :

$$\frac{\int x^{2f-1}dx:\sqrt{\left(1-x^{2g}\right)}}{\int x^{2f+g-1}dx:\sqrt{\left(1-x^{2g}\right)}} = \frac{4fg}{\pi}\left(\int x^{2f-1}dx:\sqrt{\left(1-x^{2g}\right)}\right)^2,$$

if indeed after the integration there may be put $x = 1$. Hence therefore it follows there becomes :

$$\pi = 4fg\int \frac{x^{2f-1}dx}{\sqrt{\left(1-x^{2g}\right)}} \cdot \int \frac{x^{2f+g-1}dx}{\sqrt{\left(1-x^{2g}\right)}};$$

or if on putting $2f = a$, there will be

$$\pi = 2ag\int \frac{x^{a-1}dx}{\sqrt{\left(1-x^{2g}\right)}} \cdot \int \frac{x^{a+g-1}dx}{\sqrt{\left(1-x^{2g}\right)}},$$

which reasonably is a most noteworthy theorem, since by its usefulness, the product of two integrals, of which most often neither can be demonstrated, may be able to be assigned.

13. Indeed the truth of this theorem may be made clear from these cases, in which either formula depends on the quadrature of the circle or on an allowed absolute integration. For we may put $g = 1$ and $a = 1$; certainly there will become :

$$\pi = 2\int \frac{dx}{\sqrt{\left(1-x^2\right)}} \cdot \int \frac{x\,dx}{\sqrt{\left(1-x^2\right)}};$$

for

$$2\int \frac{dx}{\sqrt{\left(1-x^2\right)}}$$

on putting $x=1$, after the integration gives that same quantity *π,* and

$$\int \frac{x\,dx}{\sqrt{(1-xx)}} = 1-\sqrt{(1-xx)}$$

on making $x=1$ becomes $=1$. In a similar manner, if $a=2$ with there remaining $g=1$, there is seen to become :

$$\pi = 4\int \frac{xdx}{\sqrt{(1-xx)}} \cdot \int \frac{xx\,dx}{\sqrt{(1-xx)}};$$

for

$$\int \frac{xdx}{\sqrt{(1-xx)}} = 1 \quad \text{and} \quad \int \frac{xx\,dx}{\sqrt{(1-xx)}} = \frac{\pi}{4};$$

from which cases the truth of the theorem known from elsewhere may be confirmed.

14. Nevertheless the remaining cases, by which the magnitude of the integral cannot be shown either actually or by the quadrature of the circle, present just as many abstruse theorems especially worthy of investigation. Thus on putting $g=2$ and $a=1$ there becomes

$$\pi = 4\int \frac{dx}{\sqrt{(1-x^4)}} \cdot \int \frac{xx\,dx}{\sqrt{(1-x^4)}},$$

where

$$\int \frac{xx\,dx}{\sqrt{(1-x^4)}}$$

shows the *y* coordinate for a rectangular elastic curve, truly

$$\int \frac{dx}{\sqrt{(1-x^4)}}$$

the corresponding arc of the elastic abscissa *x*. [See Add. I , Section 27 of Euler's *Methodus inveniendi curvas…*: *Method of finding curves* on this website] On account of which the rectangle from the corresponding elastic arc of abscissa 1 and the corresponding applied line will be equal to the area of a circle, of which the diameter is that abscissa 1; and which elastic properties will be able to be demonstrated by another method perhaps scarcely known or not.

15. But before we leave this case of the elasticity, it will help to express each integral by a series of the ordinates at least in the case where $x=1$. Indeed since there shall be

$$\frac{1}{\sqrt{(1-x^4)}} = \frac{(1+x^2)^{-\frac{1}{2}}}{\sqrt{(1-x^2)}}$$

and

$$\left(1+xx\right)^{-\frac{1}{2}}=1-\tfrac{1}{2}x^{2}+\tfrac{1\cdot 3}{2\cdot 4}x^{4}-\tfrac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^{6}+\text{etc.},$$

the individual members will depend on the quadrature of the circle. But with each absolute integration for the case $x=1$ there will be

$$\int\frac{dx}{\sqrt{\left(1-x^{4}\right)}}=\frac{\pi}{2}\left(1-\tfrac{1}{4}+\tfrac{1\cdot 9}{4\cdot 16}-\tfrac{1\cdot\ 9\cdot 25}{4\cdot 16\cdot 36}+\text{etc.}\right)$$

and

$$\int\frac{x^{2}dx}{\sqrt{\left(1-x^{4}\right)}}=\frac{\pi}{2}\left(\tfrac{1}{2}-\tfrac{1\cdot 3}{4\cdot 4}+\tfrac{1\cdot\ 9\cdot 5}{4\cdot 16\cdot 6}-\tfrac{1\cdot\ 9\cdot 25\cdot 7}{4\cdot 16\cdot 36\cdot 8}+\text{etc.}\right)$$

But hence by approximating there arises very closely

$$\int\frac{dx}{\sqrt{\left(1-x^{4}\right)}}=\frac{5}{6}\cdot\frac{\pi}{2}\quad\text{and}\quad\int\frac{xxdx}{\sqrt{\left(1-x^{4}\right)}}=\frac{3}{5}\cdot\frac{\pi}{2}$$

16. If there were $a=1$, there will become

$$\pi=2g\int\frac{dx}{\sqrt{\left(1-x^{2g}\right)}}\cdot\int\frac{x^{g}dx}{\sqrt{\left(1-x^{2g}\right)}},$$

which two integral expressions are prepared thus, so that, if

$$\int\frac{x^{g}dx}{\sqrt{\left(1-x^{2g}\right)}}$$

were the applied line [i.e. the *y* coordinate] of some curve corresponding to the abscissa *x*,

$$\int\frac{dx}{\sqrt{\left(1-x^{2g}\right)}}$$

shall become the length of the same curve. On account of which if on this curve the abscissa may be taken $x=1$, the product or the rectangle from the applied line by the length of the curve to the area of the circle, of which the diameter is the abscissa $x=1$, will be as 2 to the number *g;* which proposition is valid, as long as *g* were a positive number; for negative values may be excluded at once.

17. If $a-1$ may be taken less than *g,* thus so that the numbers *a* and *g* shall be prime relative to each other, the following noteworthy theorems will be had ; for if

$$a+g-1>2g,$$

then the integration may be able to be reduced to a simpler form.

$$\pi=2\int\frac{dx}{\sqrt{(1-x^2)}}\cdot\int\frac{xdx}{\sqrt{(1-x^2)}}$$
$$\pi=4\int\frac{dx}{\sqrt{(1-x^4)}}\cdot\int\frac{x^2dx}{\sqrt{(1-x^4)}}$$
$$\pi=6\int\frac{dx}{\sqrt{(1-x^6)}}\cdot\int\frac{x^3dx}{\sqrt{(1-x^6)}}$$
$$\pi=12\int\frac{xdx}{\sqrt{(1-x^6)}}\cdot\int\frac{x^4dx}{\sqrt{(1-x^6)}}$$
$$\pi=8\int\frac{dx}{\sqrt{(1-x^8)}}\cdot\int\frac{x^4dx}{\sqrt{(1-x^8)}}$$
$$\pi=12\int\frac{dx}{\sqrt{(1-x^{12})}}\cdot\int\frac{x^6dx}{\sqrt{(1-x^{12})}}$$
$$\pi=60\int\frac{x^4dx}{\sqrt{(1-x^{12})}}\cdot\int\frac{x^{10}dx}{\sqrt{(1-x^{12})}}$$
$$\pi=14\int\frac{dx}{\sqrt{(1-x^{14})}}\cdot\int\frac{x^7dx}{\sqrt{(1-x^{14})}}$$
$$\pi=24\int\frac{x^2dx}{\sqrt{(1-x^8)}}\cdot\int\frac{x^6dx}{\sqrt{(1-x^8)}}$$
$$\pi=10\int\frac{dx}{\sqrt{(1-x^{10})}}\cdot\int\frac{x^5dx}{\sqrt{(1-x^{10})}}$$
$$\pi=20\int\frac{xdx}{\sqrt{(1-x^{10})}}\cdot\int\frac{x^6dx}{\sqrt{(1-x^{10})}}$$
$$\pi=30\int\frac{x^2dx}{\sqrt{(1-x^{10})}}\cdot\int\frac{x^7dx}{\sqrt{(1-x^{10})}}$$
$$\pi=40\int\frac{x^3dx}{\sqrt{(1-x^{10})}}\cdot\int\frac{x^8dx}{\sqrt{(1-x^{10})}}$$
$$\pi=28\int\frac{xdx}{\sqrt{(1-x^{14})}}\cdot\int\frac{x^8dx}{\sqrt{(1-x^{14})}}$$
$$\pi=42\int\frac{x^2dx}{\sqrt{(1-x^{14})}}\cdot\int\frac{x^9dx}{\sqrt{(1-x^{14})}}$$
$$\pi=56\int\frac{x^3dx}{\sqrt{(1-x^{14})}}\cdot\int\frac{x^{10}dx}{\sqrt{(1-x^{14})}}$$
$$\pi=70\int\frac{x^4dx}{\sqrt{(1-x^{14})}}\cdot\int\frac{x^{11}dx}{\sqrt{(1-x^{14})}}.$$

18. Therefore with this found also the reduction of integral formulas towards more simple ones is significantly advanced. For since at this point only these two formulas

$$\int\frac{x^m dx}{\sqrt{(1-x^{2g})}} \quad\text{and}\quad \int\frac{x^{m+n}dx}{\sqrt{(1-x^{2g})}}$$

may have been able to be reduced in turn, if n were a multiple of the exponent $2g$, thus now the reduction also will succeed, if n were some multiple of g, in the case understood, where there becomes $x=1$. But just as if n were the product of the exponent g by an even number, the quotient, which results from the division of the one formula by the other, may be assigned easily, thus on the contrary, if n shall be a factor from g in an odd number, then the product of the formulas is assigned most easily.

19. All these return to this, so that, if the integral of the formula were known in the case,

$$\int \frac{x^m dx}{\sqrt{\left(1-x^{2g}\right)}}$$

where $x=1$, in the same case also the integral of this formula

$$\int \frac{x^{m+n} dx}{\sqrt{\left(1-x^{2g}\right)}},$$

if n shall be a multiple of g, may be able to be shown. For if A shall be the integral of the formula

$$\int \frac{x^m dx}{\sqrt{\left(1-x^{2g}\right)}}$$

in the case where there is $x=1$; the integrals of the other formula by putting g, $2g$, $3g$ etc. successively in place of n themselves will be had in the following manner :

$$\int \frac{x^m dx}{\sqrt{\left(1-x^{2g}\right)}} = A,$$

$$\int \frac{x^{m+g} dx}{\sqrt{\left(1-x^{2g}\right)}} = \frac{\pi}{2(m+1)gA},$$

$$\int \frac{x^{m+2g} dx}{\sqrt{\left(1-x^{2g}\right)}} = \frac{(m+1)A}{m+g+1},$$

$$\int \frac{x^{m+3g} dx}{\sqrt{\left(1-x^{2g}\right)}} = \frac{(m+g+1)\pi}{2(m+1)(m+2g+1)gA},$$

$$\int \frac{x^{m+4g} dx}{\sqrt{\left(1-x^{2g}\right)}} = \frac{(m+1)(m+2g+1)A}{(m+g+1)(m+3g+1)},$$

$$\int \frac{x^{m+5g} dx}{\sqrt{\left(1-x^{2g}\right)}} = \frac{(m+g+1)(m+3g+1)\pi}{2(m+1)(m+2g+1)(m+4g+1)gA},$$

etc.

20. Thereupon since this general formula

$$\int x^{m+ig} dx\left(1-x^{2g}\right)^{k-\frac{1}{2}}$$

with i and k denoting some whole numbers, may be able to be reduced to this formula

$$\int\frac{x^{m+ig}dx}{\sqrt{\left(1-x^{2g}\right)}},$$

it is understood the most general integral $\int x^{m+ig}dx\left(1-x^{2g}\right)^{k-\frac{1}{2}}$ of this formula can be assigned from the known integral

$$\int\frac{x^{m}dx}{\sqrt{\left(1-x^{2g}\right)}},$$

at least in the case where there is put $x=1$ after the integration. But the cases, in which i is an odd number, besides this integral also require the quadrature of the circle π.

21. Therefore just as I have deduced comparisons through the term of the indices $\frac{1}{2}$ of the series § 5 assumed above for these integral formulas, thus it will perhaps be worth the effort to investigate other intermediate terms in a similar manner. Therefore the term is sought, of which the index is $\frac{p}{q}$, which may be put $=z$, from which the following will be had:

$$\begin{array}{lll} \frac{p}{q} & \frac{p+q}{q} & \frac{p+2q}{q} \\ z+ & \frac{z(fq+(p+q))g}{q}+ & \frac{z(fq+(p+q)g)(fq+(p+2q)g)}{q^{2}}+\text{etc.} \end{array}$$

Now by considering in an equal manner, how this progression finally may be changed into a geometric one, the following approximations may arise for the term z:

$$\text{I.}\qquad z=1(f+g)^{\frac{p}{q}},$$

$$\text{II.}\quad \frac{z(fq+(p+q))g}{q}=(f+g)^{\frac{q-p}{q}}(f+g)^{\frac{p}{q}}(f+2g)^{\frac{p}{q}},$$

$$\text{III.}\quad z\left(f+\left(\frac{p+q}{q}\right)g\right)\left(f+\left(\frac{p+2q}{q}\right)g\right)$$
$$=(f+g)^{\frac{q-p}{q}}(f+g)^{\frac{p}{q}}(f+2g)^{\frac{q-p}{q}}(f+2g)^{\frac{p}{q}}(f+3g)^{\frac{p}{q}}.$$

Hence therefore the true value of z may be elicited

$$=\frac{(f+g)^{\frac{p}{q}}(f+g)^{\frac{q-p}{q}}(f+2g)^{\frac{p}{q}}(f+2g)^{\frac{q-p}{q}}(f+3g)^{\frac{p}{q}}(f+3g)^{\frac{q-p}{q}}}{1\left(f+\frac{p+q}{q}g\right)^{\frac{p}{q}}\left(f+\frac{p+q}{q}g\right)^{\frac{q-p}{q}}\left(f+\frac{p+2q}{q}g\right)^{\frac{p}{q}}\left(f+\frac{p+2q}{q}g\right)^{\frac{q-p}{q}}\left(f+\left(\frac{p+3q}{q}\right)g\right)^{\frac{p}{q}}}\ \text{etc.}$$

Or with a few changes, so that the infinitesimal factors may become $=1$ and the expression, where it may be allowed to be terminated, there will become

$$\frac{z}{\left(f+\frac{p}{q}g\right)^{\frac{p}{q}}}=\frac{(f+g)^{\frac{p}{q}}}{\left(f+\frac{p}{q}g\right)^{\frac{p}{q}}}\cdot\frac{(f+g)^{\frac{q-p}{q}}}{\left(f+\frac{p+q}{q}g\right)^{\frac{q-p}{q}}}\cdot\frac{(f+2g)^{\frac{p}{q}}}{\left(f+\frac{p+q}{q}g\right)^{\frac{p}{q}}}$$
$$\cdot\frac{(f+2g)^{\frac{q-p}{q}}}{\left(f+\frac{p+2q}{q}g\right)^{\frac{q-p}{q}}}\cdot\frac{(f+3g)^{\frac{p}{q}}}{\left(f+\frac{p+2q}{q}g\right)^{\frac{p}{q}}}\cdot\text{etc.},$$

the law of which expression, by which the factors are progressing, may be clear at once.

22. But intermediate terms of the same value z can be expressed with the aid of the general term; indeed there will become

$$z=\frac{g^{\frac{p+q}{q}}\int dx(-lx)^{\frac{p}{q}}}{\left(f+\frac{p+q}{q}g\right)\int x^{f:g}dx(1-x)^{\frac{p}{q}}}.$$

Whereby if there may be put

$$\int dx(-lx)^{\frac{p}{q}}=\sqrt[q]{1\cdot 2\cdot 3\cdots p\left(\frac{2p}{q}+1\right)\left(\frac{3p}{q}+1\right)\left(\frac{4p}{q}+1\right)\cdots\left(\frac{qp}{q}+1\right)}$$
$$\times\int dx\left(x-x^2\right)^{\frac{p}{q}}\cdot\int dx\left(x^2-x^3\right)^{\frac{p}{q}}\cdot\int dx\left(x^3-x^4\right)^{\frac{p}{q}}\cdots\int dx\left(x^{q-1}-x^q\right)^{\frac{p}{q}}=\sqrt[q]{P}$$

and $x=y^g$, from which there becomes

$$\int x^{f:g}dx(1-x)^{\frac{p}{q}}=g\int y^{f+g-1}dy\left(1-y^g\right)^{\frac{p}{q}}=$$
$$\frac{ggp}{fq+(p+q)g}\int\frac{y^{f+g-1}dy}{\left(1-y^g\right)^{\frac{q-p}{q}}}=\frac{pfgg}{q\left(f+\frac{p}{q}g\right)\left(f+\frac{p+q}{q}g\right)}\int\frac{y^{f-1}dy}{\left(1-y^g\right)^{\frac{q-p}{q}}},$$

again there may be put

$$\int\frac{y^{f-1}dy}{\left(1-y^g\right)^{\frac{q-p}{q}}}=Q,$$

there will be

$$z=\frac{q\left(f+\frac{p}{q}g\right)P^{\frac{1}{q}}}{pfg^{\frac{q-p}{q}}Q}.$$

23. Now with the above infinite expression substituted in place of z, and with the above powers of the exponent taken of the exponent q, this equation will be produced :

$$\frac{q^qP}{p^qf^pg^{q-p}Q^q}=\frac{f^{q-p}}{\left(f+\frac{p}{q}g\right)^{q-p}}\cdot\frac{(f+g)^p}{\left(f+\frac{p}{q}g\right)^p}\cdot\frac{(f+g)^{q-p}}{\left(f+\frac{p+q}{q}g\right)^{q-p}}$$
$$\cdot\frac{(f+2g)^p}{\left(f+\frac{p+q}{q}g\right)^p}\cdot\frac{(f+2g)^{q-p}}{\left(f+\frac{p+2q}{q}g\right)^{q-p}}\cdot\text{etc.}$$

Therefore if in a similar manner there may be put

$$\int\frac{y^{h-1}dy}{\left(1-y^g\right)^{\frac{q-p}{q}}}=R,$$

there will become

$$\frac{p^qh^pg^{q-p}R^q}{q^qP}=\frac{\left(h+\frac{p}{q}g\right)^{q-p}}{h^{q-p}}\cdot\frac{\left(h+\frac{p}{q}g\right)^p}{(h+g)^p}\cdot\frac{\left(h+\frac{p+q}{q}g\right)^{q-p}}{(h+g)^{q-p}}\cdot\text{etc.},$$

which two expressions multiplied together will give :

$$\frac{h^pR^q}{f^pQ^q}=\frac{f^{q-p}\left(h+\frac{p}{q}g\right)^q}{h^{q-p}\left(f+\frac{p}{q}g\right)^q}\cdot\frac{(f+g)^q}{(h+g)^q}\cdot\frac{\left(h+\frac{p+q}{q}g\right)^q}{\left(f+\frac{p+q}{q}g\right)^q}\cdot\frac{(f+2g)^q}{(h+2g)^q}\cdot\frac{\left(h+\frac{p+2q}{q}g\right)^q}{\left(f+\frac{p+2q}{q}g\right)^q}\cdot\text{etc.},$$

24. Therefore if both sides may be multiplied by $\frac{f^p}{h^p}$ and the root of the power q may be extracted, there will be found :

$$\frac{R}{Q}=\frac{f\left(h+\frac{p}{q}g\right)}{h\left(f+\frac{p}{q}g\right)}\cdot\frac{(f+g)}{(h+g)}\cdot\frac{\left(h+\frac{p+q}{q}g\right)}{\left(f+\frac{p+q}{q}g\right)}\cdot\frac{(f+2g)}{(h+2g)}\cdot\frac{\left(h+\frac{p+2q}{q}g\right)}{\left(f+\frac{p+2q}{q}g\right)}\cdot\text{etc.}$$
$$=\frac{\int y^{h-1}dy\left(1-y^g\right)^{\frac{p-q}{q}}}{\int y^{f-1}dy\left(1-y^g\right)^{\frac{p-q}{q}}},$$

in which integrations, since thus they will have been taken, so that they may vanish on putting $y=0$, there must become $y=1$, with which done the value of the proposed infinite expression will be had by the quadratures. Therefore with the aid of this infinite expression another quadrature will be able to be reduced to that other, if indeed there may be put $y=1$.

25. But so that we may deduce comparisons of this kind of integral, thus as from the previous case, where there was $p=1$ and $q=2$, here we may put $p=1$ and $q=3$ and there will become

$$P=\tfrac{10}{3}\int dx\left(x-x^2\right)^{\frac{1}{3}}\cdot\int dx\left(x^2-x^3\right)^{\frac{1}{3}}$$

and

$$Q=\int\frac{y^{f-1}dy}{\left(1-y^g\right)^{\frac{2}{3}}}$$

and

$$R=\int\frac{y^{h-1}dy}{\left(1-y^g\right)^{\frac{2}{3}}}.$$

therefore there will become:

$$\frac{27P}{fg^2Q^3}=\frac{ff(f+g)(f+g)(f+g)(f+2g)}{(f+\frac{1}{3}g)(f+\frac{1}{3}g)(f+\frac{1}{3}g)(f+\frac{4}{3}g)(f+\frac{4}{3}g)(f+\frac{4}{3}g)}\text{ etc.}$$

and

$$\frac{R}{Q}=\frac{f(f+\frac{1}{3}g)(f+g)(f+\frac{4}{3}g)(f+2g)(h+\frac{7}{3}g)}{h(f+\frac{1}{3}g)(h+g)(f+\frac{4}{3}g)(h+2g)(f+\frac{7}{3}g)}\text{ etc.,}$$

which two expressions, since that one depends on one revolution from three, but this agrees on one revolution from two factors, are unable to be transformed mutually into each other, whatever also may be substituted in place of h.

26. Therefore let there be

$$S=\int\frac{y^{k-1}dy}{\left(1-y^g\right)^{\frac{2}{3}}};$$

there will become

$$\frac{S}{Q}=\frac{f(k+\frac{1}{3}g)(f+g)(k+\frac{4}{3}g)(f+2g)(k+\frac{7}{3}g)}{k(f+\frac{1}{3}g)(k+g)(f+\frac{4}{3}g)(k+2g)(f+\frac{7}{3}g)}\text{ etc.,}$$

which joined together with the previous expression will give :

$$\frac{RS}{Q^2}=\frac{ff(h+\frac{1}{3}g)(k+\frac{1}{3}g)(f+g)(f+g)(h+\frac{4}{3}g)}{hk(f+\frac{1}{3}g)(f+\frac{1}{3}g)(h+g)(k+g)(f+\frac{4}{3}g)}\text{ etc.,}$$

which expression may be converted into that equal to $\frac{27P}{fg^2Q^3}$ on putting

$$h=f+\tfrac{1}{3}g \text{ and } k=f+\tfrac{2}{3}g.$$

On account of which this equation will be had :

$$\frac{27P}{fg^2}=QRS\,,$$

or with the true values substituted there will become :

$$90\int dx\left(x-x^2\right)^{\frac{1}{3}}\cdot\int dx\left(x^2-x^3\right)^{\frac{1}{3}}=fg^2\int\frac{y^{f-1}}{\left(1-y^g\right)^{\frac{2}{3}}}\cdot\int\frac{y^{f+\frac{1}{3}g-1}}{\left(1-y^g\right)^{\frac{2}{3}}}\cdot\int\frac{y^{f+\frac{2}{3}g-1}}{\left(1-y^g\right)^{\frac{2}{3}}}.$$

27. But before we may pursue this further, it may be agreed to attribute a more convenient form generally to the value of P. But on making $x=z^q$ since there shall become

$$\int dx(x^n-x^{n+1})^{\frac{p}{q}}=\frac{npq}{(n+1)\left((n+1)p+q\right)}\int\frac{z^{np-1}}{(1-z^q)^{\frac{q-p}{q}}}dz$$

after the substitution there will be produced

$$P=1\cdot2\cdot3\cdots p\frac{p^{q-1}}{q}\int\frac{z^{p-1}dz}{(1-z^q)^{\frac{q-p}{q}}}\cdot\int\frac{z^{2p-1}dz}{(1-z^q)^{\frac{q-p}{q}}}\cdot\int\frac{z^{3p-1}dz}{(1-z^q)^{\frac{q-p}{q}}}\cdots\int\frac{z^{(q-1)p-1}dz}{(1-z^q)^{\frac{q-p}{q}}}.$$

From which expression if the root of the power q may be extracted, the value of $\int dx\left(-lx\right)^{\frac{p}{q}}$ will be produced.

28. Now on putting $p=1$ and $q=3$ there will be produced :

$$P=\tfrac{1}{3}\int\frac{dz}{\left(1-z^3\right)^{\frac{2}{3}}}\cdot\int\frac{zdz}{\left(1-z^3\right)^{\frac{2}{3}}}.$$

But on putting $y=z^3$ the following equation will be obtained :

$$\int\frac{dz}{\left(1-z^3\right)^{\frac{2}{3}}}\cdot\int\frac{zdz}{\left(1-z^3\right)^{\frac{2}{3}}}=3fg^2\int\frac{z^{3f-1}dz}{\left(1-z^{3g}\right)^{\frac{2}{3}}}\cdot\int\frac{z^{3f+g-1}dz}{\left(1-z^{3g}\right)^{\frac{2}{3}}}\cdot\int\frac{z^{3f+2g-1}dz}{\left(1-z^{3g}\right)^{\frac{2}{3}}}.$$

If now there may be put $3f=a$, the following noteworthy equation will arise :

$$\int\frac{dz}{\left(1-z^3\right)^{\frac{2}{3}}}\cdot\int\frac{zdz}{\left(1-z^3\right)^{\frac{2}{3}}}=ag^2\int\frac{z^{a-1}dz}{\left(1-z^{3g}\right)^{\frac{2}{3}}}\cdot\int\frac{z^{a+g-1}dz}{\left(1-z^{3g}\right)^{\frac{2}{3}}}\cdot\int\frac{z^{a+2g-1}dz}{\left(1-z^{3g}\right)^{\frac{2}{3}}}.$$

Which compared with the above

$$\int\frac{dz}{\sqrt{\left(1-z^2\right)}}=ag\int\frac{z^{a-1}dz}{\sqrt{\left(1-z^{2g}\right)}}\cdot\int\frac{z^{a+g-1}dz}{\sqrt{\left(1-z^{2g}\right)}}$$

now indicates in a certain way, how the sequences of this kind shall themselves become equations.

29. But before I run the risk of concluding anything by induction, I may set out some actual cases. Therefore let $p=2$ and $q=3$ and hence there will be found

$$P=\frac{8}{3}\int\frac{zdz}{\left(1-z^3\right)^{\frac{1}{3}}}\cdot\int\frac{z^3dz}{\left(1-z^3\right)^{\frac{1}{3}}}=\frac{8}{9}\int\frac{dz}{\left(1-z^3\right)^{\frac{1}{3}}}\cdot\int\frac{zdz}{\left(1-z^3\right)^{\frac{1}{3}}},$$

$$Q=\int\frac{y^{f-1}dy}{\left(1-y^g\right)^{\frac{1}{3}}},\quad R=\int\frac{y^{h-1}dy}{\left(1-y^g\right)^{\frac{1}{3}}}.$$

Moreover the infinite expressions thus will be found :

$$\frac{27P}{8f^2gQ^3}=\frac{f(f+g)(f+g)(f+g)(f+2g)(f+2g)}{(f+\frac{2}{3}g)(f+\frac{2}{3}g)(f+\frac{2}{3}g)(f+\frac{5}{3}g)(f+\frac{5}{3}g)(f+\frac{5}{3}g)}\text{ etc.}$$

and

$$\frac{R}{Q}=\frac{f(h+\frac{2}{3}g)(f+g)(h+\frac{5}{3}g)(f+2g)(h+\frac{8}{3}g)}{h(f+\frac{2}{3}g)(h+g)(f+\frac{5}{3}g)(h+2g)(f+\frac{8}{3}g)}\text{ etc.}$$

Besides, there shall be

$$S=\int\frac{y^{m-1}dy}{\left(1-y^g\right)^{\frac{1}{3}}}\quad\text{and}\quad T=\int\frac{y^{n-1}dy}{\left(1-y^g\right)^{\frac{1}{3}}};$$

will be

$$\frac{T}{S}=\frac{m(n+\frac{2}{3}g)(m+g)(n+\frac{5}{3}g)(m+2g)}{n(m+\frac{2}{3}g)(n+g)(m+\frac{5}{3}g)(n+2g)}\text{ etc.,}$$

which two expressions multiplied together give :

$$\frac{RT}{QS}=\frac{fm(h+\frac{2}{3}g)(n+\frac{2}{3}g)(f+g)(m+g)(h+\frac{5}{3}g)(n+\frac{5}{3}g)}{hn(f+\frac{2}{3}g)(m+\frac{2}{3}g)(h+g)(n+g)(f+\frac{5}{3}g)(m+\frac{5}{3}g)}\text{ etc.}$$

30. But this expression, to which $\frac{27P}{8f^2gQ^3}$ has been found equal, cannot be reduced to that, unless that may be multiplied by $\frac{f}{f-\frac{1}{3}g}$, thus so that there shall become

$$\frac{27P}{8fg\left(f-\frac{1}{3}g\right)Q^3}=\frac{ff(f+g)(f+g)(f+g)(f+2g)}{(f-\frac{1}{3}g)(f+\frac{2}{3}g)(f+\frac{2}{3}g)(f+\frac{2}{3}g)(f+\frac{5}{3}g)(f+\frac{5}{3}g)}\text{ etc.}$$

now indeed the reduction will become on putting

$$m=f,\ h=f-\tfrac{1}{3}g\ \text{ and }\ n=f+\tfrac{1}{3}g.$$

Therefore with these values substituted there will be

$$\frac{27P}{8fg\left(f-\frac{1}{3}g\right)Q^3}=\frac{RT}{QS}.$$

Truly since there shall be $S=Q$ and

$$R=\int\frac{y^{f-\frac{1}{3}g-1}dy}{\left(1-y^g\right)^{\frac{1}{3}}}=\frac{f+\frac{1}{3}g}{f-\frac{1}{3}g}\int\frac{y^{f+\frac{2}{3}g-1}dy}{\left(1-y^g\right)^{\frac{1}{3}}}$$

and

$$T=\int\frac{y^{f+\frac{1}{3}g-1}dy}{\left(1-y^g\right)^{\frac{1}{3}}},$$

this equation will be obtained on putting $y=z^3$

$$\int\frac{dz}{\left(1-z^3\right)^{\frac{1}{3}}}\cdot\int\frac{zdz}{\left(1-z^3\right)^{\frac{1}{3}}}=3fg\left(3f+g\right)\int\frac{z^{3f-1}dz}{\left(1-z^{3g}\right)^{\frac{1}{3}}}\cdot\int\frac{z^{3f+g-1}dz}{\left(1-z^{3g}\right)^{\frac{1}{3}}}\cdot\int\frac{z^{3f+2g-1}dz}{\left(1-z^{3g}\right)^{\frac{1}{3}}}.$$

And if there may be put $3f=a$, there will be

$$\int\frac{dz}{\left(1-z^3\right)^{\frac{1}{3}}}\cdot\int\frac{zdz}{\left(1-z^3\right)^{\frac{1}{3}}}=ag\left(a+g\right)\int\frac{z^{a-1}dz}{\left(1-z^{3g}\right)^{\frac{1}{3}}}\cdot\int\frac{z^{a+g-1}dz}{\left(1-z^{3g}\right)^{\frac{1}{3}}}\cdot\int\frac{z^{a+2g-1}dz}{\left(1-z^{3g}\right)^{\frac{1}{3}}}.$$

31. We may put $p=1$ and $q=4$ and there will be had:

$$\frac{4^4P}{fg^3Q^4}=\frac{fff(f+g)(f+g)(f+g)}{(f+\frac{1}{4}g)(f+\frac{1}{4}g)(f+\frac{1}{4}g)(f+\frac{1}{4}g)(f+\frac{5}{4}g)(f+\frac{5}{4}g)}\text{ etc.}$$

and

$$\frac{R}{Q}=\frac{f(h+\frac{1}{4}g)(f+g)(h+\frac{5}{4}g)(f+2g)}{h(f+\frac{1}{4}g)(h+g)(f+\frac{5}{4}g)(h+2g)}\text{ etc.}$$

But if there shall be as before:

$$S=\int\frac{y^{m-1}dy}{\left(1-y^{g}\right)^{\frac{q-p}{q}}},\quad T=\int\frac{y^{n-1}dy}{\left(1-y^{g}\right)^{\frac{q-p}{q}}};$$

there will become

$$\frac{RST}{Q^{3}}=\frac{fff(h+\frac{1}{4}g)(m+\frac{1}{4}g)(n+\frac{1}{4}g)(f+g)}{hmn(f+\frac{1}{4}g)(h+\frac{1}{4}g)(f+\frac{1}{4}g)(h+g)}\text{ etc.,}$$

6 factors of which expression are required to be transformed into these four, which happens on putting

$$h=f+\tfrac{1}{4}g,\quad m=f+\tfrac{2}{4}g\quad\text{and}\quad n=f+\tfrac{3}{4}g,$$

with which done there will be had

$$4^{4}P=fg^{3}QRST.$$

Whereby since there shall be

$$P=\frac{1}{4}\int\frac{dz}{\left(1-z^{4}\right)^{\frac{3}{4}}}\cdot\int\frac{zdz}{\left(1-z^{4}\right)^{\frac{3}{4}}}\cdot\int\frac{z^{2}dz}{\left(1-z^{4}\right)^{\frac{3}{4}}},$$

if there may be put $y=z^{4}$ and $4f=a$, this equation will arise :

$$\int\frac{dz}{\left(1-z^{4}\right)^{\frac{3}{4}}}\cdot\int\frac{zdz}{\left(1-z^{4}\right)^{\frac{3}{4}}}\cdot\int\frac{z^{2}dz}{\left(1-z^{4}\right)^{\frac{3}{4}}}$$
$$=ag^{3}\int\frac{z^{a-1}dz}{\left(1-z^{4g}\right)^{\frac{3}{4}}}\cdot\int\frac{z^{a+g-1}dz}{\left(1-z^{4g}\right)^{\frac{3}{4}}}\cdot\int\frac{z^{a+2g-1}dz}{\left(1-z^{4g}\right)^{\frac{3}{4}}}\cdot\int\frac{z^{a+3g-1}dz}{\left(1-z^{4g}\right)^{\frac{3}{4}}},$$

the connection of which with the previous cases, in which there was $p=1,\ q=2$ and $p=1,\ q=3$, is easily seen.

32. Therefore from these all the equations of this kind will be allowed to be formed, which arise if there may be put $p=1$ and $q=$ to any whole positive number; clearly there will be :

I. $$\int\frac{dz}{\sqrt{(1-z^2)}}=ag\int\frac{z^{a-1}dz}{\sqrt{(1-z^{2g})}}\cdot\int\frac{z^{a+g-1}dz}{\sqrt{(1-z^{2g})}},$$

II. $$\int\frac{dz}{(1-z^3)^{\frac{2}{3}}}\cdot\int\frac{zdz}{(1-z^3)^{\frac{2}{3}}}$$
$$=ag^2\int\frac{z^{a-1}dz}{(1-z^{3g})^{\frac{2}{3}}}\cdot\int\frac{z^{a+g-1}dz}{(1-z^{3g})^{\frac{2}{3}}}\cdot\int\frac{z^{a+2g-1}dz}{(1-z^{3g})^{\frac{2}{3}}},$$

III. $$\int\frac{dz}{(1-z^4)^{\frac{3}{4}}}\cdot\int\frac{zdz}{(1-z^4)^{\frac{3}{4}}}\cdot\int\frac{z^2dz}{(1-z^4)^{\frac{3}{4}}}$$
$$=ag^3\int\frac{z^{a-1}dz}{(1-z^{4g})^{\frac{3}{4}}}\cdot\int\frac{z^{a+g-1}dz}{(1-z^{4g})^{\frac{3}{4}}}\cdot\int\frac{z^{a+2g-1}dz}{(1-z^{4g})^{\frac{3}{4}}}\cdot\int\frac{z^{a+3g-1}dz}{(1-z^{4g})^{\frac{3}{4}}},$$

IV. $$\int\frac{dz}{(1-z^5)^{\frac{4}{5}}}\cdot\int\frac{zdz}{(1-z^5)^{\frac{4}{5}}}\cdot\int\frac{z^2dz}{(1-z^5)^{\frac{4}{5}}}\cdot\int\frac{z^3dz}{(1-z^5)^{\frac{4}{5}}}$$
$$=ag^4\int\frac{z^{a-1}dz}{(1-z^{5g})^{\frac{4}{5}}}\cdot\int\frac{z^{a+g-1}dz}{(1-z^{5g})^{\frac{4}{5}}}\cdot\int\frac{z^{a+2g-1}dz}{(1-z^{5g})^{\frac{4}{5}}}\cdot\int\frac{z^{a+3g-1}dz}{(1-z^{5g})^{\frac{4}{5}}}\cdot\int\frac{z^{a+4g-1}dz}{(1-z^{5g})^{\frac{4}{5}}}$$

etc.

33. So that also we may be able to deduce these equation which arise, if p is not $=1$, we may put $p=3$ and $q=4$; with which put in place and with the rest remaining as above there will become :

$$\frac{4^4P}{3^4f^3gQ^4}=\frac{f(f+g)(f+g)(f+g)}{(f+\frac{3}{4}g)(f+\frac{3}{4}g)(f+\frac{3}{4}g)(f+\frac{3}{4}g)}\text{ etc.},$$

where the remaining constant terms may be formed from these four factors and from these the individual factors being augmented by the quantity g. Truly in a similar way there will become :

$$\frac{RST}{Q^3}=\frac{fff(h+\frac{3}{4}g)(m+\frac{3}{4}g)(n+\frac{3}{4}g)}{hmn(f+\frac{3}{4}g)(f+\frac{3}{4}g)(f+\frac{3}{4}g)}\text{ etc.},$$

where six factors make on revolution or period. But for the comparison required to be put in place it is necessary each series may be considered thus :

$$\frac{4^4P}{3^4f^2(f-\frac{1}{4}g)Q^4}=\frac{ff(f+g)(f+g)}{(f-\frac{1}{4}g)(f+\frac{3}{4}g)(f+\frac{3}{4}g)(f+\frac{3}{4}g)}\text{ etc.,}$$

$$\frac{hRST}{fQ^3}=\frac{ff(h+\frac{3}{4}g)(m+\frac{3}{4}g)(n+\frac{3}{4}g)(f+g)}{mn(f+\frac{3}{4}g)(f+\frac{3}{4}g)(f+\frac{3}{4}g)(h+g)}\text{ etc.,}$$

of which this may be transformed into that, thus so that there becomes

$$\frac{4^4P}{3^4f(f-\frac{1}{4}g)}=QRST,$$

if there may become

$$h=f+\tfrac{1}{4}g,\ \ m=f-\tfrac{1}{4}g\ \text{ and }\ n=f+\tfrac{2}{4}g.$$

34. Therefore since there shall be

$$P=\frac{3^4}{2}\int\frac{z^2dz}{\left(1-z^4\right)^{\frac{1}{4}}}\cdot\int\frac{z^5dz}{\left(1-z^4\right)^{\frac{1}{4}}}\cdot\int\frac{z^8dz}{\left(1-z^4\right)^{\frac{1}{4}}}$$
$$=\frac{3^4}{32}\int\frac{dz}{\left(1-z^4\right)^{\frac{1}{4}}}\cdot\int\frac{zdz}{\left(1-z^4\right)^{\frac{1}{4}}}\cdot\int\frac{zzdz}{\left(1-z^4\right)^{\frac{1}{4}}}$$

and

$$Q=\int\frac{y^{f-1}dy}{\left(1-y^g\right)^{\frac{1}{4}}},\quad R=\int\frac{y^{f+\frac{1}{4}g-1}dy}{\left(1-y^g\right)^{\frac{1}{4}}},$$

$$S=\int\frac{y^{f-\frac{1}{4}g-1}dy}{\left(1-y^g\right)^{\frac{1}{4}}}=\frac{f+\frac{2}{4}g}{f-\frac{1}{4}g}\int\frac{y^{f+\frac{3}{4}g-1}dy}{\left(1-y^g\right)^{\frac{1}{4}}}$$

and

$$T=\int\frac{y^{f+\frac{2}{4}g-1}dy}{\left(1-y^g\right)^{\frac{1}{4}}},$$

from these on putting $y=z^4$ and $4f=a$ the following equation is prepared :

$$\int\frac{dz}{\left(1-z^4\right)^{\frac{1}{4}}}\cdot\int\frac{zdz}{\left(1-z^4\right)^{\frac{1}{4}}}\cdot\int\frac{zzdz}{\left(1-z^4\right)^{\frac{1}{4}}}$$
$$=ag\frac{(a+g)(a+2g)}{1\cdot 2}\int\frac{z^{a-1}dz}{\left(1-z^{4g}\right)^{\frac{1}{4}}}\cdot\int\frac{z^{a+g-1}dz}{\left(1-z^{4g}\right)^{\frac{1}{4}}}\cdot\int\frac{z^{a+2g-1}dz}{\left(1-z^{4g}\right)^{\frac{1}{4}}}\cdot\int\frac{z^{a+3g-1}dz}{\left(1-z^{4g}\right)^{\frac{1}{4}}}.$$

35. By progressing in this manner the following equations may be found, when p is not $=1$; and indeed, if $p=2$, there will be found :

I.
$$\int\frac{dz}{\left(1-z^3\right)^{\frac{1}{3}}}\cdot\int\frac{zdz}{\left(1-z^3\right)^{\frac{1}{3}}}$$
$$=ag\left(a+g\right)\int\frac{z^{a-1}dz}{\left(1-z^{3g}\right)^{\frac{1}{3}}}\cdot\int\frac{z^{a+g-1}dz}{\left(1-z^{3g}\right)^{\frac{1}{3}}}\cdot\int\frac{z^{a+2g-1}dz}{\left(1-z^{3g}\right)^{\frac{1}{3}}}.$$

II.
$$\int\frac{dz}{\left(1-z^4\right)^{\frac{2}{4}}}\cdot\int\frac{zdz}{\left(1-z^4\right)^{\frac{2}{4}}}\cdot\int\frac{zzdz}{\left(1-z^4\right)^{\frac{2}{4}}}$$
$$=ag^2\left(a+g\right)\int\frac{z^{a-1}dz}{\left(1-z^{4g}\right)^{\frac{2}{4}}}\cdot\int\frac{z^{a+g-1}dz}{\left(1-z^{4g}\right)^{\frac{2}{4}}}\cdot\int\frac{z^{a+2g-1}dz}{\left(1-z^{4g}\right)^{\frac{2}{4}}}\cdot\int\frac{z^{a+3g-1}dz}{\left(1-z^{4g}\right)^{\frac{2}{4}}}.$$

Moreover generally, whatever q shall be, if there may be put

$$\frac{dz}{\left(1-z^q\right)^{\frac{q-2}{q}}}=Xdz \text{ and } \frac{z^{a-1}dz}{\left(1-z^{qg}\right)^{\frac{q-2}{q}}}=Ydz,$$

there will be

$$\int Xdz\cdot\int zXdz\cdot\int z^2Xdz\cdots\int z^{q-2}Xdz$$
$$=ag^{q-2}\left(a+g\right)\int Ydz\cdot\int z^gYdz\cdot\int z^{2g}Ydz\cdots\int z^{(q-1)g}Ydz.$$

36. In a similar manner, if there shall be $p=3$ and there may be put

$$\frac{dz}{\left(1-z^q\right)^{\frac{q-3}{q}}}=Xdz \text{ and } \frac{z^{a-1}dz}{\left(1-z^{qg}\right)^{\frac{q-3}{q}}}=Ydz,$$

the following general equation will be produced

$$\int Xdz\cdot\int zXdz\cdot\int z^2Xdz\cdots\int z^{q-2}Xdz$$
$$=ag^{q-3}\frac{(a+g)(a+g)}{1\cdot 2}\int Ydz\cdot\int z^gYdz\cdot\int z^{2g}Ydz\cdots\int z^{(q-1)g}Ydz.$$

And hence all these formulas may be allowed to be gathered together into one of the widest extent. Indeed p and q shall be any positive integers and there may be put

$$\frac{dz}{\left(1-z^q\right)^{\frac{q-p}{q}}}=Xdz \text{ and } \frac{z^{a-1}dz}{\left(1-z^{qg}\right)^{\frac{q-p}{q}}}=Ydz,$$

there will be had :

$$\int Xdz\cdot\int zXdz\cdot\int z^{2}Xdz\cdots\int z^{q-2}Xdz$$
$$=ag^{q-p}\,\frac{(a+g)(a+2g)\cdots(a+(p-1)g)}{1\cdot 2\cdots(p-1)}\int Ydz\cdot\int z^{g}Ydz\cdot\int z^{2g}Ydz\cdots\int z^{(q-1)g}Ydz.$$

37. But since there shall be:

$$\int z^{q-1}Xdz=\frac{1}{p},$$

if each may be multiplied by this, the following elegant enough equation will be arrived at :

$$\frac{(a+g)(a+2g)\cdots(a+(p-1)g)}{1\cdot 2\cdots p}g^{q-p}$$
$$=\frac{\int Xdz}{\int Ydz}\cdot\frac{\int zXdz}{\int z^{g}Ydz}\cdot\frac{\int z^{2}Xdz}{\int z^{2g}Ydz}\cdot\frac{\int z^{3}Xdz}{\int z^{3g}Ydz}\cdots\frac{\int z^{q-1}Xdz}{\int z^{(q-1)g}Ydz.},$$

which expression includes everything found up to this point and is noteworthy on account of the distinct order.

38. Now we may progress to the other method, with the aid of which it will be allowed to arrive at expressions of this kind with innumerable constant factors, which is more fitting for analysis. Indeed I have observed from the reduction of integral formulas it will be possible to obtain other expressions of this kind. Indeed this integral formula shall be proposed :

$$\int x^{m-1}dx\left(1-x^{nq}\right)^{\frac{p}{q}}$$

which may be changed without difficulty into this expression * :

$$\frac{x^{m}\left(1-x^{nq}\right)^{\frac{p+q}{q}}}{m}+\frac{m+(p+q)n}{m}\int x^{m+nq-1}dx\left(1-x^{nq}\right)^{\frac{p}{q}}.$$

Therefore if m and $\frac{p+q}{q}$ were positive numbers and the integrals may be taken thus, so that they may vanish on putting $x=0$, and then there may be put $x=1$, there will become

$$\int x^{m-1}dx\left(1-x^{nq}\right)^{\frac{p}{q}}=\frac{m+(p+q)n}{m}\int x^{m+nq-1}dx\left(1-x^{nq}\right)^{\frac{p}{q}}.$$

39. Then since in a similar manner there shall be

$$\int x^{m+nq-1}dx\left(1-x^{nq}\right)^{\frac{p}{q}}=\frac{m+(p+2q)n}{m+nq}\int x^{m+2nq-1}dx\left(1-x^{nq}\right)^{\frac{p}{q}},$$

there will be also

$$\int x^{m-1}dx\left(1-x^{nq}\right)^{\frac{p}{q}}=\frac{(m+(p+q)n)(m+(p+2q)n)}{m(m+nq)}\int x^{m+2nq-1}dx\left(1-x^{nq}\right)^{\frac{p}{q}}.$$

Therefore with this reduction continued indefinitely there will be produced :

$$\int x^{m-1}dx\left(1-x^{nq}\right)^{\frac{p}{q}}$$
$$=\frac{(m+(p+q)n)(m+(p+2q)n)(m+(p+3q)n)\cdots(m+(p+\infty q)n)}{m(m+nq)(m+2nq)\cdots(m+\infty nq)}\int x^{m+\infty nq-1}dx\left(1-x^{nq}\right)^{\frac{p}{q}}.$$

And in a similar manner there is :

$$\int x^{\mu-1}dx\left(1-x^{nq}\right)^{\frac{p}{q}}$$
$$=\frac{(\mu+(p+q)n)(\mu+(p+2q)n)(\mu+(p+3q)n)\cdots(\mu+(p+\infty q)n)}{\mu(\mu+nq)(\mu+2nq)\cdots(\mu+\infty nq)}\int x^{\mu+\infty nq-1}dx\left(1-x^{nq}\right)^{\frac{p}{q}}.$$

while m, μ, nq and $\frac{p+q}{q}$ shall be positive numbers, or greater than zero.

40. But because, if m is infinite, there becomes

$$\int x^{m}dx\left(1-x^{nq}\right)^{\frac{p}{q}}=\int x^{m+\alpha}dx\left(1-x^{nq}\right)^{\frac{p}{q}},$$

any finite number may be accepted in place of α , as is deduced from paragraph 38, there will be also :

$$\int x^{m+\infty nq-1}dx\left(1-x^{nq}\right)^{\frac{p}{q}}=\int x^{\mu+\infty nq-1}dx\left(1-x^{nq}\right)^{\frac{p}{q}}.$$

On account of which, if the former of the preceding expressions may be divided by the latter, this same equation will arise :

$$\frac{\int x^{m-1}dx\left(1-x^{nq}\right)^{\frac{p}{q}}}{\int x^{\mu-1}dx\left(1-x^{nq}\right)^{\frac{p}{q}}}=\frac{\mu(m+(p+q)n)(\mu+nq)(m+(p+2q)n)(\mu+2nq)(m+(p+3q)n)(\mu+3nq)}{m(\mu+(p+q)n)(m+nq)(\mu+(p+2q)n)(m+2nq)(\mu+(p+3q)n)(m+3nq)} \text{ etc. to infin.,}$$

the expression of which can be shown to be in agreement with the aid of innumerable products from an infinite factors, the values of which will be able to be assigned from the quadratures of curves.

41. If the other integral formula allows integration, then a suitable infinite expression for the other integral will be had. For if there shall $\mu = nq$; then there will be

$$\int x^{\mu-1}dx\left(1-x^{nq}\right)^{\frac{p}{q}}=\frac{1}{(p+q)n}$$

so that with the value substituted there will be produced :

$$\int x^{m-1}dx\left(1-x^{nq}\right)^{\frac{p}{q}}=\frac{1}{(p+q)n}\cdot\frac{nq(m+(p+q)n)2nq(m+(p+2q)n)3nq}{m(p+2q)n(m+nq)(p+3q)n(m+2nq)} \text{ etc.,}$$

with the aid of which expressions by continued factors extending to infinity are able to be found for innumerable integrals ; at least in that case, where $x=1$, certainly which may be desired chiefly in general.

42. There may be put n in place of nq and there will be produced

$$\int x^{m-1}dx\left(1-x^{n}\right)^{\frac{p}{q}}=\frac{q}{(p+q)n}\cdot\frac{n(mq+(p+q)n)2n(mq+(p+2q)n)3n(mq+(p+3q)n)}{m(p+2q)n(m+n)(p+3q)n(m+2n)(p+4q)n} \text{ etc.,}$$

which resolved into two factors becomes simpler and there arises :

$$\int x^{m-1}dx\left(1-x^{n}\right)^{\frac{p}{q}}=\frac{q}{(p+q)n}\cdot\frac{1(mq+(p+q)n)}{m(p+2q)}\cdot\frac{2(mq+(p+2q)n)}{(m+n)(p+3q)}\cdot\frac{3(mq+(p+3q)n)}{(m+2n)(p+4q)}\cdot \text{ etc.,}$$

from which the following more notable examples are deduced :

$$\int \frac{dx}{\sqrt{(1-xx)}} = 1\cdot\frac{1\cdot 4}{1\cdot 3}\cdot\frac{2\cdot 8}{3\cdot 5}\cdot\frac{3\cdot 12}{5\cdot 7}\text{etc.} = \frac{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6}{1\cdot 3\cdot 3\cdot 5\cdot 5\cdot 7}\text{ etc.}$$

$$\int \frac{xdx}{\sqrt{(1-xx)}} = 1\cdot\frac{1\cdot 6}{2\cdot 3}\cdot\frac{2\cdot 10}{4\cdot 5}\cdot\frac{3\cdot 14}{6\cdot 7}\cdot\text{etc.} = 1,$$

$$\int \frac{x^2dx}{\sqrt{(1-xx)}} = 1\cdot\frac{1\cdot 8}{3\cdot 3}\cdot\frac{2\cdot 12}{5\cdot 5}\cdot\frac{3\cdot 16}{7\cdot 7}\text{etc.} = \frac{2\cdot 4\cdot 4\cdot 6\cdot 6\cdot 8}{3\cdot 3\cdot 5\cdot 5\cdot 7\cdot 7}\text{ etc.},$$

$$\int \frac{dx}{\sqrt{(1-x^3)}} = \frac{2}{3}\cdot\frac{1\cdot 5\cdot 2\cdot 11\cdot 3\cdot 17\cdot\ 4\cdot 23\cdot\ 5\cdot 29\cdot}{1\cdot 3\cdot 4\cdot\ 5\cdot 7\cdot\ 7\cdot 10\cdot\ 9\cdot 13\cdot 11\cdot}\text{ etc.},$$

$$\int \frac{xdx}{\sqrt{(1-x^3)}} = \frac{2}{3}\cdot\frac{1\cdot 7\cdot 2\cdot 13\cdot 3\cdot 19\cdot\ 4\cdot 25\cdot\ 5\cdot 31\cdot}{2\cdot 3\cdot 5\cdot\ 5\cdot 8\cdot\ 7\cdot 11\cdot\ 9\cdot 14\cdot 11\cdot}\text{ etc.},$$

$$\int \frac{dx}{\sqrt{(1-x^4)}} = \frac{1}{2}\cdot\frac{1\cdot 6\cdot 2\cdot 14\cdot 3\cdot 22\cdot\ 4\cdot 30\cdot}{1\cdot 3\cdot 5\cdot\ 5\cdot 9\cdot\ 7\cdot 13\cdot\ 9\cdot}\text{ etc.} = \frac{1}{2}\cdot\frac{2\cdot 3\cdot 4\cdot 7\cdot 6\cdot 11\cdot\ 8\cdot 15\cdot}{1\cdot 3\cdot 5\cdot 5\cdot 9\cdot\ 7\cdot 13\cdot\ 9\cdot}\text{etc.},$$

$$\int \frac{xxdx}{\sqrt{(1-x^4)}} = \frac{1}{2}\cdot\frac{1\cdot 10\cdot 2\cdot 18\cdot\ 3\cdot 26\cdot 4\cdot 34}{3\cdot\ 3\cdot 7\cdot\ 5\cdot 11\cdot\ 7\cdot 15\cdot\ 9}\text{ etc.}$$

$$\int \frac{dx}{\sqrt[3]{(1-x^3)}} = \frac{1}{2}\cdot\frac{3\cdot 3\cdot 6\cdot 6\cdot 9\cdot\ 9\cdot 12\cdot 12\cdot}{1\cdot 5\cdot 4\cdot 8\cdot 7\cdot 11\cdot 10\cdot 14\cdot}\text{ etc.},$$

$$\int \frac{dx}{\sqrt[4]{(1-x^4)}} = \frac{1}{3}\cdot\frac{4\cdot 4\cdot 8\cdot\ 8\cdot 12\cdot 12\cdot 16\cdot 16}{1\cdot 7\cdot 5\cdot 11\cdot\ 9\cdot 15\cdot 13\cdot 19}\text{ etc.}$$

Besides these expressions there are deserved to be noted :

$$\int x^{m-1}dx(1-x^n)^{-\frac{m}{n}} = \frac{1}{n-m}\cdot\frac{n\cdot n\cdot 2n\cdot 2n\cdot 3n\cdot 3n}{m(2n-m)(m+n)(3n-m)(m+2n)(4n-m)}\text{etc.},$$

$$\int x^{m-1}dx(1-x^n)^{\frac{m-n}{n}} = \frac{1}{m}\cdot\frac{n\cdot 2m\cdot 2n(2m+n)3n(2m+2n)4n(2m+3n)}{m(m+n)(m+n)(m+2n)(m+2n)(m+3n)(m+3n)(m+4n)}\text{etc.}$$

43. But since in a like manner there will be :

$$\int x^{\mu-1}dx(1-x^\nu)^{\frac{r}{s}} = \frac{s}{(r+s)\nu}\cdot\frac{1(\mu s+(r+s)\nu)2(\mu s+(r+2s)\nu)3(\mu s+(r+3s)\nu}{\mu(r+2s)(\mu+\nu)(r+3s)(\mu+2\nu)(r+4s)}\text{etc.},$$

there will be on dividing the former expression by this :

$$\frac{\int x^{m-1}dx(1-x^n)^{\frac{p}{q}}}{\int x^{\mu-1}dx(1-x^\nu)^{\frac{r}{s}}} = \frac{(r+s)q\nu}{(p+q)sn}\cdot\frac{\mu(r+2s)(mq+(p+q)n)}{m(p+2q)(\mu s+(r+s)\nu)}\cdot\frac{(\mu+\nu)(r+3s)(mq+(p+2q)n)}{m(p+3q)(\mu s+(r+2s)\nu)}\cdot\text{etc.}$$

Therefore as often as this infinite expression has a finite value, also the summation of the other integral will be able to be reduced to another. Moreover cases of this kind exist, when the factors of the numerators cancel the factors of the denominator, thus so that after the cancellation a finite number of factors may remain. For in this expression all the general reductions of integral formulas into others will be contained.

44. But so that several expressions of this kind may be able to be compared between themselves, it has been considered to take that in this manner :

$$\frac{\int x^{a-1}dx(1-x^{b})^{c}}{\int x^{f-1}dx(1-x^{g})^{h}}=\frac{(h+1)g}{(c+1)b}\cdot\frac{f(h+2)(a+(c+1)b)}{a(c+2)(f+(h+1)g)}\cdot\frac{(f+g)(h+3)(a+(c+2)b)}{(a+b)(c+3)(f+(h+2)g)}\cdot\text{etc.}$$

In a similar manner there will be

$$\frac{\int x^{\alpha-1}dx(1-x^{\beta})^{\gamma}}{\int x^{\zeta-1}dx(1-x^{\eta})^{\theta}}=\frac{(\theta+1)\eta}{(\gamma+1)\beta}\cdot\frac{\zeta(\theta+2)(\alpha+(\gamma+1)\beta)}{\alpha(\gamma+2)(\zeta+(\theta+1)\eta)}\cdot\frac{(\zeta+\eta)(\theta+3)(\alpha+(\gamma+2)\beta)}{(\alpha+\beta)(\gamma+3)(\zeta+(\theta+2)\eta)}\cdot\text{etc.}$$

which expressions, even if they may not differ between themselves in this matter, yet, since they have a different form, they will be able to be compared between themselves.

45. Now so that we may elicit the same theorems from these expressions, which we have found above, there shall be

$$\theta=\gamma=h=c,\ \eta=\beta=g=b;$$

there will be

$$\frac{\int x^{a-1}dx(1-x^{b})^{c}}{\int x^{f-1}dx(1-x^{b})^{c}}=\frac{f(a+(c+1)b)(f+b)(a+(c+2)b)(f+2b)(a+(c+3)b)}{a(f+(c+1)b)(a+b)(f+(c+2)b)(a+2b)(f+(c+3)b)}\ \text{etc.}$$

and from the other formula

$$\frac{\int x^{\alpha-1}dx(1-x^{b})^{c}}{\int x^{\zeta-1}dx(1-x^{b})^{c}}=\frac{\zeta(\alpha+(c+1)b)(\zeta+b)(\alpha+(c+2)b)(\zeta+2b)(\alpha+(c+3)b)}{\alpha(\zeta+(c+1)b)(\alpha+b)(\zeta+(c+2)b)(\alpha+2b)(\zeta+(c+3)b)}\ \text{etc.}$$

The product of these expressions, if it may be put $=\frac{f}{a}$, will require to become

$$\frac{(a+(c+1)b)(f+b)\zeta(a+(c+1)b)}{(f+(c+1)b)(a+b)\alpha(\zeta+(c+1)b)}=1;$$

if this indeed were the case, the product of the whole infinite expressions becomes $=\frac{f}{a}$. But this will be obtained by making

$$\alpha=a+(c+1)b,\ \zeta=f+(c+1)b$$

and there becomes

$$c=-\tfrac{1}{2},$$

thus so that there shall become

$$\alpha = a + \tfrac{1}{2}b,\ \zeta = f + \tfrac{1}{2}b,$$

and thus there will become

$$\int \frac{x^{a-1}dx}{\sqrt{(1-x^b)}} \cdot \int \frac{x^{a+\frac{1}{2}b-1}dx}{\sqrt{(1-x^b)}} = \frac{f}{a} \int \frac{x^{f-1}dx}{\sqrt{(1-x^b)}} \cdot \int \frac{x^{f+\frac{1}{2}b-1}dx}{\sqrt{(1-x^b)}};$$

or if there may be put $x = z^2$, there will be

$$\int \frac{z^{a-1}dz}{\sqrt{(1-z^{2b})}} \cdot \int \frac{z^{a+b-1}dz}{\sqrt{(1-z^{2b})}} = \frac{f}{a} \int \frac{z^{f-1}dz}{\sqrt{(1-z^{2b})}} \cdot \int \frac{z^{f+b-1}dz}{\sqrt{(1-z^{2b})}};$$

on putting *a* and *f* in place of $2a$ and 2*f*. But this equation is no other than the theorem found above in § 12; indeed on making $f = b$ there becomes

$$\int \frac{z^{2b-1}dz}{\sqrt{(1-z^{2b})}} = \frac{1}{b} \text{ and } \int \frac{z^{b-1}dx}{\sqrt{(1-z^{2b})}} = \frac{\pi}{2b},$$

from which there becomes

$$\pi = 2ab \int \frac{z^{a-1}dz}{\sqrt{(1-z^{2b})}} \cdot \int \frac{z^{a+b-1}dx}{\sqrt{(1-z^{2b})}}.$$

46. In a similar manner the other theorems of this kind can be found; for if there shall be

$$g = b,\ h = c,\ \eta = \beta = b \text{ et } \theta = \gamma$$

and the case may be sought, where the product of both the expressions becomes $= 1$. But this will be obtained, if there shall be

$$\frac{f(a+(c+1)b)\zeta(a+(\gamma+1)b)}{a(f+(c+1)b)\alpha(\zeta+(\gamma+1)b)} = 1,$$

that which will happen on taking

$$\alpha = a + (c+1)b,\ f = a + (\gamma+1)b,\ \zeta = a.$$

Therefore with these values substituted, the following elegant theorem arises

$$\frac{\int x^{a-1}dx(1-x^b)^c}{\int x^{a-1}dx(1-x^b)^\gamma} \cdot \frac{\int x^{a+(c+1)b-1}dx(1-x^b)^\gamma}{\int x^{a+(\gamma+1)b-1}dx(1-x^b)^c} = 1;$$

or, if there may be put

$$c + 1 = m \text{ and } \gamma + 1 = n,$$

there will be had

$$\int\frac{x^{a-1}dx}{(1-x^b)^{1-m}}\cdot\int\frac{x^{a+mb-1}dx}{(1-x^b)^{1-n}}=\int\frac{x^{a-1}dx}{(1-x^b)^{1-n}}\cdot\int\frac{x^{a+nb-1}dx}{(1-x^b)^{1-m}}.$$

47. A more concise theorem may be able to be elicited above in another way by putting $\gamma=h$ and $\theta=c$ with $\eta=\beta=g=b$ remaining and by being effected, so that the product of the integral expressions may become $=\frac{f}{a}$ so that which may eventuate, there will be required to become

$$\frac{(a+(c+1)b)(f+b)\zeta(\alpha+(h+1)b)}{(f+(h+1)b)(a+b)\alpha(\zeta+(c+1)b)}=1.$$

Truly this may be effected by taking

$$\alpha=a+\left(c+1\right)b,\quad \zeta=f+(h+1)b,$$

from which there will be found

$$c+h+1=0 \text{ or } h=-1-c;$$

whereby there may be taken

$$c=-\tfrac{1}{2}+n \text{ and } h=-\tfrac{1}{2}-n,$$

and the following theorem will be produced

$$\frac{f}{a}=\frac{\int x^{a-1}dx(1-x^b)^{-\frac{1}{2}+n}}{\int x^{f-1}dx(1-x^b)^{-\frac{1}{2}-n}}\cdot\frac{\int x^{a+(\frac{1}{2}+n)b-1}dx(1-x^b)^{-\frac{1}{2}-n}}{\int x^{f+(\frac{1}{2}-n)b-1}dx(1-x^b)^{-\frac{1}{2}+n}}.$$

48. Now all the exponents $c,\ h,\ \gamma$ and θ shall be unequal, but $g=\beta=\eta=b,$ and the case will be required to be found, in which the product of both the expressions becomes $=\frac{(h+1)(\theta+1)}{(c+1)(\gamma+1)}$.

But this will eventuate, if this form may be returned

$$\frac{f\left(bh+2b\right)(a+(c+1)b)\zeta(b\theta+2b)(\alpha+(\gamma+1)b)}{a\left(bc+2b\right)(f+(h+1)b)\alpha(b\gamma+2b)(\zeta+(\theta+1)b)}=1,$$

which factor expressed thus, so that the individual terms in the following members may increase by the quantity b. Now there may be put

$$\zeta+(\theta+1)b=bh+2b \text{ or } \zeta=b(1+h-\theta)$$

and

$$\alpha+\left(\gamma+1\right)b=bc+2b \text{ or } \alpha=b(1+c-\gamma)..$$

Again there becomes

$$f+(h+1)b=b\theta+2b \text{ or } f=b(1+\theta-h)$$

and

$$\alpha+(c+1)b=b\gamma+2b \text{ or } \alpha=b(1+\gamma-c).$$

And then there will have to be $\alpha=f$ and $\zeta=a$, which two equations are required, so that there shall be

$$c-\gamma=\theta-h \text{ or } c+h=\gamma+\theta.$$

From which the following theorem arises

$$\frac{(h+1)(\theta+1)}{(c+1)(\gamma+1)}=\frac{\int x^{b(1+\gamma-c)-1}dx(1-x^b)^c}{\int x^{b(1+\theta-h)-1}dx(1-x^b)^h}\cdot\frac{\int x^{b(1+c-\gamma)-1}dx(1-x^b)^\gamma}{\int x^{b(1+h-\theta)-1}dx(1-x^b)^\theta},$$

while there shall become $c+h=\gamma+\theta.$

49. But that expression can be effected $=1$ in another way , by putting

$$\alpha=a+(c+1)b \text{ and } \zeta=f+(h+1)b,\ f=b(\gamma+2),\ a=b(\theta+2),$$

thus so that there shall become

$$\alpha=b(3+c+\theta) \text{ and } \zeta=b(3+h+\gamma).$$

But again there must be

$$\zeta+(\theta+1)b=bh+2b \text{ and } \alpha+(\gamma+1)b=bc+2b;$$

from which it may be demanded, that there shall be

$$\gamma+\theta+2=0.$$

Therefore there may be put

$$\gamma=-1+n \text{ and } \theta=-1-n.$$

But if it may be required, that the product of both expressions shall be $=\frac{f(h+1)(\theta+1)}{a(c+1)(\gamma+1)}$, that will be obtained by putting

$$\alpha=a+(c+1)b,\ \zeta=f+(h+1)b,\ f=b(\gamma+1),\ a=b(\theta+1),$$

from which there will become

$$\alpha=b(2+c+\theta) \text{ and } \zeta=b(2+h+\gamma).$$

Finally truly there must be

$$\gamma + \theta + 1 = 0.$$

There may be put

$$\gamma = -\tfrac{1}{2} + n \text{ and } \theta = -\tfrac{1}{2} - n$$

and this theorem will be had :

$$\frac{h+1)}{c+1} = \frac{\int x^{b(\frac{1}{2}-n)-1} dx(1-x^b)^c}{\int x^{b(\frac{1}{2}+n)-1} dx(1-x^b)^h} \cdot \frac{\int x^{b(\frac{3}{2}+c-n)-1} dx(1-x^b)^{-\frac{1}{2}+n}}{\int x^{b(\frac{3}{2}+c+n)-1} dx(1-x^b)^{-\frac{1}{2}-n}};$$

in which it is required to be observed the exponents $c,\ h, -\frac{1}{2}+n, -\frac{1}{2}-n$ are able to be certain negative numbers, but such that with unity they may become positive ; for otherwise the integrals will not obtain a finite value in the case $x = 1$.

50. Therefore just as not only the theorem found above concerning the product of the two integral formulas uncovered by this more direct method, but also it has been elicited by another new method not less noteworthy, thus , if in a like manner three expressions of this kind may in turn be multiplied together, many theorems will be produced about the products of the three integral formulas, and further it will be allowed to progress to any number of factors; but since this inquiry thus requires an extended calculation, so that also the letters may scarcely be sufficient, I will be content both with the particular theorem indicated, as well as the way it has been shown.

DE PRODUCTIS
EX INFINITIS FACTORIBUS ORTIS

Commentatio 122 indicis ENESTROEMIANI

Commentarii academiae scientiarum Petropolitanae 11 (1739), 1750, p. 3-31

1. Cum in Analysi ad eiusmodi quantitates pervenitur, quae numeris nec rationalibus nec irrationalibus exponi possunt, expressiones infinitae ad eas quantitates denotandas adhiberi solent; quae eo magis idoneae sunt censendae,quo citius earum ope ad cognitionem et aestimationem quantitatum iis expressarum pervenitur. Huiusmodi igitur expressionum maximus et amplissimus est usus ad valores quantitatum transcendentium, cuiusmodi sunt logarithmi, arcus circulares aliaeque per quadraturas curvarum determinatae quantitates, repraesentandos earumque beneficio ad tam exactam cum logarithmorum, tum arcuum circularium, tum etiam plurium aliarum quantitatum transcendentium cognitionem pertigimus. Quin etiam istiusmodi expressiones infinitae insignem afferunt utilitatem ad quantitates irrationales et radices aequationum algebraicarum per numeros rationales vero proxime definiendas; quae, si usus spectetur, veris expressionibus plerumque longe sunt anteferendae.

2. Huiusmodi autem expressionum infinitarum nonnulla genera inter se maxime diversa sunt constituenda, quorum primum in se complectitur omnes series infinitas, infinitis terminis signis + vel − iunctis constantes; quae doctrina nunc quidem iam tantopere est exculta, ut non solum plures habeantur methodi quasvis quantitates tam algebraicas quam transcendantes huiusmodi seriebus infinitis exprimendi, sed etiam proposita serie infinita investigandi, cuiusmodi quantitas ea indicetur. Duplici enim modo expressiones infinitas cuiusque generis tractari oportet, quorum alter in conversione quantitatum vel algebraicarum vel transcendentium in expressiones infinitas consistit, alter vero in indagatione illius quantitatis, quam proposita expressîo infinita designat, vicissim versatur.

3. Ad alterum genus expressionum infinitarum referri convenit eas, quae ex innumerabilibus factoribus constant; cuiusmodi expressiones quamquam iam complures sunt inventae ac cognitae, tamen nec modus ad eas perveniendi nec via earum valores dignoscendi usquam est exposita. Aeque autem dignae huius generis expressiones infinitae videntur, quae excolantur, ac priores ex infinito terminorum numero constantes, neque forte minus commodi Analysi afferetur earum per tractatione. Praeterquam enim, quod istiusmodi expressiones naturam quantitatum, quas referunt, satis distincte ob oculos ponunt et saepenumero ad valores proximos inveniendos perquam sunt accommodatae, insignem praestant usum ad logarithmos ipsarum quantitatum formandos, id quod incalculo saepissime summam affert utilitatem. Sic si quantitas quaecunque X transformata fuerit in istiusmodi expressionem

$$\frac{a}{\alpha}\cdot\frac{b}{\beta}\cdot\frac{c}{\gamma}\cdot\frac{d}{\delta}\cdot\frac{e}{\varepsilon}\cdot\text{etc.},$$

statim habebitur logarithmus quantitatis X

$$l\frac{a}{\alpha}+l\frac{b}{\beta}+l\frac{c}{\gamma}+l\frac{d}{\delta}+l\frac{e}{\varepsilon}+\text{etc.},$$

quae series eo magis convergit, quo propius factores illi ad unitatem inclinant. Hanc ob causam constitui in hac dissertatione theoriam huiusmodi expressionum infinitarum, quantum quidem observationes meae subsidii suppeditaverunt, inchoare, quo aliis facilius sit eam aliquando magis perficere.

4. Primus eiusmodi expressionem infinitis factoribus contentam protulit WALLISIUS in *Arithmetica infinitorum,* ubi ostendit, si circuli diameter sit, fore aream circuli $=1$

$$\frac{2\cdot4\cdot4\cdot6\cdot6\cdot8\cdot8\cdot10\cdot10\cdot12}{3\cdot3\cdot5\cdot7\cdot7\cdot9\cdot9\cdot11\cdot11}\ \text{etc.},$$

quam expressionem deduxit ex interpolatione seriei

$$\frac{2}{3}+\frac{2\cdot4}{3\cdot5}+\frac{2\cdot4\cdot6}{3\cdot5\cdot7}+\text{etc.},$$

cuius terminos intermedios demonstraverat a circuli quadratura pendera Cum igitur istae expressiones interpolationi serierum originem suam debeant, non incongruum fore visum est tractationem hanc de productis ex infinitis factoribus constantibus ab interpolationibus incipere. Cum enim in Tomo quinto Commentariorum nostrorum methodum tradidissem interpolationes per quadraturas curvarum perficiendi, simul constabit, cuiusmodi quantitatem transcendentem producta infinita hac ratione orta exhibeant.

5. Considero igitur sequentem progressionem

$$\overset{1}{(f+g)}+\overset{2}{(f+g)(f+2g)}+\overset{3}{(f+g)(f+2g)(f+3g)}+\overset{4}{(f+g)(f+2g)(f+3g)(f+4g)}+\text{etc.},$$

cuius quilibet terminus, cuius index est *n,* invenitur ex praecedente hunc per $f+ng$ multiplicando; ostendi autem in dissertatione allegata huius seriei terminum, cuius index est *n,* esse

$$\frac{g^{n+1}\int dx(-lx)^n}{(f+(n+1)g)\int x^{f:g}dx(1-x)^n}$$

utraque integratione ita peracta, ut integralia evanescant posito $x=0$, tumque facto $x=1$. Quamobrem ista expressio simul indicabit, a quanam quadratura singuli termini intermedii pendeant. Quanquam enim, si n sit numerus fractus, non ita facile constat, qualem quadraturam $\int dx(-lx)^n$ contineat, tamen eodem loco ostendi posito $\frac{p}{q}$ loco n

formulam $\int dx(-lx)^{\frac{p}{q}}$ congruere cum

$$\sqrt[q]{1\cdot 2\cdot 3\cdot p\left(\tfrac{2p}{q}+1\right)\left(\tfrac{3p}{q}+1\right)\left(\tfrac{4p}{q}+1\right)\cdots\left(\tfrac{qp}{q}+1\right)}$$
$$\times\int dx(x-xx)^{\frac{p}{q}}\cdot\int dx\left(x^2-x^3\right)^{\frac{p}{q}}\cdot\int dx\left(x^3-x^4\right)^{\frac{p}{q}}$$
$$\cdot\int dx\left(x^4-x^5\right)^{\frac{p}{q}}\cdots\int dx\left(x^{q-1}-x^q\right)^{\frac{p}{q}}.$$

cuius reductionis ope valor ipsius $\int dx(-lx)^{\frac{p}{q}}$ per quadraturas curvarum algebraicarum exprimi potest.

6. Si nunc in serie assumta terminus, cuius index est $=\frac{1}{2}$, ponatur *z,* ex lege seriei termini, quorum indices sunt $\frac{1}{2}$, $\frac{3}{2}$, $\frac{5}{2}$, etc., sequenti modo se habebunt:

$$\mathrm{z}+z(f+\tfrac{3}{2}g)+z(f+\tfrac{3}{2}g)(f+\tfrac{5}{2}g)+z(f+\tfrac{3}{2}g)(f+\tfrac{5}{2}g)(f+\tfrac{7}{2}g)+\text{etc.}$$

Quoniam autem progressio assumta tandem cum geometrica confunditur, hi termini interpolati evadent tandem medii proportionales inter contiguos seriei terminos. Quare si singuli termini interpolati iam ab initio tanquam medii proportionales spectentur, sequentes prodibunt approximationes ad terminum *z,*
cuius index est $\frac{1}{2}$:

$$\text{I. } z=\sqrt{(f+g)},$$
$$\text{II. } z=\sqrt{\frac{(f+g)(f+g)(f+2g)}{1\;\left(f+\frac{3}{2}g\right)\left(f+\frac{3}{2}g\right)}},$$
$$\text{III. } z=\sqrt{\frac{(f+g)(f+g)(f+2g)(f+2g)(f+3g)}{1\;\left(f+\frac{3}{2}g\right)\left(f+\frac{3}{2}g\right)\left(f+\frac{5}{2}g\right)\left(f+\frac{5}{2}g\right)}}$$
etc.

ex qua progressionis lege intelligitur terminum indicis $\frac{1}{2}$ vere esse

$$=(f+g)^{\frac{1}{2}}\sqrt{\frac{(f+g)(f+2g)(f+2g)(f+3g)(f+3g)(f+4g)(f+4g)(f+5g)(f+5g)}{\left(f+\frac{3}{2}g\right)\left(f+\frac{3}{2}g\right)\left(f+\frac{5}{2}g\right)\left(f+\frac{5}{2}g\right)\left(f+\frac{7}{2}g\right)\left(f+\frac{7}{2}g\right)\left(f+\frac{9}{2}g\right)\left(f+\frac{9}{2}g\right)\left(f+\frac{11}{2}g\right)\left(f+\frac{11}{2}g\right)}}\text{etc.}$$

7. Nunc igitur non solum certum est hac expressione infinita terminum seriei assumtae

$$\overset{1}{(f+g)}+\overset{2}{(f+g)(f+2g)}+\overset{3}{(f+g)(f+2g)(f+3g)}+\text{etc.},$$

cuius index est $=\frac{1}{2}$, exhiberi, sed etiam eadem expressio inventa ad quadraturas curvarum reducitur. Posito, enim $n=\frac{1}{2}$ ob $p=1$ et $q=2$ fit

$$\int dx(-lx)^{\frac{1}{2}}=\sqrt{1\cdot 2}\int dx\sqrt{(x-xx)};$$

quae expressio debito modo integrata dat radicem quadratam ex area circuli, cuius diameter est $=1$; vel posita $1:\pi$ ratione diametri ad peripheriam erit

$$\int dx(-lx)^{\frac{1}{2}}=\frac{\sqrt{\pi}}{2}.$$

Hinc ergo idem terminus, cuius index $=\frac{1}{2}$, quem posuimus *z,* reperitur

$$=\frac{g\sqrt{\pi g}}{(2f+3g)\int x^{f:g}dx\sqrt{(1-x)}}=\frac{\sqrt{\pi g}}{(2f+3g)\int y^{f+g-1}dy\sqrt{(1-y^g)}}$$

integrali hoc eodem tractato modo, quo ante ratione variabilis *x* est praescriptum. At per reductionem formularum huius modi integralium est

$$\int y^{f+g-1}dy\sqrt{(1-y^g)}=\frac{2fg}{(2f+g)(2f+3g)}\int\frac{y^{f-1}dy}{\sqrt{(1-y^g)}}=\frac{2f}{2f+3g}\int y^{f-1}dy\sqrt{(1-y^g)}.$$

His substitutis reperitur

$$\frac{(2f+g)(2f+3g)(2f+3g)(2f+5g)(2f+5g)(2f+7g)(2f+7g)}{(2f+2g)(2f+2g)(2f+4g)(2f+4g)(2f+6g)(2f+6g)}\text{ etc.}$$

$$=\frac{2ff(2f+g)}{\pi g}\left(\int y^{f-1}dy\sqrt{(1-y^g)}\right)^2=\frac{2ffg}{\pi(2f+g)}\left(\int\frac{y^{f-1}dy}{\sqrt{(1-y^g)}}\right)^2.$$

Per hanc igitur aequationem innumerabiles quadraturae in factores infinitos et vicissim huiusmodi factorum infinitorum valores in quadraturas curvarum transformari possunt.

8. Ut hanc aequalitatem exemplis illustremus, sit $g=1$ eritque

$$\int y^{f-1}dy\sqrt{(1-y)}=\frac{2\cdot 2\cdot 4\cdot 6\cdot 8\cdot 10\cdots(2f-2)}{3\cdot 5\cdot 7\cdot 9\cdot 11\cdot 13\cdots(2f+1)}\text{ etc.}$$

Unde fiet

$$\frac{2ff(2f+1)2\cdot 2\cdot 2\cdot 2\cdot 4\cdot 4\cdots(2f-2)(2f-2)}{\pi\cdot 3\cdot 3\cdot 5\cdot 5\cdot 7\cdot 7\cdots(2f+1)(2f+1)}\text{ etc.}=\frac{(2f+1)(2f+3)(2f+3)}{(2f+2)(2f+2)(2f+4)}\text{ etc.,}$$

quae expressio ordinata seu ad continuitatem reducta dat

$$\pi = 4 \cdot \frac{2\cdot4\cdot4\cdot6\cdot6\cdot8\cdot8\cdot10\cdot10}{3\cdot3\cdot5\cdot5\cdot7\cdot7\cdot9\cdot9\cdot11\cdot11} \text{ etc.,}$$

quae est ipsa formula WALLISIANA prodiitque, quicunque numerus integer affirmativus loco f substituatur. Haec eadem expressio autem prodit, si ponatur $g = 2$ et $f =$ numero cuicunque impari integro.

9. Cum igitur sit

$$\frac{fg}{\pi}\left(\int \frac{y^{f-1}dy}{\sqrt{(1-y^{g})}}\right)^{2} = \frac{(2f+g)(2f+g)(2f+3g)(2f+3g)(2f+5g)(2f+5g)}{2f(2f+2g)(2f+2g)(2f+4g)(2f+4g)(2f+6g)} \text{ etc.}$$

erit pari modo

$$\frac{hk}{\pi}\left(\int \frac{y^{h-1}dy}{\sqrt{(1-y^{k})}}\right)^{2} = \frac{(2h+k)(2h+k)(2h+3k)(2h+3k)(2h+5k)(2h+5k)}{2h(2h+2k)(2h+2k)(2h+4k)(2h+4k)(2h+6k)} \text{ etc.}$$

Quare illa expressione per hanc divisa obtinebitur sequens aequatio libera a peripheria circuli π

$$\frac{fg\left(\int y^{f-1}dy:\sqrt{(1-y^{g})}\right)^{2}}{hk\left(\int y^{h-1}dy:\sqrt{(1-y^{k})}\right)^{2}} = \frac{2h(2f+g)^{2}(2h+2k)^{2}(2f+3g)^{2}(2h+4k)^{2}(2f+5g)^{2}}{2f(2h+k)^{2}(2f+2g)^{2}(2h+3k)^{2}(2f+4g)^{2}(2h+5k)^{2}} \text{ etc.}$$

quae radice quadrata extracta praebet hanc aequationem

$$\frac{\int y^{f-1}dy:\sqrt{(1-y^{g})}}{\int y^{h-1}dy:\sqrt{(1-y^{k})}}\cdot\sqrt{\frac{g}{k}} = \frac{2h(2f+g)(2h+2k)(2f+3g)(2h+4k)(2f+5g)}{2f(2h+k)(2f+2g)(2h+3k)(2f+4g)(2h+5k)} \text{ etc.}$$

10. Haec autem expressio infinita valorem constantem non habet; nam, etiamsi in infinitum continuetur, tamen alium habet valorem, si numerus factorum capiatur par, alium, si numerus impar. Quamobrem, nisi sit $k = g$, quo casu perinde est, ubi multiplicatio abrumpatur, bini factores coniunctim sunt accipiendi, quo facto binae obtinebuntur aequationes, prout numerus factorum capiatur par sive impar. Primo autem accurate evoluta expressione generali obtinebitur

$$\frac{g\int y^{f-1}dy:\sqrt{(1-y^{g})}}{k\int y^{h-1}dy:\sqrt{(1-y^{k})}} = \frac{2h(2f+g)}{2f(2h+k)}\cdot\frac{(2h+2k)(2f+3g)}{(2f+2g)(2h+3k)}\cdot\frac{(2h+4k)(2f+5g)}{(2f+4g)(2h+5k)}\cdot\frac{(2h+6k)(2f+7g)}{(2f+6g)(2h+7k)}\cdot \text{ etc.}$$

Sumendis autem alteris terminorum paribus erit

$$\frac{f\int y^{f-1}dy:\sqrt{\left(1-y^{g}\right)}}{h\int y^{h-1}dy:\sqrt{\left(1-y^{k}\right)}}$$
$$=\frac{(2f+g)(2h+2k)}{(2h+k)(2f+2g)}\cdot\frac{(2f+3g)(2h+4k)}{(2h+3k)(2f+4g)}\cdot\frac{(2f+5g)(2h+6k)}{(2h+5k)(2f+6g)}\cdot\frac{(2f+7g)(2h+8k)}{(2h+7k)(2f+8g)}\cdot\text{etc.},$$

in quibus expressionibus loca, ubi operationem abrumpere licet, punctis sunt distincta.

11. Consideremus autem attentius casum, quo est $k=g$, quippe quo expressio infinita tanquam ex simplicibus factoribus constans concipi potest, eritque

$$\frac{\int y^{f-1}dy:\sqrt{\left(1-y^{g}\right)}}{\int y^{h-1}dy:\sqrt{\left(1-y^{g}\right)}}=\frac{2h(2f+g)(2h+2g)(2f+3g)(2h+4g)}{2f(2h+g)(2f+2g)(2h+3g)(2f+4g)}\text{ etc.};$$

quae expressio quo minus cum praecedente ob easdem litteras confundatur, ponamus hic $2f=a$ et $2h=b$ atque $y=x^{2}$, quo substituto prodibit

$$\frac{\int x^{a-1}dx:\sqrt{\left(1-x^{g}\right)}}{\int x^{b-1}dx:\sqrt{\left(1-x^{g}\right)}}=\frac{b(a+g)(b+2g)(a+3g)(b+4g)(a+5g)}{a(b+g)(a+2g)(b+3g)(a+4g)(b+5g)}\text{ etc.},$$

quae expressio cum priori § 9 data, quae facto pariter $y=x^{2}$ transit in hanc

$$\frac{4fg}{\pi}\left(\int\frac{x^{2f-1}dx}{\sqrt{\left(1-x^{2g}\right)}}\right)^{2}=\frac{(2f+g)(2f+g)(2f+3g)(2f+3g)(2f+5g)(2f+5g)}{2f(2f+2g)(2f+2g)(2f+4g)(2f+4g)(2f+6g)}\text{ etc.}$$

comparata insignes manifestabit proprietates, quarum veritas alias vix ostendi poterit.

12. Statim enim patet, si ponatur $a=2f$, $b=2f+g$, illam expressionem infinitam in hanc transmutari; quamobrem etiam expressiones illis aequales, quadraturas curvarum continentes, hoc casu fient aequales, ex quo sequens emergit aequalitas

$$\frac{\int x^{2f-1}dx:\sqrt{\left(1-x^{2g}\right)}}{\int x^{2f+g-1}dx:\sqrt{\left(1-x^{2g}\right)}}=\frac{4fg}{\pi}\left(\int x^{2f-1}dx:\sqrt{\left(1-x^{2g}\right)}\right)^{2}$$

si quidem ponatur post integrationem $x=1$. Hinc igitur sequitur fore

$$\pi=4fg\int\frac{x^{2f-1}dx}{\sqrt{\left(1-x^{2g}\right)}}\cdot\int\frac{x^{2f+g-1}dx}{\sqrt{\left(1-x^{2g}\right)}};$$

sive posito $2f=a$ erit

$$\pi = 2ag\int \frac{x^{a-1}dx}{\sqrt{\left(1-x^{2g}\right)}} \cdot \int \frac{x^{a+g-1}dx}{\sqrt{\left(1-x^{2g}\right)}},$$

quod sane est theorema maxime notatu dignum, cum eius beneficio productum duorum integralium, quorum saepissime neutrum exhiberi potest, assignari queat.

13. Veritas huius theorematis quidem facile declaratur iis casibus, quibus altera formula integralis vel absolute integrationem admittit vel a circuli quadratura pendet. Ponamus enim $g=1$ et $a=1$; utique erit

$$\pi = 2\int \frac{dx}{\sqrt{\left(1-x^{2}\right)}} \cdot \int \frac{x\ dx}{\sqrt{\left(1-x^{2}\right)}};$$

nam

$$2\int \frac{dx}{\sqrt{\left(1-x^{2}\right)}}$$

posito post integrationem $x=1$ dat ipsam quantitatem π atque

$$\int \frac{x\ dx}{\sqrt{(1-xx)}} = 1-\sqrt{(1-xx)}$$

facto $x=1$ fit $=1$. Simili modo, si $a=2$ manente $g=1$, perspicitur fore

$$\pi = 4\int \frac{xdx}{\sqrt{(1-xx)}} \cdot \int \frac{xx\ dx}{\sqrt{(1-xx)}};$$

nam est

$$\int \frac{xdx}{\sqrt{(1-xx)}} = 1 \quad \text{et} \quad \int \frac{xx\ dx}{\sqrt{(1-xx)}} = \frac{\pi}{4};$$

quibus casibus theorematis veritas aliunde cognita confirmatur.

14. Reliqui autem casus, quibus neutra quantitas integralis vel actu vel per quadraturam circuli exhiberi potest, totidem praebent theoremata maxime abstrusae indaginis. Ita posito $g=2$ et $a=1$ fiet

$$\pi = 4\int \frac{dx}{\sqrt{\left(1-x^{4}\right)}} \cdot \int \frac{xx\ dx}{\sqrt{\left(1-x^{4}\right)}},$$

ubi

$$\int \frac{xx\ dx}{\sqrt{\left(1-x^{4}\right)}}$$

exhibet applicatam in curva elastica rectangula,

$$\int \frac{dx}{\sqrt{\left(1-x^4\right)}}$$

vero arcum elasticae abscissae x respondentem. Quocirca rectangulum ex arcu elasticae abscissae 1 respondente et applicata respondente aequabitur areae circuli, cuius diameter est abscissa illa 1; quae proprietas elasticae fortasse alia methodo vix ac ne vix quidem cognosci demonstrarique poterit.

15. Antequam autem hunc elasticae casum relinquam, iuvabit utrumque integrale per seriem ordinariam exprimere casu saltem, quo $x=1$. Cum enim sit

$$\frac{1}{\sqrt{\left(1-x^4\right)}} = \frac{\left(1+x^2\right)^{-\frac{1}{2}}}{\sqrt{\left(1-x^2\right)}}$$

atque

$$\left(1+x^2\right)^{-\frac{1}{2}} = 1-\tfrac{1}{2}x^2+\tfrac{1\cdot 3}{2\cdot 4}x^4-\tfrac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^6+\text{etc.},$$

singula membra a circuli quadratura pendebunt. Absoluta autem utraque integratione pro casu $x=1$ erit

$$\int \frac{dx}{\sqrt{\left(1-x^4\right)}} = \frac{\pi}{2}\left(1-\tfrac{1}{4}+\tfrac{1\cdot 9}{4\cdot 16}-\tfrac{1\cdot 9\cdot 25}{4\cdot 16\cdot 36}+\text{etc.}\right)$$

atque

$$\int \frac{x^2dx}{\sqrt{\left(1-x^4\right)}} = \frac{\pi}{2}\left(\tfrac{1}{2}-\tfrac{1\cdot 3}{4\cdot 4}+\tfrac{1\cdot 9\cdot 5}{4\cdot 16\cdot 6}-\tfrac{1\cdot\ 9\cdot 25\cdot 7}{4\cdot 16\cdot 36\cdot 8}+\text{etc.}\right)$$

Hinc autem approximando prodit tam prope

$$\int \frac{dx}{\sqrt{\left(1-x^4\right)}} = \frac{5}{6}\cdot\frac{\pi}{2} \quad \text{et} \quad \int \frac{xxdx}{\sqrt{\left(1-x^4\right)}} = \frac{3}{5}\cdot\frac{\pi}{2}$$

16. Si fuerit $a=1$, erit

$$\pi = 2g\int \frac{dx}{\sqrt{\left(1-x^{2g}\right)}}\cdot\int \frac{x^g dx}{\sqrt{\left(1-x^{2g}\right)}},$$

quae duae expressiones integrales ita sunt comparatae, ut, si fuerit

$$\int \frac{x^g dx}{\sqrt{\left(1-x^{2g}\right)}}$$

applicata curvae cuiusdam abscissae x respondens, futura sit

$$\int \frac{dx}{\sqrt{\left(1-x^{2g}\right)}}$$

ipsa eiusdem curvae longitudo. Quamobrem si in hac curva sumatur abscissa $x=1$, erit productum seu rectangulum ex applicata in longitudinem curvae ad aream circuli, cuius diameter est abscissa $x=1$, uti se habet 2 ad numerum *g;* quae propositio locum habet, dummodo *g* fuerit numerus affirmativus; valores negativi enim sponte excipiuntur.

17. Si $a-1$ minor accipiatur quam *g,* ita ut numeri *a* et *g* sint primi inter se, sequentia habebuntur theoremata notatu digna; nam si

$$a+g-1>2g,$$

tum integratio ad formulam simpliciorem reduci posset.

$$\pi = 2\int \frac{dx}{\sqrt{\left(1-x^{2}\right)}} \cdot \int \frac{xdx}{\sqrt{\left(1-x^{2}\right)}} \qquad \pi = 24\int \frac{x^2dx}{\sqrt{\left(1-x^{8}\right)}} \cdot \int \frac{x^6dx}{\sqrt{\left(1-x^{8}\right)}}$$

$$\pi = 4\int \frac{dx}{\sqrt{\left(1-x^{4}\right)}} \cdot \int \frac{x^2dx}{\sqrt{\left(1-x^{4}\right)}} \qquad \pi = 10\int \frac{dx}{\sqrt{\left(1-x^{10}\right)}} \cdot \int \frac{x^5dx}{\sqrt{\left(1-x^{10}\right)}}$$

$$\pi = 6\int \frac{dx}{\sqrt{\left(1-x^{6}\right)}} \cdot \int \frac{x^3dx}{\sqrt{\left(1-x^{6}\right)}} \qquad \pi = 20\int \frac{xdx}{\sqrt{\left(1-x^{10}\right)}} \cdot \int \frac{x^6dx}{\sqrt{\left(1-x^{10}\right)}}$$

$$\pi = 12\int \frac{xdx}{\sqrt{\left(1-x^{6}\right)}} \cdot \int \frac{x^4dx}{\sqrt{\left(1-x^{6}\right)}} \qquad \pi = 30\int \frac{x^2dx}{\sqrt{\left(1-x^{10}\right)}} \cdot \int \frac{x^7dx}{\sqrt{\left(1-x^{10}\right)}}$$

$$\pi = 8\int \frac{dx}{\sqrt{\left(1-x^{8}\right)}} \cdot \int \frac{x^4dx}{\sqrt{\left(1-x^{8}\right)}} \qquad \pi = 40\int \frac{x^3dx}{\sqrt{\left(1-x^{10}\right)}} \cdot \int \frac{x^8dx}{\sqrt{\left(1-x^{10}\right)}}$$

$$\pi = 12\int \frac{dx}{\sqrt{\left(1-x^{12}\right)}} \cdot \int \frac{x^6dx}{\sqrt{\left(1-x^{12}\right)}} \qquad \pi = 28\int \frac{xdx}{\sqrt{\left(1-x^{14}\right)}} \cdot \int \frac{x^8dx}{\sqrt{\left(1-x^{14}\right)}}$$

$$\pi = 60\int \frac{x^4dx}{\sqrt{\left(1-x^{12}\right)}} \cdot \int \frac{x^{10}dx}{\sqrt{\left(1-x^{12}\right)}} \qquad \pi = 42\int \frac{x^2dx}{\sqrt{\left(1-x^{14}\right)}} \cdot \int \frac{x^9dx}{\sqrt{\left(1-x^{14}\right)}}$$

$$\pi = 14\int \frac{dx}{\sqrt{\left(1-x^{14}\right)}} \cdot \int \frac{x^7dx}{\sqrt{\left(1-x^{14}\right)}} \qquad \pi = 56\int \frac{x^3dx}{\sqrt{\left(1-x^{14}\right)}} \cdot \int \frac{x^{10}dx}{\sqrt{\left(1-x^{14}\right)}}$$

$$\pi = 70\int \frac{x^4dx}{\sqrt{\left(1-x^{14}\right)}} \cdot \int \frac{x^{11}dx}{\sqrt{\left(1-x^{14}\right)}}.$$

18. Hoc ipso igitur invento reductio etiam formularum integralium ad simpliciores insigniter est promota. Cum enim adhuc duae istae formulae

$$\int \frac{x^m dx}{\sqrt{(1-x^{2g})}} \quad \text{et} \quad \int \frac{x^{m+n} dx}{\sqrt{(1-x^{2g})}}$$

ad se invicem tantum reduci potuissent, si n erat multiplum exponentis 2g, ita nunc reductio etiam succedit, si n tantum ipsius g fuerit multiplum, casu intellige, quo fit $x=1$. Quemadmodum autem, si n est productum exponentis g per numerum parem, quotus, qui resultat ex divisione alterius formulae per alteram, facile assignatur, ita e contrario, si n sit factum ex g in numerum imparem, tum productum formularum facillime assignatur.

19. Haec omnia ergo huc redeunt, ut, si cognitum fuerit integrale formulae casu,

$$\int \frac{x^m dx}{\sqrt{(1-x^{2g})}}$$

quo $x=1$, eodem casu etiam huius formulae

$$\int \frac{x^{m+n} dx}{\sqrt{(1-x^{2g})}}$$

integrale, si sit n multiplum ipsius g, exhiberi queat. Sit enim A integrale formulae

$$\int \frac{x^m dx}{\sqrt{(1-x^{2g})}}$$

casu, quo est $x=1$; integralia alterius formulae ponendo g, $2g$, $3g$ etc. successive loco n sequenti modo se habebunt:

$$\int\frac{x^{m}dx}{\sqrt{\left(1-x^{2g}\right)}}=A,$$

$$\int\frac{x^{m+g}dx}{\sqrt{\left(1-x^{2g}\right)}}=\frac{\pi}{2(m+1)gA},$$

$$\int\frac{x^{m+2g}dx}{\sqrt{\left(1-x^{2g}\right)}}=\frac{(m+1)A}{m+g+1},$$

$$\int\frac{x^{m+3g}dx}{\sqrt{\left(1-x^{2g}\right)}}=\frac{(m+g+1)\pi}{2(m+1)(m+2g+1)gA},$$

$$\int\frac{x^{m+4g}dx}{\sqrt{\left(1-x^{2g}\right)}}=\frac{(m+1)(m+2g+1)A}{(m+g+1)(m+3g+1)},$$

$$\int\frac{x^{m+5g}dx}{\sqrt{\left(1-x^{2g}\right)}}=\frac{(m+g+1)(m+3g+1)\pi}{2(m+1)(m+2g+1)(m+4g+1)gA},$$

etc.

20. Cum deinde haec formula generalis

$$\int x^{m+ig}dx\left(1-x^{2g}\right)^{k-\frac{1}{2}}$$

denotantibus *i* et *k* numeros integros quoscunque reduci queat ad hanc formulam

$$\int\frac{x^{m+ig}dx}{\sqrt{\left(1-x^{2g}\right)}},$$

intelligitur illius formulae latissime patentis $\int x^{m+ig}dx\left(1-x^{2g}\right)^{k-\frac{1}{2}}$ integrale assignari posse ex integrali

$$\int\frac{x^{m}dx}{\sqrt{\left(1-x^{2g}\right)}},$$

cognito, casu saltem, quo post integrationem fit $x=1$. Casus autem, quibus *i* est numerus impar, praeter hoc integrale etiam circuli quadraturam π requirunt.

21. Quemadmodum igitur per terminum indicis $\frac{1}{2}$ seriei supra § 5 assumtae ad istas formularum integralium comparationes sum deductus, ita operae pretium forte erit alios terminos intermedios simili modo investigare. Quaeratur igitur terminus, cuius index est $\frac{p}{q}$, qui ponatur $=z$, ex quo sequentes *q* ita se habebunt:

$$\begin{array}{llll} \frac{p}{q} & \frac{p+q}{q} & \frac{p+2q}{q} & \\ z+ & \frac{z(fq+(p+q))g}{q}+ & \frac{z(fq+(p+q)g)(fq+(p+2q)g)}{q^2} & +\text{etc.} \end{array}$$

Considerando nunc pari modo, quod haec progressio tandem in geometricam abeat, sequentes orientur approximationes ad terminum z:

$$\text{I.}\qquad z=1(f+g)^{\frac{p}{q}},$$

$$\text{II.}\quad \frac{z(fq+(p+q))g}{q}=(f+g)^{\frac{q-p}{q}}(f+g)^{\frac{p}{q}}(f+2g)^{\frac{p}{q}},$$

$$\text{III.}\quad z\left(f+\left(\tfrac{p+q}{q}\right)g\right)\left(f+\left(\tfrac{p+2q}{q}\right)g\right)$$
$$=(f+g)^{\frac{q-p}{q}}(f+g)^{\frac{p}{q}}(f+2g)^{\frac{q-p}{q}}(f+2g)^{\frac{p}{q}}(f+3g)^{\frac{p}{q}}.$$

Hinc igitur elicietur verus valor ipsius z

$$=\frac{(f+g)^{\frac{p}{q}}(f+g)^{\frac{q-p}{q}}(f+2g)^{\frac{p}{q}}(f+2g)^{\frac{q-p}{q}}(f+3g)^{\frac{p}{q}}(f+3g)^{\frac{q-p}{q}}}{1\left(f+\frac{p+q}{q}g\right)^{\frac{p}{q}}\left(f+\frac{p+q}{q}g\right)^{\frac{q-p}{q}}\left(f+\frac{p+2q}{q}g\right)^{\frac{p}{q}}\left(f+\frac{p+2q}{q}g\right)^{\frac{q-p}{q}}\left(f+\left(\frac{p+3q}{q}\right)g\right)^{\frac{p}{q}}}\ \text{etc.}$$

Vel paucis mutatis, ut factores infinitesimi fiant $=1$ et expressio, ubi libuerit, abrumpi queat, erit

$$\frac{z}{\left(f+\frac{p}{q}g\right)^{\frac{p}{q}}}=\frac{(f+g)^{\frac{p}{q}}}{\left(f+\frac{p}{q}g\right)^{\frac{p}{q}}}\cdot\frac{(f+g)^{\frac{q-p}{q}}}{\left(f+\frac{p+q}{q}g\right)^{\frac{q-p}{q}}}\cdot\frac{(f+2g)^{\frac{p}{q}}}{\left(f+\frac{p+q}{q}g\right)^{\frac{p}{q}}}$$
$$\cdot\frac{(f+2g)^{\frac{q-p}{q}}}{\left(f+\frac{p+2q}{q}g\right)^{\frac{q-p}{q}}}\cdot\frac{(f+3g)^{\frac{p}{q}}}{\left(f+\frac{p+2q}{q}g\right)^{\frac{p}{q}}}\cdot\text{etc.},$$

cuius expressionis lex, qua factores progrediuntur, sponte elucet.

22. Eiusdem autem termini intermedii z valor ope termini generalis huius seriei exprimi potest; fiet enim

$$z=\frac{g^{\frac{p+q}{q}}\int dx(-lx)^{\frac{p}{q}}}{\left(f+\frac{p+q}{q}g\right)\int x^{f:g}dx(1-x)^{\frac{p}{q}}}.$$

Quare si ponatur

$$\int dx(-lx)^{\frac{p}{q}} = \sqrt[q]{1\cdot 2\cdot 3\cdots p\left(\tfrac{2p}{q}+1\right)\left(\tfrac{3p}{q}+1\right)\left(\tfrac{4p}{q}+1\right)\cdots\left(\tfrac{qp}{q}+1\right)}$$
$$\times\int dx\left(x-x^2\right)^{\frac{p}{q}}\cdot\int dx\left(x^2-x^3\right)^{\frac{p}{q}}\cdot\int dx\left(x^3-x^4\right)^{\frac{p}{q}}\cdots\int dx\left(x^{q-1}-x^q\right)^{\frac{p}{q}} = \sqrt[q]{P}$$

atque $x = y^g$, quo fit

$$\int x^{f:g}dx(1-x)^{\frac{p}{q}} = g\int y^{f+g-1}dy\left(1-y^g\right)^{\frac{p}{q}} =$$
$$\frac{ggp}{fq+(p+q)g}\int\frac{y^{f+g-1}dy}{\left(1-y^g\right)^{\frac{q-p}{q}}} = \frac{pfgg}{q\left(f+\frac{p}{q}g\right)\left(f+\frac{p+q}{q}g\right)}\int\frac{y^{f-1}dy}{\left(1-y^g\right)^{\frac{q-p}{q}}},$$

ponatur porro

$$\int\frac{y^{f-1}dy}{\left(1-y^g\right)^{\frac{q-p}{q}}} = Q,$$

erit

$$z = \frac{q\left(f+\frac{p}{q}g\right)P^{\frac{1}{q}}}{pfg^{\frac{q-p}{q}}Q}.$$

23. Substituta nunc loco z superiore expressione infinita sumtisque potestatibus exponentis q prodibit ista aequatio

$$\frac{q^qP}{p^qf^pg^{q-p}Q^q} = \frac{f^{q-p}}{\left(f+\frac{p}{q}g\right)^{q-p}}\cdot\frac{(f+g)^p}{\left(f+\frac{p}{q}g\right)^p}\cdot\frac{(f+g)^{q-p}}{\left(f+\frac{p+q}{q}g\right)^{q-p}}$$
$$\cdot\frac{(f+2g)^p}{\left(f+\frac{p+q}{q}g\right)^p}\cdot\frac{(f+2g)^{q-p}}{\left(f+\frac{p+2q}{q}g\right)^{q-p}}\cdot\text{etc.}$$

Si igitur pari modo ponatur

$$\int\frac{y^{h-1}dy}{\left(1-y^g\right)^{\frac{q-p}{q}}} = R,$$

erit

$$\frac{p^qh^pg^{q-p}R^q}{q^q} = \frac{\left(h+\frac{p}{q}g\right)^{q-p}}{h^{q-p}}\cdot\frac{\left(h+\frac{p}{q}g\right)^p}{(h+g)^p}\cdot\frac{\left(h+\frac{p+q}{q}g\right)^{q-p}}{(h+g)^{q-p}}\cdot\text{etc.},$$

quae duae expressiones in se mutuo ductae dabunt

$$\frac{h^pR^q}{f^pQ^q}=\frac{f^{q-p}\left(h+\frac{p}{q}g\right)^q}{h^{q-p}\left(f+\frac{p}{q}g\right)^q}\cdot\frac{(f+g)^q}{(h+g)^q}\cdot\frac{\left(h+\frac{p+q}{q}g\right)^q}{\left(f+\frac{p+q}{q}g\right)^q}\cdot\frac{(f+2g)^q}{(h+2g)^q}\cdot\frac{\left(h+\frac{p+2q}{q}g\right)^q}{\left(f+\frac{p+2q}{q}g\right)^q}\cdot\text{etc.},$$

24. Si ergo utrinque multiplicetur per $\frac{f^p}{h^p}$ atque radix potestatis q extrahatur, reperietur

$$\frac{R}{Q}=\frac{f\left(h+\frac{p}{q}g\right)}{h\left(f+\frac{p}{q}g\right)}\cdot\frac{(f+g)}{(h+g)}\cdot\frac{\left(h+\frac{p+q}{q}g\right)}{\left(f+\frac{p+q}{q}g\right)}\cdot\frac{(f+2g)}{(h+2g)}\cdot\frac{\left(h+\frac{p+2q}{q}g\right)}{\left(f+\frac{p+2q}{q}g\right)}\cdot\text{etc.}$$
$$=\frac{\int y^{h-1}dy\left(1-y^g\right)^{\frac{p-q}{q}}}{\int y^{f-1}dy\left(1-y^g\right)^{\frac{p-q}{q}}},$$

in quibus integralibus, cum ita fuerint accepta, ut evanescant posito $y=0$, fieri debet $y=1$, quo facto habebitur per quadraturas valor expressionis infinitae propositae. Ope huius igitur expressionis infinitae altera quadratura ad alteram, si quidem ponatur $y=1$, reduci poterit.

25. Ut autem hinc eiusmodi integralium comparationes deducamus, sicuti ex priori casu, quo erat $p=1$ et $q=2$, ponamus hic $p=1$ et $q=3$ fietque

$$P=\tfrac{10}{3}\int dx\left(x-x^2\right)^{\frac{1}{3}}\cdot\int dx\left(x^2-x^3\right)^{\frac{1}{3}}$$

et

$$Q=\int\frac{y^{f-1}dy}{\left(1-y^g\right)^{\frac{2}{3}}}$$

atque

$$R=\int\frac{y^{h-1}dy}{\left(1-y^g\right)^{\frac{2}{3}}}.$$

Erit ergo

$$\frac{27P}{fg^2Q^3}=\frac{ff(f+g)(f+g)(f+g)(f+2g)}{(f+\frac{1}{3}g)(f+\frac{1}{3}g)(f+\frac{1}{3}g)(f+\frac{4}{3}g)(f+\frac{4}{3}g)(f+\frac{4}{3}g)}\ \text{etc.}$$

atque

$$\frac{R}{Q}=\frac{f(f+\frac{1}{3}g)(f+g)(f+\frac{4}{3}g)(f+2g)(h+\frac{7}{3}g)}{h(f+\frac{1}{3}g)(h+g)(f+\frac{4}{3}g)(h+2g)(f+\frac{7}{3}g)}\ \text{etc.},$$

quae duae expressiones, cum in illa una revolutio ex tribus, hic autem ex duobus factoribus constet, in se mutuo transformari nequeunt, quicquid etiam loco h substituatur.

26. Sit igitur

$$S=\int\frac{y^{k-1}dy}{\left(1-y^{g}\right)^{\frac{2}{3}}};$$

erit

$$\frac{S}{Q}=\frac{f(k+\frac{1}{3}g)(f+g)(k+\frac{4}{3}g)(f+2g)(k+\frac{7}{3}g)}{k(f+\frac{1}{3}g)(k+g)(f+\frac{4}{3}g)(k+2g)(f+\frac{7}{3}g)}\text{ etc.,}$$

quae expressio cum praecedente coniuncta dabit

$$\frac{RS}{Q^{2}}=\frac{ff(h+\frac{1}{3}g)(k+\frac{1}{3}g)(f+g)(f+g)(h+\frac{4}{3}g)}{hk(f+\frac{1}{3}g)(f+\frac{1}{3}g)(h+g)(k+g)(f+\frac{4}{3}g)}\text{ etc.,}$$

quae expressio in illam ipsi $\frac{27P}{fg^{2}Q^{3}}$ aequalem convertetur ponendo

$$h=f+\tfrac{1}{3}g\quad\text{et}\quad k=f+\tfrac{2}{3}g.$$

Quamobrem habebitur ista aequatio

$$\frac{27P}{fg^{2}}=QRS$$

seu substitutis veris valoribus erit

$$90\int dx\left(x-x^{2}\right)^{\frac{1}{3}}\cdot\int dx\left(x^{2}-x^{3}\right)^{\frac{1}{3}}=fg^{2}\int\frac{y^{f-1}}{\left(1-y^{g}\right)^{\frac{2}{3}}}\cdot\int\frac{y^{f+\frac{1}{3}g-1}}{\left(1-y^{g}\right)^{\frac{2}{3}}}\cdot\int\frac{y^{f+\frac{2}{3}g-1}}{\left(1-y^{g}\right)^{\frac{2}{3}}}.$$

27. Antequam autem haec ulterius prosequamur, conveniet valori ipsius *P* commodiorem formam generaliter tribui. Facto autem $x=z^{q}$ cum sit

$$\int dx\left(x^{n}-x^{n+1}\right)^{\frac{p}{q}}=\frac{npq}{(n+1)\left((n+1)p+q\right)}\int\frac{z^{np-1}}{\left(1-z^{q}\right)^{\frac{q-p}{q}}}dz$$

post substitutionem prodibit

$$P=1\cdot2\cdot3\cdots p\frac{p^{q-1}}{q}\int\frac{z^{p-1}dz}{\left(1-z^{q}\right)^{\frac{q-p}{q}}}\cdot\int\frac{z^{2p-1}dz}{\left(1-z^{q}\right)^{\frac{q-p}{q}}}\cdot\int\frac{z^{3p-1}dz}{\left(1-z^{q}\right)^{\frac{q-p}{q}}}\cdots\int\frac{z^{(q-1)p-1}dz}{\left(1-z^{q}\right)^{\frac{q-p}{q}}}.$$

Ex qua expressione si extrahatur radix potestatis *q,* prodibit valor ipsius $\int dx\left(-lx\right)^{\frac{p}{q}}$.

28. Posito nunc $p=1$ et $q=3$ prodibit

$$P=\frac{1}{3}\int\frac{dz}{\left(1-z^3\right)^{\frac{2}{3}}}\cdot\int\frac{zdz}{\left(1-z^3\right)^{\frac{2}{3}}}.$$

Facto autem $y=z^3$ obtinebitur sequens aequatio

$$\int\frac{dz}{\left(1-z^3\right)^{\frac{2}{3}}}\cdot\int\frac{zdz}{\left(1-z^3\right)^{\frac{2}{3}}}=3fg^2\int\frac{z^{3f-1}dz}{\left(1-z^{3g}\right)^{\frac{2}{3}}}\cdot\int\frac{z^{3f+g-1}dz}{\left(1-z^{3g}\right)^{\frac{2}{3}}}\cdot\int\frac{z^{3f+2g-1}dz}{\left(1-z^{3g}\right)^{\frac{2}{3}}}.$$

Si nunc ponatur $3f=a$, orietur sequens aequatio notatu digna

$$\int\frac{dz}{\left(1-z^3\right)^{\frac{2}{3}}}\cdot\int\frac{zdz}{\left(1-z^3\right)^{\frac{2}{3}}}=ag^2\int\frac{z^{a-1}dz}{\left(1-z^{3g}\right)^{\frac{2}{3}}}\cdot\int\frac{z^{a+g-1}dz}{\left(1-z^{3g}\right)^{\frac{2}{3}}}\cdot\int\frac{z^{a+2g-1}dz}{\left(1-z^{3g}\right)^{\frac{2}{3}}}.$$

Quae cum superiore

$$\int\frac{dz}{\sqrt{\left(1-z^2\right)}}=ag\int\frac{z^{a-1}dz}{\sqrt{\left(1-z^{2g}\right)}}\cdot\int\frac{z^{a+g-1}dz}{\sqrt{\left(1-z^{2g}\right)}}$$

comparata iam quodammodo indicat, quomodo sequentes huius generis aequationes se sint habiturae.

29. Antequam autem per inductionem quicquam concludendi periculum faciam, casus nonnullos actu evolvam. Sit igitur $p=2$ et $q=3$ hincque reperietur

$$P=\frac{8}{3}\int\frac{zdz}{\left(1-z^3\right)^{\frac{1}{3}}}\cdot\int\frac{z^3dz}{\left(1-z^3\right)^{\frac{1}{3}}}=\frac{8}{9}\int\frac{dz}{\left(1-z^3\right)^{\frac{1}{3}}}\cdot\int\frac{zdz}{\left(1-z^3\right)^{\frac{1}{3}}},$$

$$Q=\int\frac{y^{f-1}dy}{\left(1-y^g\right)^{\frac{1}{3}}},\quad R=\int\frac{y^{h-1}dy}{\left(1-y^g\right)^{\frac{1}{3}}}.$$

Expressiones autem infinitae ita se habebunt:

$$\frac{27P}{8f^2gQ^3}=\frac{f(f+g)(f+g)(f+g)(f+2g)(f+2g)}{(f+\frac{2}{3}g)(f+\frac{2}{3}g)(f+\frac{2}{3}g)(f+\frac{5}{3}g)(f+\frac{5}{3}g)(f+\frac{5}{3}g)}\text{ etc.}$$

et

$$\frac{R}{Q}=\frac{f(h+\frac{2}{3}g)(f+g)(h+\frac{5}{3}g)(f+2g)(h+\frac{8}{3}g)}{h(f+\frac{2}{3}g)(h+g)(f+\frac{5}{3}g)(h+2g)(f+\frac{8}{3}g)}\text{ etc.}$$

Sit praeterea

$$S=\int\frac{y^{m-1}dy}{\left(1-y^{g}\right)^{\frac{1}{3}}}\quad \text{et}\quad T=\int\frac{y^{n-1}dy}{\left(1-y^{g}\right)^{\frac{1}{3}}}.$$

erit

$$\frac{T}{S}=\frac{m(n+\frac{2}{3}g)(m+g)(n+\frac{5}{3}g)(m+2g)}{n(m+\frac{2}{3}g)(n+g)(m+\frac{5}{3}g)(n+2g)}\text{ etc.,}$$

quae duae expressiones in se ductae dant

$$\frac{RT}{QS}=\frac{fm(h+\frac{2}{3}g)(n+\frac{2}{3}g)(f+g)(m+g)(h+\frac{5}{3}g)(n+\frac{5}{3}g)}{hn(f+\frac{2}{3}g)(m+\frac{2}{3}g)(h+g)(n+g)(f+\frac{5}{3}g)(m+\frac{5}{3}g)}\text{ etc.}$$

30. Haec autem expressio ad illam, cui $\frac{27P}{8f^{2}gQ^{3}}$ aequale est inventum, reduci non potest, nisi illa multiplicetur per $\frac{f}{f-\frac{1}{3}g}$, ita ut sit

$$\frac{27P}{8fg\left(f-\frac{1}{3}g\right)Q^{3}}=\frac{ff(f+g)(f+g)(f+g)(f+2g)}{(f-\frac{1}{3}g)(f+\frac{2}{3}g)(f+\frac{2}{3}g)(f+\frac{2}{3}g)(f+\frac{5}{3}g)(f+\frac{5}{3}g)}\text{ etc.}$$

nunc enim fiet reductio ponendo

$$m=f,\ h=f-\tfrac{1}{3}g\ \text{ et }\ n=f+\tfrac{1}{3}g.$$

His igitur valoribus substitutis erit

$$\frac{27P}{8fg\left(f-\frac{1}{3}g\right)Q^{3}}=\frac{RT}{QS}$$

Cum vero sit $S=Q$ et

$$R=\int\frac{y^{f-\frac{1}{3}g-1}dy}{\left(1-y^{g}\right)^{\frac{1}{3}}}=\frac{f+\frac{1}{3}g}{f-\frac{1}{3}g}\int\frac{y^{f+\frac{2}{3}g-1}dy}{\left(1-y^{g}\right)^{\frac{1}{3}}}$$

et

$$T=\int\frac{y^{f+\frac{1}{3}g-1}dy}{\left(1-y^{g}\right)^{\frac{1}{3}}},$$

obtinebitur haec aequatio posito $y=z^{3}$

$$\int\frac{dz}{\left(1-z^{3}\right)^{\frac{1}{3}}}\cdot\int\frac{zdz}{\left(1-z^{3}\right)^{\frac{1}{3}}}=3fg\left(3f+g\right)\int\frac{z^{3f-1}dz}{\left(1-z^{3g}\right)^{\frac{1}{3}}}\cdot\int\frac{z^{3f+g-1}dz}{\left(1-z^{3g}\right)^{\frac{1}{3}}}\cdot\int\frac{z^{3f+2g-1}dz}{\left(1-z^{3g}\right)^{\frac{1}{3}}}.$$

Ac si ponatur $3f=a$, erit

$$\int\frac{dz}{\left(1-z^{3}\right)^{\frac{1}{3}}}\cdot\int\frac{zdz}{\left(1-z^{3}\right)^{\frac{1}{3}}}=ag\left(a+g\right)\int\frac{z^{a-1}dz}{\left(1-z^{3g}\right)^{\frac{1}{3}}}\cdot\int\frac{z^{a+g-1}dz}{\left(1-z^{3g}\right)^{\frac{1}{3}}}\cdot\int\frac{z^{a+2g-1}dz}{\left(1-z^{3g}\right)^{\frac{1}{3}}}.$$

31. Ponamus $p=1$ et $q=4$ habebiturque

$$\frac{4^4P}{fg^3Q^4}=\frac{fff(f+g)(f+g)(f+g)}{(f+\frac{1}{4}g)(f+\frac{1}{4}g)(f+\frac{1}{4}g)(f+\frac{1}{4}g)(f+\frac{5}{4}g)(f+\frac{5}{4}g)}\text{ etc.}$$

et

$$\frac{R}{Q}=\frac{f(h+\frac{1}{4}g)(f+g)(h+\frac{5}{4}g)(f+2g)}{h(f+\frac{1}{4}g)(h+g)(f+\frac{5}{4}g)(h+2g)}\text{ etc.}$$

Sit vero ut ante

$$S=\int\frac{y^{m-1}dy}{\left(1-y^g\right)^{\frac{q-p}{q}}},\quad T=\int\frac{y^{n-1}dy}{\left(1-y^g\right)^{\frac{q-p}{q}}};$$

erit

$$\frac{RST}{Q^3}=\frac{fff(h+\frac{1}{4}g)(m+\frac{1}{4}g)(n+\frac{1}{4}g)(f+g)}{hmn(f+\frac{1}{4}g)(h+\frac{1}{4}g)(f+\frac{1}{4}g)(h+g)}\text{ etc.},$$

cuius expressionis 6 factores in illius quatuor sunt transmutandi, quod fiet ponendo

$$h=f+\tfrac{1}{4}g,\quad m=f+\tfrac{2}{4}g\quad\text{et}\quad n=f+\tfrac{3}{4}g,$$

quo facto habebitur

$$4^4P=fg^3QRST.$$

Quare cum sit

$$P=\frac{1}{4}\int\frac{dz}{\left(1-z^4\right)^{\frac{3}{4}}}\cdot\int\frac{zdz}{\left(1-z^4\right)^{\frac{3}{4}}}\cdot\int\frac{z^2dz}{\left(1-z^4\right)^{\frac{3}{4}}},$$

si ponatur $y=z^4$ et $4f=a$, orietur ista aequatio

$$\int\frac{dz}{\left(1-z^4\right)^{\frac{3}{4}}}\cdot\int\frac{zdz}{\left(1-z^4\right)^{\frac{3}{4}}}\cdot\int\frac{z^2dz}{\left(1-z^4\right)^{\frac{3}{4}}}$$
$$=ag^3\int\frac{z^{a-1}dz}{\left(1-z^{4g}\right)^{\frac{3}{4}}}\cdot\int\frac{z^{a+g-1}dz}{\left(1-z^{4g}\right)^{\frac{3}{4}}}\cdot\int\frac{z^{a+2g-1}dz}{\left(1-z^{4g}\right)^{\frac{3}{4}}}\cdot\int\frac{z^{a+3g-1}dz}{\left(1-z^{4g}\right)^{\frac{3}{4}}},$$

cuius cum praecedentibus casibus, quibus erat $p=1,\ q=2$ et $p=1,\ q=3$, connexio facile perspicitur.

32. Ex his igitur licebit omnes istiusmodi aequationes, quae orientur, si ponatur $p=1$ et $q=$ numero cuicunque affirmativo integro, formare;
erit scilicet

I. $$\int\frac{dz}{\sqrt{(1-z^2)}}=ag\int\frac{z^{a-1}dz}{\sqrt{(1-z^{2g})}}\cdot\int\frac{z^{a+g-1}dz}{\sqrt{(1-z^{2g})}},$$

II. $$\int\frac{dz}{(1-z^3)^{\frac{2}{3}}}\cdot\int\frac{zdz}{(1-z^3)^{\frac{2}{3}}}$$
$$=ag^2\int\frac{z^{a-1}dz}{(1-z^{3g})^{\frac{2}{3}}}\cdot\int\frac{z^{a+g-1}dz}{(1-z^{3g})^{\frac{2}{3}}}\cdot\int\frac{z^{a+2g-1}dz}{(1-z^{3g})^{\frac{2}{3}}},$$

III. $$\int\frac{dz}{(1-z^4)^{\frac{3}{4}}}\cdot\int\frac{zdz}{(1-z^4)^{\frac{3}{4}}}\cdot\int\frac{z^2dz}{(1-z^4)^{\frac{3}{4}}}$$
$$=ag^3\int\frac{z^{a-1}dz}{(1-z^{4g})^{\frac{3}{4}}}\cdot\int\frac{z^{a+g-1}dz}{(1-z^{4g})^{\frac{3}{4}}}\cdot\int\frac{z^{a+2g-1}dz}{(1-z^{4g})^{\frac{3}{4}}}\cdot\int\frac{z^{a+3g-1}dz}{(1-z^{4g})^{\frac{3}{4}}},$$

IV. $$\int\frac{dz}{(1-z^5)^{\frac{4}{5}}}\cdot\int\frac{zdz}{(1-z^5)^{\frac{4}{5}}}\cdot\int\frac{z^2dz}{(1-z^5)^{\frac{4}{5}}}\cdot\int\frac{z^3dz}{(1-z^5)^{\frac{4}{5}}}$$
$$=ag^4\int\frac{z^{a-1}dz}{(1-z^{5g})^{\frac{4}{5}}}\cdot\int\frac{z^{a+g-1}dz}{(1-z^{5g})^{\frac{4}{5}}}\cdot\int\frac{z^{a+2g-1}dz}{(1-z^{5g})^{\frac{4}{5}}}\cdot\int\frac{z^{a+3g-1}dz}{(1-z^{5g})^{\frac{4}{5}}}\cdot\int\frac{z^{a+4g-1}dz}{(1-z^{5g})^{\frac{4}{5}}}$$

etc.

33. Quo etiam eas aequationes, quae oriuntur, si p non $=1$, colligere queamus, ponamus $p=3$ et $q=4$; quo posito et reliquis manentibus ut supra erit

$$\frac{4^4P}{3^4f^3gQ^4}=\frac{f(f+g)(f+g)(f+g)}{(f+\frac{3}{4}g)(f+\frac{3}{4}g)(f+\frac{3}{4}g)(f+\frac{3}{4}g)}\text{ etc.},$$

ubi reliqua membra ex quaternis factoribus constantia ex his formantur singulos factores quantitate g augendo. Simili vero modo erit

$$\frac{RST}{Q^3}=\frac{fff(h+\frac{3}{4}g)(m+\frac{3}{4}g)(n+\frac{3}{4}g)}{hmn(f+\frac{3}{4}g)(f+\frac{3}{4}g)(f+\frac{3}{4}g)}\text{ etc.},$$

ubi seni factores unam revolutionem seu periodum constituunt. Ad comparationem autem instituendam necesse est utramque seriem ita contemplari

$$\frac{4^4 P}{3^4 f^2 (f-\frac{1}{4}g)Q^4} = \frac{ff(f+g)(f+g)}{(f-\frac{1}{4}g)(f+\frac{3}{4}g)(f+\frac{3}{4}g)(f+\frac{3}{4}g)} \text{ etc.,}$$

$$\frac{hRST}{fQ^3} = \frac{ff(h+\frac{3}{4}g)(m+\frac{3}{4}g)(n+\frac{3}{4}g)(f+g)}{mn(f+\frac{3}{4}g)(f+\frac{3}{4}g)(f+\frac{3}{4}g)(h+g)} \text{ etc.,}$$

quarum haec transmutatur in illam, ita ut fiat

$$\frac{4^4 P}{3^4 f(f-\frac{1}{4}g)} = QRST,$$

si fiat

$$h = f + \tfrac{1}{4}g, \ \ m = f - \tfrac{1}{4}g \ \text{ et } \ n = f + \tfrac{2}{4}g.$$

34. Cum igitur sit

$$P = \frac{3^4}{2}\int \frac{z^2 dz}{\left(1-z^4\right)^{\frac{1}{4}}} \cdot \int \frac{z^5 dz}{\left(1-z^4\right)^{\frac{1}{4}}} \cdot \int \frac{z^8 dz}{\left(1-z^4\right)^{\frac{1}{4}}}$$
$$= \frac{3^4}{32}\int \frac{dz}{\left(1-z^4\right)^{\frac{1}{4}}} \cdot \int \frac{zdz}{\left(1-z^4\right)^{\frac{1}{4}}} \cdot \int \frac{zzdz}{\left(1-z^4\right)^{\frac{1}{4}}}$$

et

$$Q = \int \frac{y^{f-1}dy}{\left(1-y^g\right)^{\frac{1}{4}}}, \quad R = \int \frac{y^{f+\frac{1}{4}g-1}dy}{\left(1-y^g\right)^{\frac{1}{4}}},$$

$$S = \int \frac{y^{f-\frac{1}{4}g-1}dy}{\left(1-y^g\right)^{\frac{1}{4}}} = \frac{f+\frac{2}{4}g}{f-\frac{1}{4}g}\int \frac{y^{f+\frac{3}{4}g-1}dy}{\left(1-y^g\right)^{\frac{1}{4}}}$$

atque

$$T = \int \frac{y^{f+\frac{2}{4}g-1}dy}{\left(1-y^g\right)^{\frac{1}{4}}},$$

ex his posito $y = z^4$ et $4f = a$ sequens conficitur aequatio

$$\int \frac{dz}{\left(1-z^4\right)^{\frac{1}{4}}} \cdot \int \frac{zdz}{\left(1-z^4\right)^{\frac{1}{4}}} \cdot \int \frac{zzdz}{\left(1-z^4\right)^{\frac{1}{4}}}$$
$$= ag\frac{(a+g)(a+2g)}{1\cdot 2}\int \frac{z^{a-1}dz}{\left(1-z^{4g}\right)^{\frac{1}{4}}} \cdot \int \frac{z^{a+g-1}dz}{\left(1-z^{4g}\right)^{\frac{1}{4}}} \cdot \int \frac{z^{a+2g-1}dz}{\left(1-z^{4g}\right)^{\frac{1}{4}}} \cdot \int \frac{z^{a+3g-1}dz}{\left(1-z^{4g}\right)^{\frac{1}{4}}}.$$

35. Hoc modo progrediendo reperientur sequentes aequationes, quando p non est $=1$; et quidem, si $p=2$, invenietur

I. $$\int\frac{dz}{\left(1-z^3\right)^{\frac{1}{3}}}\cdot\int\frac{zdz}{\left(1-z^3\right)^{\frac{1}{3}}}$$
$$=ag\left(a+g\right)\int\frac{z^{a-1}dz}{\left(1-z^{3g}\right)^{\frac{1}{3}}}\cdot\int\frac{z^{a+g-1}dz}{\left(1-z^{3g}\right)^{\frac{1}{3}}}\cdot\int\frac{z^{a+2g-1}dz}{\left(1-z^{3g}\right)^{\frac{1}{3}}}.$$

II. $$\int\frac{dz}{\left(1-z^4\right)^{\frac{2}{4}}}\cdot\int\frac{zdz}{\left(1-z^4\right)^{\frac{2}{4}}}\cdot\int\frac{zzdz}{\left(1-z^4\right)^{\frac{2}{4}}}$$
$$=ag^2\left(a+g\right)\int\frac{z^{a-1}dz}{\left(1-z^{4g}\right)^{\frac{2}{4}}}\cdot\int\frac{z^{a+g-1}dz}{\left(1-z^{4g}\right)^{\frac{2}{4}}}\cdot\int\frac{z^{a+2g-1}dz}{\left(1-z^{4g}\right)^{\frac{2}{4}}}\cdot\int\frac{z^{a+3g-1}dz}{\left(1-z^{4g}\right)^{\frac{2}{4}}}.$$

Generaliter autem, quicquid sit q, si ponatur

$$\frac{dz}{\left(1-z^q\right)^{\frac{q-2}{q}}}=Xdz \quad\text{et}\quad \frac{z^{a-1}dz}{\left(1-z^{qg}\right)^{\frac{q-2}{q}}}=Ydz,$$

erit

$$\int Xdz\cdot\int zXdz\cdot\int z^2Xdz\cdots\int z^{q-2}Xdz$$
$$=ag^{q-2}\left(a+g\right)\int Ydz\cdot\int z^gYdz\cdot\int z^{2g}Ydz\cdots\int z^{(q-1)g}Ydz.$$

36. Simili modo, si sit $p=3$ ac ponatur

$$\frac{dz}{\left(1-z^q\right)^{\frac{q-3}{q}}}=Xdz \quad\text{et}\quad \frac{z^{a-1}dz}{\left(1-z^{qg}\right)^{\frac{q-3}{q}}}=Ydz,$$

prodibit sequens aequatio generalis

$$\int Xdz\cdot\int zXdz\cdot\int z^2Xdz\cdots\int z^{q-2}Xdz$$
$$=ag^{q-3}\frac{(a+g)(a+g)}{1\cdot2}\int Ydz\cdot\int z^gYdz\cdot\int z^{2g}Ydz\cdots\int z^{(q-1)g}Ydz.$$

Atque hinc omnes has formulas in unam latissime patentem colligi licet. Sint enim p et q numeri quicunque integri affirmativi ac ponatur

$$\frac{dz}{\left(1-z^q\right)^{\frac{q-p}{q}}}=Xdz \quad\text{et}\quad \frac{z^{a-1}dz}{\left(1-z^{qg}\right)^{\frac{q-p}{q}}}=Ydz,$$

habebitur

$$\int Xdz\cdot\int zXdz\cdot\int z^2Xdz\cdots\int z^{q-2}Xdz$$
$$=ag^{q-p}\,\frac{(a+g)(a+2g)\cdots(a+(p-1)g)}{1\cdot 2\cdots(p-1)}\int Ydz\cdot\int z^gYdz\cdot\int z^{2g}Ydz\cdots\int z^{(q-1)g}Ydz.$$

37. Cum autem sit

$$\int z^{q-1}Xdz=\frac{1}{p}$$

si per hunc factorem utrinque multiplicetur, proveniet sequens aequatio satis elegans

$$\frac{(a+g)(a+2g)\cdots(a+(p-1)g)}{1\cdot 2\cdots p}g^{q-p}$$
$$=\frac{\int Xdz}{\int Ydz}\cdot\frac{\int zXdz}{\int z^gYdz}\cdot\frac{\int z^2Xdz}{\int z^{2g}Ydz}\cdot\frac{\int z^3Xdz}{\int z^{3g}Ydz}\cdots\frac{\int z^{q-1}Xdz}{\int z^{(q-1)g}Ydz.},$$

quae expressio omnes hactenus inventas in se complectitur atque ob insignem ordinem est notatu digna.

38. Progrediar nunc ad aliam methodum, cuius ope ad huiusmodi expressiones ex factoribus innumerabilibus constantes pervenire licet, quae magis ad analysin est accommodata. Observavi enim ex reductione formularum integralium ad alias istiusmodi expressiones obtineri posse. Sit enim proposita ista formula integralis

$$\int x^{m-1}dx\left(1-x^{nq}\right)^{\frac{p}{q}}$$

quae non difficulter transmutatur in hanc expressionem

$$\frac{x^m\left(1-x^{nq}\right)^{\frac{p+q}{q}}}{m}+\frac{m+(p+q)n}{m}\int x^{m+nq-1}dx\left(1-x^{nq}\right)^{\frac{p}{q}}.$$

Si ergo m et $\frac{p+q}{q}$ fuerint numeri affirmativi atque integralia ita capiantur, ut evanescant posito $x=0$, tumque ponatur $x=1$, fiet

$$\int x^{m-1}dx\left(1-x^{nq}\right)^{\frac{p}{q}}=\frac{m+(p+q)n}{m}\int x^{m+nq-1}dx\left(1-x^{nq}\right)^{\frac{p}{q}}.$$

39. Cum deinde simili modo sit

$$\int x^{m+nq-1}dx\left(1-x^{nq}\right)^{\frac{p}{q}}=\frac{m+(p+2q)n}{m+nq}\int x^{m+2nq-1}dx\left(1-x^{nq}\right)^{\frac{p}{q}},$$

erit quoque

$$\int x^{m-1}dx\left(1-x^{nq}\right)^{\frac{p}{q}}=\frac{(m+(p+q)n)(m+(p+2q)n)}{m(m+nq)}\int x^{m+2nq-1}dx\left(1-x^{nq}\right)^{\frac{p}{q}}.$$

Hac ergo reductione in infinitum continuata prodibit

$$\int x^{m-1}dx\left(1-x^{nq}\right)^{\frac{p}{q}}$$
$$=\frac{(m+(p+q)n)(m+(p+2q)n)(m+(p+3q)n)\cdots(m+(p+\infty q)n)}{m(m+nq)(m+2nq)\cdots(m+\infty nq)}\int x^{m+\infty nq-1}dx\left(1-x^{nq}\right)^{\frac{p}{q}}.$$

Ac simili modo est

$$\int x^{\mu-1}dx\left(1-x^{nq}\right)^{\frac{p}{q}}$$
$$=\frac{(\mu+(p+q)n)(\mu+(p+2q)n)(\mu+(p+3q)n)\cdots(\mu+(p+\infty q)n)}{\mu(\mu+nq)(\mu+2nq)\cdots(\mu+\infty nq)}\int x^{\mu+\infty nq-1}dx\left(1-x^{nq}\right)^{\frac{p}{q}}.$$

dummodo m et μ et nq et $\frac{p+q}{q}$ sint numeri affirmativi seu nihilo maiores.

40. Quoniam autem, si m est infinitum, fit

$$\int x^{m}dx\left(1-x^{nq}\right)^{\frac{p}{q}}=\int x^{m+\alpha}dx\left(1-x^{nq}\right)^{\frac{p}{q}},$$

quicunque numerus finitus loco α accipiatur, uti ex paragrapho 38 colligitur, erit quoque

$$\int x^{m+\infty nq-1}dx\left(1-x^{nq}\right)^{\frac{p}{q}}=\int x^{\mu+\infty nq-1}dx\left(1-x^{nq}\right)^{\frac{p}{q}}.$$

Quamobrem si praecedentium expressionum altera per alteram dividatur,
proveniet ista aequatio

$$\frac{\int x^{m-1}dx\left(1-x^{nq}\right)^{\frac{p}{q}}}{\int x^{\mu-1}dx\left(1-x^{nq}\right)^{\frac{p}{q}}}=\frac{\mu(m+(p+q)n)(\mu+nq)(m+(p+2q)n)(\mu+2nq)(m+(p+3q)n)(\mu+3nq)}{m(\mu+(p+q)n)(m+nq)(\mu+(p+2q)n)(m+2nq)(\mu+(p+3q)n)(m+3nq)}\text{ etc. in infin.,}$$

cuius expressionis ope innumerabilla producta ex infinitis factoribus constantia exhiberi possunt, quorum valores per quadraturas curvarum assignari poterunt.

41. Si altera formula integralis admittat integrationem, tum commoda expressio infinita pro altero integrali habebitur. Sit enim $\mu=nq$; erit

$$\int x^{\mu-1}dx\left(1-x^{nq}\right)^{\frac{p}{q}}=\frac{1}{(p+q)n}$$

quo valore substituto prodibit

$$\int x^{m-1}dx\left(1-x^{nq}\right)^{\frac{p}{q}}=\frac{1}{(p+q)n}\cdot\frac{nq\left(m+(p+q)n\right)2nq\left(m+(p+2q)n\right)3nq}{m(p+2q)n(m+nq)(p+3q)n(m+2nq)}\text{ etc.,}$$

cuius ope pro innumerabilibus integralibus expressiones per continuos factores in infinitum excurrentes inveniri possunt; eo saltem casu, quo $x=1$, quippe qui plerumque potissimum desideratur.

42. Ponatur *n* loco *nq* et prodibit

$$\int x^{m-1}dx\left(1-x^{n}\right)^{\frac{p}{q}}=\frac{q}{(p+q)n}\cdot\frac{n\left(mq+(p+q)n\right)2n\left(mq+(p+2q)n\right)3n\left(mq+(p+3q)n\right)}{m(p+2q)n(m+n)(p+3q)n(m+2n)(p+4q)}\text{ etc.,}$$

quae in binos factores resoluta fit simplicior evaditque

$$\int x^{m-1}dx\left(1-x^{n}\right)^{\frac{p}{q}}=\frac{q}{(p+q)n}\cdot\frac{1\left(mq+(p+q)n\right)}{m(p+2q)}\cdot\frac{2\left(mq+(p+2q)n\right)}{(m+n)(p+3q)}\cdot\frac{3\left(mq+(p+3q)n\right)}{(m+2n)(p+4q)}\cdot\text{ etc.,}$$

unde sequentia exempla notabiliora deducuntur:

$$\int\frac{dx}{\sqrt{(1-xx)}}=1\cdot\frac{1\cdot4}{1\cdot3}\cdot\frac{2\cdot8}{3\cdot5}\cdot\frac{3\cdot12}{5\cdot7}\text{etc.}=\frac{2\cdot2\cdot4\cdot4\cdot6\cdot6}{1\cdot3\cdot3\cdot5\cdot5\cdot7}\text{ etc.}$$

$$\int\frac{xdx}{\sqrt{(1-xx)}}=1\cdot\frac{1\cdot6}{2\cdot3}\cdot\frac{2\cdot10}{4\cdot5}\cdot\frac{3\cdot14}{6\cdot7}\cdot\text{etc.}=1,$$

$$\int\frac{x^2dx}{\sqrt{(1-xx)}}=1\cdot\frac{1\cdot8}{3\cdot3}\cdot\frac{2\cdot12}{5\cdot5}\cdot\frac{3\cdot16}{7\cdot7}\text{etc.}=\frac{2\cdot4\cdot4\cdot6\cdot6\cdot8}{3\cdot3\cdot5\cdot5\cdot7\cdot7}\text{ etc.},$$

$$\int\frac{dx}{\sqrt{(1-x^3)}}=\frac{2}{3}\cdot\frac{1\cdot5\cdot2\cdot11\cdot3\cdot17\cdot4\cdot23\cdot5\cdot29\cdot}{1\cdot3\cdot4\cdot5\cdot7\cdot7\cdot10\cdot9\cdot13\cdot11\cdot}\text{ etc.},$$

$$\int\frac{xdx}{\sqrt{(1-x^3)}}=\frac{2}{3}\cdot\frac{1\cdot7\cdot2\cdot13\cdot3\cdot19\cdot4\cdot25\cdot5\cdot31\cdot}{2\cdot3\cdot5\cdot5\cdot8\cdot7\cdot11\cdot9\cdot14\cdot11\cdot}\text{ etc.},$$

$$\int\frac{dx}{\sqrt{(1-x^4)}}=\frac{1}{2}\cdot\frac{1\cdot6\cdot2\cdot14\cdot3\cdot22\cdot4\cdot30\cdot}{1\cdot3\cdot5\cdot5\cdot9\cdot7\cdot13\cdot9\cdot}\text{ etc.}=\frac{1}{2}\cdot\frac{2\cdot3\cdot4\cdot7\cdot6\cdot11\cdot8\cdot15\cdot}{1\cdot3\cdot5\cdot5\cdot9\cdot7\cdot13\cdot9\cdot}\text{etc.},$$

$$\int\frac{xxdx}{\sqrt{(1-x^4)}}=\frac{1}{2}\cdot\frac{1\cdot10\cdot2\cdot18\cdot3\cdot26\cdot4\cdot34}{3\cdot3\cdot7\cdot5\cdot11\cdot7\cdot15\cdot9}\text{ etc.}$$

$$\int\frac{dx}{\sqrt[3]{(1-x^3)}}=\frac{1}{2}\cdot\frac{3\cdot3\cdot6\cdot6\cdot9\cdot9\cdot12\cdot12\cdot}{1\cdot5\cdot4\cdot8\cdot7\cdot11\cdot10\cdot14\cdot}\text{ etc.},$$

$$\int\frac{dx}{\sqrt[4]{(1-x^4)}}=\frac{1}{3}\cdot\frac{4\cdot4\cdot8\cdot8\cdot12\cdot12\cdot16\cdot16}{1\cdot7\cdot5\cdot11\cdot9\cdot15\cdot13\cdot19}\text{ etc.}$$

Praeterea hae expressiones notari merentur

$$\int x^{m-1}dx(1-x^n)^{-\frac{m}{n}}=\frac{1}{n-m}\cdot\frac{n\cdot n\cdot2n\cdot2n\cdot3n\cdot3n}{m(2n-m)(m+n)(3n-m)(m+2n)(4n-m)}\text{etc.},$$

$$\int x^{m-1}dx(1-x^n)^{\frac{m-n}{n}}=\frac{1}{m}\cdot\frac{n\cdot2m\cdot2n(2m+n)3n(2m+2n)4n(2m+3n)}{m(m+n)(m+n)(m+2n)(m+2n)(m+3n)(m+3n)(m+4n)}\text{etc.}$$

43. Cum autem pari modo sit

$$\int x^{\mu-1}dx(1-x^\nu)^{\frac{r}{s}}=\frac{s}{(r+s)\nu}\cdot\frac{1(\mu s+(r+s)\nu)2(\mu s+(r+2s)\nu)3(\mu s+(r+3s)\nu}{\mu(r+2s)(\mu+\nu)(r+3s)(\mu+2\nu)(r+4s)}\text{etc.},$$

erit priorem expressionem per hanc dividendo

$$\frac{\int x^{m-1}dx(1-x^n)^{\frac{p}{q}}}{\int x^{\mu-1}dx(1-x^\nu)^{\frac{r}{s}}}=\frac{(r+s)q\nu}{(p+q)sn}\cdot\frac{\mu(r+2s)(mq+(p+q)n)}{m(p+2q)(\mu s+(r+s)\nu)}\cdot\frac{(\mu+\nu)(r+3s)(mq+(p+2q)n)}{m(p+3q)(\mu s+(r+2s)\nu)}\cdot\text{etc.}$$

Haec igitur expressio infinita quoties habet valorem finitum, toties summatio alterius integralis ad alterum reduci poterit. Huiusmodi autem casus existunt, quando factores numeratoris destruunt factores denominatoris, ita ut post destructionem finitus factorum numerus supersit. Continentur enim in hac expressione omnes omnino reductiones formularum integralium ad alias.

44. Quo autem plures istiusmodi expressiones inter se comparari queant, eam hoc modo accipere visum est:

$$\frac{\int x^{a-1}dx(1-x^b)^c}{\int x^{f-1}dx(1-x^g)^h}=\frac{(h+1)g}{(c+1)b}\cdot\frac{f(h+2)(a+(c+1)b)}{a(c+2)(f+(h+1)g)}\cdot\frac{(f+g)(h+3)(a+(c+2)b)}{(a+b)(c+3)(f+(h+2)g)}\cdot\text{etc.}$$

Simili modo erit

$$\frac{\int x^{\alpha-1}dx(1-x^\beta)^\gamma}{\int x^{\zeta-1}dx(1-x^\eta)^\theta}=\frac{(\theta+1)\eta}{(\gamma+1)\beta}\cdot\frac{\zeta(\theta+2)(\alpha+(\gamma+1)\beta)}{\alpha(\gamma+2)(\zeta+(\theta+1)\eta)}\cdot\frac{(\zeta+\gamma)(\theta+3)(\alpha+(\gamma+2)\beta)}{(\alpha+\beta)(\gamma+3)(\zeta+(\theta+2)\eta)}\cdot\text{etc.}$$

quae expressiones, etsi re non inter se differunt, tamen, quoniam habent formam diversam, inter se comparari poterunt.

44. Quo autem plures istiusmodi expressiones inter se comparari queant, eam hoc modo accipere visum est:

$$\frac{\int x^{a-1}dx(1-x^b)^c}{\int x^{f-1}dx(1-x^g)^h}=\frac{(h+1)g}{(c+1)b}\cdot\frac{f(h+2)(a+(c+1)b)}{a(c+2)(f+(h+1)g)}\cdot\frac{(f+g)(h+3)(a+(c+2)b)}{(a+b)(c+3)(f+(h+2)g)}\cdot\text{etc.}$$

Simili modo erit

$$\frac{\int x^{\alpha-1}dx(1-x^\beta)^\gamma}{\int x^{\zeta-1}dx(1-x^\eta)^\theta}=\frac{(\theta+1)\eta}{(\gamma+1)\beta}\cdot\frac{\zeta(\theta+2)(\alpha+(\gamma+1)\beta)}{\alpha(\gamma+2)(\zeta+(\theta+1)\eta)}\cdot\frac{(\zeta+\gamma)(\theta+3)(\alpha+(\gamma+2)\beta)}{(\alpha+\beta)(\gamma+3)(\zeta+(\theta+2)\eta)}\cdot\text{etc.}$$

quae expressiones, etsi re non inter se differunt, tamen, quoniam habent formam diversam, inter se comparari poterunt.

45. Ut nunc ex his expressionibus eadem theoremata eliciamus, quae supra invenimus, sit erit

$$\theta=\gamma=h=c,\ \ \eta=\beta=g=b;$$

$$\frac{\int x^{a-1}dx(1-x^b)^c}{\int x^{f-1}dx(1-x^b)^c}=\frac{f(a+(c+1)b)(f+b)(a+(c+2)b)(f+2b)(a+(c+3)b)}{a(f+(c+1)b)(a+b)(f+(c+2)b)(a+2b)(f+(c+3)b)}\ \text{etc.}$$

atque altera formula

$$\frac{\int x^{\alpha-1}dx(1-x^b)^c}{\int x^{\eta-1}dx(1-x^b)^c}=\frac{\zeta(\alpha+(c+1)b)(\zeta+b)(\alpha+(c+2)b)(\zeta+2b)(\alpha+(c+3)b)}{\alpha(\zeta+(c+1)b)(\alpha+b)(\zeta+(c+2)b)(\alpha+2b)(\zeta+(c+3)b)}\ \text{etc.}$$

Harum expressionum productum si ponatur $=\frac{f}{a}$, oportet esse

$$\frac{(a+(c+1)b)(f+b)\zeta(a+(c+1)b)}{(f+(c+1)b)(a+b)\alpha(\zeta+(c+1)b)}=1;$$

hoc enim si fuerit, totarum expressionum infinitarum productum fiet $=\frac{f}{a}$.
At hoc obtinebitur faciendo

$$\alpha=a+(c+1)b,\ \zeta=f+(c+1)b$$

fietque

$$c=-\tfrac{1}{2},$$

ita ut sit

$$\alpha=a+\tfrac{1}{2}b,\ \zeta=f+\tfrac{1}{2}b,$$

eritque ideo

$$\int\frac{x^{\alpha-1}dx}{\sqrt{(1-x^b)}}\cdot\int\frac{x^{\alpha+\frac{1}{2}b-1}dx}{\sqrt{(1-x^b)}}=\frac{f}{a}\int\frac{x^{f-1}dx}{\sqrt{(1-x^b)}}\cdot\int\frac{x^{f+\frac{1}{2}b-1}dx}{\sqrt{(1-x^b)}};$$

seu si ponatur $x=z^2$, erit

$$\int\frac{z^{a-1}dz}{\sqrt{(1-z^{2b})}}\cdot\int\frac{x^{a+b-1}dz}{\sqrt{(1-z^{2b})}}=\frac{f}{a}\int\frac{z^{f-1}dz}{\sqrt{(1-z^{2b})}}\cdot\int\frac{z^{f+b-1}dz}{\sqrt{(1-z^{2b})}};$$

positis a et f loco $2a$ et $2f$. Haec autem aequatio nil aliud est nisi Theorema supra inventum § 12; facto enim $f=b$ fit

$$\int\frac{z^{2b-1}dz}{\sqrt{(1-z^{2b})}}=\frac{1}{b}\ \text{ et }\ \int\frac{z^{b-1}dx}{\sqrt{(1-z^{2b})}}=\frac{\pi}{2b},$$

unde fiet

$$\pi=2ab\int\frac{z^{\alpha-1}dz}{\sqrt{(1-z^{2b})}}\cdot\int\frac{z^{\alpha+b-1}dx}{\sqrt{(1-z^{2b})}}.$$

46. Simili modo alia huius generis Theoremata inveniri possunt; sit enim

$$g=b,\ h=c,\ \eta=\beta=b\ \text{ et }\ \theta=\gamma$$

quaeraturque casus, quo productum ambarum expressionum fiat $=1$. Hoc autem obtinebitur, si sit

$$\frac{f(a+(c+1)b)\zeta(a+(\gamma+1)b)}{a(f+(c+1)b)\alpha(\zeta+(\gamma+1)b)}=1,$$

id quod fiet capiendo

$$\alpha=a+\left(c+1\right)b,\ f=a+(\gamma+1)b,\ \zeta=a.$$

His igitur valoribus substitutis orietur sequens Theorema non inelegans

$$\frac{\int x^{a-1}dx(1-x^b)^c}{\int x^{\alpha-1}dx(1-x^b)^\gamma}\cdot\frac{\int x^{a+(c+1)b-1}dx(1-x^b)^\gamma}{\int x^{\alpha+(\gamma+1)b-1}dx(1-x^b)^c}=1;$$

sive, si ponatur

$$c+1=m \text{ et } \gamma+1=n,$$

habebitur

$$\int\frac{x^{a-1}dx}{(1-x^b)^{1-m}}\cdot\int\frac{x^{a+mb-1}dx}{(1-x^b)^{1-n}}=\int\frac{x^{a-1}dx}{(1-x^b)^{1-n}}\cdot\int\frac{x^{a+nb-1}dx}{(1-x^b)^{1-m}}.$$

47. Alio insuper modo concinnum Theorema elici poterit ponendo $\gamma=h$ et $\theta=c$ manente $\eta=\beta=g=b$ atque efficiendo, ut productum expressionum integralium fiat $=\frac{f}{a}$ quod quo eveniat, oportet esse

$$\frac{(a+(c+1)b)(f+b)\zeta(\alpha+(h+1)b)}{(f+(h+1)b)(a+b)\alpha(\zeta+(c+1)b)}=1.$$

Hoc vero efficietur capiendo

$$\alpha=a+(c+1)b,\quad \zeta=f+(h+1)b,$$

ex quo reperietur

$$c+h+1=0 \text{ seu } h=-1-c;$$

quare sumatur

$$c=-\tfrac{1}{2}+n \text{ et } h=-\tfrac{1}{2}-n,$$

atque sequens prodibit Theorema

$$\frac{f}{a}=\frac{\int x^{a-1}dx(1-x^b)^{-\frac{1}{2}+n}}{\int x^{f-1}dx(1-x^b)^{-\frac{1}{2}-n}}\cdot\frac{\int x^{a+(\frac{1}{2}+n)b-1}dx(1-x^b)^{-\frac{1}{2}-n}}{\int x^{f+(\frac{1}{2}-n)b-1}dx(1-x^b)^{-\frac{1}{2}+n}}.$$

48. Sint nunc omnes exponentes $c,\ h,\ \gamma$ et θ inaequales, at $g=\beta=\eta=b$, quaeranturque casus, quibus productum ambarum expressionum fiat $=\frac{(h+1)(\theta+1)}{(c+1)(\gamma+1)}$. Hoc autem eveniet, si reddatur haec forma

$$\frac{f(bh+2b)(a+(c+1)b)\zeta(b\theta+2b)(\alpha+(\gamma+1)b)}{a(bc+2b)(f+(h+1)b)\alpha(b\gamma+2b)(\zeta+(\theta+1)b)}=1,$$

quos factores ita expressi, ut singuli in sequentibus membris quantitate *b* crescant. Ponatur iam

$$\zeta+(\theta+1)b=bh+2b \text{ seu } \zeta=b(1+h-\theta)$$

et

$$\alpha + (\gamma + 1)b = bc + 2b \text{ seu } \alpha = b(1 + c - \gamma)..$$

Porro fiat

$$f + (h+1)b = b\theta + 2b \text{ seu } f = b(1 + \theta - h)$$

et

$$\alpha + (c+1)b = b\gamma + 2b \text{ seu } \alpha = b(1 + \gamma - c).$$

Denique debebit esse $\alpha = f$ et $\zeta = a$, quae duae aequationes requirunt, ut sit

$$c - \gamma = \theta - h \text{ sive } c + h = \gamma + \theta.$$

Unde sequens orietur Theorema

$$\frac{(h+1)(\theta+1)}{(c+1)(\gamma+1)} = \frac{\int x^{b(1+\gamma-c)-1}dx(1-x^b)^c}{\int x^{b(1+\theta-h)-1}dx(1-x^b)^h} \cdot \frac{\int x^{b(1+c-\gamma)-1}dx(1-x^b)^\gamma}{\int x^{b(1+h-\theta)-1}dx(1-x^b)^\theta},$$

dummodo sit $c + h = \gamma + \theta$.

49. Alio autem insuper modo expressio illa effici potest $=1$, ponendo

$$\alpha = a + (c+1)b \text{ et } \zeta = f + (h+1)b,\ f = b(\gamma+2),\ a = b(\theta+2),$$

ita ut sit

$$\alpha = b(3 + c + \theta) \text{ et } \zeta = b(3 + h + \gamma).$$

Porro autem debet esse

$$\zeta + (\theta+1)b = bh + 2b \text{ et } \alpha + (\gamma+1)b = bc + 2b;$$

quibus postulatur, ut sit

$$\gamma + \theta + 2 = 0.$$

Ponatur ergo

$$\gamma = -1 + n \text{ et } \theta = -1 - n.$$

At si requiratur, ut productum ambarum expressionum sit $= \frac{f(h+1)(\theta+1)}{a(c+1)(\gamma+1)}$,, id obtinebitur ponendo

$$\alpha = a + (c+1)b,\ \zeta = f + (h+1)b,\ f = b(\gamma+1),\ a = b(\theta+1),$$

unde erit

$$\alpha = b(2 + c + \theta) \text{ et } \zeta = b(2 + h + \gamma).$$

Tandem vero debebit esse

$$\gamma + \theta + 1 = 0.$$

Ponatur

$$\gamma = -\tfrac{1}{2} + n \text{ et } \theta = -\tfrac{1}{2} - n$$

atque habebitur hoc Theorema

$$\frac{h+1}{c+1} = \frac{\int x^{b(\frac{1}{2}-n)-1} dx(1-x^b)^c}{\int x^{b(\frac{1}{2}+n)-1} dx(1-x^b)^h} \cdot \frac{\int x^{b(\frac{3}{2}+c-n)-1} dx(1-x^b)^{-\frac{1}{2}+n}}{\int x^{b(\frac{3}{2}+c+n)-1} dx(1-x^b)^{-\frac{1}{2}-n}};$$

in quo notandum est exponentes $c,\ h, -\frac{1}{2}+n, -\frac{1}{2}-n$ numeros negativos quidem esse posse, sed tales, ut cum unitate ad affirmativos transeant; alioquin enim integralia valorem finitum non obtinerent casu $x=1$.

50. Quemadmodum igitur non solum Theorema supra inventum circa duarum formularum integralium producta detexi hac methodo magis directa, sed etiam alia nova elicui non minus notatu digna, ita, si pari modo tres eiusmodi expressiones in se invicem ducantur, Theoremata complura circa producta trium formularum integralium prodibunt atque ultra ad quotcunque factorum numerum progredi licebit; sed cum haec inquisitio adeo prolixum calculum requirat, ut etiam litterae vix sufficiant, cum ipsis Theorematis praecipuis indicatis tum via monstrata contentus ero.