CONCERNING THE RESOLUTION OF TRANSCENDING FRACTIONS INTO INFINITELY MANY SIMPLE FRACTIONS

Opuscula analytica 2, 1785, p. 102-137; [E592]

1. For some proposed algebraic function

$$\frac{P}{Q},$$

of which both the numerator P as well as the denominator Q shall be rational integral functions of the magnitude z, I have shown some time ago now[e.g. *Introductionem in analysin*…Bk I, Ch. II], how that may be able to be resolved into simple fractions, the denominators of which may be equal to simple factors of the denominator Q, the numerators truly shall be constants, if indeed the variable z in the denominator Q may have more dimensions than in the numerator P. Indeed I have shown also, how for any simple factor of the denominator the simple fraction corresponding may be able to be found without being had with respect to the remaining factors. Thus if it may be agreed the denominator Q to include the simple factor $z-a$, a simple fraction thence arises, which will be of this form

$$\frac{\alpha}{z-a},$$

is defined most easily in this manner. There may be put

$$\frac{P}{Q}=\frac{\alpha}{z-a}+R,$$

where R includes all the simple fractions arising from the rest ; each side may be multiplied by $z-a$, so that there may become

$$\frac{P(z-a)}{Q}=\alpha+R(z-a),$$

and because α is a constant quantity, that will retain the same value always, whatever value may be attributed to the variable z ; on account of which there may become everywhere $z=a$, so that the ratio of the remaining simple fractions shall vanish, and there will be had

$$\alpha=\frac{P(z-a)}{Q},$$

if indeed there may become $z=a$ in this formula ; but then the numerator $P(z-a)$ will be changed into zero; truly, because $z-a$ is a factor of the denominator Q, also the denominator Q will become. Hence therefore by the customary rule [*i.e.* L'Hopital's Rule; see also Euler's *Differential Calculus*, Ch. XV] in place of the numerator and denominator their differentials may be substituted, since even now there will be

$$\frac{Pdz+(z-a)dP}{dQ}=\alpha,$$

if indeed here everywhere there may be written a in place of z. Therefore we may put in this case $z=a$ to become

$$P=A \text{ and } \frac{dQ}{dz}=C,$$

therefore which quantities A and C are found more easily; therefore then the numerator sought will be produced

$$\alpha=\frac{A}{C},$$

thus so that the simple fraction shall arise from the factor of the denominator $z-a$

$$=\frac{A}{C(z-a)},$$

thus so that there shall be no need to know the remaining factors of the denominator. Moreover in a similar manner the simple fractions corresponding to the individual remaining factors will be determined, of which the sum of all will be equal to the proposed fraction $\frac{P}{Q}$, provided the variable z may have fewer dimensions in the numerator P than in the denominator Q.

2. Therefore we may assume transcending functions for the denominator Q following these principles of this kind, which shall be allowed to be resolved into infinitely many simple factors, because that happens, if these may avoid being equal to zero in an infinitely many cases. Truly besides it is necessary, that all these factors shall be unequal to each other, since equal factors demand a particular resolution. But in the first place it is required, that the product of all such factors may represent that function Q in its entirety, since as it were imaginary factors are themselves able to be mixed together. Just as if $Q=\text{tang.}\varphi$ may be taken, that certainly vanishes in all the same cases as in which this function $\sin.\varphi$, and hence both these functions involve the same simple factors, even if they shall never be equal to each other. Truly thence the numerator P is required to be prepared thus, so that it may have no common factors with the denominator Q. But in the first place it is required to be warned, lest variable quantities in the numerator may rise to the same or more dimensions than in the denominator. But since that in the denominator shall be agreed to rise to infinite dimensions, that inconvenience will not be required to be worrying, as long as only a finite number may be contained in the numerator. But if its powers may also rise to infinity, often it will be judged with difficulty, whether the number of the dimension shall be greater or less than in the denominator. Yet meanwhile also in these cases the proposed fraction $\frac{P}{Q}$ will contain all the simple fractions, to which our method leads. Truly it can happen, that besides these also certain parts may emerge as if whole numbers. Therefore with these noted beforehand we may present the following cases.

I. THERE MAY BE TAKEN $Q = \sin.\varphi$ SO THAT THE FRACTION REQUIRING TO BE RESOLVED SHALL BE $\frac{P}{\sin.\varphi}$.

3. Because the formula $\sin.\varphi$, with π denoting the semi periphery of the circle of which the radius $=1$, or an angle equal to two right angles, will vanish for all these cases

$$\varphi = 0,\ \varphi = \pm\pi,\ \varphi = \pm 2\pi,\ \varphi = \pm 3\pi \text{ etc. and in general } \varphi = \pm i\pi,$$

the number of which factors will be infinite :

$$\varphi,\ (\varphi \pm \pi),\ (\varphi \pm 2\pi),\ \ (\varphi \pm 3\pi), \text{ etc. and in general, } (\varphi \pm i\pi).$$

But from elsewhere it is certain this formula $\sin.\varphi$, besides these factors, to involve no other factors either real or imaginary ; for since there shall be

$$\sin.\varphi = \varphi - \frac{\varphi^3}{1\cdot 2\cdot 3} + \frac{\varphi^5}{1\cdot 2\cdot 3\cdot 4\cdot 5} - \text{etc.},$$

it is agreed this series to be equal to this infinite product

$$\varphi(1-\tfrac{\varphi}{\pi})(1+\tfrac{\varphi}{\pi})(1-\tfrac{\varphi}{2\pi})(1+\tfrac{\varphi}{2\pi})(1-\tfrac{\varphi}{3\pi})(1+\tfrac{\varphi}{3\pi}) \text{ etc.}$$

4. Therefore we will consider some factor $\varphi - i\pi$ of our denominator $Q = \sin.\varphi$, where plainly i will denote all the whole numbers both positive and negative with zero not excepted, and a partial fraction hence arises

$$\frac{\alpha}{\varphi - i\pi}.$$

Towards finding its numerator initially α may be put in the numerator *P,* and everywhere $\varphi = i\pi$ and thence the resulting quantity shall be $= A$; then since there shall be $Q = \sin.\varphi$, there will be

$$dQ = d\varphi\cos.\varphi \ \text{ or } \ \frac{dQ}{d\varphi} = \cos.\varphi,$$

where in place of φ likewise it will be required to write $i\pi$, so that we may obtain *C,* from which it is apparent to become

$$C = \cos.i\pi,$$

thus so that there shall be

$$C = \pm 1,$$

where the sign + will prevail for even numbers, truly the sign $-$ for odd numbers assumed in place of i . Therefore in this manner the numerator of our fraction will be

$$\alpha = \pm A$$

and the fraction itself sought :

$$\pm\tfrac{A}{\varphi - i\pi}.$$

But hence it is not allowed to progress further, as long as we may consider the numerator in general; from which in place of this we may take several determined and singular values we may establish in the following examples.

1°. $P=1$ AND THE PROPOSED FRACTION SHALL BE $\frac{1}{\sin.\varphi}$

5. Therefore here there shall be always $A=1$ and any simple fraction

$$= \tfrac{\pm 1}{\varphi - i\pi},$$

the upper sign prevails, if i shall be an even number, truly the lower, if i shall be odd. Therefore we may substitute for i successively all its values in order

$$0,\ +1,\ -1,\ +2,\ -2,\ 3,\ -3,\ +\ 4,\ -4 \text{ etc.}$$

and the resolution of our fraction $\frac{1}{\sin.\varphi}$ into simple fractions thus will itself be had :

$$\tfrac{1}{\sin.\varphi} = +\tfrac{1}{\varphi} - \tfrac{1}{\varphi-\pi} - \tfrac{1}{\varphi+\pi} + \tfrac{1}{\varphi-2\pi} + \tfrac{1}{\varphi+2\pi} - \tfrac{1}{\varphi-3\pi} - \tfrac{1}{\varphi+3\pi} + \text{etc.}$$

which may be reduced into this form:

$$\tfrac{1}{\sin.\varphi} = +\tfrac{1}{\varphi} + \tfrac{1}{\pi-\varphi} - \tfrac{1}{\pi+\varphi} - \tfrac{1}{2\pi-\varphi} + \tfrac{1}{2\pi+\varphi} + \tfrac{1}{3\pi-\varphi} - \tfrac{1}{3\pi+\varphi} - \text{etc.}$$

After the first term the pairs following may be combined, so that we may obtain this series :

$$\tfrac{1}{\sin.\varphi} = +\tfrac{1}{\varphi} + \tfrac{2\varphi}{\pi\pi-\varphi\varphi} - \tfrac{2\varphi}{4\pi\pi-\varphi\varphi} + \tfrac{2\varphi}{9\pi\pi-\varphi\varphi} - \tfrac{2\varphi}{16\pi\pi-\varphi\varphi} + \text{etc.},$$

from which the following noteworthy series is deduced

$$\tfrac{1}{2\varphi\sin.\varphi} - \tfrac{1}{2\varphi\varphi} = \tfrac{1}{\pi\pi-\varphi\varphi} - \tfrac{1}{4\pi\pi-\varphi\varphi} + \tfrac{1}{9\pi\pi-\varphi\varphi} - \tfrac{1}{16\pi\pi-\varphi\varphi} + \text{etc.},$$

6. Indeed now at one time I had pursued these series more fully [see : *Introduction to Analysis…*, Vol. 1, Ch. X.] ; yet meanwhile here it will be useful to repeat the following transformations for the following cases. Therefore initially we may put $\varphi = \lambda\pi$, so that the letter π may be removed from the series, and hence we will obtain:

$$\frac{\pi}{\sin.\lambda\pi} = \frac{1}{\lambda} - \frac{1}{\lambda-1} - \frac{1}{\lambda+1} + \frac{1}{\lambda-2} + \frac{1}{\lambda+2} - \frac{1}{\lambda-3} - \frac{1}{\lambda+3} + \text{etc.}$$

and

$$\frac{\pi}{2\lambda\sin.\lambda\pi} - \frac{1}{2\lambda\lambda} = \frac{1}{1-\lambda\lambda} - \frac{1}{4-\lambda\lambda} + \frac{1}{9-\lambda\lambda} - \frac{1}{16-\lambda\lambda} + \text{etc.}$$

and hence by differentiation by regarding λ as a variable quantity we will be able to elicit infinitely many other most noteworthy series. Evidently from the first equation above we will obtain:

$$\frac{\pi\pi\cos.\lambda\pi}{\sin^2.\lambda\pi} = \frac{1}{\lambda\lambda} - \frac{1}{(\lambda-1)^2} - \frac{1}{(\lambda+1)^2} + \frac{1}{(\lambda-2)^2} + \frac{1}{(\lambda+2)^2} - \frac{1}{(\lambda-3)^2} - \text{etc.}$$

Hence therefore there follows, if $\lambda = \frac{1}{2}$, to become

$$0 = 1 - 1 - \tfrac{1}{9} + \tfrac{1}{9} + \tfrac{1}{25} - \tfrac{1}{25} - \text{etc.},$$

which is indeed evident. But if $\lambda = \frac{1}{3}$, there will be

$$\frac{2\pi\pi}{27} = 1 - \frac{1}{4} - \frac{1}{16} + \frac{1}{25} + \frac{1}{49} - \frac{1}{64} - \frac{1}{100} + \frac{1}{121} + \frac{1}{169} - \text{etc.}$$

If $\lambda = \frac{2}{3}$, the preceding series arises.

If $\lambda = \frac{1}{4}$, this summation will be produced :

$$\frac{\pi\pi}{8\sqrt{2}} = 1 - \frac{1}{9} - \frac{1}{25} + \frac{1}{49} + \frac{1}{81} - \frac{1}{121} - \frac{1}{169} + \text{etc.}$$

But if we may differentiate anew, the following summation will be obtained :

$$\frac{\pi^3}{\sin^3.\lambda\pi} - \frac{\pi^3}{2\sin.\lambda\pi} = \frac{1}{\lambda^3} - \frac{1}{(\lambda-1)^3} - \frac{1}{(\lambda+1)^3} + \frac{1}{(\lambda-2)^3} + \frac{1}{(\lambda+2)^3} - \frac{1}{(\lambda-3)^2} - \text{etc.}$$

and thus it is allowed to be continually progressing further.

7. In a similar manner also we may differentiate the other form also: which reduced presents:

$$\frac{1}{2\lambda^4} - \frac{\pi}{4\lambda^3\sin.\lambda\pi} - \frac{\pi\pi\cos.\lambda\pi}{4\lambda\lambda\sin^2.\lambda\pi} = \frac{1}{(1-\lambda\lambda)^2} - \frac{1}{(4-\lambda\lambda)^2} + \frac{1}{(9-\lambda\lambda)^2} - \frac{1}{(16-\lambda\lambda)^3} + \text{etc.}$$

But if now we may suppose $\lambda = \frac{1}{2}$, it will produce this summation:

$$\frac{1}{2} - \frac{\pi}{8} \;=\; \frac{1}{3^2} - \frac{1}{15^2} + \frac{1}{35^2} - \frac{1}{63^2} + \frac{1}{99^2} - \text{etc.}$$

which evidently new series deserves all attention ; but nor is there hence a need to put in place a new differentiation.

8. But we may consider the latter summation

$$\frac{\pi}{2\lambda \sin.\lambda\pi} - \frac{1}{2\lambda\lambda} = \frac{1}{1-\lambda\lambda} - \frac{1}{4-\lambda\lambda} + \frac{1}{9-\lambda\lambda} - \frac{1}{16-\lambda\lambda} + \text{etc.}$$

more carefully and indeed at first, since that must be real always, whatever we may assume for λ, we may assume $\lambda = 0$. But since in this case the left hand member will be changed into $\infty - \infty$, λ may be treated rather as a minimum quantity, and since there shall be $\sin.\lambda\pi = \lambda\pi - \frac{1}{6}\lambda^3\pi^3$, that member itself emerges :

$$\frac{\pi}{2\lambda(\lambda\pi - \frac{1}{6}\lambda^3\pi^3)} - \frac{1}{2\lambda\lambda},$$

which fractions reduced to a common denominator give :

$$\frac{1-1+\frac{1}{6}\lambda\lambda\pi\pi}{2\lambda\lambda(1-\frac{1}{6}\lambda\lambda\pi\pi)} = \frac{\pi\pi}{12-2\lambda\lambda\pi\pi}.$$

Therefore now on making $\lambda = 0$ the product itself will be $= \frac{\lambda\lambda}{12}$, but in that case this series itself will emerge

$$1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} - \text{etc.},$$

of which $\frac{\pi\pi}{12}$ is agreed to be the sum.

9. Again it is evident, as often as a whole number may be taken for λ, one term of the series and thus also that series itself becomes infinite, which agrees especially with the sum found, since in this case there becomes $\sin.\lambda\pi = 0$. And hence this question arises : if that term of the series going to infinity may be transferred to the left hand side, how great shall the sum of the remaining terms become ? Clearly we may put $\lambda = 1$, and the first term of the series becomes infinite ; which therefore transferred to the left hand side will give

$$\frac{\pi}{2\lambda \sin.\lambda\pi} - \frac{1}{2\lambda\lambda} - \frac{1}{1-\lambda\lambda} = -\frac{1}{3} + \frac{1}{8} - \frac{1}{15} + \frac{1}{24} - \frac{1}{35} + \frac{1}{48} - \text{etc.}$$

Now for finding the value of this series λ may be put in place only approximately equal to unity, by putting $\lambda = 1-\omega$, and there will be

$$\sin.\lambda\pi = \sin.(\pi - \pi\omega) = \sin.\pi\omega;$$

truly there is :

$$\sin.\pi\omega = \pi\omega - \tfrac{1}{6}\pi^3\omega^3,$$

with which value substituted there will be produced

$$\frac{1}{2(1-\omega)\omega(1-\frac{1}{6}\pi^2\omega^2)} - \frac{1}{2(1-\omega)^2} - \frac{1}{2\omega-\omega\omega}.$$

But the first part :

$$\frac{1}{2(1-\omega)\omega(1-\frac{1}{6}\pi^2\omega^2)},$$

on account of :

$$\tfrac{1}{1-\omega} = 1 + \omega + \omega^2$$

and

$$\frac{1}{1-\frac{1}{6}\pi^2\omega^2} = 1 + \tfrac{1}{6}\pi^2\omega^2$$

with the powers of ω higher than with the square may be transformed into this form :

$$\tfrac{1}{2\omega}(1 + \omega + \omega^2 + \tfrac{1}{6}\pi\pi\omega\omega);$$

but the third part:

$$-\frac{1}{2\omega(1-\frac{1}{2}\omega)}$$

on account of

$$\frac{1}{1-\frac{1}{2}\omega} = 1 + \tfrac{1}{2}\omega + \tfrac{1}{4}\omega\omega$$

will be changed into

$$-\tfrac{1}{2\omega}(1 + \tfrac{1}{2}\omega + \tfrac{1}{4}\omega\omega),$$

from which the first and third part together make :

$$\tfrac{1}{2\omega}(\tfrac{1}{2}\omega + \tfrac{3}{5}\omega\omega + \tfrac{1}{6}\pi\pi\omega\omega) = \tfrac{1}{4} + \tfrac{3}{8}\omega + \tfrac{1}{12}\pi\pi\omega;$$

which with the value put $\omega = 0$ becomes $= \frac{1}{4}$, from which the second part, which will be $-\frac{1}{2}$, added on will give the whole sum sought $-\frac{1}{4}$, thus so that with a change of signs

$$\tfrac{1}{4} = \tfrac{1}{3} - \tfrac{1}{8} + \tfrac{1}{15} - \tfrac{1}{24} + \tfrac{1}{35} - \tfrac{1}{48} + \tfrac{1}{63} - \text{etc.},$$

the reasoning of which is evident, since there shall be

$$\tfrac{1}{3}=\tfrac{1}{2}(1-\tfrac{1}{3}),\ \tfrac{1}{8}=\tfrac{1}{2}(\tfrac{1}{2}-\tfrac{1}{4}),\ \tfrac{1}{15}=\tfrac{1}{2}(\tfrac{1}{3}-\tfrac{1}{5}),\ \tfrac{1}{24}=\tfrac{1}{2}(\tfrac{1}{4}-\tfrac{1}{6})\text{ etc.;}$$

for with these values substituted, and with the remaining terms cancelling each other, there will become

$$\tfrac{1}{4}=\tfrac{1}{2}-\tfrac{1}{2\cdot 2}=\tfrac{1}{4}.$$

10. But concerning the same series a more difficult question occurs, when the sum of the series is sought, if $\lambda\lambda$ were a negative number, and thus λ were an imaginary quantity. Therefore there may be put

$$\lambda\lambda=-\mu\mu \text{ or } \lambda=\mu\sqrt{-1}$$

and the series nevertheless will be real, clearly :

$$\frac{1}{1+\mu\mu}-\frac{1}{4+\mu\mu}+\frac{1}{9+\mu\mu}-\frac{1}{16+\mu\mu}+\frac{1}{25+\mu\mu}-\text{etc.;}$$

of which the sum therefore will be :

$$\frac{\pi}{2\mu\sqrt{-1}\cdot\sin.\pi\mu\sqrt{-1}}+\frac{1}{2\mu\mu},$$

of which therefore a real value is sought, if indeed there is no doubt, why the value of such a series may not be real.

11. In the doctrine of angles it is customarily shown [E61] :

$$\sin.\varphi=\frac{e^{\varphi\sqrt{-1}}-e^{-\varphi\sqrt{-1}}}{2\sqrt{-1}}.$$

Therefore there may become $\varphi=\mu\pi\sqrt{-1}$ and there will be:

$$\varphi\sqrt{-1}=-\mu\pi \text{ and } \varphi\pi\sqrt{-1}=+\mu\pi,$$

from which it is concluded:

$$\sin.\mu\pi\sqrt{-1}=\frac{e^{-\mu\pi}-e^{+\mu\pi}}{2\sqrt{-1}},$$

from which the sum sought will be :

$$\frac{\pi}{\mu(e^{-\mu\pi}-e^{+\mu\pi})}+\frac{1}{2\mu\mu}.$$

Therefore there will become :

$$\frac{1}{1+\mu\mu}-\frac{1}{4+\mu\mu}+\frac{1}{9+\mu\mu}-\frac{1}{16+\mu\mu}+\frac{1}{25+\mu\mu}-\text{etc.}=\frac{1}{2\mu^2}-\frac{\pi}{\mu(e^{\mu\pi}-e^{-\mu\pi})}.$$

2°. $P=\varphi$ AND THE PROPOSED FRACTION SHALL BE $\frac{\varphi}{\sin.\varphi}$

12. Therefore here on account of the numerator $P=\varphi$ the first factor φ of the denominator is removed, just as our resolution also provides the corresponding numerator itself equality to zero. Therefore here for the denominator $\varphi-i\pi$ the numerator becomes $\frac{i\pi}{\cos.i\pi}=\pm i\pi$, where the upper sign prevails, if the number i were even, the lower, if odd. So that if therefore there were $i=2n$, the fraction thence produced will be

$$\frac{2n\pi}{\varphi-2n\pi};$$

but if $i=-2n$, the fraction will be

$$-\frac{2n\pi}{\varphi+2n\pi};$$

but if there were $i=2n-1$, the fraction will be

$$-\frac{(2n-1)\pi}{\varphi-(2n-1)\pi};$$

finally from $i=2n-1$ there becomes:

$$\frac{(2n-1)\pi}{\varphi+(2n-1)\pi},$$

on account of which the series found will be

$$\frac{\varphi}{\sin.\varphi}=-\frac{\pi}{\varphi-\pi}+\frac{\pi}{\varphi+\pi}+\frac{2\pi}{\varphi-2\pi}-\frac{2\pi}{\varphi+2\pi}-\frac{3\pi}{\varphi-3\pi}+\frac{3\pi}{\varphi+3\pi}+\frac{4\pi}{\varphi-4\pi}-\text{etc.}$$

or

$$\frac{\varphi}{\sin.\varphi}=\frac{\pi}{\pi-\varphi}+\frac{\pi}{\pi+\varphi}-\frac{2\pi}{2\pi-\varphi}-\frac{2\pi}{2\pi+\varphi}+\frac{3\pi}{3\pi-\varphi}+\frac{3\pi}{3\pi+\varphi}-\frac{4\pi}{4\pi-\varphi}-\text{etc.},$$

from which, if the double terms may be contracted into one, there will be

$$\frac{\varphi}{\sin.\varphi}=\frac{2\pi\pi}{\pi\pi-\varphi\varphi}-\frac{8\pi\pi}{4\pi\pi-\varphi\varphi}+\frac{18\pi\pi}{9\pi\pi-\varphi\varphi}-\frac{32\pi\pi}{16\pi\pi-\varphi\varphi}+\text{etc.},$$

which divided by $2\pi\pi$ produces this summation :

$$\frac{\varphi}{2\pi\pi\sin.\varphi}=\frac{1}{\pi\pi-\varphi\varphi}-\frac{4}{4\pi\pi-\varphi\varphi}+\frac{9}{9\pi\pi-\varphi\varphi}-\frac{16}{16\pi\pi-\varphi\varphi}+\text{etc.}$$

And if there may be put $\varphi = \lambda\pi$, there will be produced:

$$\frac{\lambda\pi}{2\sin.\lambda\pi} = \frac{1}{1-\lambda\lambda} - \frac{4}{4-\lambda\lambda} + \frac{9}{9-\lambda\lambda} - \frac{16}{16-\lambda\lambda} + \text{etc.},$$

from which, if there were

$$\lambda\lambda = -\mu\mu \text{ or } \lambda = \mu\sqrt{-1},$$

on account of

$$\sin.\mu\pi\sqrt{-1} = \frac{e^{-\mu\pi} - e^{+\mu\pi}}{2\sqrt{-1}}$$

there will be

$$\frac{\mu\pi}{e^{\mu\pi} - e^{-\mu\pi}} = \frac{1}{1+\mu\mu} - \frac{1}{4+\mu\mu} + \frac{1}{9+\mu\mu} - \frac{1}{16+\mu\mu} + \text{etc.},$$

and hence by differentiation infinitely many other summations will be allowed to be deduced.

3°. THE NUMERATOR SHALL BE $P = \varphi^2$ AND THE FRACTION $\frac{\varphi\varphi}{\sin.\varphi}$

13. Therefore for the denominator $\varphi - i\pi$ the numerator will be $\pm ii\pi\pi$, where the upper sign prevails for an even number i , truly the lower for odd. Hence if in place of i the numbers may be written successively

$$+1, \ -1, \ +2, \ -2, \ +3, \ -3 \text{ etc.},$$

the resulting series will be

$$\frac{\varphi\varphi}{\sin.\varphi} = -\frac{\pi\pi}{\varphi-\pi} - \frac{\pi\pi}{\varphi+\pi} + \frac{4\pi\pi}{\varphi-2\pi} + \frac{4\pi\pi}{\varphi+2\pi} - \frac{9\pi}{\varphi-3\pi} - \frac{9\pi}{\varphi+3\pi} + \text{etc.}$$

or

$$\frac{\varphi\varphi}{\sin.\varphi} = \frac{\pi\pi}{\pi-\varphi} - \frac{\pi\pi}{\pi+\varphi} - \frac{4\pi\pi}{2\pi-\varphi} + \frac{4\pi\pi}{2\pi+\varphi} + \frac{9\pi\pi}{3\pi-\varphi} - \frac{9\pi\pi}{3\pi+\varphi} - \text{etc.}$$

Therefore with the pairs of terms contracted together there will become :

$$\frac{\varphi\varphi}{\sin.\varphi} = \frac{2\pi\pi\varphi}{\pi\pi-\varphi\varphi} - \frac{8\pi\pi\varphi}{4\pi\pi-\varphi\varphi} + \frac{18\pi\pi\varphi}{9\pi\pi-\varphi\varphi} - \frac{32\pi\pi\varphi}{16\pi\pi-\varphi\varphi} + \text{etc.}$$

or

$$\frac{\varphi}{2\sin.\varphi} = \frac{\pi\pi}{\pi\pi-\varphi\varphi} - \frac{4\pi\pi}{4\pi\pi-\varphi\varphi} + \frac{9\pi\pi}{9\pi\pi-\varphi\varphi} - \frac{16\pi\pi}{16\pi\pi-\varphi\varphi} + \text{etc.}$$

But if now any term of the series may be divided into two parts, of which the first is 1 always, the following double series will be produced :

$$\frac{\varphi}{2\sin.\varphi}=\begin{cases}+1 \qquad -1 \qquad +1 \qquad -1 \qquad +1 \qquad -\text{etc.}\\ \frac{\varphi\varphi}{\pi\pi-\varphi\varphi}-\frac{\varphi\varphi}{4\pi\pi-\varphi\varphi}+\frac{\varphi\varphi}{9\pi\pi-\varphi\varphi}-\frac{\varphi\varphi}{16\pi\pi-\varphi\varphi}+\frac{\varphi\varphi}{25\pi\pi-\varphi\varphi}-\text{etc.}\end{cases}$$

But the sum of the series

$$+1-1+1-1+1-1+\text{etc.}$$

has been known to be $=\frac{1}{2}$ [see E247]; which moved to the other side and divided by $\varphi\varphi$ will produce

$$\frac{1}{2\varphi\sin.\varphi}-\frac{1}{2\varphi\varphi}=\frac{1}{\pi\pi-\varphi\varphi}-\frac{1}{4\pi\pi-\varphi\varphi}+\frac{1}{9\pi\pi-\varphi\varphi}-\text{etc.},$$

which agrees at once with the series found in §. 5.

4°. LET $P=\varphi^{\gamma}$, WITH γ DENOTING SOME ODD POSITIVE NUMBER, SO THAT THE FRACTION PROPOSED SHALL BE $\frac{\varphi^{\gamma}}{\sin.\varphi}$

14. Therefore since for the denominator $\varphi-i\pi$ there may become $A=i^{\gamma}\pi^{\gamma}$ and $C=\pm1$, the numerator will be $\pm i^{\gamma}\pi^{\gamma}$, from which, since γ shall be an odd number, the signs of our terms proceed by the same rule and in the case $P=\varphi$, where $\gamma=1$; from which hence there will arise :

$$\frac{\varphi^{\gamma}}{\sin.\varphi}=\frac{\pi^{\gamma}}{\pi-\varphi}+\frac{\pi^{\gamma}}{\pi+\varphi}-\frac{2^{\gamma}\pi^{\gamma}}{2\pi-\varphi}-\frac{2^{\gamma}\pi^{\gamma}}{2\pi+\varphi}+\frac{3^{\gamma}\pi^{\gamma}}{3\pi-\varphi}+\frac{3^{\gamma}\pi^{\gamma}}{3\pi+\varphi}-\text{etc.},$$

which divided by π^{γ} gives :

$$\frac{\varphi^{\gamma}}{\pi^{\gamma}\sin.\varphi}=\frac{1}{\pi-\varphi}+\frac{1}{\pi+\varphi}-\frac{2^{\gamma}}{2\pi-\varphi}-\frac{2^{\gamma}}{2\pi+\varphi}+\frac{3^{\gamma}}{3\pi-\varphi}+\frac{3^{\gamma}}{3\pi+\varphi}-\text{etc.},$$

and with the pairs of terms contracted there will become:

$$\frac{\varphi^{\gamma}}{2\pi^{\gamma+1}\sin.\varphi}=\frac{1}{\pi\pi-\varphi\varphi}-\frac{2^{\gamma}}{4\pi\pi-\varphi\varphi}+\frac{3^{\gamma}}{9\pi\pi-\varphi\varphi}-\frac{4^{\gamma}}{16\pi\pi-\varphi\varphi}+\text{etc.}$$

Now we may take $\varphi=\lambda\pi$ and there will be

$$\frac{\lambda^{\gamma}\pi}{2\sin.\varphi}=\frac{1}{1-\lambda\lambda}-\frac{2^{\gamma}}{4-\lambda\lambda}+\frac{3^{\gamma}}{9-\lambda\lambda}-\frac{4^{\gamma}}{16-\lambda\lambda}+\text{etc.}$$

15. Hence if there were $\lambda = \mu\sqrt{-1}$, as before there will be

$$\sin.\mu\pi\sqrt{-1} = \frac{e^{-\mu\pi} - e^{+\mu\pi}}{2\sqrt{-1}}.$$

But for the value of the power λ^{γ} it will be required to set out two cases, just as there were

$$\gamma = 4n+1 \text{ and } \gamma = 4n-1.$$

In the first case there will be

$$\lambda^{\gamma} = (\mu\sqrt{-1})^{4n+1} = (\mu\sqrt{-1})^{4n} \cdot \mu\sqrt{-1}$$

Truly there is

$$(\mu\sqrt{-1})^{4n} = \mu^{4n}$$

from which there will be

$$\lambda^{\gamma} = \mu^{4n+1}\sqrt{-1},$$

and hence the following real summation is produced

$$\frac{\mu^{4n+1}\pi}{e^{\mu\pi} - e^{-\mu\pi}} = \frac{1}{1+\mu\mu} - \frac{2^{4n+1}}{4+\mu\mu} + \frac{3^{4n+1}}{9+\mu\mu} - \frac{4^{4n+1}}{16+\mu\mu} + \text{etc.}$$

But in the other case, where $\gamma = 4n-1$, the first member must be taken negative and there will be

$$-\frac{\mu^{4n-1}\pi}{e^{\mu\pi} - e^{-\mu\pi}} = \frac{1}{1+\mu\mu} - \frac{2^{4n-1}}{4+\mu\mu} + \frac{3^{4n-1}}{9+\mu\mu} - \frac{4^{4n-1}}{16+\mu\mu} + \text{etc.}$$

But it is not able to show easily these summations to be true, unless γ shall be an odd integer and indeed positive.

5°. THE NUMERATOR SHALL BE $P = \varphi^{\delta}$ WITH δ DENOTING SOME POSITIVE EVEN NUMBER AND THE FRACTION $\frac{\varphi^{\delta}}{\sin.\varphi}$

15 [a]. Therefore with the denominator $\varphi - i\pi$, the numerator will be $\pm i^{\delta}\pi^{\delta}$, with the sign ambiguity maintaining the same rule. Therefore in this case the reckoning of the signs likewise will be had between themselves, and as in the case $P = \varphi\varphi$ there will be therefore :

$$\frac{\varphi^{\delta}}{\sin.\varphi} = \frac{\pi^{\delta}}{\pi-\varphi} - \frac{\pi^{\delta}}{\pi+\varphi} - \frac{2^{\delta}\pi^{\delta}}{2\pi-\varphi} + \frac{2^{\delta}\pi^{\delta}}{2\pi+\varphi} + \frac{3^{\delta}\pi^{\delta}}{3\pi-\varphi} - \frac{3^{\delta}\pi^{\delta}}{3\pi+\varphi} - \text{etc.}$$

Whereby if we may put $\varphi = \lambda\pi$, there will be this series :

$$\frac{\lambda^{\delta}\pi}{\sin.\lambda\pi} = \frac{1}{1-\lambda} - \frac{1}{1+\lambda} - \frac{2^{\delta}}{2-\lambda} + \frac{2^{\delta}}{2+\lambda} + \frac{3^{\delta}}{3-\lambda} - \frac{3^{\delta}}{3+\lambda} - \text{etc.};$$

hence with the pairs of terms combined there will become :

$$\frac{\lambda^{\delta-1}\pi}{2\sin.\lambda\pi}=\frac{1}{1-\lambda\lambda}-\frac{2^{\delta}}{2-\lambda\lambda}+\frac{3^{\delta}}{3-\lambda\lambda}-\frac{4^{\delta}}{4-\lambda\lambda}+\text{etc.}$$

16. Now we may put also $\lambda=\mu\sqrt{-1}$, so that there shall become

$$\sin.\lambda\pi=\frac{e^{-\mu\pi}-e^{+\mu\pi}}{2\sqrt{-1}}$$

But for the value of $\lambda^{\delta-1}$ itself it will be necessary to set out two cases, as there were either $\delta=4n$ or $\delta=4n+2$.
In the first case, where $\delta=4n,$ there will be $\lambda^{4n}=\mu^{4n}$, and thus

$$\lambda^{4n-1}=\frac{\mu^{4n-1}}{\sqrt{-1}};$$

and hence this summation will arise :

$$\frac{-\mu^{4n-1}\pi}{e^{\mu\pi}-e^{-\mu\pi}}=\frac{1}{1+\mu\mu}-\frac{2^{4n}}{4+\mu\mu}+\frac{3^{4n}}{9+\mu\mu}-\frac{4^{4n}}{16+\mu\mu}+\text{etc.}$$

But for the other case $\delta=4n+2$ thus this summation itself will be had

$$\frac{+\mu^{4n+1}\pi}{e^{\mu\pi}-e^{-\mu\pi}}=\frac{1}{1+\mu\mu}-\frac{2^{4n+2}}{4+\mu\mu}+\frac{3^{4n+2}}{9+\mu\mu}-\frac{4^{4n+2}}{16+\mu\mu}+\text{etc.}$$

17. But these summations only will be agreed to be true so far as they have been defined, provided the exponents γ and δ may be taken to be integers, and nothing hinders however small or great they may be assumed. Indeed since the denominator

$$\sin.\varphi=\varphi-\tfrac{1}{6}\varphi^{3}+\tfrac{1}{120}\varphi^{5}-\text{etc.}$$

may rise to infinite dimensions of φ, provided the maximum power in the numerator may not become infinite, the resolution into fractions always leads to the truth. But if these exponents shall not be positive integers, but fractions or indeed to be negative, the resolution into partial fractions clearly is unable to be made. On account of which if in place of the numerator P we may establish functions of φ this kind, which also may rise to infinite dimensions, then we will no longer be certain of the sum found. Truly it can happen, so that for the partial fractions found above, certain parts may be added to whole numbers. Therefore we will establish some cases of this kind.

6°. LET THE NUMERATOR BE $P=\cos.\varphi$ AND THE FRACTION $=\frac{\cos.\varphi}{\sin.\varphi}$

18. Since there shall be

$$\cos.\varphi=1-\tfrac{1}{2}\varphi\varphi+\tfrac{1}{24}\varphi^4-\tfrac{1}{720}\varphi^6+\text{etc.},$$

the powers of φ in the numerator and in the denominator rise equally to infinity, from which it may happen, that this fraction may involve a whole number part; which since it may be found, if there may be taken $\varphi=\infty$, this integral part itself to become $=\frac{\cos.\infty}{\sin.\infty}=\cot.\infty$, which moreover evidently itself being indeterminate. Yet meanwhile, by rightly assuming since just as many positive cases as negative can arise, the middle value can be seen to be zero $=0$; remaining doubt may be removed by the following expansion. Since for the denominator $\varphi-i\pi$ there may become $A=\cos.i\pi$ and $C=\cos.i\pi$, the numerator of this fraction $=1$; therefore hence the following series will arise

$$\frac{\cos.\varphi}{\sin.\varphi}=\frac{1}{\varphi}+\frac{1}{\varphi-\pi}+\frac{1}{\varphi+\pi}+\frac{1}{\varphi-2\pi}+\frac{1}{\varphi+2\pi}+\text{etc.}$$

or

$$\cot.\varphi=\frac{1}{\varphi}-\frac{1}{\pi-\varphi}+\frac{1}{\pi+\varphi}-\frac{1}{2\pi-\varphi}+\frac{1}{2\pi+\varphi}-\text{etc.}$$

Therefore on putting $\varphi=\lambda\pi$ this series will adopt this form :

$$\pi\cot.\lambda\pi=\frac{1}{\lambda}-\frac{1}{1-\lambda}+\frac{1}{1+\lambda}-\frac{1}{2-\lambda}+\frac{1}{2+\lambda}-\frac{1}{3-\lambda}+\text{etc.};$$

or which summation shall be true, by the cases we may investigate. And indeed initially if λ may denote a whole number, the truth may be confirmed ; indeed always if any other term of the series becomes infinite, truly the sum also becomes infinite. Moreover we may suppose $\lambda=\frac{1}{2}$; there will become $\pi\cot.\frac{\pi}{2}=0$, but that series produces

$$\frac{2}{1}-\frac{2}{1}+\frac{2}{3}-\frac{2}{3}+\frac{2}{5}-\frac{2}{5}+\text{etc.},$$

where all the terms evidently cancel each other. But we may assume above $\lambda=\frac{1}{4}$, and there will be produced

$$\pi=\frac{4}{1}-\frac{4}{3}+\frac{4}{5}-\frac{4}{7}+\frac{4}{9}-\frac{4}{11}+\text{etc.},$$

which is the most noteworthy series of Leibniz. And thus any doubt about the truth of this summation vanishes.

19. We may contract the double terms, with the first excepted, into single terms and we will obtain

$$\pi \cot.\lambda\pi = \frac{1}{\lambda} - \frac{2\lambda}{1-\lambda\lambda} - \frac{2\lambda}{4-\lambda\lambda} - \frac{2\lambda}{9-\lambda\lambda} - \frac{2\lambda}{16-\lambda\lambda} - \text{etc.},$$

which series is reduced to this form

$$\frac{1}{2\lambda\lambda} - \frac{\pi\cot.\lambda\pi}{2\lambda} = \frac{\lambda}{1-\lambda\lambda} + \frac{\lambda}{4-\lambda\lambda} + \frac{\lambda}{9-\lambda\lambda} + \frac{\lambda}{16-\lambda\lambda} + \text{etc.}$$

But if here again we may put $\lambda = \mu\sqrt{-1}$, on account of

$$\cos.\mu\pi\sqrt{-1} = \frac{e^{-\mu\pi} + e^{+\mu\pi}}{2}$$

and

$$\sin.\lambda\pi\sqrt{-1} = \frac{e^{-\mu\pi} - e^{+\mu\pi}}{2\sqrt{-1}}$$

this summation is obtained

$$-\frac{1}{2\mu\mu} + \frac{\pi(e^{+\mu\pi} + e^{-\mu\pi})}{2\mu(e^{+\mu\pi} - e^{-\mu\pi})} = \frac{1}{1+\mu\mu} + \frac{1}{4+\mu\mu} + \frac{1}{9+\mu\mu} + \frac{1}{16+\mu\mu} + \text{etc.}$$

But now it is itself evident the differentiation in a similar manner as above an infinitude of other summations can be obtained.

II. THERE MAY BE TAKEN $Q = \cos.\zeta - \cos.\varphi$ SO THAT THE FRACTION BEING RESOLVED SHALL BE $\frac{P}{\cos.\zeta - \cos.\varphi}$

20. Since the denominator shall be $Q = \cos.\zeta - \cos.\varphi$, where the angle ζ may be considered as given and to be constant, this will vanish in the following cases

$$\varphi = \pm\zeta,\ \varphi = \pm 2\pi \pm \zeta,\ \varphi = \pm 4\pi \pm \zeta,$$
$$\varphi = \pm 6\pi \pm \zeta,\ \varphi = \pm 8\pi \pm \zeta \text{ etc.}$$

and thus in general

$$\varphi = \pm i\pi \pm \zeta,$$

where i denotes all the even numbers both positive and negative ; from which the denominators of the simple fractions, which we seek, will be

$$\varphi - \zeta,\ \varphi + \zeta,\ \varphi - 2\pi - \zeta,\ \varphi - 2\pi + \zeta,\ \varphi + 2\pi - \zeta,\ \varphi + 2\pi + \zeta \text{ etc.}$$

and in this manner we will find all the simple fractions, of which the sum of all must be equal to the fraction proposed :

$$\frac{P}{\cos.\zeta - \cos.\varphi}.$$

21. We will now consider initially in general the simple denominator $\varphi - i\pi - \zeta$ and by putting $\varphi = i\pi + \zeta$ the numerator P may be changed into A. Then since from the denominator there may become $\frac{dQ}{d\varphi} = \sin.\varphi$, there will be $C = \sin(i\pi + \zeta) = \sin.\zeta$ from which the numerator of this fraction will be $\frac{A}{C} = \frac{A}{\sin.\zeta}$, and thus the fraction hence arises

$$\frac{A}{\sin.\zeta\cdot(\varphi - i\pi - \zeta)}.$$

But truly for the denominator $\varphi - i\pi - \zeta$, if there may be put $\varphi = i\pi - \zeta$, the quantity B appears in the numerator P ; but with the denominator there will become $C = \sin(i\pi - \zeta) = -\sin.\zeta$, from which this fraction shall arise

$$-\frac{B}{\sin.\zeta(\varphi - i\pi + \zeta)}.$$

Now therefore there is only the need, so that in place of i successively all the even numbers both positive as well as negative may be substituted.

1°. THE NUMERATOR SHALL BE $P = 1$

AND THE PROPOSED FRACTION $\frac{1}{\cos.\zeta - \cos.\varphi}$

22. Therefore with the two general formulas there will be both $A = 1$ as well as $B = 1$, from which these general fractions will become

$$\frac{1}{\sin.\zeta(\varphi - i\pi - \zeta)} - \frac{1}{\sin.\zeta(\varphi - i\pi + \zeta)} = \frac{2\zeta}{\sin.\zeta((\varphi - i\pi)^2 - \zeta\zeta)};$$

hence consequently we deduce the following summation :

$$\frac{1}{\cos.\zeta - \cos.\varphi} = \frac{2\zeta}{\sin.\zeta(\varphi\varphi - \zeta\zeta)} + \frac{2\zeta}{\sin.\zeta((\varphi - 2\pi)^2 - \zeta\zeta)} + \frac{2\zeta}{\sin.\zeta((\varphi + 2\pi)^2 - \zeta\zeta)}$$
$$+ \frac{2\zeta}{\sin.\zeta((\varphi - 4\pi)^2 - \zeta\zeta)} + \frac{2\zeta}{\sin.\zeta((\varphi + 4\pi)^2 - \zeta\zeta)} + \text{etc.}$$

or we will have

$$\frac{\sin.\zeta}{2\zeta(\cos.\zeta - \cos.\varphi)} = \frac{1}{(\varphi\varphi - \zeta\zeta)} + \frac{1}{((\varphi - 2\pi)^2 - \zeta\zeta)} + \frac{1}{((\varphi + 2\pi)^2 - \zeta\zeta)}$$
$$+ \frac{1}{((\varphi - 4\pi)^2 - \zeta\zeta)} + \frac{1}{((\varphi + 4\pi)^2 - \zeta\zeta)} + \text{etc.}$$

23. But if therefore there were $\zeta = 0$, there will become

$$\frac{1}{2-2\cos.\varphi}=\frac{1}{\varphi^2}+\frac{1}{(\varphi-2\pi)^2}+\frac{1}{(\varphi+2\pi)^2}+\frac{1}{(\varphi-4\pi)^2}+\frac{1}{(\varphi+4\pi)^2}+\text{etc.}$$

Now again there may be $\varphi=\frac{\pi}{2}$; there will be this summation

$$\frac{\pi\pi}{8}=1+\frac{1}{9}+\frac{1}{25}+\frac{1}{49}+\frac{1}{81}+\frac{1}{121}+\text{etc.},$$

as indeed agrees well enough. Again we may put $\varphi=\pi$, and this summation will be produced :

$$\frac{\pi\pi}{4}=1+1+\frac{1}{9}+\frac{1}{9}+\frac{1}{25}+\frac{1}{25}+\frac{1}{49}+\frac{1}{49}+\text{etc.},$$

which agrees with the preceding series. But if there may be put $\varphi=\lambda\pi$, there will be :

$$\frac{\pi\pi}{2(1-\cos.\lambda\pi)}=\frac{1}{\lambda\lambda}+\frac{1}{(\lambda-2)^2}+\frac{1}{(\lambda+2)^2}+\frac{1}{(\lambda-4)^2}+\frac{1}{(\lambda+4)^2}+\text{etc.},$$

which sum is also :

$$\frac{\pi\pi}{4(\sin.\frac{1}{2}\lambda\pi)^2}.$$

24. Moreover we may put in general

$$\zeta=\alpha\pi \text{ and } \varphi=\lambda\pi,$$

so that this summation may be obtained

$$\frac{\pi\sin.\alpha\pi}{2\alpha(\cos.\alpha\pi-\cos.\lambda\pi)}=\frac{1}{\lambda\lambda-\alpha\alpha}+\frac{1}{(\lambda-2)^2-\alpha\alpha}+\frac{1}{(\lambda+2)^2-\alpha\alpha}$$
$$+\frac{1}{(\lambda-4)^2-\alpha\alpha}+\frac{1}{(\lambda+4)^2-\alpha\alpha}+\text{etc.}$$

But if now α were an imaginary quantity or $\alpha=\beta\sqrt{-1}$, this summation will become :

$$\frac{\pi(e^{+\beta\pi}+e^{-\beta\pi})}{2\beta((e^{+\beta\pi}-e^{-\beta\pi})-2\cos.\lambda\pi)}=\frac{1}{\lambda\lambda+\beta\beta}+\frac{1}{(\lambda-2)^2+\beta\beta}+\frac{1}{(\lambda+2)^2+\beta\beta}$$
$$+\frac{1}{(\lambda-4)^2+\beta\beta}+\frac{1}{(\lambda+4)^2+\beta\beta}+\text{etc.}$$

25. Hence if this fraction may be proposed being required to be resolved into a series :

$$\frac{1}{a-\cos.\varphi} \text{ or } \frac{1}{a-\cos.\lambda\pi},$$

two cases will be required to be considered, just as a were either greater or less than one. There shall be $a<1$, so that there may be able to become $a=\cos.\alpha\pi$; from which there becomes :

$$\alpha = \frac{\text{A}\cos.a}{\pi}$$

and with α found there will become:

$$\frac{1}{a-\cos.\lambda\pi} = \frac{2\alpha}{\pi\sqrt{(1-aa)}}\left(\frac{1}{\lambda\lambda-\alpha\alpha}+\frac{1}{(\lambda-2)^2-\alpha\alpha}+\frac{1}{(\lambda+2)^2-\alpha\alpha}+\frac{1}{(\lambda-4)^2-\alpha\alpha}+\text{etc.}\right)$$

But if there were $a>1$, β must be found, so that there may become :

$$\frac{e^{+\beta\pi}+e^{-\beta\pi}}{2} = a.$$

Hence therefore there will become $e^{+2\beta\pi}+1=2ae^{+\beta\pi}$, from which with the root extracted there is found :

$$e^{+\beta\pi} = a+\sqrt{(aa-1)}$$

and hence

$$e^{-\beta\pi} = a-\sqrt{(aa-1)}\ ,$$

from which again there will become

$$\beta\pi = l(a+\sqrt{(aa-1)}),$$

therefore

$$\beta = \frac{1}{\pi}l(a+\sqrt{(aa-1)}).$$

Therefore with this number β found the latter formula provides this series

$$\frac{\pi\sqrt{(aa-1)}}{2\beta(a-\cos.\lambda\pi)} = \frac{1}{\lambda\lambda+\beta\beta}+\frac{1}{(\lambda-2)^2+\beta\beta}+\frac{1}{(\lambda+2)^2+\beta\beta}+\frac{1}{(\lambda-4)^2+\beta\beta}+\text{etc.};$$

consequently we will have :

$$\frac{1}{a-\cos.\lambda\pi}=$$
$$=\frac{2\beta}{\pi\sqrt{(aa-1)}}\left(\frac{1}{\lambda\lambda+\beta\beta}+\frac{1}{(\lambda-2)^2+\beta\beta}+\frac{1}{(\lambda+2)^2+\beta\beta}+\frac{1}{(\lambda-4)^2+\beta\beta}+\text{etc.}\right).$$

But in the middle case, where $a=1$, there becomes $\alpha=0$; then truly there may be put $a=1-\omega$ and there will be:

$$\text{Acos.}(1-\omega)=\text{Asin.}\sqrt{(2\omega-\omega\omega)}=\sqrt{(2\omega-\omega\omega)}.$$

Truly also there will be

$$\sqrt{(1-aa)}=\sqrt{(2\omega-\omega\omega)},$$

from which for this case the sum of the series will be :

$$\frac{1}{1-\cos.\lambda\pi}=\frac{2}{\pi\pi}\left(\frac{1}{\lambda\lambda}+\frac{1}{(\lambda-2)^2}+\frac{1}{(\lambda+2)^2}+\frac{1}{(\lambda-4)^2}+\text{etc.}\right).$$

Therefore since there shall be :

$$1-\cos.\lambda\pi=2\sin^2.\tfrac{1}{2}\lambda\pi,$$

we will have this summation:

$$\frac{\pi\pi}{4\sin^2.\frac{1}{2}\lambda\pi,}=\frac{1}{\lambda\lambda}+\frac{1}{(\lambda-2)^2}+\frac{1}{(\lambda+2)^2}+\frac{1}{(\lambda-4)^2}+\text{etc.},$$

which series now has been found above in § 23.

2°. NOW THERE SHALL BE $P=\sin.\varphi$

AND THE PROPOSED FRACTION $\frac{\sin.\varphi}{\cos.\zeta-\cos.\varphi}$

26. Now since there shall be $P=\sin.\varphi$, on taking $\varphi=i\pi+\zeta$ there will become:

$$A=\sin.(i\pi+\zeta)=\sin.\zeta;$$

but on putting $\varphi=i\pi-\zeta$ there appears $B=-\sin.\zeta$; hence the two resulting fractions will become thence:

$$\frac{1}{\varphi-i\pi-\zeta}+\frac{1}{\varphi-i\pi+\zeta}=\frac{2\varphi-2i\pi}{(\varphi-i\pi)^2-\zeta\zeta}.$$

Whereby if in place of i we may write successively all its values, we will obtain the following series :

$$\frac{\sin.\varphi}{\cos.\zeta-\cos.\varphi}=\frac{2\varphi}{\varphi\varphi-\zeta\zeta}+\frac{2(\varphi-2\pi)}{(\varphi-2\pi)^2-\zeta\zeta}+\frac{2(\varphi+2\pi)}{(\varphi+2\pi)^2-\zeta\zeta}+\frac{2(\varphi-4\pi)}{(\varphi-4\pi)^2-\zeta\zeta}+\text{etc.}$$

or

$$\frac{\sin.\varphi}{2(\cos.\zeta-\cos.\varphi)}=\frac{\varphi}{\varphi\varphi-\zeta\zeta}+\frac{\varphi-2\pi}{(\varphi-2\pi)^2-\zeta\zeta}+\frac{\varphi+2\pi}{(\varphi+2\pi)^2-\zeta\zeta}+\frac{\varphi-4\pi}{(\varphi-4\pi)^2-\zeta\zeta}+\text{etc.}$$

27. Hence, if there were $\zeta=0$, there will become:

$$\frac{\sin.\varphi}{2(1-\cos.\varphi)}=\frac{1}{\varphi}+\frac{1}{\varphi-2\pi}+\frac{1}{\varphi+2\pi}+\frac{1}{\varphi-4\pi}+\frac{1}{\varphi+4\pi}+\text{etc.},$$

therefore the sum of which series is :

$$\tfrac{1}{2}\cot.\tfrac{1}{2}\varphi.$$

Hence, if we may put $\varphi=\lambda\pi$, there will become:

$$\tfrac{1}{2}\pi\cot.\tfrac{1}{2}\lambda\pi=\frac{1}{\lambda}+\frac{1}{\lambda-2}+\frac{1}{\lambda+2}+\frac{1}{\lambda-4}+\frac{1}{\lambda+4}+\text{etc.}$$

and with the two terms contracted

$$\tfrac{1}{2}\pi\cot.\tfrac{1}{2}\lambda\pi=\frac{1}{\lambda}+\frac{2\lambda}{\lambda\lambda-4}+\frac{2\lambda}{\lambda\lambda-16}+\frac{2\lambda}{\lambda\lambda-36}+\text{etc.}$$

and hence

$$\frac{1}{2\lambda\lambda}-\frac{1}{2}\,\frac{\pi\cot.\frac{1}{2}\lambda\pi}{4\lambda}=\frac{1}{4-\lambda\lambda}+\frac{1}{16-\lambda\lambda}+\frac{1}{36-\lambda\lambda}+\text{etc.}$$

But here in place of λ, if we may write 2λ, we will have :

$$\frac{1}{8\lambda\lambda}-\frac{\pi\cot.\lambda\pi}{8\lambda}=\frac{1}{4-4\lambda\lambda}+\frac{1}{16-4\lambda\lambda}+\frac{1}{36-4\lambda\lambda}+\text{etc.}$$

or

$$\frac{1}{2\lambda\lambda}-\frac{\pi\cot.\lambda\pi}{2\lambda}=\frac{1}{1-\lambda\lambda}+\frac{1}{4-\lambda\lambda}+\frac{1}{9-\lambda\lambda}+\frac{1}{16-\lambda\lambda}+\text{etc.}$$

which series clearly is the same, as we found above in § 19 .

28. Now we may put as above :

$$\zeta=\alpha\pi \text{ and } \varphi=\lambda\pi,$$

so that the following series may be obtained:

$$\frac{\pi\sin.\lambda\pi}{2(\cos.\alpha\pi-\cos.\lambda\pi)}=\frac{\lambda}{(\lambda\lambda-\alpha\alpha)}+\frac{\lambda-2}{(\lambda-2)^2-\alpha\alpha}+\frac{\lambda+2}{(\lambda+2)^2-\alpha\alpha}+\frac{\lambda-4}{(\lambda-4)^2-\alpha\alpha}+\text{etc.}$$

But if here there may be put $\alpha=\beta\sqrt{-1}$, this series will adopt the following form :

$$\frac{\pi\sin.\lambda\pi}{e^{+\beta\pi}+e^{-\beta\pi}-2\cos.\lambda\pi}=\frac{\lambda}{\lambda\lambda+\beta\beta}+\frac{\lambda-2}{(\lambda-2)^2+\beta\beta}+\frac{\lambda+2}{(\lambda+2)^2+\beta\beta}+\frac{\lambda-4}{(\lambda-4)^2+\beta\beta}+\text{etc.}$$

29. So that if therefore this fraction were proposed :

$$\frac{\sin.\varphi}{a-\cos.\varphi}\text{ or }\frac{\sin.\lambda\pi}{a-\cos.\lambda\pi},$$

again it is agreed two cases to arise, the one where $a<1$, the other, where $a>1$.

Indeed in the first case, where $a<1$, there may be put $\cos.\alpha\pi=a$, from which there becomes

$$\alpha=\frac{\text{Acos}.a}{\pi},$$

with which found there will be

$$\frac{\sin.\lambda\pi}{a-\cos.\lambda\pi}=\frac{2}{\pi}\left(\frac{\lambda}{\lambda\lambda-\alpha\alpha}+\frac{\lambda-2}{(\lambda-2)^2-\alpha\alpha}+\frac{\lambda+2}{(\lambda+2)^2-\alpha\alpha}+\text{etc.}\right).$$

But if there were $a>1$, β must be found, thus so that as before there shall be :

$$\beta=\frac{1}{\pi}l(a+\sqrt{(aa-1)}),$$

with which value found there will be

$$\frac{\sin.\lambda\pi}{a-\cos.\lambda\pi}=\frac{2}{\pi}\left(\frac{\lambda}{\lambda\lambda+\beta\beta}+\frac{\lambda-2}{(\lambda-2)^2+\beta\beta}+\frac{\lambda+2}{(\lambda+2)^2+\beta\beta}+\text{etc.}\right).$$

But if there were $a=1$, then there becomes both $\alpha=0$ as well as $\beta=0$, and the same series results, as we have elicited above from the case $\zeta=0$. Hence therefore, if there may be taken $\lambda=\frac{1}{2}$, this series will be produced:

$$\frac{\pi}{2\cos.\alpha\pi}=\frac{2}{1-4\alpha\alpha}-\frac{6}{9-4\alpha\alpha}+\frac{10}{25-4\alpha\alpha}-\frac{14}{49-4\alpha\alpha}+\text{etc.}$$

or also this:

$$\frac{\pi}{e^{+\beta\pi}-e^{-\beta\pi}}=\frac{2}{1+4\beta\beta}-\frac{6}{9+4\beta\beta}+\frac{10}{25+4\beta\beta}-\frac{14}{49+4\beta\beta}+\text{etc.}$$

Furthermore it is understood of course, many other series can be formed by differentiation.

III. THE FRACTION BEING RESOLVED SHALL BE $\frac{1}{\cos.\varphi-\cos.2\varphi}$.

30. Therefore here before everything else it must be inquired, for which cases this denominator may vanish. Since therefore there will be in general $\cos.\varphi=\cos.(i\pi\pm\varphi)$ with *i* denoting an even number and in a similar manner $\cos.2\varphi=\cos.(i'\pi\pm 2\varphi)$, we will have $i\pi\pm\varphi=i'\pi\pm 2\varphi$, from which on account of the ambiguity of the sines the following cases are to be elicited :

$$\varphi=i\pi,\quad \varphi=\tfrac{i\pi}{3}.$$

But here it is to be required to be understood properly the first cases to occur twice or the factors hence arising $\varphi-i\pi$ require to be taken together twice, thus so that the factor of the denominator shall be $(\varphi-i\pi)^2$. Since because it may appear to be less clear, thus we may show : since in general there is

$$\cos.a-\cos.b=2\sin.\tfrac{a+b}{2}.\sin.\tfrac{b-a}{2},$$

our denominator will be $2\sin.\tfrac{1}{2}\varphi\sin.\tfrac{3}{2}\varphi$, which therefore vanishes, since when $\sin.\tfrac{1}{2}\varphi=0$, as well as when $\sin.\tfrac{3}{2}\varphi=0.$ But there shall be $\sin.\tfrac{1}{2}\varphi=0$, as often as $\tfrac{1}{2}\varphi=i\pi$ with *i* denoting all the whole numbers and thus $\varphi=2i\pi$. And in a similar manner $\sin.\tfrac{3}{2}\varphi$ vanishes , if $\tfrac{3}{2}\varphi=i\pi$ and thus $\varphi=\tfrac{2i\pi}{3}$, which latter formula, as often as $\tfrac{i}{3}$ is a whole number, satisfies the former cases, and thus it is evident to occur in all the square factors $(\varphi-i\pi)^2$. Truly the remaining factors $\varphi-\tfrac{2i\pi}{3}$ will be simple, when *i* is not divisible by 3.

31. Therefore since the formula $(\varphi-2i\pi)^2$ shall be a factor of our denominator $\cos.\varphi-\cos.2\varphi$, we may state the following rule for cases of this kind

$$\frac{1}{\cos.\varphi-\cos.2\varphi}=\frac{\alpha}{(\varphi-2i\pi)^2}+\frac{\beta}{\varphi-2i\pi}+R,$$

where *R* includes all the remaining fractions. Now we may multiply each side by $(\varphi-2i\pi)^2$ and we will have

$$\frac{(\varphi-2i\pi)^2}{\cos.\varphi-\cos.2\varphi}=\alpha+\beta(\varphi-2i\pi)+R(\varphi-2i\pi)^2.$$

We may make $\varphi=2i\pi$ and there will become

$$\alpha = \frac{(\varphi - 2i\pi)^2}{\cos.\varphi - \cos.2\varphi},$$

of which fraction the numerator and denominator vanish; hence with the differentials substituted there will become

$$\alpha = \frac{2(\varphi - 2i\pi)}{-\sin.\varphi + 2\sin.2\varphi};$$

where since the numerator et denominator again vanish, the differentials may be written anew in place of these and there will be

$$\alpha = \frac{2}{-\cos.\varphi + 4\cos.2\varphi}.$$

Now therefore on putting $\varphi = 2i\pi$, $\alpha = \frac{2}{3}$ will be found .

32. Now in the equation

$$\frac{(\varphi - 2i\pi)^2}{\cos.\varphi - \cos.2\varphi} = \alpha + \beta(\varphi - 2i\pi) + R(\varphi - 2i\pi)^2.$$

the term $\alpha = \frac{2}{3}$ may be transferred to the other side and may be reduced to the same denominator and this equation will result :

$$\frac{(\varphi - 2i\pi)^2 - \frac{2}{3}(\cos.\varphi - \cos.2\varphi)}{\cos.\varphi - \cos.2\varphi} = \beta(\varphi - 2i\pi) + R(\varphi - 2i\pi)^2,$$

from which on dividing by $\varphi - 2i\pi$ there will become :

$$\frac{(\varphi - 2i\pi)^2 - \frac{2}{3}(\cos.\varphi - \cos.2\varphi)}{(\varphi - 2i\pi)(\cos.\varphi - \cos.2\varphi)} = \beta + R(\varphi - 2i\pi).$$

Because if now there may be put $\varphi = 2i\pi$, β will be equal to the fraction, of which both the numerator as well as the denominator vanish three times, so that thus there shall be need of a threefold differentiation.

Moreover the first differentiation will give :

$$\beta = \frac{2(\varphi - 2i\pi) + \frac{2}{3}(\sin.\varphi - 2\sin.2\varphi)}{\cos.\varphi - \cos.2\varphi - (\varphi - 2i\pi)(\sin.\varphi - 2\sin.2\varphi)}.$$

The second differentiation will give:

$$\beta = \frac{2+\frac{2}{3}(\cos.\varphi-4\cos.2\varphi)}{-2\sin.\varphi+4\sin.2\varphi-(\varphi-2i\pi)(\cos.\varphi-4\cos.2\varphi)}.$$

Finally the third differentiation gives :

$$\beta = \frac{-\frac{2}{3}(\sin.\varphi-8\sin.2\varphi)}{-3\cos.\varphi+12\cos.2\varphi+(\varphi-2i\pi)(\sin.\varphi-8\sin.2\varphi)}.$$

But now with the factor $\varphi = 2i\pi$ the numerator indeed vanishes again, truly the denominator becomes 9, thus so that there shall be $\beta = 0$.

33. But truly this value for β can be elicited more easily without differentiation by putting $\varphi = 2i\pi + \omega$ with ω being infinitely small; but then there will be

$$\cos.\varphi = \cos.\omega \text{ and } \cos.2\varphi = \cos.2\omega;$$

moreover the equation will become :

$$\frac{\omega\omega}{\cos.\omega-\cos.2\omega} = \frac{2}{3} + \beta\omega + R\omega\omega.$$

Now we may show both the cosines approximately by proceeding as far as to the fourth power of ω, and since there shall be

$$\cos.\omega = 1 - \tfrac{1}{2}\omega\omega + \tfrac{1}{24}\omega^4$$

and

$$\cos.2\omega = 1 - 2\omega\omega + \tfrac{16}{24}\omega^4,$$

there will become

$$\cos.\omega - \cos.2\omega = \tfrac{3}{2}\omega\omega - \tfrac{5}{8}\omega^4 = \tfrac{3}{2}\omega\omega\left(1 - \tfrac{5}{12}\omega\omega\right),$$

with which value substituted, we will have

$$\frac{2}{3\omega\omega\left(1-\frac{5}{12}\omega\omega\right)} = \frac{2}{3}(1 + \frac{5}{12}\omega\omega) = \frac{2}{3} + \beta\omega + R\omega\omega,$$

and hence there becomes $\beta = \frac{2}{3}\cdot\frac{5}{12}\omega$; and thus by making $\omega = 0$ also there will be $\beta = 0$.

34. For this reason, according to the square factor of the denominator $(\varphi - 2i\pi)^2$, on account of $\alpha = \frac{2}{3}$, the fraction thence produced will be

$$\frac{2}{3(\varphi-2i\pi)^2}.$$

But for the remaining simple factors $\varphi-\frac{2}{3}i\pi$ we may put

$$\frac{1}{\cos.\varphi-\cos.2\varphi}=\frac{\alpha}{\varphi-\frac{2}{3}i\pi}+R,$$

which equation may be multiplied by $\varphi-\frac{2}{3}i\pi=\omega$, so that there may be produced

$$\frac{\omega}{\cos.\varphi-\cos.2\varphi}=\alpha+R\omega.$$

Where it may be observed the number i not to be divisible by 3, from which $\frac{2i\pi}{3}$ will express the following angles:

$$\frac{2}{3}\pi,\ \frac{4}{3}\pi,\ \frac{8}{3}\pi,\ \frac{10}{3}\pi,\ \frac{14}{3}\pi \text{ etc.};$$

but the values of the angle $\frac{4i\pi}{3}$ are :

$$\frac{4}{3}\pi,\ \frac{8}{3}\pi,\ \frac{16}{3}\pi,\ \frac{20}{3}\pi,\ \frac{28}{3}\pi \text{ etc.},$$

of which the cosine of the angles is the same $-\frac{1}{2}$. But the sine of these angles are :

$$\sin.\frac{2i\pi}{3}=\pm\frac{\sqrt{3}}{2},$$

where the upper sign prevails, if i shall be $3n+1$, truly the lower, if there were $i=3n+2$; but truly $\sin.\frac{4i\pi}{3}$ is always $\mp\frac{\sqrt{3}}{2}$, where again the upper sign prevails , if $i=3n+1$, truly the lower, if $i=3n+2$. And this rule always prevails, if n shall be a positive or negative number.

35. With these noted, there will be

$$\cos.\varphi=-\frac{1}{2}\cos.\omega\mp\frac{\sqrt{3}}{2}\sin.\omega$$

and

$$\cos.2\varphi=-\frac{1}{2}\cos.2\omega\mp\frac{\sqrt{3}}{2}\sin.2\omega,$$

from which truly we will have approximately

$$\cos.\varphi=-\frac{1}{2}\left(1-\frac{1}{2}\omega\omega\right)\mp\frac{\sqrt{3}}{2}\omega$$

and

$$\cos.2\varphi=-\frac{1}{2}\left(1-\frac{1}{2}\omega\omega\right)\pm\frac{\sqrt{3}}{2}\cdot2\omega,$$

where the upper signs prevail constantly, if $i = 3n+1$, but the lower signs, if $i = 3n+2$. Hence therefore our denominator will be:

$$\cos.\varphi - \cos.2\varphi = -\tfrac{3}{4}\omega\omega \mp \tfrac{3\sqrt{3}}{2}\omega,$$

from which there becomes :

$$\frac{1}{-\frac{3}{4}\omega \mp \frac{3\sqrt{3}}{2}} = \alpha.$$

Therefore on putting $\omega = 0$ there will be

$$\alpha = \mp\frac{2}{3\sqrt{3}},$$

thus this fraction may arise from the factor $\varphi - \frac{2i\pi}{3}$

$$\mp\frac{2}{3\sqrt{3}} \cdot \frac{1}{\varphi - \frac{2i\pi}{3}} = \mp\frac{2}{(3\varphi - 2i\pi)\sqrt{3}}.$$

36. Therefore in the first place we may establish all the terms of the series arising from the twin factors $(\varphi - 2i\pi)^2$, and since the numerator may have been $\frac{2}{3}$, if in place of i we may write successively all the whole numbers both positive as well as negative, the following series will arise :

$$\frac{2}{3\varphi\varphi} + \frac{2}{3(\varphi-2\pi)^2} + \frac{2}{3(\varphi+2\pi)^2} + \frac{2}{3(\varphi-4\pi)^2} + \frac{2}{3(\varphi+4\pi)^2} + \frac{2}{3(\varphi-6\pi)^2} + \text{etc.}$$

Initially for the other series there shall be $i = 3n+1$ and hence the fraction will become

$$-\frac{2}{(3\varphi - 2(3n+1)\pi)\sqrt{3}};$$

but if there shall be $i = -3n-1$, the lower sign will prevail and the fraction will be :

$$+\frac{2}{(3\varphi + 2(3n+1)\pi)\sqrt{3}},$$

which two terms contracted together provide

$$-\frac{8(3n+1)\pi}{(9\varphi\varphi - 4(3n+1)^2\pi\pi)\sqrt{3}};$$

but if there were $i = 3n+2$, then also truly $i = -3n-2$, the two fractions contracted into one provide :

$$+\frac{8(3n+2)\pi}{(9\varphi\varphi - 4(3n+1)^2\pi\pi)\sqrt{3}}.$$

Whereby since the negatives values of i we have now complex, in place of n only all the positive numbers 0, 1, 2, 3, 4, 5 etc. will be required to be put in place, from which the following series will result :

$$-\frac{8\pi}{(9\varphi\varphi-4\pi\pi)\sqrt{3}}-\frac{8\cdot4\pi}{(9\varphi\varphi-4\cdot16\pi\pi)\sqrt{3}}-\frac{8\cdot7\pi}{(9\varphi\varphi-4\cdot49\pi\pi)\sqrt{3}}-\text{etc.}$$
$$+\frac{8\pi}{(9\varphi\varphi-16\pi\pi)\sqrt{3}}+\frac{8\cdot5\pi}{(9\varphi\varphi-4\cdot25\pi\pi)\sqrt{3}}+\frac{8\cdot8\pi}{(9\varphi\varphi-4\cdot64\pi\pi)\sqrt{3}}+\text{etc.}$$

37. Therefore the proposed fraction

$$\frac{1}{\cos.\varphi-\cos.2\varphi}$$

is resolved into these two series :

$$\frac{2}{3}\left(\frac{1}{\varphi^2}+\frac{1}{(\varphi-2\pi)^2}+\frac{1}{(\varphi+2\pi)^2}+\frac{1}{(\varphi-4\pi)^2}+\frac{1}{(\varphi+4\pi)^2}+\text{etc}\right),$$
$$-\frac{8\pi}{\sqrt{3}}\left(\frac{1}{9\varphi\varphi-4\cdot1^2\pi\pi}+\frac{4}{9\varphi\varphi-4\cdot4^2\pi\pi}+\frac{7}{9\varphi\varphi-4\cdot7^2\pi\pi}+\frac{10}{9\varphi\varphi-4\cdot10^2\pi\pi}+\text{etc.}\right)$$
$$+\frac{8\pi}{\sqrt{3}}\left(\frac{2}{9\varphi\varphi-4\cdot2^2\pi\pi}+\frac{5}{9\varphi\varphi-4\cdot5^2\pi\pi}+\frac{8}{9\varphi\varphi-4\cdot8^2\pi\pi}+\frac{11}{9\varphi\varphi-4\cdot11^2\pi\pi}+\text{etc.}\right)$$

Hence therefore if we may make so that above $\varphi=\lambda\pi$, the fraction will be

$$\frac{\pi\pi}{\cos.\lambda\pi-\cos.2\lambda\pi}=$$
$$\frac{2}{3}\left(\frac{1}{\lambda^2}+\frac{1}{(\lambda-2\pi)^2}+\frac{1}{(\lambda+2\pi)^2}+\frac{1}{(\lambda-4\pi)^2}+\frac{1}{(\lambda+4\pi)^2}+\text{etc}\right),$$
$$-\frac{8\pi}{\sqrt{3}}\left(\frac{1}{9\lambda\lambda-4\cdot1^2\pi\pi}+\frac{4}{9\lambda\lambda-4\cdot4^2\pi\pi}+\frac{7}{9\lambda\lambda-4\cdot7^2\pi\pi}+\frac{10}{9\lambda\lambda-4\cdot10^2\pi\pi}+\text{etc.}\right)$$
$$+\frac{8\pi}{\sqrt{3}}\left(\frac{2}{9\lambda\lambda-4\cdot2^2\pi\pi}+\frac{5}{9\lambda\lambda-4\cdot5^2\pi\pi}+\frac{8}{9\lambda\lambda-4\cdot8^2\pi\pi}+\frac{11}{9\lambda\lambda-4\cdot11^2\pi\pi}+\text{etc.}\right)$$

38. So that we may advance an example, let $\lambda=\frac{1}{3}$, so that there may become $9\lambda\lambda=1$, and this summation will be produced :

$$\pi\pi=\frac{2}{3}\left(\frac{9}{1^2}+\frac{9}{5^2}+\frac{9}{7^2}+\frac{9}{11^2}+\frac{9}{13^2}+\frac{9}{17^2}+\text{etc}\right)$$
$$+\frac{8\pi}{\sqrt{3}}\left(\frac{1}{4\cdot1^2-1}+\frac{4}{4\cdot4^2-1}+\frac{7}{4\cdot7^2-1}+\frac{10}{4\cdot10^2-1}+\frac{13}{4\cdot13^2-1}+\text{etc.}\right)$$
$$-\frac{8\pi}{\sqrt{3}}\left(\frac{2}{4\cdot2^2-1}+\frac{5}{4\cdot5^2-1}+\frac{8}{4\cdot8^2-1}+\frac{11}{4\cdot11^2-1}+\frac{14}{4\cdot14^2-1}+\text{etc.}\right),$$

which summation also can be referred to in this way :

$$\pi\pi = 6\left(\frac{1}{1^2}+\frac{1}{5^2}+\frac{1}{7^2}+\frac{1}{11^2}+\frac{1}{13^2}+\frac{1}{17^2}+\frac{1}{19^2}+\text{etc}\right)$$
$$+\frac{4\pi}{\sqrt{3}}\left(\frac{2}{2^2-1}-\frac{4}{4^2-1}+\frac{8}{8^2-1}-\frac{10}{10^2-1}+\frac{14}{14^2-1}-\frac{16}{16^2-1}+\text{etc.}\right);$$

but a more elegant form will be the following :

$$\pi\pi = 6\left(\frac{1}{1^2}+\frac{1}{5^2}+\frac{1}{7^2}+\frac{1}{11^2}+\frac{1}{13^2}+\frac{1}{17^2}+\frac{1}{19^2}+\text{etc.}\right)$$
$$+\frac{2\pi}{\sqrt{3}}\left(1-\tfrac{1}{5}+\tfrac{1}{7}-\tfrac{1}{11}+\tfrac{1}{13}-\tfrac{1}{17}+\tfrac{1}{19}-\text{etc.}\right).$$

39. Because square factors occur in this case, we will be able also to resolve fractions of this kind, of which these denominators are squares and thus involve pure simple square factors. And thus it will be allowed to extend this resolution to cubic denominators and of a higher power, but only if this precept may be called into help, which I gave at one time for resolutions of this kind.

IV. THE PROPOSED FRACTION REQUIRING TO BE RESOLVED SHALL BE $\frac{1}{\sin^2.\varphi}$

40. Therefore since here all the square factors of the denominator may be contained in this form

$$\frac{1}{(\varphi-i\pi)^2}$$

with *i* denoting all the whole numbers, both positive and negative, we may put for the general resolution :

$$\frac{1}{\sin^2.\varphi}=\frac{\alpha}{(\varphi-i\pi)^2}+\frac{\beta}{\varphi-i\pi}+R,$$

where *R* includes all the remaining fractions. Hence on multiplying by $(\varphi-i\pi)^2$ there will be

$$\frac{(\varphi-i\pi)^2}{\sin^2.\varphi}=\alpha+\beta(\varphi-i\pi)+R(\varphi-i\pi)^2.$$

Now there may become $\varphi=i\pi$, and since in this case the numerator and denominator of our fraction vanish, we me put in place $\varphi-i\pi=\omega$ and there will become

$$\sin.\varphi=\sin.(i\pi+\omega)=\sin.i\pi\cos.\omega+\sin.\omega\cos.i\pi=\pm\sin.\omega$$

on account of $\sin.i\pi = 0$ and $\cos.i\pi = \pm 1;$ where the superior sign prevails, if *i* shall be even , truly the lower, if odd, yet because here the distinction does not come into consideration, since there shall be $\sin^2.\varphi = \sin^2\omega$. Hence therefore there will be

$$\frac{\omega\omega}{\sin^2.\omega} = \alpha + \beta\omega + R\omega\omega.$$

Therefore since there shall be

$$\sin.\omega = \omega - \tfrac{1}{6}\omega^3 = \omega(1 \ -\tfrac{1}{6}\omega\omega),$$

there will be

$$\frac{1}{(1 - \frac{1}{6}\omega\omega)^2} = 1 + \tfrac{1}{3}\omega\omega = \alpha + \beta\omega + R\omega\omega,$$

from which there becomes at once $\alpha = 1$. Then truly the equation will be

$$\tfrac{1}{3}\omega = \beta + R\omega,$$

and thus by making $\omega = 0$ there becomes $\beta = 0$; consequently from the factor of the denominator $(\varphi - i\pi)^2$ this fraction arises $\frac{1}{(\varphi - i\pi)^2}$.

41. Now all the due values for *i* may be attributed and this series will be found

$$\frac{1}{\sin^2.\varphi} = \frac{1}{\varphi\varphi} + \frac{1}{(\varphi-\pi)^2} + \frac{1}{(\varphi+\pi)^2} + \frac{1}{(\varphi-2\pi)^2} + \frac{1}{(\varphi+2\pi)^2} + \frac{1}{(\varphi-3\pi)^2} + \text{etc.};$$

indeed which series may be able to be deduced from §18, where we have found

$$\frac{\cos.\varphi}{\sin.\varphi} = \frac{1}{\varphi} + \frac{1}{\varphi-\pi} + \frac{1}{\varphi+\pi} + \frac{1}{\varphi-2\pi} + \frac{1}{\varphi+2\pi} + \text{etc.},$$

from which by differentiation with the signs changed this series itself arises, as we find here.

42. But if this fraction were proposed

$$\frac{\cos^2.\varphi}{\sin^2.\varphi}$$

and the resolution were put in place in the same manner, on account of

$$\cos.(i\pi + \omega) = \pm\cos.\omega$$

and thus

$$\cos^2.\varphi = \cos^2.\omega = 1 - \omega\omega,$$

since the second powers of ω do not enter into the computation, the numerator would become as in the preceding case $=1$ and thus plainly the same series would be produced, because that certainly would be absurd. But above now we have noticed the truth of resolutions of this kind not to be agreed upon, unless the variable quantity, unless the variable quantity φ in the numerator may have fewer dimensions than in the denominator, since otherwise besides the series of fractional parts would be approaching integers, because that clearly happens in this case, since there shall be

$$\frac{\cos^2.\varphi}{\sin^2.\varphi}=\frac{1}{\sin^2.\varphi}-1,$$

thus so that the integral part in this case shall be $=-1$.

V. THE FRACTION REQUIRING TO BE RESOLVED

SHALL BE $\frac{1}{\sin^3.\varphi}$

43. Therefore for this case it will be required to have put

$$\frac{1}{\sin^3.\varphi}=\frac{\alpha}{(\varphi-i\pi)^3}+\frac{\beta}{(\varphi-i\pi)^2}+\frac{\gamma}{\varphi-i\pi}+R.$$

Now again we may put $\varphi=i\pi+\omega$, and since there shall be

$$\frac{1}{\sin^3.\varphi}=\pm\frac{1}{\omega^3}\left(1+\tfrac{1}{2}\omega\omega\right),$$

where the account of the signs obeys the above given rule, this equation results, after being multiplied by ω^3,

$$\pm\frac{\omega^3(1+\frac{1}{2}\omega\omega)}{\omega^3}=\alpha+\beta\omega+\gamma\omega\omega+R\omega^3=\pm(1+\tfrac{1}{2}\omega\omega),$$

from which it is clear there becomes $\alpha=\pm1$, then truly

$$\beta+\gamma\omega+R\omega\omega=\pm\tfrac{1}{2}\omega$$

and thus there will be $\beta=0$ and $\gamma=\pm\frac{1}{2}$. Therefore in this manner from the denominator with the cubic factor $(\varphi-i\pi)^3$, these two fractions will arise :

$$\pm\frac{1}{(\varphi-i\pi)^3}\pm\frac{1}{2(\varphi-i\pi)}.$$

44. Therefore we may attribute to the letter i all the successive values both positive and negative, and we will obtain the following resolution :

$$\frac{1}{\sin^3.\varphi}=\frac{1}{\varphi^3}-\frac{1}{(\varphi-\pi)^3}-\frac{1}{(\varphi+\pi)^3}+\frac{1}{(\varphi-2\pi)^3}+\frac{1}{(\varphi+2\pi)^3}-\text{etc.}$$
$$+\frac{1}{2\varphi}-\frac{1}{2(\varphi-\pi)}-\frac{1}{2(\varphi+\pi)}+\frac{1}{2(\varphi-2\pi)}+\frac{1}{2(\varphi+2\pi)}-\text{etc.}$$

Here it will help to be noting the lower series now to be found above in the first example [§ 5] ; from which we understand to become the sum of this series

$$=\frac{1}{2\sin.\varphi};$$

on account of which the above series of the cubes will be equal to this formula only :

$$\frac{1}{\sin^3.\varphi}-\frac{1}{2\sin.\varphi}.$$

45. This agrees likewise outstandingly well with the principles established above, from which by continually differentiation we are led to elicit new series. Indeed since there shall be

$$\frac{1}{\sin.\varphi}=\frac{1}{\varphi}-\frac{1}{\varphi-\pi}-\frac{1}{\varphi+\pi}+\frac{1}{\varphi-2\pi}+\frac{1}{\varphi+2\pi}-\text{etc.},$$

hence by differentiation there is deduced

$$-\frac{\cos.\varphi}{\sin^2.\varphi}=-\frac{1}{\varphi^2}+\frac{1}{(\varphi-\pi)^2}+\frac{1}{(\varphi+\pi)^2}-\frac{1}{(\varphi-2\pi)^2}-\frac{1}{(\varphi+2\pi)^2}+\text{etc.},$$

and hence by differentiation anew,

$$\frac{1}{\sin.\varphi}+\frac{2\cos^2.\varphi}{\sin^3.\varphi}=\frac{2}{\varphi^3}-\frac{2}{(\varphi-\pi)^3}-\frac{2}{(\varphi+\pi)^3}+\frac{2}{(\varphi-2\pi)^3}+\frac{2}{(\varphi+2\pi)^3}-\text{etc.},$$

which is reduced to this form

$$\frac{2}{\sin^3.\varphi}-\frac{1}{\sin.\varphi},$$

that which agrees exceedingly well with the preceding value.

VI. THE PROPOSED FRACTION REQUIRING TO BE RESOLVED

SHALL BE $=\frac{1}{\text{tang}.\varphi-\sin.\varphi}$

46. This denominator $\text{tang}.\varphi-\sin.\varphi$ evidently vanishes in the cases, in which $\varphi=i\pi$ with i denoting all the integers, both positive and negative, from which the simple fractions, of which the denominators contain this factor $\varphi-i\pi$, emerge infinite in the case $\varphi=i\pi$, while the remaining fractions retain finite values. And this consideration uncovers for us a new method for investigating all the simple fractions. Indeed for any such vanishing factor the value of its proposed fraction may be sought ; which since it may become infinite, these will have to be equal to these terms of the series, which become infinite in the same case. This being the case, it will be required to put $\varphi-i\pi=\omega$ with ω denoting an indefinitely small angle. With which done the proposed fraction certainly will become some function of this ω, as it will be appropriate to establish according to its dimensions.

47. Hence therefore likewise we must distinguish the two following cases, just as i may be an even or odd number, since in the first case there becomes $\sin.\varphi=\sin.\omega$, but truly in the second case there becomes $\sin.\varphi=-\sin.\omega$, while in each case there remains :

$$\text{tang}.\varphi=\text{tang}.\omega.$$

Therefore in the first case i shall be an odd number and in this case there will be $\varphi=i\pi$ with our fraction

$$\frac{1}{\text{tang}.\omega+\sin.\omega}.$$

Truly there is approximately,

$$\text{tang}.\omega=\omega+\tfrac{1}{3}\omega^3 \text{ and } \sin.\omega=\omega-\tfrac{1}{6}\omega^3,$$

from which this fraction will become

$$\frac{1}{2\omega+\frac{1}{6}\omega^3}=\frac{1}{2\omega(1+\frac{1}{12}\omega\omega)}=\frac{1}{2\omega}(1-\tfrac{1}{12}\omega\omega).$$

Now this expression at once provides these two fractions $\frac{1}{2\omega}-\frac{1}{24}\omega$, from which on account of $\omega=\varphi-i\pi$ this simple fraction arises for this factor itself

$$\frac{1}{2(\varphi-i\pi)},$$

because the other part vanishes. Whereby if now in place of i we may write the odd numbers in turn, we will arrive at the following series of fractions :

$$\frac{1}{2(\varphi-\pi)}+\frac{1}{2(\varphi+\pi)}+\frac{1}{2(\varphi-3\pi)}+\frac{1}{2(\varphi+3\pi)}+\frac{1}{2(\varphi-5\pi)}+\text{etc.}$$

48. Now also i shall be an even number, from which there becomes :

$$\text{tang.}\varphi=\text{tang.}\omega \text{ et } \sin.\varphi=\sin.\omega;$$

hence our fraction will become:

$$\frac{1}{\text{tang.}\omega-\sin.\omega}$$

where from the expansion made the first terms cancel each other ω, thus so that in this denominator the lowest power of ω shall become ω^3. And on account of this reason a further approximation will be required to be continued as in the preceding case. To this end in place of $\text{tang.}\omega$ we may write

$$\frac{\sin.\omega}{\cos.\omega},$$

so that our fraction shall be

$$\frac{\cos.\omega}{\sin.\omega-\sin.\omega\cos.\omega}.$$

Now since there shall be

$$\sin.\omega\cos.\omega=\tfrac{1}{2}\sin.2\omega,$$

there will be by the series

$$\sin.\omega=\omega-\tfrac{1}{6}\omega^3+\tfrac{1}{120}\omega^5$$

and

$$\sin.2\omega=2\omega-\tfrac{8}{6}\omega^3+\tfrac{32}{120}\omega^5,$$

from which the total denominator will become

$$+\tfrac{1}{2}\omega^3-\tfrac{1}{8}\omega^5=\tfrac{1}{2}\omega^3(1-\tfrac{1}{4}\omega\omega);$$

truly the numerator is $\cos.\omega=1-\tfrac{1}{2}\omega\omega$, from which our total fraction will become :

$$\frac{1-\frac{1}{2}\omega\omega}{\frac{1}{2}\omega^3(1-\frac{1}{4}\omega\omega);}=\frac{1-\frac{1}{4}\omega\omega}{\frac{1}{2}\omega^3};$$

and hence the resulting parts will be

$$\frac{2}{\omega^3}-\frac{1}{2\omega},$$

which both become infinite in the case $\omega = 0$. But it is readily apparent, if we may extend the approximation further, in the following term the letter ω now is going to be transferred into the numerator. Therefore we may write $\varphi - i\pi$ in place of ω and the parts arising from this factor of the denominator will be

$$\frac{2}{(\varphi-i\pi)^3} - \frac{1}{2(\varphi-i\pi)},$$

from which in place of i successively by writing all the even numbers this twin series itself will be produced :

$$\frac{2}{\varphi^3} + \frac{2}{(\varphi-2\pi)^3} + \frac{2}{(\varphi+2\pi)^3} + \frac{2}{(\varphi-4\pi)^3} + \frac{2}{(\varphi+4\pi)^3} + \frac{2}{(\varphi-6\pi)^3} + \text{etc.}$$
$$-\frac{1}{2\varphi} - \frac{1}{2(\varphi-2\pi)} - \frac{1}{2(\varphi+2\pi)} - \frac{1}{2(\varphi-4\pi)} - \frac{1}{2(\varphi+4\pi)} - \text{etc}$$

49. Therefore we may join together these two series deduced from each case, and the fraction proposed

$$\frac{1}{\text{tang.}\varphi - \sin.\varphi}$$

is found to be resolved into the three following series :

$$\frac{1}{2(\varphi-\pi)} + \frac{1}{2(\varphi+\pi)} + \frac{1}{2(\varphi-3\pi)} + \frac{1}{2(\varphi+3\pi)} + \frac{1}{2(\varphi-5\pi)} + \text{etc.},$$
$$-\frac{1}{2\varphi} - \frac{1}{2(\varphi-2\pi)} - \frac{1}{2(\varphi+2\pi)} - \frac{1}{2(\varphi-4\pi)} - \frac{1}{2(\varphi+4\pi)} - \text{etc.},$$
$$+\frac{2}{\varphi^3} + \frac{2}{(\varphi-2\pi)^3} + \frac{2}{(\varphi+2\pi)^3} + \frac{2}{(\varphi-4\pi)^3} + \frac{2}{(\varphi+4\pi)^3} + \text{etc.}$$

50. Anyone readily here may perceive this method to surpass greatly that one, which we have used before, since in this way at once we have obtained fractions arising from any factor of the denominator and nor was there be a need to designate the numerators of these by indefinite letters. Besides also for this reason there was no need to be concerned finding, how many individual factors may be contained in the denominator, if indeed our method may declare this at once.

51. But in general series of this kind, where the denominators of whatever terms in a certain case vanish and thus these terms increase to infinity, it is customary to inquire, with these terms removed how great the sum of the remaining terms shall become. Thus for the case where i is an odd number, the term

$$\frac{1}{2(\varphi-i\pi)}$$

shall be infinite in the case $\varphi = i\pi$. Therefore with this term removed, how great shall the sum of the remaining terms become in the case $\varphi = i\pi$. Towards resolving this question there may be put $\varphi - i\pi = \omega$ and from § 47 there is to become :

$$\frac{1}{2\omega}-\frac{1}{24}\omega=\frac{1}{2(\varphi-i\pi)}+R,$$

where R includes all the remaining terms, the sum of which may be wished in the case $\varphi=i\pi$. Therefore the term

$$\frac{1}{2(\varphi-i\pi)}=\frac{1}{2\omega}$$

may be transferred to the other side, and at once to be elicited

$$R=-\frac{1}{24}\omega=0 \text{ on account of } \omega=0,$$

thus so that with that infinite sum omitted the sum of all the remaining in the case $\varphi=i\pi$ always shall be 0.

52. But when i is an even number, the same conclusion will be had, towards showing which it is necessary for the approximation used to be continued further. But then the numerator will be

$$\cos.\omega=1-\frac{1}{2}\omega\omega+\frac{1}{24}\omega^4;$$

and truly for the denominator

$$\sin.\omega=\omega-\frac{1}{6}\omega^3+\frac{1}{120}\omega^5-\frac{1}{5040}\omega^7$$

and

$$\sin.2\omega=2\omega-\frac{8}{6}\omega^3+\frac{32}{120}\omega^5-\frac{128}{5040}\omega^7,$$

from which the denominator itself becomes:

$$\frac{1}{2}\omega^3-\frac{1}{8}\omega^5-\frac{1}{80}\omega^7=\frac{1}{2}\omega^3\left(1-\frac{1}{4}\omega\omega+\frac{1}{40}\omega^4\right);$$

hence the latter factor moved into the numerator gives

$$1+\frac{1}{4}\omega\omega+\frac{3}{40}\omega^4$$

and hence the total fraction now will be

$$\frac{1-\frac{1}{4}\omega\omega-\frac{11}{240}\omega^4}{\frac{1}{2}\omega^3},$$

which must be equal to the whole series on putting $\varphi=i\pi$, that is with the terms found

$$\frac{2}{(\varphi-i\pi)^3}-\frac{1}{2(\varphi-i\pi)}$$

with all the remaining terms R, from which there is elicited $R=-\frac{11}{120}\omega=0$; from which it is apparent in these cases the sum of all the rest to be $=0$.

53. But if therefore we may accept $\varphi=0$ and we may delete the terms increasing indefinitely, the terms remaining will be

$$\begin{aligned}&-\frac{1}{2\pi}+\frac{1}{2\pi}-\frac{1}{6\pi}+\frac{1}{6\pi}-\frac{1}{10\pi}+\frac{1}{10\pi}-\text{etc.}\\&+\frac{1}{4\pi}-\frac{1}{4\pi}+\frac{1}{8\pi}-\frac{1}{8\pi}+\frac{1}{12\pi}-\frac{1}{12\pi}+\text{etc.}\\&-\frac{2}{8\pi^3}+\frac{2}{8\pi^3}-\frac{2}{64\pi^3}+\frac{2}{64\pi^3}-\frac{2}{216\pi^3}+\frac{2}{216\pi^3}-\text{etc.,}\end{aligned}$$

where all the terms evidently cancel each other, that which also pertains in the remaining cases for which there is put $\varphi=i\pi$.

54. But if we may draw together the pairs of contiguous terms , these series will be produced

$$\begin{aligned}&-\frac{1}{2\varphi}+\frac{\varphi}{\varphi\varphi-\pi\pi}-\frac{\varphi}{\varphi\varphi-4\pi\pi}+\frac{\varphi}{\varphi\varphi-9\pi\pi}-\frac{\varphi}{\varphi\varphi-16\pi\pi}+\frac{\varphi}{\varphi\varphi-25\pi\pi}-\text{etc.,}\\&+\frac{1}{2\varphi^3}+\frac{4\varphi(\varphi\varphi+3\cdot4\pi\pi)}{(\varphi\varphi-4\pi\pi)^3}+\frac{4\varphi(\varphi\varphi+3\cdot16\pi\pi)}{(\varphi\varphi-16\pi\pi)^3}+\frac{4\varphi(\varphi\varphi+3\cdot36\pi\pi)}{(\varphi\varphi-36\pi\pi)^3}+\text{etc.,}\end{aligned}$$

of which the sum of the series is

$$\frac{1}{\text{tang.}\omega-\sin.\omega}.$$

So that if now here we may put $\varphi=0$ or $\varphi=\omega$, since all the terms are divisible by $\varphi=0$ but the sum found is $-\frac{11}{120}\omega$, if we may divide each side by ω , the sum will be $-\frac{11}{120}$; but these sums emerge

$$\begin{aligned}&-\frac{1}{\pi\pi}+\frac{1}{4\pi\pi}-\frac{1}{9\pi\pi}+\frac{1}{16\pi\pi}-\frac{1}{25\pi\pi}+\text{etc.,}\\&-\frac{3\cdot4}{(4\pi\pi)^2}-\frac{3\cdot4}{(16\pi\pi)^2}-\frac{3\cdot4}{(36\pi\pi)^2}-\frac{3\cdot4}{(64\pi\pi)^2}-\text{etc.}\end{aligned}$$

55. Therefore with the signs changed and with the terms reduced to the simplest form we will obtain this summation :

$$\begin{aligned}&\frac{11}{120}=\frac{1}{\pi\pi}\left(1-\frac{1}{4}+\frac{1}{9}-\frac{1}{16}+\frac{1}{25}-\frac{1}{36}+\text{etc.}\right)\\&+\frac{3}{4\pi^4}\left(1+\frac{1}{4^2}+\frac{1}{9^2}+\frac{1}{25^2}+\frac{1}{36^2}+\text{etc.}\right).\end{aligned}$$

But there is known to be

$$1-\frac{1}{4}+\frac{1}{9}-\frac{1}{16}+\frac{1}{25}-\frac{1}{36}+\text{etc.}=\frac{\pi\pi}{12}$$

and

$$1+\frac{1}{4^2}+\frac{1}{9^2}+\frac{1}{25^2}+\frac{1}{36^2}+\text{etc.}=\frac{\pi^4}{90},$$

thence this equality arising can be seen at once.

DE RESOLUTIONE FRACTIONUM TRANSCENDENTIUM

IN INFINITAS FRACTIONES SIMPLICES

Opuscula analytica 2, 1785, p. 102-137; [E592]

1. Proposita fractione quacunque algebraica

$$\frac{P}{Q},$$

cuius tam numerator P quam denominator Q sint functiones rationales integrae quantitatis z, iam pridem ostendi, quomodo ea in fractiones simplices resolvi possit, quarum denominatores aequentur factoribus simplicibus denominatoris Q, numeratores vero sint constantes, siquidem variabilis z in denominatore Q plures habeat dimensiones quam in numeratore P. Quin etiam ostendi, quemadmodum pro quolibet factore simplici denominatoris fractio simplex respondens reperiri queat sine ullo respectu ad reliquos factores habito. Ita si constet denominatorem Q factorem complecti simplicem $z-a$, fractio simplex inde nata, quae erit huius formae

$$\frac{\alpha}{z-a},$$

facillime hoc modo definitur. Statuatur

$$\frac{P}{Q}=\frac{\alpha}{z-a}+R,$$

ubi R complectatur omnes fractiones simplices ex reliquis oriendas; multiplicetur utrinque per $z-a$, ut fiat

$$\frac{P(z-a)}{Q}=\alpha+R(z-a),$$

et quia α est quantitas constans, ea semper eundem retinebit valorem, quicunque valor variabili z tribuatur; quamobrem fiat ubique $z=a$, ut reliquarum fractionum simplicium ratio ex calculo excedat, et habebitur

$$\alpha=\frac{P(z-a)}{Q},$$

siquidem in hac formula fiat $z=a$; tum autem numerator $P(z-a)$ in nihilum abit; verum, quia $z-a$ est factor denominatoris Q, etiam denominator Q in nihilum abibit. Hinc igitur per regulam consuetam loco numeratoris ac denominatoris sua differentialia substituantur, quandoquidem etiamnunc erit

$$\frac{Pdz+(z-a)dP}{dQ}=\alpha,$$

siquidem hic ubique loco *z* scribatur *a*. Ponamus igitur hoc casu $z=a$ fieri

$$P=A \text{ et } \frac{dQ}{dz}=C,$$

quae ergo quantitates *A* et *C* facillime inveniuntur; tum igitur prodibit numerator quaesitus

$$\alpha=\frac{A}{C},$$

ita ut fractio simplex ex denominatoris factore $z-a$ oriunda sit

$$=\frac{A}{C(z-a)},$$

ita ut non opus sit reliquos factores denominatoris nosse. Simili autem modo pro singulis reliquis factoribus fractiones simplices respondentes determinabuntur, quarum omnium summa aequabitur fractioni propositae $\frac{P}{Q}$, dummodo variabilis *z* pauciores habeat dimensiones in numeratore *P* quam in denominatore *Q*.

2. Haec igitur principia sequentes pro denominatore *Q* eiusmodi assumamus functiones transcendentes, quas in infinitos factores simplices resolvere liceat, id quod evenit, si eae infinitis casibus nihilo aequales evadant. Praeterea vero necesse est, ut omnes isti factores inter se sint inaequales, quandoquidem factores aequales peculiarem resolutionem postulant. Imprimis autem requiritur, ut productum omnium talium factorum ipsam functionem *Q* penitus exhauriat, quoniam quandoque factores imaginarii se intermiscere possent. Veluti si sumatur $Q=\text{tang.}\varphi$, ea utique omnibus iisdem casibus evanescit, quibus haec functio $\sin.\varphi$, hincque ambae istae functiones eosdem factores simplices involvunt, etiamsi inter se neutiquam sint aequales. Deinde vero numeratorem *P* ita comparatum esse oportet, ut cum denominatore *Q* nullos habeat factores communes. Imprimis autem cavendum est, ne quantitas variabilis in numeratore ad totidem vel plures dimensiones assurgat quam in denominatore. Cum autem ea in denominatore ad infinitas dimensiones assurgere sit censenda, istud incommodum non erit pertimescendum, quamdiu variabilis in numeratore tantum finito dimensionum numero continetur. Sin autem eius potestates etiam in infinitum ascendant, saepenumero difficile erit iudicare, num dimensionum numerus maior sit vel minor quam in denominatore. Interim tamen etiam his casibus fractio proposita $\frac{P}{Q}$ omnes continebit fractiones simplices, ad quas methodus nostra perducit. Verum evenire potest, ut praeter eas etiam quasdam partes quasi integras evolvat. His igitur praenotatis sequentes casus evolvamus.

I. SUMATUR $Q=\sin.\varphi$
UT FRACTIO RE SOLVENDA SIT $\frac{P}{\sin.\varphi}$.

3. Quoniam formula $\sin.\varphi$ denotante π semiperipheriam circuli, cuius radius $=1$, seu angulum duobus rectis aequalem, omnibus his casibus evanescit

$$\varphi=0, \varphi=\pm\pi,\ \varphi=\pm 2\pi,\ \varphi=\pm 3\pi \text{ etc. et in genere } \varphi=\pm i\pi,$$

eius factores numero infiniti erunt

$$\varphi,\ (\varphi\pm\pi),\ (\varphi\pm 2\pi),\ \ (\varphi\pm 3\pi), \text{ etc. et in genere } (\varphi\pm i\pi).$$

Aliunde autem certum est hanc formulam $\sin.\varphi,$ praeter istos factores nullos alios sive reales sive imaginarios involvere ; cum enim sit

$$\sin.\varphi=\varphi-\frac{\varphi^3}{1\cdot 2\cdot 3}+\frac{\varphi^5}{1\cdot 2\cdot 3\cdot 4\cdot 5}-\text{etc.},$$

constat hanc seriem aequari huic producto infinito

$$\varphi(1-\tfrac{\varphi}{\pi})(1+\tfrac{\varphi}{\pi})(1-\tfrac{\varphi}{2\pi})(1+\tfrac{\varphi}{2\pi})(1-\tfrac{\varphi}{3\pi})(1+\tfrac{\varphi}{3\pi}) \text{ etc.}$$

4. Consideremus igitur nostri denominatoris $Q=\sin.\varphi$ factorem quemcunque $\varphi-i\pi$, ubi i denotet omnes plane numeros integros tam positivos quam negativos cyphra non excepta, sitque fractio partialis hinc oriunda

$$\frac{\alpha}{\varphi-i\pi}.$$

Ad eius numeratorem α inveniendum statuatur primo in numeratore P ubique $\varphi=i\pi$ sitque quantitas inde resultans $=A$; deinde cum sit $Q=\sin.\varphi$, erit

$$dQ=d\varphi\cos.\varphi \text{ sive } \frac{dQ}{d\varphi}=\cos.\varphi,$$

ubi loco φ itidem scribi oportet *in,* ut obtineamus *C,* unde patet fore

$$C=\cos.i\pi,$$

ita ut sit

$$C=\pm 1,$$

ubi signum $+$ valebit pro numeris paribus, signum vero $-$ pro imparibus numeris loco i assumtis. Hoc igitur modo numerator fractionis nostrae erit

$$\alpha=\pm A$$

ipsaque fractio quaesita

$$\pm \frac{A}{\varphi - i\pi}.$$

Hinc autem ulterius progredi non licet, quamdiu numeratorem in genere spectamus; unde eius loco plures valores determinatos accipiamus singulosque in sequentibus exemplis evolvamus.

1°. SIT $P = 1$ ET FRACTIO PROPOSITA $\frac{1}{\sin.\varphi}$

5. Hic igitur semper erit $A = 1$ et fractio simplex quaecunque

$$= \frac{\pm 1}{\varphi - i\pi},$$

signum superius valet, si *i* numerus par, inferius vero, si impar. Substituamus igitur successive pro *i* omnes eius valores ordine

$$0,\ +1,\ -1,\ +2,\ -2,\ 3,\ -3,\ +\ 4,\ -4 \text{ etc.}$$

et resolutio nostrae fractionis $\frac{1}{\sin.\varphi}$ in fractiones simplices ita se habebit

$$\frac{1}{\sin.\varphi} = +\frac{1}{\varphi} - \frac{1}{\varphi-\pi} - \frac{1}{\varphi+\pi} + \frac{1}{\varphi-2\pi} + \frac{1}{\varphi+2\pi} - \frac{1}{\varphi-3\pi} - \frac{1}{\varphi+3\pi} + \text{etc.}$$

quae in hanc formam reducatur

$$\frac{1}{\sin.\varphi} = +\frac{1}{\varphi} + \frac{1}{\pi-\varphi} - \frac{1}{\pi+\varphi} - \frac{1}{2\pi-\varphi} + \frac{1}{2\pi+\varphi} + \frac{1}{3\pi-\varphi} - \frac{1}{3\pi+\varphi} - \text{etc.}$$

Contrahantur post primum terminum bini sequentium, ut nanciscamur hanc seriem

$$\frac{1}{\sin.\varphi} = +\frac{1}{\varphi} + \frac{2\varphi}{\pi\pi-\varphi\varphi} - \frac{2\varphi}{4\pi\pi-\varphi\varphi} + \frac{2\varphi}{9\pi\pi-\varphi\varphi} - \frac{2\varphi}{16\pi\pi-\varphi\varphi} + \text{etc.},$$

unde deducitur sequens series memoratu digna

$$\frac{1}{2\varphi\sin.\varphi} - \frac{1}{2\varphi\varphi} = \frac{1}{\pi\pi-\varphi\varphi} - \frac{1}{4\pi\pi-\varphi\varphi} + \frac{1}{9\pi\pi-\varphi\varphi} - \frac{1}{16\pi\pi-\varphi\varphi} + \text{etc.},$$

6. Has quidem series iam olim fusius sum prosecutus ; interim tamen pro sequentibus casibus haud inutile erit sequentes transformationes hic repetere. Ponamus igitur primo $\varphi = \lambda\pi$ *,* ut littera π ex seriebus elidatur, atque hinc nanciscemur

$$\frac{\pi}{\sin.\lambda\pi} = \frac{1}{\lambda} - \frac{1}{\lambda-1} - \frac{1}{\lambda+1} + \frac{1}{\lambda-2} + \frac{1}{\lambda+2} - \frac{1}{\lambda-3} - \frac{1}{\lambda+3} + \text{etc.}$$

et

$$\frac{\pi}{2\lambda\sin.\lambda\pi}-\frac{1}{2\lambda\lambda}=\frac{1}{1-\lambda\lambda}-\frac{1}{4-\lambda\lambda}+\frac{1}{9-\lambda\lambda}-\frac{1}{16-\lambda\lambda}+\text{etc.}$$

atque hinc per differentiationem spectando λ tanquam quantitatem variabilem infinitas alias series notatu dignissimas elicere poterimus. Ex priore scilicet nanciscemur

$$\frac{\pi\pi\cos.\lambda\pi}{\sin^2.\lambda\pi}=\frac{1}{\lambda\lambda}-\frac{1}{(\lambda-1)^2}-\frac{1}{(\lambda+1)^2}+\frac{1}{(\lambda-2)^2}+\frac{1}{(\lambda+2)^2}-\frac{1}{(\lambda-3)^2}-\text{etc.}$$

Hinc igitur sequitur, si $\lambda=\frac{1}{2}$, fore

$$0=1-1-\tfrac{1}{9}+\tfrac{1}{9}+\tfrac{1}{25}-\tfrac{1}{25}-\text{etc.},$$

quod quidem est manifestum. At si $\lambda=\frac{1}{3}$, erit

$$\frac{2\pi\pi}{27}=1-\frac{1}{4}-\frac{1}{16}+\frac{1}{25}+\frac{1}{49}-\frac{1}{64}-\frac{1}{100}+\frac{1}{121}+\frac{1}{169}-\text{etc.}$$

Si $\lambda=\frac{2}{3}$, oritur series praecedens.

Si $\lambda=\frac{1}{4}$, prodit haec summatio

$$\frac{\pi\pi}{8\sqrt{2}}=1-\frac{1}{9}-\frac{1}{25}+\frac{1}{49}+\frac{1}{81}-\frac{1}{121}-\frac{1}{169}+\text{etc.}$$

Quodsi denuo differentiemus, obtinebitur sequens summatio

$$\frac{\pi^3}{\sin^3.\lambda\pi}-\frac{\pi^3}{2\sin.\lambda\pi}=\frac{1}{\lambda^3}-\frac{1}{(\lambda-1)^3}-\frac{1}{(\lambda+1)^3}+\frac{1}{(\lambda-2)^3}+\frac{1}{(\lambda+2)^3}-\frac{1}{(\lambda-3)^2}-\text{etc.}$$

sicque continuo ulterius progredi licet.

7. Simili modo etiam alteram formam differentiemus, quae reducta praebet

$$\frac{1}{2\lambda^4}-\frac{\pi}{4\lambda^3\sin.\lambda\pi}-\frac{\pi\pi\cos.\lambda\pi}{4\lambda\lambda\sin^2.\lambda\pi}=\frac{1}{(1-\lambda\lambda)^2}-\frac{1}{(4-\lambda\lambda)^2}+\frac{1}{(9-\lambda\lambda)^2}-\frac{1}{(16-\lambda\lambda)^3}+\text{etc.}$$

Quodsi nunc sumamus $\lambda=\frac{1}{2}$, prodibit ista summatio

$$\frac{1}{2}-\frac{\pi}{8}\;=\;\frac{1}{3^2}-\frac{1}{15^2}+\frac{1}{35^2}-\frac{1}{63^2}+\frac{1}{99^2}-\text{etc.}$$

quae series prorsus nova omnem attentionem meretur; neque autem opus est hinc novam differentiationem instituere.

8. Posteriorem autem summationem

$$\frac{\pi}{2\lambda\sin.\lambda\pi}-\frac{1}{2\lambda\lambda}=\frac{1}{1-\lambda\lambda}-\frac{1}{4-\lambda\lambda}+\frac{1}{9-\lambda\lambda}-\frac{1}{16-\lambda\lambda}+\text{etc.}$$

accuratius perpendamus ac primo quidem, cum ea semper debeat esse vera, quicquid pro λ assumatur, sumamus $\lambda=0$. Quia autem hoc casu membrum sinistrum abit in $\infty-\infty$, tractetur λ ut quantitas quam minima, et cum sit $sin.\lambda\pi=\lambda\pi-\frac{1}{6}\lambda^3\pi^3$, istud membrum evadet

$$\frac{\pi}{2\lambda(\lambda\pi-\frac{1}{6}\lambda^3\pi^3)}-\frac{1}{2\lambda\lambda},$$

quae fractiones ad communem denominatorem perductae dant

$$\frac{1-1+\frac{1}{6}\lambda\lambda\pi\pi}{2\lambda\lambda(1-\frac{1}{6}\lambda\lambda\pi\pi)}=\frac{\pi\pi}{12-2\lambda\lambda\pi\pi}.$$

Nunc igitur facto $\lambda=0$ eius factor erit $=\frac{\lambda\lambda}{12}$, series autem ipsa hoc casu evadet

$$1-\frac{1}{4}+\frac{1}{9}-\frac{1}{16}+\frac{1}{25}-\text{etc.},$$

cuius summam constat $\frac{\pi\pi}{12}$.

9. Manifestum porro est, quoties pro λ accipiatur numerus integer, unum terminum seriei ideoque etiam ipsam seriem fieri infinitam, quod egregie convenit cum summa inventa, quandoquidem hoc casu fit $\sin.\lambda\pi=0$. Atque hinc nata est ista quaestio: si ille terminus seriei in infinitum abiens ad sinistram partem transferatur, quanta futura sit reliquorum terminorum summa. Ponamus scilicet esse $\lambda=1$, et primus seriei terminus evadet infinitus; qui ergo ad sinistram partem translatus dabit

$$\frac{\pi}{2\lambda\sin.\lambda\pi}-\frac{1}{2\lambda\lambda}-\frac{1}{1-\lambda\lambda}=-\frac{1}{3}+\frac{1}{8}-\frac{1}{15}+\frac{1}{24}-\frac{1}{35}+\frac{1}{48}-\text{etc.}$$

Nunc ad valorem huius seriei investigandum statuatur λ unitati tantum proxime aequale ponendo $\lambda=1-\omega$ eritque

$$\sin.\lambda\pi=\sin.(\pi-\pi\omega)=\sin.\pi\omega;$$

est vero

$$\sin.\pi\omega=\pi\omega-\frac{1}{6}\pi^3\omega^3,$$

quo valore substituto prodibit

$$\frac{1}{2(1-\omega)\omega(1-\frac{1}{6}\pi^2\omega^2)}-\frac{1}{2(1-\omega)^2}-\frac{1}{2\omega-\omega\omega}.$$

Primum autem membrum

$$\frac{1}{2(1-\omega)\omega(1-\frac{1}{6}\pi^2\omega^2)}$$

ob

$$\frac{1}{1-\omega}=1+\omega+\omega^2$$

et

$$\frac{1}{1-\frac{1}{6}\pi^2\omega^2}=1+\frac{1}{6}\pi^2\omega^2$$

negligendo potestates ipsius ω quadrato altiores transmutatur in hanc formam

$$\frac{1}{2\omega}(1+\omega+\omega^2+\frac{1}{6}\pi\pi\omega\omega);$$

tertium autem membrum

$$-\frac{1}{2\omega(1-\frac{1}{2}\omega)}$$

ob

$$\frac{1}{1-\frac{1}{2}\omega}=1+\frac{1}{2}\omega+\frac{1}{4}\omega\omega$$

abit in

$$-\frac{1}{2\omega}(1+\frac{1}{2}\omega+\frac{1}{4}\omega\omega),$$

unde primum et tertium membrum simul faciunt

$$\frac{1}{2\omega}(\frac{1}{2}\omega+\frac{3}{5}\omega\omega+\frac{1}{6}\pi\pi\omega\omega)=\frac{1}{4}+\frac{3}{8}\omega+\frac{1}{12}\pi\pi\omega;$$

qui valor posito $\omega=0$ fit $=\frac{1}{4}$, unde secundum membrum, quod erit $-\frac{1}{2}$, iunctum dabit totam summam quaesitam $-\frac{1}{4}$, ita ut sit mutatis signis

$$\frac{1}{4}=\frac{1}{3}-\frac{1}{8}+\frac{1}{15}-\frac{1}{24}+\frac{1}{35}-\frac{1}{48}+\frac{1}{63}-\text{etc.},$$

cuius ratio est manifesta, cum sit

$$\frac{1}{3}=\frac{1}{2}(1-\frac{1}{3}),\ \frac{1}{8}=\frac{1}{2}(\frac{1}{2}-\frac{1}{4}),\ \frac{1}{15}=\frac{1}{2}(\frac{1}{3}-\frac{1}{5}),\ \frac{1}{24}=\frac{1}{2}(\frac{1}{4}-\frac{1}{6})\ \text{etc.};$$

his enim valoribus substitutis et sublatis terminis se destruentibus fiet

$$\frac{1}{4}=\frac{1}{2}-\frac{1}{2\cdot 2}=\frac{1}{4}.$$

10. Circa eandem autem seriem quaestio magis ardua occurrit, qua quaeritur summa seriei, si λλ fuerit numerus negativus ideoque λ quantitas imaginaria. Ponatur igitur

$$\lambda\lambda = -\mu\mu \text{ sive } \lambda = \mu\sqrt{-1}$$

ac series nihilominus erit realis, scilicet

$$\frac{1}{1+\mu\mu} - \frac{1}{4+\mu\mu} + \frac{1}{9+\mu\mu} - \frac{1}{16+\mu\mu} + \frac{1}{25+\mu\mu} - \text{etc.};$$

cuius ergo summa erit

$$\frac{\pi}{2\mu\sqrt{-1}\cdot\sin.\pi\mu\sqrt{-1}} + \frac{1}{2\mu\mu},$$

cuius ergo valor realis quaeritur, siquidem nullum est dubium, quin seriei valor fiat realis.

11. In doctrina angulorum ostendi solet esse

$$\sin.\varphi = \frac{e^{\varphi\sqrt{-1}} - e^{-\varphi\sqrt{-1}}}{2\sqrt{-1}}.$$

Fiat igitur $\varphi = \mu\pi\sqrt{-1}$ eritque

$$\varphi\sqrt{-1} = -\mu\pi \text{ et } \varphi\pi\sqrt{-1} = +\mu\pi,$$

unde concluditur

$$\sin.\mu\pi\sqrt{-1} = \frac{e^{-\mu\pi} - e^{+\mu\pi}}{2\sqrt{-1}}.$$

unde summa quaesita erit

$$\frac{\pi}{\mu(e^{-\mu\pi} - e^{+\mu\pi})} + \frac{1}{2\mu\mu}.$$

Erit igitur

$$\frac{1}{1+\mu\mu} - \frac{1}{4+\mu\mu} + \frac{1}{9+\mu\mu} - \frac{1}{16+\mu\mu} + \frac{1}{25+\mu\mu} - \text{etc.} = \frac{1}{2\mu^2} - \frac{\pi}{\mu(e^{\mu\pi} - e^{-\mu\pi})}.$$

2°. SIT $P = \varphi$ ET FRACTIO PROPOSITA $\frac{\varphi}{\sin.\varphi}$

12. Hic ergo ob numeratorem $P = \varphi$ factor denominatoris primus φ tollitur, quemadmodum etiam nostra resolutio numeratorem ipsi respondentem nihilo praebet aequalem. Hic igitur pro denominatore $\varphi - i\pi$ fit numerator $\frac{i\pi}{\cos.i\pi} = \pm i\pi$, ubi signum superius valet, si i numerus par, inferius, si impar. Quodsi ergo fuerit $i = 2n$, fractio inde nata erit

$$\frac{2n\pi}{\varphi - 2n\pi};$$

at si $i = -2n$, fractio erit

$$-\frac{2n\pi}{\varphi+2n\pi};$$

at si fuerit $i=2n-1$, fractio erit

$$-\frac{(2n-1)\pi}{\varphi-(2n-1)\pi};$$

denique ex $i=2n-1$ oritur

$$\frac{(2n-1)\pi}{\varphi+(2n-1)\pi},$$

quocirca series inventa erit

$$\frac{\varphi}{\sin.\varphi}=-\frac{\pi}{\varphi-\pi}+\frac{\pi}{\varphi+\pi}+\frac{2\pi}{\varphi-2\pi}-\frac{2\pi}{\varphi+2\pi}-\frac{3\pi}{\varphi-3\pi}+\frac{3\pi}{\varphi+3\pi}+\frac{4\pi}{\varphi-4\pi}-\text{etc.}$$

sive

$$\frac{\varphi}{\sin.\varphi}=\frac{\pi}{\pi-\varphi}+\frac{\pi}{\pi+\varphi}-\frac{2\pi}{2\pi-\varphi}-\frac{2\pi}{2\pi+\varphi}+\frac{3\pi}{3\pi-\varphi}+\frac{3\pi}{3\pi+\varphi}-\frac{4\pi}{4\pi-\varphi}-\text{etc.},$$

unde, si bini termini in unum contrahantur, erit

$$\frac{\varphi}{\sin.\varphi}=\frac{2\pi\pi}{\pi\pi-\varphi\varphi}-\frac{8\pi\pi}{4\pi\pi-\varphi\varphi}+\frac{18\pi\pi}{9\pi\pi-\varphi\varphi}-\frac{32\pi\pi}{16\pi\pi-\varphi\varphi}+\text{etc.},$$

quae per $2\pi\pi$ divisa producit hanc summationem

$$\frac{\varphi}{2\pi\pi\sin.\varphi}=\frac{1}{\pi\pi-\varphi\varphi}-\frac{4}{4\pi\pi-\varphi\varphi}+\frac{9}{9\pi\pi-\varphi\varphi}-\frac{16}{16\pi\pi-\varphi\varphi}+\text{etc.}$$

Ac si ponatur $\varphi=\lambda\pi$, prodibit

$$\frac{\lambda\pi}{2\sin.\lambda\pi}=\frac{1}{1-\lambda\lambda}-\frac{4}{4-\lambda\lambda}+\frac{9}{9-\lambda\lambda}-\frac{16}{16-\lambda\lambda}+\text{etc.},$$

unde, si fuerit

$$\lambda\lambda=-\mu\mu \text{ seu } \lambda=\mu\sqrt{-1},$$

ob

$$\sin.\mu\pi\sqrt{-1}=\frac{e^{-\mu\pi}-e^{+\mu\pi}}{2\sqrt{-1}}$$

erit

$$\frac{\mu\pi}{e^{\mu\pi}-e^{-\mu\pi}}=\frac{1}{1+\mu\mu}-\frac{1}{4+\mu\mu}+\frac{1}{9+\mu\mu}-\frac{1}{16+\mu\mu}+\text{etc.}$$

atque hinc per differentiationem infinitas alias summationes deducere licebit.

3°. SIT NUMERATOR $P=\varphi^2$ ET FRACTIO $\frac{\varphi\varphi}{\sin.\varphi}$

13. Pro denominatore igitur $\varphi-i\pi$ numerator erit *,* ubi signum $\pm ii\pi\pi$ superius valet pro *i* numero pari, inferius vero pro impari. Hinc si loco *i* successive scribantur numeri

$$+1,\ -1,\ +2,\ -2,\ +3,\ -3 \text{ etc.},$$

series resultans erit

$$\frac{\varphi\varphi}{\sin.\varphi} = -\frac{\pi\pi}{\varphi-\pi} - \frac{\pi\pi}{\varphi+\pi} + \frac{4\pi\pi}{\varphi-2\pi} + \frac{4\pi\pi}{\varphi+2\pi} - \frac{9\pi}{\varphi-3\pi} - \frac{9\pi}{\varphi+3\pi} + \text{etc.}$$

sive

$$\frac{\varphi\varphi}{\sin.\varphi} = \frac{\pi\pi}{\pi-\varphi} - \frac{\pi\pi}{\pi+\varphi} - \frac{4\pi\pi}{2\pi-\varphi} + \frac{4\pi\pi}{2\pi+\varphi} + \frac{9\pi\pi}{3\pi-\varphi} - \frac{9\pi\pi}{3\pi+\varphi} - \text{etc.}$$

Contractis igitur binis terminis fiet

$$\frac{\varphi\varphi}{\sin.\varphi} = \frac{2\pi\pi\varphi}{\pi\pi-\varphi\varphi} - \frac{8\pi\pi\varphi}{4\pi\pi-\varphi\varphi} + \frac{18\pi\pi\varphi}{9\pi\pi-\varphi\varphi} - \frac{32\pi\pi\varphi}{16\pi\pi-\varphi\varphi} + \text{etc.}$$

sive

$$\frac{\varphi}{2\sin.\varphi} = \frac{\pi\pi}{\pi\pi-\varphi\varphi} - \frac{4\pi\pi}{4\pi\pi-\varphi\varphi} + \frac{9\pi\pi}{9\pi\pi-\varphi\varphi} - \frac{16\pi\pi}{16\pi\pi-\varphi\varphi} + \text{etc.}$$

Quodsi nunc quilibet terminus huius seriei in duas partes discerpatur, quarum prior semper est 1, binae sequentes series nascentur

$$\frac{\varphi}{2\sin.\varphi} = \begin{Bmatrix} +1 \quad -1 \quad +1 \quad -1 \quad +1 \quad -\text{etc.} \\ \frac{\varphi\varphi}{\pi\pi-\varphi\varphi} - \frac{\varphi\varphi}{4\pi\pi-\varphi\varphi} + \frac{\varphi\varphi}{9\pi\pi-\varphi\varphi} - \frac{\varphi\varphi}{16\pi\pi-\varphi\varphi} + \frac{\varphi\varphi}{25\pi\pi-\varphi\varphi} - \text{etc.} \end{Bmatrix}$$

Notum autem est seriei

$$+1-1+1-1+1-1+\text{etc.}$$

summam esse $=\frac{1}{2}$; qua ad alteram partem translata et per $\varphi\varphi$ divisa prodibit

$$\frac{1}{2\varphi\sin.\varphi} - \frac{1}{2\varphi\varphi} = \frac{1}{\pi\pi-\varphi\varphi} - \frac{1}{4\pi\pi-\varphi\varphi} + \frac{1}{9\pi\pi-\varphi\varphi} - \text{etc.},$$

quae prorsus convenit cum serie in § 5 inventa.

4°. SIT $P=\varphi^{\gamma}$ DENOTANTE γ NUMERUM IMPAREM QUEMCUNQUE POSITIVUM UT FRACTIO PROPOSITA SIT $\frac{\varphi^{\gamma}}{\sin.\varphi}$

14. Cum igitur pro denominatore $\varphi-i\pi$ fiat $A=i^{\gamma}\pi^{\gamma}$ et $C=\pm 1$, erit numerator $\pm i^{\gamma}\pi^{\gamma}$, unde, cum γ sit numerus impar, signa nostrorum terminorum eadem lege procedent atque in casu $P=\varphi$, ubi $\gamma=1$; unde series hinc nata erit

$$\frac{\varphi^{\gamma}}{\sin.\varphi}=\frac{\pi^{\gamma}}{\pi-\varphi}+\frac{\pi^{\gamma}}{\pi+\varphi}-\frac{2^{\gamma}\pi^{\gamma}}{2\pi-\varphi}-\frac{2^{\gamma}\pi^{\gamma}}{2\pi+\varphi}+\frac{3^{\gamma}\pi^{\gamma}}{3\pi-\varphi}+\frac{3^{\gamma}\pi^{\gamma}}{3\pi+\varphi}-\text{etc.},$$

quae per π^{γ} divisa praebet

$$\frac{\varphi^{\gamma}}{\pi^{\gamma}\sin.\varphi}=\frac{1}{\pi-\varphi}+\frac{1}{\pi+\varphi}-\frac{2^{\gamma}}{2\pi-\varphi}-\frac{2^{\gamma}}{2\pi+\varphi}+\frac{3^{\gamma}}{3\pi-\varphi}+\frac{3^{\gamma}}{3\pi+\varphi}-\text{etc.},$$

et binis terminis contractis erit

$$\frac{\varphi^{\gamma}}{2\pi^{\gamma+1}\sin.\varphi}=\frac{1}{\pi\pi-\varphi\varphi}-\frac{2^{\gamma}}{4\pi\pi-\varphi\varphi}+\frac{3^{\gamma}}{9\pi\pi-\varphi\varphi}-\frac{4^{\gamma}}{16\pi\pi-\varphi\varphi}+\text{etc.}$$

Statuamus nunc $\varphi=\lambda\pi$ eritque

$$\frac{\lambda^{\gamma}\pi}{2\sin.\varphi}=\frac{1}{1-\lambda\lambda}-\frac{2^{\gamma}}{4-\lambda\lambda}+\frac{3^{\gamma}}{9-\lambda\lambda}-\frac{4^{\gamma}}{16-\lambda\lambda}+\text{etc.}$$

15. Hinc si fuerit $\lambda=\mu\sqrt{-1}$, erit ut ante

$$\sin.\mu\pi\sqrt{-1}=\frac{e^{-\mu\pi}-e^{+\mu\pi}}{2\sqrt{-1}}.$$

Pro valore autem potestatis λ^{γ} duos casus evolvi oportet, prouti fuerit

$$\gamma=4n+1 \text{ vel } \gamma=4n-1.$$

Priore casu erit

$$(\mu\sqrt{-1})^{4n+1}=(\mu\sqrt{-1})^{4n}\cdot\mu\sqrt{-1}$$

Est vero

$$(\mu\sqrt{-1})^{4n}=\mu^{4n}$$

unde erit

$$\lambda^{\gamma}=\mu^{4n+1}\sqrt{-1},$$

hincque prodit sequens summatio realis

$$\frac{\mu^{4n+1}\pi}{e^{\mu\pi}-e^{-\mu\pi}}=\frac{1}{1+\mu\mu}-\frac{2^{4n+1}}{4+\mu\mu}+\frac{3^{4n+1}}{9+\mu\mu}-\frac{4^{4n+1}}{16+\mu\mu}+\text{etc.}$$

Altero autem casu, quo $\gamma=4n-1$, prius membrum capi debet negative eritque

$$-\frac{\mu^{4n-1}\pi}{e^{\mu\pi}-e^{-\mu\pi}}=\frac{1}{1+\mu\mu}-\frac{2^{4n-1}}{4+\mu\mu}+\frac{3^{4n-1}}{9+\mu\mu}-\frac{4^{4n-1}}{16+\mu\mu}+\text{etc.}$$

Has autem summationes facile patet veras esse non posse, nisi γ sit numerus integer impar et quidem positivus.

5°. SIT NUMERATOR $P=\varphi^{\delta}$ DENOTANTE δ NUMERUM PAREM POSITIVUM QUEMCUNQUE ET FRACTIO $\frac{\varphi^{\delta}}{\sin.\varphi}$

15 [a]. Pro denominatore ergo $\varphi-i\pi$ numerator erit $\pm i^{\delta}\pi^{\delta}$ ambiguitate signorum eandem legem tenante. Hoc igitur casu ratio signorum perinde se habebit ac casu $P=\varphi\varphi$ eritque idcirco

$$\frac{\varphi^{\delta}}{\sin.\varphi}=\frac{\pi^{\delta}}{\pi-\varphi}-\frac{\pi^{\delta}}{\pi+\varphi}-\frac{2^{\delta}\pi^{\delta}}{2\pi-\varphi}+\frac{2^{\delta}\pi^{\delta}}{2\pi+\varphi}+\frac{3^{\delta}\pi^{\delta}}{3\pi-\varphi}-\frac{3^{\delta}\pi^{\delta}}{3\pi+\varphi}-\text{etc.}$$

Quare si ponamus $\varphi=\lambda\pi$, erit haec series

$$\frac{\lambda^{\delta}\pi}{\sin.\lambda\pi}=\frac{1}{1-\lambda}-\frac{1}{1+\lambda}-\frac{2^{\delta}}{2-\lambda}+\frac{2^{\delta}}{2+\lambda}+\frac{3^{\delta}}{3-\lambda}-\frac{3^{\delta}}{3+\lambda}-\text{etc.};$$

hinc binis terminis in unum contrahendis fiet

$$\frac{\lambda^{\delta-1}\pi}{2\sin.\lambda\pi}=\frac{1}{1-\lambda\lambda}-\frac{2^{\delta}}{2-\lambda\lambda}+\frac{3^{\delta}}{3-\lambda\lambda}-\frac{4^{\delta}}{4-\lambda\lambda}+\text{etc.}$$

16. Statuamus nunc etiam $\lambda=\mu\sqrt{-1}$, ut sit

$$\sin.\lambda\pi=\frac{e^{-\mu\pi}-e^{+\mu\pi}}{2\sqrt{-1}}$$

Pro valore autem ipsius $\lambda^{\delta-1}$ duos iterum casus evolvi oportet, prout fuerit vel $\delta=4n$ vel $\delta=4n+2$.
Priore casu, quo $\delta=4n$, erit $\lambda^{4n}=\mu^{4n}$, ideoque

$$\lambda^{4n-1}=\frac{\mu^{4n-1}}{\sqrt{-1}};$$

atque hinc orietur ista summatio

$$\frac{-\mu^{4n-1}\pi}{e^{\mu\pi}-e^{-\mu\pi}}=\frac{1}{1+\mu\mu}-\frac{2^{4n}}{4+\mu\mu}+\frac{3^{4n}}{9+\mu\mu}-\frac{4^{4n}}{16+\mu\mu}+\text{etc.}$$

Pro altero autem casu $\delta=4n+2$ summatio ita se habebit

$$\frac{+\mu^{4n+1}\pi}{e^{\mu\pi}-e^{-\mu\pi}}=\frac{1}{1+\mu\mu}-\frac{2^{4n+2}}{4+\mu\mu}+\frac{3^{4n+2}}{9+\mu\mu}-\frac{4^{4n+2}}{16+\mu\mu}+\text{etc.}$$

17. Hae autem summationes eatenus tantum veritati erunt consentaneae, quatenus pro exponentibus γ et δ numeri integri, prouti sunt definiti, accipiantur, nihilque impedit, quominus quantumvis magni assumantur. Cum enim denominator

$$\sin.\varphi=\varphi-\tfrac{1}{6}\varphi^{3}+\tfrac{1}{120}\varphi^{5}-\text{etc.}$$

ad dimensiones infinitas ipsius φ assurgat, dummodo maxima potestas in numeratore non fiat infinita, resolutio in fractiones semper ad veritatem perducit. Sin autem exponentes illi non essent integri positivi, sed fracti vel adeo negativi, resolutio in fractiones partiales locum plane habere nequit. Quamobrem si loco numeratoris P eiusmodi functiones ipsius φ statuamus, quae etiam ad infinitum dimensionum numerum adsurgant, tum de summa inventa non amplius erimus certi. Verum fieri potest, ut ad fractiones partiales inventas insuper quaedam partes integrae adiici debeant. Huiusmodi igitur casus aliquos evolvamus.

6°. SIT NUMERATOR $P=\cos.\varphi$ ET FRACTIO $=\frac{\cos.\varphi}{\sin.\varphi}$

18. Cum sit

$$\cos.\varphi=1-\tfrac{1}{2}\varphi\varphi+\tfrac{1}{24}\varphi^4-\tfrac{1}{720}\varphi^6+\text{etc.},$$

potestates ipsius φ in numeratore aeque in infinitum exsurgunt atque in denominatore, unde fieri posset, ut haec fractio partem integram involveret; quae cum reperiatur, si sumatur $\varphi=\infty$, foret ista pars integra $=\frac{\cos.\infty}{\sin.\infty}=\cot.\infty$, quae autem in se prorsus est indeterminata. Interim tamen, quia totidem casibus evadere potest negativa atque positiva, medium sumendo valor recte videri potest $=0$; ceterum dubium per sequentem evolutionem tolletur. Cum pro denominatore $\varphi-i\pi$ fiat $A=\cos.i\pi$ et $C=\cos.i\pi$, erit numerator huius fractionis $=1$; hinc ergo nascetur sequens series

$$\frac{\cos.\varphi}{\sin.\varphi}=\frac{1}{\varphi}+\frac{1}{\varphi-\pi}+\frac{1}{\varphi+\pi}+\frac{1}{\varphi-2\pi}+\frac{1}{\varphi+2\pi}+\text{etc.}$$

sive

$$\cot.\varphi=\frac{1}{\varphi}-\frac{1}{\pi-\varphi}+\frac{1}{\pi+\varphi}-\frac{1}{2\pi-\varphi}+\frac{1}{2\pi+\varphi}-\text{etc.}$$

Posito igitur $\varphi=\lambda\pi$ haec series induet hanc formam

$$\pi\cot.\lambda\pi=\frac{1}{\lambda}-\frac{1}{1-\lambda}+\frac{1}{1+\lambda}-\frac{1}{2-\lambda}+\frac{1}{2+\lambda}-\frac{1}{3-\lambda}+\text{etc.};$$

quae summatio an vera sit, per casus investigemus. Ac primo quidem si λ denotet numerum integrum, veritas confirmatur; semper enim aliquis seriei terminus fit infinitus, summa vero quoque fit infinita. Sumamus autem $\lambda=\frac{1}{2}$; erit $\pi\cot.\frac{\pi}{2}=0$, ipsa autem series prodit

$$\frac{2}{1}-\frac{2}{1}+\frac{2}{3}-\frac{2}{3}+\frac{2}{5}-\frac{2}{5}+\text{etc.},$$

ubi omnes termini se manifesto destruunt. Sumamus autem insuper $\lambda=\frac{1}{4}$ prodibitque

$$\pi=\frac{4}{1}-\frac{4}{3}+\frac{4}{5}-\frac{4}{7}+\frac{4}{9}-\frac{4}{11}+\text{etc.},$$

quae est series notissima LEIBNIZIANA. Sicque omne dubium circa veritatem huius summationis evanescit.

19. Contrahamus binos terminas primo excepto in singulos et obtinebimus

$$\pi\cot.\lambda\pi=\tfrac{1}{\lambda}-\tfrac{2\lambda}{1-\lambda\lambda}-\tfrac{2\lambda}{4-\lambda\lambda}-\tfrac{2\lambda}{9-\lambda\lambda}-\tfrac{2\lambda}{16-\lambda\lambda}-\text{etc.},$$

quae series reducitur ad hanc formam

$$\tfrac{1}{2\lambda\lambda}-\tfrac{\pi\cot.\lambda\pi}{2\lambda}=\tfrac{\lambda}{1-\lambda\lambda}+\tfrac{\lambda}{4-\lambda\lambda}+\tfrac{\lambda}{9-\lambda\lambda}+\tfrac{\lambda}{16-\lambda\lambda}+\text{etc.}$$

Quodsi hic iterum statuamus $\lambda=\mu\sqrt{-1}$, ob

$$\cos.\mu\pi\sqrt{-1}=\tfrac{e^{-\mu\pi}+e^{+\mu\pi}}{2}$$

et

$$\sin.\lambda\pi\sqrt{-1}=\tfrac{e^{-\mu\pi}-e^{+\mu\pi}}{2\sqrt{-1}}$$

haec obtinebitur summatio

$$-\tfrac{1}{2\mu\mu}+\tfrac{\pi(e^{+\mu\pi}+e^{-\mu\pi})}{2\mu(e^{+\mu\pi}-e^{-\mu\pi})}=\tfrac{1}{1+\mu\mu}+\tfrac{1}{4+\mu\mu}+\tfrac{1}{9+\mu\mu}+\tfrac{1}{16+\mu\mu}+\text{etc.}$$

Nunc autem per se est manifestum per differentiationem simili modo ut supra infinitas alias summationes obtineri posse.

II. SUMATUR $Q=\cos.\zeta-\cos.\varphi$ UT FRACTIO RESOLVENDA SIT $\frac{P}{\cos.\zeta-\cos.\varphi}$

20. Cum sit denominator $Q=\cos.\zeta-\cos.\varphi$, ubi angulus ζ ut datus et constans spectatur, is sequentibus easibus evanescit

$$\varphi=\pm\zeta,\ \varphi=\pm2\pi\pm\zeta,\ \varphi=\pm4\pi\pm\zeta,$$
$$\varphi=\pm6\pi\pm\zeta,\ \varphi=\pm8\pi\pm\zeta\ \text{etc.}$$

ideoque in genere

$$\varphi=\pm i\pi\pm\zeta,$$

ubi *i* denotat omnes numeros pares tam negativos quam positivos; unde denominatores fractionum simplicium, quas quaerimus, erunt

$$\varphi-\zeta,\ \varphi+\zeta,\ \varphi-2\pi-\zeta,\ \varphi-2\pi+\zeta,\ \varphi+2\pi-\zeta,\ \varphi+2\pi+\zeta\ \text{etc.}$$

hocque modo omnes fractiones simplices reperiemus, quarum omnium summa aequalis esse debet fractioni propositae

$$\frac{P}{\cos.\zeta - \cos.\varphi}.$$

21. Consideremus nunc primo denominatorem simplicem in genere $\varphi - i\pi - \zeta$ ac posito $\varphi = i\pi + \zeta$ abeat numerator P in A. Deinde cum ex denominatore fiat $\frac{dQ}{d\varphi} = \sin.\varphi$, erit $C = \sin(i\pi + \zeta) = \sin.\zeta$ unde numerator huius fractionis erit $\frac{A}{C} = \frac{A}{\sin.\zeta}$ ideoque fractio hinc nata

$$\frac{A}{\sin.\zeta(\varphi - i\pi - \zeta)}.$$

At vero pro denominatore $\varphi - i\pi - \zeta$, si in numeratore P ponatur $\varphi = i\pi - \zeta$, prodit quantitas B; ex denominatore autem fiet $C = \sin(i\pi - \zeta) = -\sin.\zeta$, unde orietur ista fractio

$$-\frac{B}{\sin.\zeta(\varphi - i\pi + \zeta)}.$$

Nunc igitur tantum opus est, ut loco i successive omnes numeri pares tam positivi quam negativi substituantur.

1°. SIT NUMERATOR $P = 1$ ET FRACTIO PROPOSITA $\frac{1}{\cos.\zeta - \cos.\varphi}$

22. Pro binis igitur formulis generalibus erit tam $A = 1$ quam $B = 1$, unde istae fractiones generales erunt

$$\frac{1}{\sin.\zeta(\varphi - i\pi - \zeta)} - \frac{1}{\sin.\zeta(\varphi - i\pi + \zeta)} = \frac{2\zeta}{\sin.\zeta((\varphi - i\pi)^2 - \zeta\zeta)};$$

consequenter hinc deducemus sequentem summationem

$$\frac{1}{\cos.\zeta - \cos.\varphi} = \frac{2\zeta}{\sin.\zeta(\varphi\varphi - \zeta\zeta)} + \frac{2\zeta}{\sin.\zeta((\varphi - 2\pi)^2 - \zeta\zeta)} + \frac{2\zeta}{\sin.\zeta((\varphi + 2\pi)^2 - \zeta\zeta)}$$
$$+ \frac{2\zeta}{\sin.\zeta((\varphi - 4\pi)^2 - \zeta\zeta)} + \frac{2\zeta}{\sin.\zeta((\varphi + 4\pi)^2 - \zeta\zeta)} + \text{etc.}$$

sive habebimus

$$\frac{\sin.\zeta}{2\zeta(\cos.\zeta - \cos.\varphi)} = \frac{1}{(\varphi\varphi - \zeta\zeta)} + \frac{1}{((\varphi - 2\pi)^2 - \zeta\zeta)} + \frac{1}{((\varphi + 2\pi)^2 - \zeta\zeta)}$$
$$+ \frac{1}{((\varphi - 4\pi)^2 - \zeta\zeta)} + \frac{1}{((\varphi + 4\pi)^2 - \zeta\zeta)} + \text{etc.}$$

23. Quodsi ergo fuerit $\zeta = 0$, erit

$$\frac{1}{2 - 2\cos.\varphi} = \frac{1}{\varphi^2} + \frac{1}{(\varphi - 2\pi)^2} + \frac{1}{(\varphi + 2\pi)^2} + \frac{1}{(\varphi - 4\pi)^2} + \frac{1}{(\varphi + 4\pi)^2} + \text{etc.}$$

Sit nunc porro $\varphi = \frac{\pi}{2}$; erit haec summatio

$$\frac{\pi\pi}{8} = 1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \frac{1}{81} + \frac{1}{121} + \text{etc.},$$

uti quidem satis constat. Ponamus porro $\varphi = \pi$ prodibitque haec summatio

$$\frac{\pi\pi}{4} = 1 + 1 + \frac{1}{9} + \frac{1}{9} + \frac{1}{25} + \frac{1}{25} + \frac{1}{49} + \frac{1}{49} + \text{etc.},$$

quae series cum praecedente congruit. Sin autem ponatur $\varphi = \lambda\pi$, erit

$$\frac{\pi\pi}{2(1-\cos.\lambda\pi)} = \frac{1}{\lambda\lambda} + \frac{1}{(\lambda-2)^2} + \frac{1}{(\lambda+2)^2} + \frac{1}{(\lambda-4)^2} + \frac{1}{(\lambda+4)^2} + \text{etc.},$$

quae summa etiam est

$$\frac{\pi\pi}{4(\sin.\frac{1}{2}\lambda\pi)^2}.$$

24. Ponamus autem in genere

$$\zeta = \alpha\pi \text{ et } \varphi = \lambda\pi,$$

ut obtineatur ista summatio

$$\frac{\pi\sin.\alpha\pi}{2\alpha(\cos.\alpha\pi-\cos.\lambda\pi)} = \frac{1}{\lambda\lambda-\alpha\alpha} + \frac{1}{(\lambda-2)^2-\alpha\alpha} + \frac{1}{(\lambda+2)^2-\alpha\alpha} + \frac{1}{(\lambda-4)^2-\alpha\alpha} + \frac{1}{(\lambda+4)^2-\alpha\alpha} + \text{etc.}$$

Quodsi iam α fuerit quantitas imaginaria sive $\alpha = \beta\sqrt{-1}$, summatio haec erit

$$\frac{\pi(e^{+\beta\pi}+e^{-\beta\pi})}{2\beta((e^{+\beta\pi}-e^{-\beta\pi})-2\cos.\lambda\pi)} = \frac{1}{\lambda\lambda+\beta\beta} + \frac{1}{(\lambda-2)^2+\beta\beta} + \frac{1}{(\lambda+2)^2+\beta\beta} + \frac{1}{(\lambda-4)^2+\beta\beta} + \frac{1}{(\lambda+4)^2+\beta\beta} + \text{etc.}$$

25. Hinc si proponatur haec fractio in seriem resolvenda

$$\frac{1}{a-\cos.\varphi} \text{ sive } \frac{1}{a-\cos.\lambda\pi},$$

duos casus considerari oportet, prouti a fuerit vel unitate minor vel maior.
Sit $a < 1$, ut fieri queat $a = \cos.\alpha\pi$; unde fit

$$\alpha = \frac{\text{A}\cos.a}{\pi}$$

atque inventa α reperietur

$$\frac{1}{a-\cos.\lambda\pi}=\frac{2\alpha}{\pi\sqrt{(1-aa)}}\left(\frac{1}{\lambda\lambda-\alpha\alpha}+\frac{1}{(\lambda-2)^2-\alpha\alpha}+\frac{1}{(\lambda+2)^2-\alpha\alpha}+\frac{1}{(\lambda-4)^2-\alpha\alpha}+\text{etc.}\right)$$

Sin autem fuerit $a>1$, quaeri debet β, ut fiat

$$\frac{e^{+\beta\pi}+e^{-\beta\pi}}{2}=a.$$

Hinc ergo fiet $e^{+2\beta\pi}+1=2ae^{+\beta\pi}$, unde radice extracta reperitur

$$e^{+\beta\pi}=a+\sqrt{(aa-1)}$$

hincque

$$e^{-\beta\pi}=a-\sqrt{(aa-1)}\ ,$$

unde porro fiet

$$\beta\pi=l(a+\sqrt{(aa-1)}),$$

ergo

$$\beta=\frac{1}{\pi}l(a+\sqrt{(aa-1)}).$$

Inventa igitur hoc numero β postrema formula praebet hanc seriem

$$\frac{\pi\sqrt{(aa-1)}}{2\beta(a-\cos.\lambda\pi)}=\frac{1}{\lambda\lambda+\beta\beta}+\frac{1}{(\lambda-2)^2+\beta\beta}+\frac{1}{(\lambda+2)^2+\beta\beta}+\frac{1}{(\lambda-4)^2+\beta\beta}+\text{etc.};$$

consequenter habebimus

$$\frac{1}{a-\cos.\lambda\pi}=$$

$$=\frac{2\beta}{\pi\sqrt{(aa-1)}}\left(\frac{1}{\lambda\lambda+\beta\beta}+\frac{1}{(\lambda-2)^2+\beta\beta}+\frac{1}{(\lambda+2)^2+\beta\beta}+\frac{1}{(\lambda-4)^2+\beta\beta}+\text{etc.}\right).$$

Casu autem medio, quo $a=1$, fit $\alpha=0$; tum vero ponatur $a=1-\omega$ eritque

$$\text{Acos.}(1-\omega)=\text{Asin.}\sqrt{(2\omega-\omega\omega)}=\sqrt{(2\omega-\omega\omega)}.$$

Est vero etiam

$$\sqrt{(1-aa)}=\sqrt{(2\omega-\omega\omega)},$$

unde pro hoc casu seriei summatio erit

$$\frac{1}{a-\cos.\lambda\pi}=\frac{2}{\pi\pi}\left(\frac{1}{\lambda\lambda}+\frac{1}{(\lambda-2)^2}+\frac{1}{(\lambda+2)^2}+\frac{1}{(\lambda-4)^2}+\text{etc.}\right).$$

Cum igitur sit

$$1-\cos.\lambda\pi = 2\sin^2.\tfrac{1}{2}\lambda\pi,$$

habebimus hanc summationem

$$\frac{\pi\pi}{4\sin^2.\frac{1}{2}\lambda\pi,} = \frac{1}{\lambda\lambda} + \frac{1}{(\lambda-2)^2} + \frac{1}{(\lambda+2)^2} + \frac{1}{(\lambda-4)^2} + \text{etc.},$$

quae series iam supra § 23 est inventa.

2°. SIT NUNC $P = \sin.\varphi$ ET FRACTIO PROPOSITA $\frac{\sin.\varphi}{\cos.\zeta-\cos.\varphi}$

26. Cum igitur sit $P = \sin.\varphi$, sumto $\varphi = i\pi + \zeta$ erit

$$A = \sin.(i\pi + \zeta) = \sin.\zeta;$$

at posito $\varphi = i\pi - \zeta$ prodit $B = -\sin.\zeta$; hinc binae fractiones inde resultantes erunt

$$\frac{1}{\varphi-i\pi-\zeta} + \frac{1}{\varphi-i\pi+\zeta} = \frac{2\varphi-2i\pi}{(\varphi-i\pi)^2-\zeta\zeta}.$$

Quare si loco *i* successive omnes eius valores scribamus, nanciscemur sequentem seriem:

$$\frac{\sin.\varphi}{\cos.\zeta-\cos.\varphi} = \frac{2\varphi}{\varphi\varphi-\zeta\zeta} + \frac{2(\varphi-2\pi)}{(\varphi-2\pi)^2-\zeta\zeta} + \frac{2(\varphi+2\pi)}{(\varphi+2\pi)^2-\zeta\zeta} + \frac{2(\varphi-4\pi)}{(\varphi-4\pi)^2-\zeta\zeta} + \text{etc.}$$

sive

$$\frac{\sin.\varphi}{2(\cos.\zeta-\cos.\varphi)} = \frac{\varphi}{\varphi\varphi-\zeta\zeta} + \frac{\varphi-2\pi}{(\varphi-2\pi)^2-\zeta\zeta} + \frac{\varphi+2\pi}{(\varphi+2\pi)^2-\zeta\zeta} + \frac{\varphi-4\pi}{(\varphi-4\pi)^2-\zeta\zeta} + \text{etc.}$$

27. Hinc, si fuerit $\zeta = 0$, erit

$$\frac{\sin.\varphi}{2(1-\cos.\varphi)} = \frac{1}{\varphi} + \frac{1}{\varphi-2\pi} + \frac{1}{\varphi+2\pi} + \frac{1}{\varphi-4\pi} + \frac{1}{\varphi+4\pi} + \text{etc.},$$

cuius igitur seriei summa est

$$\tfrac{1}{2}\cot.\tfrac{1}{2}\varphi.$$

Hinc, si ponamus $\varphi = \lambda\pi$, erit

$$\tfrac{1}{2}\pi\cot.\tfrac{1}{2}\lambda\pi = \frac{1}{\lambda} + \frac{1}{\lambda-2} + \frac{1}{\lambda+2} + \frac{1}{\lambda-4} + \frac{1}{\lambda+4} + \text{etc.}$$

et contrahendis binis terminis

$$\tfrac{1}{2}\pi\cot.\tfrac{1}{2}\lambda\pi = \frac{1}{\lambda} + \frac{2\lambda}{\lambda\lambda-4} + \frac{2\lambda}{\lambda\lambda-16} + \frac{2\lambda}{\lambda\lambda-36} + \text{etc.}$$

hincque

$$\frac{1}{2\lambda\lambda}-\frac{1}{2}\frac{\pi\cot.\frac{1}{2}\lambda\pi}{4\lambda}=\frac{1}{4-\lambda\lambda}+\frac{1}{16-\lambda\lambda}+\frac{1}{36-\lambda\lambda}+\text{etc.}$$

Quodsi hic loco λ scribamus 2λ, habebimus

$$\frac{1}{8\lambda\lambda}-\frac{\pi\cot.\lambda\pi}{8\lambda}=\frac{1}{4-4\lambda\lambda}+\frac{1}{16-4\lambda\lambda}+\frac{1}{36-4\lambda\lambda}+\text{etc.}$$

sive

$$\frac{1}{2\lambda\lambda}-\frac{\pi\cot.\lambda\pi}{2\lambda}=\frac{1}{1-\lambda\lambda}+\frac{1}{4-\lambda\lambda}+\frac{1}{9-\lambda\lambda}+\frac{1}{16-\lambda\lambda}+\text{etc.}$$

quae series est plane eadem, quam supra § 19 invenimus.

28. Ponamus nunc ut supra

$$\zeta=\alpha\pi \text{ et } \varphi=\lambda\pi,$$

ut obtineatur sequens series:

$$\frac{\pi\sin.\lambda\pi}{2(\cos.\alpha\pi-\cos.\lambda\pi)}=\frac{\lambda}{(\lambda\lambda-\alpha\alpha)}+\frac{\lambda-2}{(\lambda-2)^2-\alpha\alpha}+\frac{\lambda+2}{(\lambda+2)^2-\alpha\alpha}+\frac{\lambda-4}{(\lambda-4)^2-\alpha\alpha}+\text{etc.}$$

Sin autem hic ponatur $\alpha=\beta\sqrt{-1}$, ista series sequentem induet formam:

$$\frac{\pi\sin.\lambda\pi}{e^{+\beta\pi}+e^{-\beta\pi}-2\cos.\lambda\pi}=\frac{\lambda}{\lambda\lambda+\beta\beta}+\frac{\lambda-2}{(\lambda-2)^2+\beta\beta}+\frac{\lambda+2}{(\lambda+2)^2+\beta\beta}+\frac{\lambda-4}{(\lambda-4)^2+\beta\beta}+\text{etc.}$$

29. Quodsi igitur proposita fuerit haec fractio

$$\frac{\sin.\varphi}{a-\cos.\varphi} \text{ sive } \frac{\sin.\lambda\pi}{a-\cos.\lambda\pi},$$

iterum duos casus evolvi convenit, alterum, quo $a<1$, alterum, quo $a>1$.
Priore quidem casu, quo $a<1$, statuatur $\cos.\alpha\pi=a$, unde fit

$$\alpha=\frac{\text{Acos.}a}{\pi},$$

quo invente erit

$$\frac{\sin.\lambda\pi}{a-\cos.\lambda\pi}=\frac{2}{\pi}\left(\frac{\lambda}{\lambda\lambda-\alpha\alpha}+\frac{\lambda-2}{(\lambda-2)^2-\alpha\alpha}+\frac{\lambda+2}{(\lambda+2)^2-\alpha\alpha}+\text{etc.}\right).$$

Sin autem $a>1$, quaeri debet β, ita ut sit ut ante

$$\beta=\frac{1}{\pi}l(a+\sqrt{(aa-1)}),$$

quo valere invente erit

$$\frac{\sin.\lambda\pi}{a-\cos.\lambda\pi}=\frac{2}{\pi}\left(\frac{\lambda}{\lambda\lambda+\beta\beta}+\frac{\lambda-2}{(\lambda-2)^2+\beta\beta}+\frac{\lambda+2}{(\lambda+2)^2+\beta\beta}+\text{etc.}\right).$$

Sin autem fuerit $a=1$, tum fit tam $\alpha=0$ quam $\beta=0$ eademque series resultat, quam supra ex casu $\zeta=0$ elicuimus. Hinc ergo, si sumatur $\lambda=\frac{1}{2}$, prodibit series

$$\frac{\pi}{2\cos.\alpha\pi}=\frac{2}{1-4\alpha\alpha}-\frac{6}{9-4\alpha\alpha}+\frac{10}{25-4\alpha\alpha}-\frac{14}{49-4\alpha\alpha}+\text{etc.}$$

vel etiam haec

$$\frac{\pi}{e^{+\beta\pi}-e^{-\beta\pi}}=\frac{2}{1+4\beta\beta}-\frac{6}{9+4\beta\beta}+\frac{10}{25+4\beta\beta}-\frac{14}{49+4\beta\beta}+\text{etc.}$$

Ceterum per se intelligitur per differentiationem plurimas alias series formari posse.

III. SIT FRACTIO RESOLVENDA $\frac{1}{\cos.\varphi-\cos.2\varphi}$.

30. Ante omnia igitur hic quaeri debet, quibusnam casibus iste denominator evanescat. Cum igitur in genere sit $\cos.\varphi=\cos.(i\pi\pm\varphi)$ denotante *i* numerum parem similique modo $\cos.2\varphi=\cos.(i'\pi\pm 2\varphi)$, habebimus $i\pi\pm\varphi=i'\pi\pm 2\varphi$, unde ob ambiguitatem signorum sequentes casus eruuntur:

$$\varphi=i\pi,\quad \varphi=\frac{i\pi}{3}.$$

Hic autem probe est observandum casus priores bis occurrere seu factores hinc natos $\varphi-i\pi$ bis esse collocandos, ita ut factor denominatoris sit $(\varphi-i\pi)^2$. Quod cum minus clare appareat, ita ostendamus: quoniam in genere est

$$\cos.a-\cos.b=2\sin.\tfrac{a+b}{2}.\sin.\tfrac{b-a}{2},$$

erit noster denominator $2\sin.\frac{1}{2}\varphi\sin.\frac{3}{2}\varphi$, qui igitur evanescit , tam quando $\sin.\frac{1}{2}\varphi=0$, quam quando $\sin.\frac{3}{2}\varphi=0.$ Fit autem $\sin.\frac{1}{2}\varphi=0$, quoties $\frac{\varphi}{2}=i\pi$ denotante *i* omnes numeros integros ideoque $\varphi=2i\pi$. Similique modo $\sin.\frac{3\varphi}{2}$ evanescit, si $\frac{3\varphi}{2}=i\pi$ ideoque $\varphi=\frac{2i\pi}{3}$, quae posterior formula, quoties $\frac{i}{3}$ est numerus integer, priores casus suppeditat, sicque manifestum est in factoribus occurrere omnia quadrata $(\varphi-i\pi)^2$. Reliqui vero factores $\varphi-\frac{2i\pi}{3}$, quando *I* per 3 non est divisibile, erunt simplices.

31. Cum igitur formula $(\varphi-2i\pi)^2$ sit factor nostri denominatoris $\cos.\varphi-\cos.2\varphi$, secundum regulam pro huiusmodi casibus statuamus

$$\frac{1}{\cos.\varphi-\cos.2\varphi}=\frac{\alpha}{(\varphi-2i\pi)^2}+\frac{\beta}{\varphi-2i\pi}+R,$$

ubi R complectitur omnes reliquas fractiones. Nunc utrinque multiplicemus per $(\varphi-2i\pi)^2$ et habebimus

$$\frac{(\varphi-2i\pi)^2}{\cos.\varphi-\cos.2\varphi}=\alpha+\beta(\varphi-2i\pi)+R(\varphi-2i\pi)^2.$$

Faciamus $\varphi=2i\pi$ fietque

$$\alpha=\frac{(\varphi-2i\pi)^2}{\cos.\varphi-\cos.2\varphi},$$

cuius fractionis numerator et denominator evanescent; hinc differentialibus substitutis fiet

$$\alpha=\frac{2(\varphi-2i\pi)}{-\sin.\varphi+2\sin.2\varphi};$$

ubi cum numerator et denominator iterum evanescant, denuo eorum loco differentialia scribantur eritque

$$\alpha=\frac{2}{-\cos.\varphi+4\cos.2\varphi}.$$

Nunc igitur posito $\varphi=2i\pi$ reperietur $\alpha=\frac{2}{3}$.

32. Iam in aequatione

$$\frac{(\varphi-2i\pi)^2}{\cos.\varphi-\cos.2\varphi}=\alpha+\beta(\varphi-2i\pi)+R(\varphi-2i\pi)^2.$$

terminus $\alpha=\frac{2}{3}$ ad alteram partem transferatur et ad eandem denominationem reducatur et resultabit haec aequatio

$$\frac{(\varphi-2i\pi)^2-\frac{2}{3}(\cos.\varphi-\cos.2\varphi)}{\cos.\varphi-\cos.2\varphi}=\beta(\varphi-2i\pi)+R(\varphi-2i\pi)^2,$$

unde per $\varphi-2i\pi$ dividendo fiet

$$\frac{(\varphi-2i\pi)^2-\frac{2}{3}(\cos.\varphi-\cos.2\varphi)}{(\varphi-2i\pi)(\cos.\varphi-\cos.2\varphi)}=\beta+R(\varphi-2i\pi),$$

Quod si iam statuatur $\varphi=2i\pi$, β aequabitur fractioni, cuius tam numerator quam denominator ter evanescit, ita ut triplici differentiatione sit opus.

Prima autem differentiatio dabit

$$\beta = \frac{2(\varphi-2i\pi)+\frac{2}{3}(\sin.\varphi-2\sin.2\varphi)}{\cos.\varphi-\cos.2\varphi-(\varphi-2i\pi)(\sin.\varphi-2\sin.2\varphi)}.$$

Secunda differentiatio dabit

$$\beta = \frac{2+\frac{2}{3}(\cos.\varphi-4\cos.2\varphi)}{-2\sin.\varphi+4\sin.2\varphi-(\varphi-2i\pi)(\cos.\varphi-4\cos.2\varphi)}.$$

Tertia denique differentiatio dat

$$\beta = \frac{-\frac{2}{3}(\sin.\varphi-8\sin.2\varphi)}{-3\cos.\varphi+12\cos.2\varphi+(\varphi-2i\pi)(\sin.\varphi-8\sin.2\varphi)}.$$

Nunc autem facto $\varphi = 2i\pi$ numerator quidem iterum evanescit, denominator vero evadit 9, ita ut sit $\beta = 0$.

33. At vero iste valor pro β sine differentiatione facilius erui potest ponendo $\varphi = 2i\pi + \omega$ existente ω infinite parvo; tum autem erit

$$\cos.\varphi = \cos.\omega \text{ et } \cos.2\varphi = \cos.2\omega;$$

aequatio autem fiet

$$\frac{\omega\omega}{\cos.\omega-\cos.2\omega} = \frac{2}{3} + \beta\omega + R\omega\omega.$$

Nunc ambos cosinus proxima exhibeamus usque ad quartam potestatem ipsius ω procedendo, et cum sit

$$\cos.\omega = 1 - \frac{1}{2}\omega\omega + \frac{1}{24}\omega^4$$

et

$$\cos.2\omega = 1 - 2\omega\omega + \frac{16}{24}\omega^4,$$

erit

$$\cos.\omega - \cos.2\omega = \frac{3}{2}\omega\omega - \frac{5}{8}\omega^4 = \frac{3}{2}\omega\omega\left(1 - \frac{5}{12}\omega\omega\right),$$

quo valore substituto habebimus

$$\frac{2}{3\omega\omega\left(1-\frac{5}{12}\omega\omega\right)} = \frac{2}{3}(1 + \frac{5}{12}\omega\omega) = \frac{2}{3} + \beta\omega + R\omega\omega,$$

hincque fit $\beta = \frac{2}{3}\cdot\frac{5}{12}\omega$; sicque facto $\omega = 0$ erit etiam $\beta = 0$.

34. Hanc ob rem pro denominatoris factore quadrato $(\varphi - 2i\pi)^2$ ob $\alpha = \frac{2}{3}$ fractio inde nata erit

$$\frac{2}{3(\varphi-2i\pi)^2}.$$

Pro reliquis autem factoribus simplicibus $\varphi-\frac{2}{3}i\pi$ statuamus

$$\frac{1}{\cos.\varphi-\cos.2\varphi}=\frac{\alpha}{\varphi-\frac{2}{3}i\pi}+R,$$

quae aequatio multiplicetur per $\varphi-\frac{2}{3}i\pi=\omega$, ut prodeat

$$\frac{\omega}{\cos.\varphi-\cos.2\varphi}=\alpha+R\omega.$$

Ubi notetur numerum i non esse per 3 divisibilem, unde $\frac{2i\pi}{3}$ sequentes angulos exprimet:

$$\tfrac{2}{3}\pi,\ \tfrac{4}{3}\pi,\ \tfrac{8}{3}\pi,\ \tfrac{10}{3}\pi,\ \tfrac{14}{3}\pi \text{ etc.;}$$

at anguli $\frac{4i\pi}{3}$ valores sunt

$$\tfrac{4}{3}\pi,\ \tfrac{8}{3}\pi,\ \tfrac{16}{3}\pi,\ \tfrac{20}{3}\pi,\ \tfrac{28}{3}\pi \text{ etc.,}$$

quorum angulorum cosinus est idem $-\frac{1}{2}$. Sinus autem horum angulorum sunt

$$\sin.\tfrac{2i\pi}{3}=\pm\frac{\sqrt{3}}{2},$$

ubi signum superius valet, si i sit $3n+1$, inferius vero, si fuerit $i=3n+2$; at vero $\sin.\frac{4i\pi}{3}$ semper est $\mp\frac{\sqrt{3}}{2}$, ubi iterum signum superius valet, si $i=3n+1$, inferius vero, si $i=3n+2$. Haecque regula semper valet, sive n sit numerus positivus sive negativus.

35. His praenotatis erit

$$\cos.\varphi=-\tfrac{1}{2}\cos.\omega\mp\frac{\sqrt{3}}{2}\sin.\omega$$

et

$$\cos.2\varphi=-\tfrac{1}{2}\cos.2\omega\mp\frac{\sqrt{3}}{2}\sin.2\omega,$$

unde vero proxime habebimus

$$\cos.\varphi=-\tfrac{1}{2}\left(1-\tfrac{1}{2}\omega\omega\right)\mp\frac{\sqrt{3}}{2}\omega$$

et

$$\cos.2\varphi = -\tfrac{1}{2}\left(1-\tfrac{1}{2}\omega\omega\right) \pm \tfrac{\sqrt{3}}{2}\cdot 2\omega,$$

ubi perpetuo signa superiora valent, si $i = 3n+1$, inferiora autem, si $i = 3n+\ 2$. Hinc igitur erit noster denominator

$$\cos.\varphi - \cos.2\varphi = -\tfrac{3}{4}\omega\omega \mp \tfrac{3\sqrt{3}}{2}\omega,$$

unde fit

$$\frac{1}{-\frac{3}{4}\omega \mp \frac{3\sqrt{3}}{2}} = \alpha.$$

Posito igitur $\omega = 0$ erit

$$\alpha = \mp \tfrac{2}{3\sqrt{3}},$$

ita ex factore $\varphi - \frac{2i\pi}{3}$ nascatur ista fractio

$$\mp \tfrac{2}{3\sqrt{3}} \cdot \frac{1}{\varphi - \frac{2i\pi}{3}} = \mp \tfrac{2}{(3\varphi - 2i\pi)\sqrt{3}}.$$

36. Evolvamus igitur primo omnes terminos seriei ex factoribus geminatis $(\varphi - 2i\pi)^2$ natos, et cum numerator fuisset $\frac{2}{3}$, si loco *i* successive omnes scribamus numeros integros tam positives quam negatives, series orietur sequens:

$$\tfrac{2}{3\varphi\varphi} + \tfrac{2}{3(\varphi-2\pi)^2} + \tfrac{2}{3(\varphi+2\pi)^2} + \tfrac{2}{3(\varphi-4\pi)^2} + \tfrac{2}{3(\varphi+4\pi)^2} + \tfrac{2}{3(\varphi-6\pi)^2} + \text{etc.}$$

Pro altera serie sit primo $i = 3n+1$ hincque fractio fiet

$$-\tfrac{2}{(3\varphi - 2(3n+1)\pi)\sqrt{3}};$$

sin autem sit $i = -3n-1$, signum inferius valebit et fractio erit

$$+\tfrac{2}{(3\varphi + 2(3n+1)\pi)\sqrt{3}},$$

qui duo termini contracti praebent

$$-\tfrac{8(3n+1)\pi}{(9\varphi\varphi - 4(3n+1)^2\pi\pi)\sqrt{3}};$$

sin autem fuerit $i = 3n+2$, tum vero etiam $i = -3n-2$, binae fractiones in unam contractae praebent

$$+\frac{8(3n+2)\pi}{(9\varphi\varphi-4(3n+1)^2\pi\pi)\sqrt{3}}.$$

Quare cum valores negatives ipsius *i* iam simus complexi, loco *n* tantum omnes numeros positivos 0, 1, 2, 3, 4, 5 etc. poni oportet, unde sequens resultabit series:

$$-\frac{8\pi}{(9\varphi\varphi-4\pi\pi)\sqrt{3}}-\frac{8\cdot 4\pi}{(9\varphi\varphi-4\cdot 16\pi\pi)\sqrt{3}}-\frac{8\cdot 7\pi}{(9\varphi\varphi-4\cdot 49\pi\pi)\sqrt{3}}-\text{etc.}$$
$$+\frac{8\pi}{(9\varphi\varphi-16\pi\pi)\sqrt{3}}+\frac{8\cdot 5\pi}{(9\varphi\varphi-4\cdot 25\pi\pi)\sqrt{3}}+\frac{8\cdot 8\pi}{(9\varphi\varphi-4\cdot 64\pi\pi)\sqrt{3}}+\text{etc.}$$

37. Proposita igitur fractio

$$\frac{1}{\cos.\varphi-\cos.2\varphi}$$

resolvitur in has duas series

$$\frac{2}{3}\left(\frac{1}{\varphi^2}+\frac{1}{(\varphi-2\pi)^2}+\frac{1}{(\varphi+2\pi)^2}+\frac{1}{(\varphi-4\pi)^2}+\frac{1}{(\varphi+4\pi)^2}+\text{etc}\right),$$
$$-\frac{8\pi}{\sqrt{3}}\left(\frac{1}{9\varphi\varphi-4\cdot 1^2\pi\pi}+\frac{4}{9\varphi\varphi-4\cdot 4^2\pi\pi}+\frac{7}{9\varphi\varphi-4\cdot 7^2\pi\pi}+\frac{10}{9\varphi\varphi-4\cdot 10^2\pi\pi}+\text{etc.}\right)$$
$$+\frac{8\pi}{\sqrt{3}}\left(\frac{2}{9\varphi\varphi-4\cdot 2^2\pi\pi}+\frac{5}{9\varphi\varphi-4\cdot 5^2\pi\pi}+\frac{8}{9\varphi\varphi-4\cdot 8^2\pi\pi}+\frac{11}{9\varphi\varphi-4\cdot 11^2\pi\pi}+\text{etc.}\right)$$

Hinc ergo si faciamus ut supra $\varphi=\lambda\pi$, erit fractio

$$\frac{\pi\pi}{\cos.\lambda\pi-\cos.2\lambda\pi}=$$
$$\frac{2}{3}\left(\frac{1}{\lambda^2}+\frac{1}{(\lambda-2\pi)^2}+\frac{1}{(\lambda+2\pi)^2}+\frac{1}{(\lambda-4\pi)^2}+\frac{1}{(\lambda+4\pi)^2}+\text{etc}\right),$$
$$-\frac{8\pi}{\sqrt{3}}\left(\frac{1}{9\lambda\lambda-4\cdot 1^2\pi\pi}+\frac{4}{9\lambda\lambda-4\cdot 4^2\pi\pi}+\frac{7}{9\lambda\lambda-4\cdot 7^2\pi\pi}+\frac{10}{9\lambda\lambda-4\cdot 10^2\pi\pi}+\text{etc.}\right)$$
$$+\frac{8\pi}{\sqrt{3}}\left(\frac{2}{9\lambda\lambda-4\cdot 2^2\pi\pi}+\frac{5}{9\lambda\lambda-4\cdot 5^2\pi\pi}+\frac{8}{9\lambda\lambda-4\cdot 8^2\pi\pi}+\frac{11}{9\lambda\lambda-4\cdot 11^2\pi\pi}+\text{etc.}\right)$$

38. Ut exemplum afferamus, sit $\lambda=\frac{1}{3}$, ut fiat $9\lambda\lambda=1$, ac prodibit ista summatio

$$\pi\pi =$$
$$\frac{2}{3}\left(\frac{9}{1^2}+\frac{9}{5^2}+\frac{9}{7^2}+\frac{9}{11^2}+\frac{9}{13^2}+\frac{9}{17^2}+\text{etc}\right)$$
$$+\frac{8\pi}{\sqrt{3}}\left(\frac{1}{4\cdot1^2-1}+\frac{4}{4\cdot4^2-1}+\frac{7}{4\cdot7^2-1}+\frac{10}{4\cdot10^2-1}+\frac{13}{4\cdot13^2-1}+\text{etc.}\right)$$
$$-\frac{8\pi}{\sqrt{3}}\left(\frac{2}{4\cdot2^2-1}+\frac{5}{4\cdot5^2-1}+\frac{8}{4\cdot8^2-1}+\frac{11}{4\cdot11^2-1}+\frac{14}{4\cdot14^2-1}+\text{etc.}\right),$$

quae summatio etiam hoc modo referri potest

$$\pi\pi = 6\left(\frac{1}{1^2}+\frac{1}{5^2}+\frac{1}{7^2}+\frac{1}{11^2}+\frac{1}{13^2}+\frac{1}{17^2}+\frac{1}{19^2}+\text{etc}\right)$$
$$+\frac{4\pi}{\sqrt{3}}\left(\frac{2}{2^2-1}-\frac{4}{4^2-1}+\frac{8}{8^2-1}-\frac{10}{10^2-1}+\frac{14}{14^2-1}-\frac{16}{16^2-1}+\text{etc.}\right);$$

elegantior autem forma erit sequens:

$$\pi\pi = 6\left(\frac{1}{1^2}+\frac{1}{5^2}+\frac{1}{7^2}+\frac{1}{11^2}+\frac{1}{13^2}+\frac{1}{17^2}+\frac{1}{19^2}+\text{etc.}\right)$$
$$+\frac{2\pi}{\sqrt{3}}\left(1-\tfrac{1}{5}+\tfrac{1}{7}-\tfrac{1}{11}+\tfrac{1}{13}-\tfrac{1}{17}+\tfrac{1}{19}-\text{etc.}\right).$$

39. Quoniam hoc casu occurrerunt factores quadrati, etiam eiusmodi fractiones resolvere poterimus, quarum denominatores ipsi sunt quadrati ideoque meros factores simplices quadratos involvunt. Atque adeo hanc resolutionem extendere licebit ad denominatores cubicos altiorumque potestatum, si modo in subsidium vocentur ea praecepta, quae pro huiusmodi resolutionibus olim dedi.

IV. SIT FRACTIO RESOLVENDA PROPOSITA $\frac{1}{\sin^2.\varphi}$

40. Cum igitur hic omnes factores quadrati denominatoris in hac forma contineantur

$$\frac{1}{(\varphi-i\pi)^2}$$

denotante *i* omnes numeros integras tam positivos quam negativos, ponamus pro resolutione generali

$$\frac{1}{\sin^2.\varphi}=\frac{\alpha}{(\varphi-i\pi)^2}+\frac{\beta}{\varphi-i\pi}+R,$$

ubi *R* complectitur omnes fractiones reliquas. Hinc per $(\varphi-i\pi)^2$ multiplicando erit

$$\frac{(\varphi-i\pi)^2}{\sin^2.\varphi}=\alpha+\beta(\varphi-i\pi)+R(\varphi-i\pi)^2.$$

Iam fiat $\varphi=i\pi$, et quoniam hoc casu numerator ac denominator nostrae fractionis evanescunt, statuamus $\varphi-i\pi=\omega$ eritque

$$\sin.\varphi=\sin.(i\pi+\omega)=\sin.i\pi\cos.\omega+\sin.\omega\cos.i\pi=\pm\sin.\omega$$

ob $\sin.i\pi=0$ et $\cos.i\pi=\pm1$; ubi signum superius valet, si *i* sit numerus par, inferius vero, si impar, quod tamen discrimen hic non in censum venit, cum sit $\sin^2.\varphi=\sin^2\omega$. Hinc igitur erit

$$\frac{\omega\omega}{\sin^2.\omega}=\alpha+\beta\omega+R\omega\omega.$$

Cum igitur sit

$$\sin.\omega=\omega-\tfrac{1}{6}\omega^3=\omega(1-\tfrac{1}{6}\omega\omega),$$

erit

$$\frac{1}{(1-\frac{1}{6}\omega\omega)^2}=1+\tfrac{1}{3}\omega\omega=\alpha+\beta\omega+R\omega\omega,$$

unde fit statim $\alpha=1$. Tum vero aequatio erit

$$\tfrac{1}{3}\omega=\beta+R\omega,$$

sicque facto $\omega=0$ fit $\beta=0$; consequenter ex denominatoris factore $(\varphi-i\pi)^2$ oritur haec fractio $\frac{1}{(\varphi-i\pi)^2}$.

41. Tribuantur nunc ipsi *i* omnes valores debiti ac reperietur haec series

$$\frac{1}{\sin^2.\varphi}=\frac{1}{\varphi\varphi}+\frac{1}{(\varphi-\pi)^2}+\frac{1}{(\varphi+\pi)^2}+\frac{1}{(\varphi-2\pi)^2}+\frac{1}{(\varphi+2\pi)^2}+\frac{1}{(\varphi-3\pi)^2}+\text{etc.};$$

quae quidem series deduci potuisset ex §18, ubi invenimus

$$\frac{\cos.\varphi}{\sin.\varphi}=\frac{1}{\varphi}+\frac{1}{\varphi-\pi}+\frac{1}{\varphi+\pi}+\frac{1}{\varphi-2\pi}+\frac{1}{\varphi+2\pi}+\text{etc.},$$

unde per differentiationem signis mutatis ea ipsa oritur series, quam hic invenimus.

42. Quodsi fractio fuisset proposita

$$\frac{\cos^2.\varphi}{\sin^2.\varphi}$$

et eodem modo resolutio instituatur, ob

$$\cos.(i\pi+\omega)=\pm\cos.\omega$$

ideoque

$$\cos^2.\varphi=\cos^2.\omega=1-\omega\omega,$$

quoniam secundae potestates ipsius ω non in computum veniunt, numerator foret ut in casu praecedente $=1$ ideoque eadem plane series prodiisset, id quod utique foret absurdum. Supra autem iam animadvertimus huiusmodi resolutiones veritati non esse consentaneas, nisi quantitas variabilis φ in numeratore pauciores habeat dimensiones quam in denominatore, quia alioquin praeter seriem fractionum partes integrae essent accessurae, id quod hoc casu manifesta evenit, cum sit

$$\frac{\cos^2.\varphi}{\sin^2.\varphi}=\frac{1}{\sin^2.\varphi}-1,$$

ita ut pars integra hoc casu sit $=-1$.

V. SIT FRACTIO RESOLVENDA $\frac{1}{\sin^3.\varphi}$

43. Pro hoc ergo casu poni oportebit

$$\frac{1}{\sin^3.\varphi}=\frac{\alpha}{(\varphi-i\pi)^3}+\frac{\beta}{(\varphi-i\pi)^2}+\frac{\gamma}{\varphi-i\pi}+R.$$

Ponamus nunc iterum $\varphi=i\pi+\omega$, et cum sit

$$\frac{1}{\sin^3.\varphi}=\pm\frac{1}{\omega^3}\left(1+\frac{1}{2}\omega\omega\right),$$

ubi ratio signorum legem supra datam servet, haec resultat aequatio, postquam per ω^3 fuerit multiplicata,

$$\pm\frac{\omega^3(1+\frac{1}{2}\omega\omega)}{\omega^3}=\alpha+\beta\omega+\gamma\omega\omega+R\omega^3=\pm(1+\tfrac{1}{2}\omega\omega),$$

unde manifesta fit $\alpha=\pm1$, tum vero

$$\beta+\gamma\omega+R\omega\omega=\pm\tfrac{1}{2}\omega$$

sicque erit $\beta=0$ et $\gamma=\pm\frac{1}{2}$. Hoc igitur modo ex denominatoris factore cubico $(\varphi-i\pi)^3$ nascentur hae duae fractiones

$$\pm\frac{1}{(\varphi-i\pi)^3}\pm\frac{1}{2(\varphi-i\pi)}.$$

44. Tribuamus igitur litterae *i* successive omnes valores tam positivos quam negativos atque obtinebimus sequentem resolutionem

$$\frac{1}{\sin^3.\varphi}=\frac{1}{\varphi^3}-\frac{1}{(\varphi-\pi)^3}-\frac{1}{(\varphi+\pi)^3}+\frac{1}{(\varphi-2\pi)^3}+\frac{1}{(\varphi+2\pi)^3}-\text{etc.}$$
$$+\frac{1}{2\varphi}-\frac{1}{2(\varphi-\pi)}-\frac{1}{2(\varphi+\pi)}+\frac{1}{2(\varphi-2\pi)}+\frac{1}{2(\varphi+2\pi)}-\text{etc.}$$

Hic observasse iuvabit inferiorem seriem iam supra [§ 5] in primo exemplo esse inventam; unde intelligimus fore summam huius seriei

$$=\frac{1}{2\sin.\varphi};$$

quamobrem series superior cuborum sola aequabitur huic formulae

$$\frac{1}{\sin^3.\varphi}-\frac{1}{2\sin.\varphi}.$$

45. Egregie hoc quoque convenit cum principiis supra stabilitis, ex quibus per differentiationem continuo alias novas series eruere docuimus. Cum enim sit

$$\frac{1}{\sin.\varphi}=\frac{1}{\varphi}-\frac{1}{\varphi-\pi}-\frac{1}{\varphi+\pi}+\frac{1}{\varphi-2\pi}+\frac{1}{\varphi+2\pi}-\text{etc.},$$

hinc deducitur differentiando

$$-\frac{\cos.\varphi}{\sin^2.\varphi}=-\frac{1}{\varphi^2}+\frac{1}{(\varphi-\pi)^2}+\frac{1}{(\varphi+\pi)^2}-\frac{1}{(\varphi-2\pi)^2}-\frac{1}{(\varphi+2\pi)^2}+\text{etc.},$$

atque hinc denuo differentiando

$$\frac{1}{\sin.\varphi}+\frac{2\cos^2.\varphi}{\sin^3.\varphi}=\frac{2}{\varphi^3}-\frac{2}{(\varphi-\pi)^3}-\frac{2}{(\varphi+\pi)^3}+\frac{2}{(\varphi-2\pi)^3}+\frac{2}{(\varphi+2\pi)^3}-\text{etc.},$$

quae reducitur ad hanc formam

$$\frac{2}{\sin^3.\varphi}-\frac{1}{\sin.\varphi},$$

id quod egregie convenit cum valore praecedente.

VI. SIT FRACTIO PROPOSITA RESOLVENDA $=\frac{1}{\text{tang}.\varphi-\sin.\varphi}$

46. Denominator iste $\text{tang}.\varphi-\sin.\varphi$ manifesto evanescit casibus, quibus $\varphi=i\pi$ denotante *i* omnes numeros integros tam positivos quam negativos, unde fractiones simplices, quarum denominatores continent istum factorem $\varphi-i\pi$, evadunt infiniti casu

$\varphi = i\pi$, dum reliquae fractiones retinent valorem finitum. Haecque consideratio nobis aperit novam methodum omnes fractiones simplices investigandi. Pro quovis enim tali factore evanescente quaeratur valor ipsius fractionis propositae; qui cum fiat infinitus, ei aequales esse debebunt ii seriei termini, qui eodem casu evadunt infiniti. Hanc ob rem statui oportebit $\varphi - i\pi = \omega$ denotante ω angulum infinite parvum. Quo facto fractio proposita fiet certa quaedam functio ipsius ω, quam secundum eius dimensiones evolvi conveniet.

47. Hanc igitur ideam sequentes duos casus distinguera debemus, prouti i fuerit numerus vel par vel impar, quoniam priore casu fit $\sin.\varphi = \sin.\omega$, posteriore vero casu fit $\sin.\varphi = -\sin.\omega$, dum utroque casu manet

$$\text{tang.}\varphi = \text{tang.}\omega.$$

Sit igitur primo i numerus impar et erit casu $\varphi = i\pi$ nostra fractio

$$= \frac{1}{\text{tang.}\omega + \sin.\omega}.$$

Est vero proxime

$$\text{tang.}\omega = \omega + \tfrac{1}{3}\omega^3 \text{ et } \sin.\omega = \omega - \tfrac{1}{6}\omega^3,$$

unde ista fractio fiet

$$\frac{1}{2\omega + \frac{1}{6}\omega^3} = \frac{1}{2\omega(1 + \frac{1}{12}\omega\omega)} = \frac{1}{2\omega}(1 - \tfrac{1}{12}\omega\omega).$$

Haec iam expressio sponte praebet has duas fractiones $\frac{1}{2\omega} - \frac{1}{24}\omega$, unde ob $\omega = \varphi - i\pi$ pro isto factore oritur haec fractio simplex

$$\frac{1}{2(\varphi - i\pi)},$$

quia altera pars evanescit. Quare si nunc loco i ordine scribamus numeros impares, sequentem fractionum seriem adipiscemur

$$\frac{1}{2(\varphi-\pi)} + \frac{1}{2(\varphi+\pi)} + \frac{1}{2(\varphi-3\pi)} + \frac{1}{2(\varphi+3\pi)} + \frac{1}{2(\varphi-5\pi)} + \text{etc.}$$

48. Sit nunc etiam i numerus par, unde fit

$$\text{tang.}\varphi = \text{tang.}\omega \text{ et } \sin.\varphi = \sin.\omega;$$

hinc fractio nostra erit

$$\frac{1}{\text{tang.}\omega - \sin.\omega}$$

ubi facta evolutione primi termini ω se tollunt, ita ut in hoc denominatore infima potestas ipsius ω futura sit ω^3. Atque ob hanc causam approximationem ulterius continuari oportet quam casu praecedente. Hunc in finem loco tang.ω scribamus

$$\frac{\sin.\omega}{\cos.\omega},$$

ut fractio nostra sit

$$\frac{\cos.\omega}{\sin.\omega - \sin.\omega\cos.\omega}.$$

Cum iam sit

$$\sin.\omega\cos.\omega = \tfrac{1}{2}\sin.2\omega,$$

erit per series

$$\sin.\omega = \omega - \tfrac{1}{6}\omega^3 + \tfrac{1}{120}\omega^5$$

et

$$\sin.2\omega = 2\omega - \tfrac{8}{6}\omega^3 + \tfrac{32}{120}\omega^5,$$

unde totus denominator erit

$$+\tfrac{1}{2}\omega^3 - \tfrac{1}{8}\omega^5 = \tfrac{1}{2}\omega^3(1 - \tfrac{1}{4}\omega\omega);$$

numerator vero est $\cos.\omega = 1 - \frac{1}{2}\omega\omega$, unde tota fractio nostra erit

$$\frac{1-\frac{1}{2}\omega\omega}{\frac{1}{2}\omega^3(1-\frac{1}{4}\omega\omega);} = \frac{1-\frac{1}{4}\omega\omega}{\frac{1}{2}\omega^3};$$

hincque partes resultantes erunt

$$\frac{2}{\omega^3} - \frac{1}{2\omega},$$

quae ambae casu $\omega = 0$ fiunt infinitae. Facile autem patet, si approximationem ulterius extendissemus, in sequenti termino litteram ω iam in numeratorem transituram fuisse. Scribatur igitur $\varphi - i\pi$ loco ω et partes ex hoc factore denominatoris oriundae erunt

$$\frac{2}{(\varphi-i\pi)^3} - \frac{1}{2(\varphi-i\pi)},$$

unde loco *i* successive omnes numeros pares scribendo ista prodibit series geminata

$$\frac{2}{\varphi^3} + \frac{2}{(\varphi-2\pi)^3} + \frac{2}{(\varphi+2\pi)^3} + \frac{2}{(\varphi-4\pi)^3} + \frac{2}{(\varphi+4\pi)^3} + \frac{2}{(\varphi-6\pi)^3} + \text{etc.}$$
$$-\frac{1}{2\varphi} - \frac{1}{2(\varphi-2\pi)} - \frac{1}{2(\varphi+2\pi)} - \frac{1}{2(\varphi-4\pi)} - \frac{1}{2(\varphi+4\pi)} - \text{etc}$$

49. Iungamus igitur has series ex utroque casu deductas et fractio proposita

$$\frac{1}{\text{tang.}\varphi - \sin.\varphi}$$

resolvi reperitur in ternas series sequentes:

$$\frac{1}{2(\varphi-\pi)}+\frac{1}{2(\varphi+\pi)}+\frac{1}{2(\varphi-3\pi)}+\frac{1}{2(\varphi+3\pi)}+\frac{1}{2(\varphi-5\pi)}+\text{etc.},$$
$$-\frac{1}{2\varphi}-\frac{1}{2(\varphi-2\pi)}-\frac{1}{2(\varphi+2\pi)}-\frac{1}{2(\varphi-4\pi)}-\frac{1}{2(\varphi+4\pi)}-\text{etc.},$$
$$+\frac{1}{2\varphi^3}+\frac{1}{2(\varphi-2\pi)^3}+\frac{1}{2(\varphi+2\pi)^3}+\frac{1}{2(\varphi-4\pi)^3}+\frac{1}{2(\varphi+4\pi)^3}+\text{etc.}$$

50. Quilibet hic facile sentiet istam methodum non parum antecellere illi, qua ante usi sumus, quandoquidem hoc modo statim fractiones ex quolibet denominatoris factore oriundas nacti sumus neque opus fuerat earum numeratores per litteras indefinitas designare. Praeterea etiam hac ratione non opus erat sollicite inquirere, quoties singuli factores simplices in denominatore contineantur, siquidem nostra methodus hoc sponte declarat.

51. In huiusmodi autem seriebus generalibus, ubi quorundam terminorum denominatores certo casu evanescunt ideoque hi termini in infinitum excrescunt, quaeri solet, his terminis sublatis quanta futura sit summa reliquorum terminorum. Ita pro casu, quo i est numerus impar, terminus

$$\frac{1}{2(\varphi-i\pi)}$$

fit infinitus casu $\varphi=i\pi$. Hoc igitur termino deleto quaeritur, quanta futura sit summa reliquorum terminorum casu $\varphi=i\pi$. Ad hanc quaestionem solvendam ponatur $\varphi-i\pi=\omega$ atque ex § 47 patet fore

$$\frac{1}{2\omega}-\frac{1}{24}\omega=\frac{1}{2(\varphi-i\pi)}+R,$$

ubi R complectitur omnes reliquos terminas, quorum summa desideratur casu $\varphi=i\pi$. Transferatur igitur terminus

$$\frac{1}{2(\varphi-i\pi)}=\frac{1}{2\omega}$$

in alteram partem ac statim elucet fore

$$R=-\frac{1}{24}\omega=0 \text{ ob } \omega=0,$$

ita ut omisso termina illo infinito summa omnium reliquorum casu $\varphi=i\pi$ semper sit 0.

52. Quando autem i est numerus par, eadem conclusio locum habebit, ad quod ostendendum necesse est approximationem adhibitam ulterius continuare. Tum autem erit numerator

$$\cos.\omega = 1 - \tfrac{1}{2}\omega\omega + \tfrac{1}{24}\omega^4;$$

pro denominatore vero

$$\sin.\omega = \omega - \tfrac{1}{6}\omega^3 + \tfrac{1}{120}\omega^5 - \tfrac{1}{5040}\omega^7$$

et

$$\sin.2\omega = 2\omega - \tfrac{8}{6}\omega^3 + \tfrac{32}{120}\omega^5 - \tfrac{128}{5040}\omega^7,$$

unde fit ipse denominator

$$\tfrac{1}{2}\omega^3 - \tfrac{1}{8}\omega^5 - \tfrac{1}{80}\omega^7 = \tfrac{1}{2}\omega^3\left(1 - \tfrac{1}{4}\omega\omega + \tfrac{1}{40}\omega^4\right);$$

hinc factor posterior in numeratorem translatus praebet

$$1 + \tfrac{1}{4}\omega\omega + \tfrac{3}{40}\omega^4$$

hincque tota fractio iam erit

$$\frac{1 - \frac{1}{4}\omega\omega - \frac{11}{240}\omega^4}{\frac{1}{2}\omega^3},$$

quae aequari debet toti seriei posito $\varphi = i\pi$, hoc est terminis inventis

$$\frac{2}{(\varphi - i\pi)^3} - \frac{1}{2(\varphi - i\pi)}$$

cum omnibus reliquis *R,* unde elicitur $R = -\frac{11}{120}\omega = 0;$ unde patet etiam his casibus summam omnium reliquorum esse $= 0$.

53. Quodsi ergo sumamus $\varphi = 0$ et terminos m infinitum excrescentes deleamus, termini remanentes erunt

$$-\frac{1}{2\pi} + \frac{1}{2\pi} - \frac{1}{6\pi} + \frac{1}{6\pi} - \frac{1}{10\pi} + \frac{1}{10\pi} - \text{etc.}$$
$$+\frac{1}{4\pi} - \frac{1}{4\pi} + \frac{1}{8\pi} - \frac{1}{8\pi} + \frac{1}{12\pi} - \frac{1}{12\pi} + \text{etc.}$$
$$-\frac{2}{8\pi^3} + \frac{2}{8\pi^3} - \frac{2}{64\pi^3} + \frac{2}{64\pi^3} - \frac{2}{216\pi^3} + \frac{2}{216\pi^3} - \text{etc.},$$

ubi omnes termini manifesto se tollunt, id quod etiam omnibus reliquis casibus, quibus ponitur $\varphi = i\pi$, contingit.

54. Sin autem binos terminas contiguos contraxissemus, hae series prodiissent

$$-\frac{1}{2\varphi}+\frac{\varphi}{\varphi\varphi-\pi\pi}-\frac{\varphi}{\varphi\varphi-4\pi\pi}+\frac{\varphi}{\varphi\varphi-9\pi\pi}-\frac{\varphi}{\varphi\varphi-16\pi\pi}+\frac{\varphi}{\varphi\varphi-25\pi\pi}-\text{etc.},$$
$$+\frac{1}{2\varphi^3}+\frac{4\varphi(\varphi\varphi+3\cdot 4\pi\pi)}{(\varphi\varphi-4\pi\pi)^3}+\frac{4\varphi(\varphi\varphi+3\cdot 16\pi\pi)}{(\varphi\varphi-16\pi\pi)^3}+\frac{4\varphi(\varphi\varphi+3\cdot 36\pi\pi)}{(\varphi\varphi-36\pi\pi)^3}+\text{etc.},$$

quarum serierum summa est

$$\frac{1}{\text{tang.}\,\omega-\sin.\omega}.$$

Quodsi nunc hic ponamus $\varphi=0$ sive $\varphi=\omega$, quoniam omnes termini sunt divisibiles per $\varphi=0$ eorumque autem summa inventa est $-\frac{11}{120}\omega$, si utrinque per ω dividamus, summa erit $-\frac{11}{120}$; ipsae autem series evadent

$$-\frac{1}{\pi\pi}+\frac{1}{4\pi\pi}-\frac{1}{9\pi\pi}+\frac{1}{16\pi\pi}-\frac{1}{25\pi\pi}+\text{etc.},$$
$$-\frac{3\cdot 4}{(4\pi\pi)^2}-\frac{3\cdot 4}{(16\pi\pi)^2}-\frac{3\cdot 4}{(36\pi\pi)^2}-\frac{3\cdot 4}{(64\pi\pi)^2}-\text{etc.}$$

55. Mutatis igitur signis et reductis terminis ad formam simplicissimam impetrabimus hanc summationem

$$\frac{11}{120}=\frac{1}{\pi\pi}\left(1-\frac{1}{4}+\frac{1}{9}-\frac{1}{16}+\frac{1}{25}-\frac{1}{36}+\text{etc.}\right)$$
$$+\frac{3}{4\pi^4}\left(1+\frac{1}{4^2}+\frac{1}{9^2}+\frac{1}{25^2}+\frac{1}{36^2}+\text{etc.}\right).$$

Notum autem est esse

$$1-\frac{1}{4}+\frac{1}{9}-\frac{1}{16}+\frac{1}{25}-\frac{1}{36}+\text{etc.}=\frac{\pi\pi}{12}$$

et

$$1+\frac{1}{4^2}+\frac{1}{9^2}+\frac{1}{25^2}+\frac{1}{36^2}+\text{etc.}=\frac{\pi^4}{90},$$

unde haec aequalitas manifesta in oculos incurrit.