CONCERNING THE SUM OF THE SERIES OF THE FORM

$$\frac{1}{3}-\frac{1}{5}+\frac{1}{7}-\frac{1}{11}-\frac{1}{17}+\frac{1}{19}+\frac{1}{23}-\frac{1}{29}+\frac{1}{31}-\text{etc.},$$

WHERE PRIME NUMBERS OF THE FORM $4n-1$ HAVE A POSITIVE SIGN,

AND $4n+1$ A NEGATIVE SIGN

[E596]

Opuscula analytica 2, 1785, p. 240-256

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1. Just as Euclid had shown the number of prime numbers actually to be infinite, now some time ago I have shown the sum of the reciprocals of the prime numbers to be infinite also, as it were

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{11}+\frac{1}{17}+\frac{1}{19}+\text{etc.},$$

to be an infinite magnitude, and thus a logarithm to refer to the sum of the harmonic series [*i.e.* $\ln(1+x)$ with $x=-1$;]

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\text{etc.},$$

as that may be considered quite wonderful, since generally the sum of the harmonic series may be accustomed to refer, as it were, to the smallest kind of infinite quantities. However, since not only shall the logarithm of an infinite number also be infinite, but also the logarithms of these themselves even now shall be infinite, evidently there is to be given in addition an infinitude of lesser orders of infinity. Thus, so that if A may denote the sum of the series of the reciprocals of the prime numbers, also lA besides will be an infinite magnitude, but it is required to be considered to belong to a lesser order of infinitudes ; then truly even now these formulas : llA, $lllA$, $llllA$ etc. will be infinite, though any of these infinite quantities shall be less than the preceding one.

2. Again besides since the prime numbers, except of two, may be distinguished as if naturally into two classes, precisely as if they were either of the form $4n+1$ or of the form $4n-1$, while all the former are the sum of two squares, the latter truly are completely excluded by this property; clearly the reciprocal series formed from each class :

$$\frac{1}{5}+\frac{1}{13}+\frac{1}{17}+\frac{1}{29}+\text{etc. and } \frac{1}{3}+\frac{1}{7}+\frac{1}{11}+\frac{1}{19}+\frac{1}{23}+\text{etc.},$$

both will be equally infinite, that which also is required to be held concerning all kinds of prime numbers. Thus so that if only these may be selected from the prime numbers, which are of the form $100n+1$, 101, 401, 601, 701 etc. shall be of this kind, not only the number of these is infinite, but also the sum of this series formed from these, evidently :

$$\frac{1}{101}+\frac{1}{401}+\frac{1}{601}+\frac{1}{701}+\frac{1}{1201}+\frac{1}{1301}+\frac{1}{1601}+\frac{1}{1801}+\frac{1}{1901}+\text{etc.},$$

also is infinite.

3. But in the first place, we will discriminate here between prime numbers of the form $4n+1$ et $4n-1$, and since both series from each order are infinite, and as if of the same order, there is no doubt, why the differences of these may not have a determined value. On this account we may attribute the + sign to terms of the form $4n-1$, truly the − sign to the remainder, so that this series will arise:

$$\frac{1}{3}-\frac{1}{5}+\frac{1}{7}-\frac{1}{11}-\frac{1}{13}-\frac{1}{17}+\frac{1}{19}+\frac{1}{23}-\frac{1}{29}+\frac{1}{31}-\text{etc.},$$

in which no order in the signs is discerned. Yet we may not consider this perhaps a hindrance, why truly its sum may not be able to be assigned approximately. Indeed it may appear most probable this same sum, if it were neither rational nor irrational, at least must belong to some notable kind of transcending quantity. Yet meanwhile, since not only in terms of the signs, but truly also much less for the fractions themselves plainly no order is evident, in the first place no way may seem to be apparent from observation, by which this sum may be permitted to be arrived at.

4. But when we contemplate the most noteworthy series of Leibnitz given for the quadrature of the circle

$$\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\text{etc.},$$

we see in that all the odd numbers of the form $4n+1$ to have the sign +, truly the rest of the form $4n-1$ the sign −, from which with the signs changed all the terms of our series occur with their signs in that series of Leibnitz. So that if therefore we may remove all the composite terms from that series, finally except for unity our series itself will remain, with the contrary signs. On account of which we will obtain the sum of our series, if from the Liebnitz series we may exclude successively all the composite numbers, when indeed of all the terms, which by whatever operation are excluded, the sum will be easy to assign.

5. Therefore we may begin from that Leibnitz series itself, by putting in place

$$A=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\frac{1}{13}-\frac{1}{15}+\frac{1}{17}-\frac{1}{19}+\text{etc.},$$

thus so that there shall be $A=\frac{\pi}{4}$, and from that in the first place we may exclude all the composite numbers divisible by 3. Which in the end hence we may form into this series :

$$(A-1)\tfrac{1}{3}=-\tfrac{1}{9}+\tfrac{1}{15}-\tfrac{1}{21}+\tfrac{1}{27}-\tfrac{1}{33}+\tfrac{1}{39}-\text{etc.},$$

which certainly contains all the composite numbers divisible by 3, from which if this series may be added to that, all these composite numbers will be excluded and there will become

$$\tfrac{4}{3}A-\tfrac{1}{3}=1-\tfrac{1}{3}+\tfrac{1}{5}-\tfrac{1}{7}-\tfrac{1}{11}+\tfrac{1}{13}+\tfrac{1}{17}-\tfrac{1}{19}-\tfrac{1}{23}+\tfrac{1}{25}+\text{etc.}=B,$$

thus so that there shall be $B=\tfrac{4}{3}A-\tfrac{1}{3}$. Therefore in this series, of which we know the sum, no further composite numbers divisible by 3 occur, but the first composite term occurring here is $+\tfrac{1}{25}$.

6. Now therefore from this latter series we may exclude all the terms divisible by 5, which in the end we will form into this series :

$$(B-1+\tfrac{1}{3})\tfrac{1}{5}=\tfrac{1}{25}-\tfrac{1}{35}-\tfrac{1}{55}+\tfrac{1}{65}+\tfrac{1}{85}-\tfrac{1}{95}-\text{etc.},$$

which includes all the terms divisible by five, which hitherto were entering into the series B , and indeed associated with the same signs, whereby with this series taken from that one this will remain :

$$\tfrac{4}{5}B+\tfrac{1}{5}(1-\tfrac{1}{3})=1-\tfrac{1}{3}+\tfrac{1}{5}-\tfrac{1}{7}-\tfrac{1}{11}+\tfrac{1}{13}+\tfrac{1}{17}-\tfrac{1}{19}-\tfrac{1}{23}+\text{etc.},$$

where now from the beginning all the terms are prime numbers, and the first which is not, which will occur here will be $\tfrac{1}{49}$, the following truly $\tfrac{1}{77},-\tfrac{1}{91}$. But we may put the sum of this series $=C$, so that there shall be :

$$C=\tfrac{4}{5}B+\tfrac{1}{5}(1-\tfrac{1}{3}).$$

7. Now therefore we may exclude all the terms, which hitherto were divisible by seven, which the following form will include:

$$(C-1+\tfrac{1}{3}-\tfrac{1}{5})\tfrac{1}{7}=-\tfrac{1}{49}-\tfrac{1}{77}+\tfrac{1}{91}+\text{etc.},$$

of which the signs of the terms are the opposite, whereby this series added to the series C will give :

$$\tfrac{8}{7}C-\tfrac{1}{7}(1-\tfrac{1}{3}+\tfrac{1}{5})=1-\tfrac{1}{3}+\tfrac{1}{5}-\tfrac{1}{7}-\tfrac{1}{11}+\tfrac{1}{13}+\tfrac{1}{17}-\text{etc.},$$

in which the first composite term will be $+\tfrac{1}{121}$, the following truly $-\tfrac{1}{11\cdot13},-\tfrac{1}{11\cdot17}$ etc. Moreover we may call this series *D,* so that there shall be:

$$D = \tfrac{8}{7}C - \tfrac{1}{7}(1 - \tfrac{1}{3} + \tfrac{1}{5}).$$

8. Now from this series in the manner found we may remove the terms D, which hitherto have been divisible by 11 , which this form will include:

$$(D - (1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7})\tfrac{1}{11}) = -\tfrac{1}{121} + \tfrac{1}{143} + \tfrac{1}{187} - \text{etc.},$$

which terms have the opposite signs in the series D ; on account of which if this series may be added to that, these same terms will be excluded and there will be produced :

$$\tfrac{12}{11}D - \tfrac{1}{11}(1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7}) = 1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} - \tfrac{1}{11} + \text{etc.},$$

in which the first non prime is $\tfrac{1}{169}$; but we will designate that same series by the letter *E,* thus so that there shall become:

$$E = \tfrac{12}{11}D - \tfrac{1}{11}(1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7}).$$

9. Therefore from this series we may exclude all the terms, which hitherto are divisible by 13, which therefore will be contained by this form:

$$(E - 1 + \tfrac{1}{3} - \tfrac{1}{5} + \tfrac{1}{7} + \tfrac{1}{11})\tfrac{1}{13} = \tfrac{1}{169} + \tfrac{1}{121} - \text{etc.},$$

and these terms have the same sign as in the same series E. Therefore this series must be subtracted from that, from which there is produced

$$\tfrac{12}{13}E + \tfrac{1}{13}(1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} - \tfrac{1}{11}) = 1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} - \tfrac{1}{11} + \text{etc.},$$

where the first non-prime is $\tfrac{1}{289}$. Moreover we will designate this whole series by the letter *F,* so that there shall be :

$$F = \tfrac{12}{13}E + \tfrac{1}{13}(1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} - \tfrac{1}{11}).$$

10. So that if now we may continue these operations further, then successively hence we exclude the terms hitherto divisible by 17, then truly by 19, by 23 etc., finally there will be left only the series of prime numbers following after unity, which if it may be designated by the letter *Z,* which as it will be required to consider the infinitesimal, certainly there will be

$$Z = 1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} - \tfrac{1}{11} + \tfrac{1}{13} + \tfrac{1}{17} - \tfrac{1}{19} - \text{etc.},$$

consequently the sum of the series proposed in the title will be $1-Z$. And it is evident for these formulas to approach closer to this value :

$$1-A,\ 1-B,\ 1-C,\ 1-D,\ 1-E,\ 1-F \text{ etc.}$$

11. But just as the values of all these letters must be deduced successively from the preceding ones, it will become evident from the following formulas:

$$B=\tfrac{4}{3}A-\tfrac{1}{3}\cdot 1$$
$$C=\tfrac{4}{5}B-\tfrac{1}{5}(1-\tfrac{1}{3})$$
$$D=\tfrac{8}{7}C-\tfrac{1}{7}(1-\tfrac{1}{3}+\tfrac{1}{5})$$
$$E=\tfrac{12}{11}D-\tfrac{1}{11}(1-\tfrac{1}{3}+\tfrac{1}{5}-\tfrac{1}{7})$$
$$F=\tfrac{12}{13}E+\tfrac{1}{13}(1-\tfrac{1}{3}+\tfrac{1}{5}-\tfrac{1}{7}-\tfrac{1}{11})$$
$$G=\tfrac{16}{17}F+\tfrac{1}{17}(1-\tfrac{1}{3}+\tfrac{1}{5}-\tfrac{1}{7}-\tfrac{1}{11}+\tfrac{1}{13})$$
$$H=\tfrac{20}{19}G-\tfrac{1}{19}(1-\tfrac{1}{3}+\tfrac{1}{5}-\tfrac{1}{7}-\tfrac{1}{11}+\tfrac{1}{13}+\tfrac{1}{17})$$
$$I=\tfrac{24}{23}H-\tfrac{1}{23}(1-\tfrac{1}{3}+\tfrac{1}{5}-\tfrac{1}{7}-\tfrac{1}{11}+\tfrac{1}{13}+\tfrac{1}{17}-\tfrac{1}{19})$$

etc.

Where it is required to be observed, if the first denominator were of the form $4n+1$, then the numerator of the first part to become less by one, or $4n$, truly the other part must be added. But if the first denominator were $4n-1$, then the numerator of the first part becomes greater by one, or $4n$, truly the other part in this case must be subtracted.

12. So that now we may express all these values in decimal fractions, first of all there may be observed to be

$$A=\tfrac{\pi}{4}=0,7853981634.$$

But for the remaining letters the following values will have to be computed :

$$1-\tfrac{1}{3} = b = 0,6666666666$$
$$1-\tfrac{1}{3}+\tfrac{1}{5} = c = 0,8666666666$$
$$1-\tfrac{1}{3}+\tfrac{1}{5}-\tfrac{1}{7} = d = 0,7238095238$$
$$1-\tfrac{1}{3}+\tfrac{1}{5}-\tfrac{1}{7}-\tfrac{1}{11} = e = 0,6329004329$$
$$1-\tfrac{1}{3}+\tfrac{1}{5}-\tfrac{1}{7}-\tfrac{1}{11}+\tfrac{1}{13} = f = 0,7098235098$$
$$G1-\tfrac{1}{3}+\tfrac{1}{5}-\tfrac{1}{7}-\tfrac{1}{11}+\tfrac{1}{13}+\tfrac{1}{17} = g = 0,7686470392$$
$$1-\tfrac{1}{3}+\tfrac{1}{5}-\tfrac{1}{7}-\tfrac{1}{11}+\tfrac{1}{13}+\tfrac{1}{17}-\tfrac{1}{19} = h = 0,7160154603$$
$$1-\tfrac{1}{3}+\tfrac{1}{5}-\tfrac{1}{7}-\tfrac{1}{11}+\tfrac{1}{13}+\tfrac{1}{17}-\tfrac{1}{19}-\tfrac{1}{23} = i = 0,6725371994,$$

in which order the first term a is equal to unity.

13. But for the computation of these letters *A, B, C, D, E* etc. it will be better to use the following formulas, from which likewise we may ascribe the numerical values of these letters :

$$B = A+\tfrac{1}{3}(A-a) = 0,713864$$
$$C = B-\tfrac{1}{5}(B-b) = 0,704424$$
$$D = C+\tfrac{1}{7}(C-c) = 0,681247$$
$$E = D+\tfrac{1}{11}(D-d) = 0,677377$$
$$F = E-\tfrac{1}{13}(E-e) = 0,673956$$
$$G = F-\tfrac{1}{17}(F-f) = 0,676066$$
$$H = G+\tfrac{1}{19}(G-g) = 0,671193$$
$$I = \mathrm{H}+\tfrac{1}{23}(H-h) = 0,669245$$
$$K = I-\tfrac{1}{29}(I-i) = 0,669358\ .$$

14. But though we have produced the calculation as far as here, yet we are unable to be sure concerning the sum of our series beyond the third decimal figure, and thus we are forced to remain in doubt, whether this same sum shall be a little greater or smaller than 0,669. But if we assume this value to be true, that same series proposed

$$\tfrac{1}{3}-\tfrac{1}{5}+\tfrac{1}{7}+\tfrac{1}{11}-\tfrac{1}{13}-\tfrac{1}{17}+\tfrac{1}{19}+\text{etc.}$$

will have the sum 0,331, and thus here the value will become a little smaller than $\frac{1}{3}$. Because truly with $\frac{1}{5}$ removed, again there must be added $\frac{1}{7}+\frac{1}{11}$, of which fraction the sum is greater than $\frac{1}{5}$, certainly it may be able to happen, so that it may increase towards the value $\frac{1}{3}$, that which in this place is required to remain in doubt. Truly another much more accurate method may be given of enquiring into the sum of this series, than we have set out here, since it may seem to be worth the effort to have got to know the true sum of this series more closely.

15. By the same method, by which we have expelled successively composite terms here from the first Leibnitz series, clearly if we may remove all the terms except unity, we will find

$$\frac{\pi}{4}=\frac{3}{4}\cdot\frac{5}{4}\cdot\frac{7}{8}\cdot\frac{11}{12}\cdot\frac{13}{12}\cdot\frac{17}{16}\cdot\frac{19}{20}\cdot\text{etc.},$$

where in the numerators all the prime numbers occur except 2, truly the denominators equally are numbers greater or less by unity. Then truly if this same series of reciprocal odd squares :

$$1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\frac{1}{9^2}+\frac{1}{11^2}+\frac{1}{13^2}+\text{etc.},$$

of which the sum has been shown to be $=\frac{\pi\pi}{8}$, may be treated in the same way, there will be found

$$\frac{\pi\pi}{8}=\frac{3\cdot3}{2\cdot4}\cdot\frac{5\cdot5}{4\cdot6}\cdot\frac{7\cdot7}{6\cdot8}\cdot\frac{11\cdot11}{10\cdot12}\cdot\frac{13\cdot13}{12\cdot14}\cdot\text{etc.},$$

where again all the prime numbers occur in the numerators twice, in the denominators truly the same either increased or diminished by one. Whereby if we may divide this expression by the square of that, which is

$$\frac{\pi\pi}{16}=\frac{3\cdot3}{4\cdot4}\cdot\frac{5\cdot5}{4\cdot4}\cdot\frac{7\cdot7}{8\cdot8}\cdot\frac{11\cdot11}{12\cdot12}\cdot\frac{13\cdot13}{12\cdot12}\cdot\text{etc.},$$

the quotient will be

$$2=\frac{4}{2}\cdot\frac{4}{6}\cdot\frac{8}{6}\cdot\frac{12}{10}\cdot\frac{12}{14}\cdot\text{etc.},$$

where all the prime numbers occur increased or decreased by unity, constituted equally with even numbers in the numerator, truly unequally with even numbers in the denominator.

16. This latter expression therefore will be able to be shown in this way :

$$2=\frac{3+1}{3-1}\cdot\frac{5-1}{5+1}\cdot\frac{7+1}{7-1}\cdot\frac{11+1}{11-1}\cdot\frac{13-1}{13+1}\cdot\text{etc.};$$

hence therefore with the logarithms taken we will have:

$$l2=l\frac{3+1}{3-1}+l\frac{5-1}{5+1}+l\frac{7+1}{7-1}+l\frac{11+1}{11-1}+l\frac{13-1}{13+1}+\text{etc.}$$

But it may be established in general by the infinite series

$$\frac{1}{2}l\frac{a+1}{a-1}=\frac{1}{a}+\frac{1}{3a^3}+\frac{1}{5a^5}+\frac{1}{7a^7}+\frac{1}{9a^9}+\text{etc.}$$

and hence :

$$\frac{1}{2}l\frac{a-1}{a+1}=-\frac{1}{a}-\frac{1}{3a^3}-\frac{1}{5a^5}-\frac{1}{7a^7}-\frac{1}{9a^9}-\text{etc.}$$

Therefore so that if with the aid of these formulas we may convert these logarithms into infinite series, indeed we will obtain innumerable infinite series, but which will be permitted to be reduced to easily handled series.

17. Initially therefore it will be required to take half of all these logarithms, and because here it is acted on by hyperbolic logarithms, on account of

$$l2=0,6931471805$$

there will be

$$\frac{1}{2}l2=0,3465735902,$$

but from the other part the logarithms may be put in order thus :

$$\begin{aligned}
\frac{1}{2}l\frac{3+1}{3-1}&=\quad\frac{1}{3}+\frac{1}{3\cdot3^3}+\frac{1}{5\cdot3^5}+\frac{1}{7\cdot3^7}+\frac{1}{9\cdot3^9}+\text{etc.}\\
\frac{1}{2}l\frac{5-1}{5+1}&=-\frac{1}{5}-\frac{1}{3\cdot5^3}-\frac{1}{5\cdot5^5}-\frac{1}{7\cdot5^7}-\frac{1}{9\cdot5^9}-\text{etc.}\\
\frac{1}{2}l\frac{7+1}{7-1}&=\quad\frac{1}{7}+\frac{1}{3\cdot7^3}+\frac{1}{5\cdot7^5}+\frac{1}{7\cdot7^7}+\frac{1}{9\cdot7^9}+\text{etc.}\\
\frac{1}{2}l\frac{11+1}{11-1}&=\quad\frac{1}{11}+\frac{1}{3\cdot11^3}+\frac{1}{5\cdot11^5}+\frac{1}{7\cdot11^7}+\frac{1}{9\cdot11^9}+\text{etc.}\\
\frac{1}{2}l\frac{13-1}{13+1}&=-\frac{1}{13}-\frac{1}{3\cdot13^3}-\frac{1}{5\cdot13^5}-\frac{1}{7\cdot13^7}-\frac{1}{9\cdot13^9}-\text{etc.}
\end{aligned}$$

etc.

18. Hence also by descending vertically we will consider the following equally infinite series :

$$\begin{aligned}
O&=\frac{1}{3}-\frac{1}{5}+\frac{1}{7}+\frac{1}{11}-\frac{1}{13}-\frac{1}{17}+\frac{1}{19}+\text{etc.}\\
P&=\frac{1}{3^3}-\frac{1}{5^3}+\frac{1}{7^3}+\frac{1}{11^3}-\frac{1}{13^3}-\frac{1}{17^3}+\frac{1}{19^3}+\text{etc.}\\
Q&=\frac{1}{3^5}-\frac{1}{5^5}+\frac{1}{7^5}+\frac{1}{11^5}-\frac{1}{13^5}-\frac{1}{17^5}+\frac{1}{19^5}+\text{etc.}\\
R&=\frac{1}{3^7}-\frac{1}{5^7}+\frac{1}{7^7}+\frac{1}{11^7}-\frac{1}{13^7}-\frac{1}{17^7}+\frac{1}{19^7}+\text{etc.}\\
S&=\frac{1}{3^9}-\frac{1}{5^9}+\frac{1}{7^9}+\frac{1}{11^9}-\frac{1}{13^9}-\frac{1}{17^9}+\frac{1}{19^9}+\text{etc.}
\end{aligned}$$

etc.

The first of which series O is that itself, of which the sum has been proposed here by us to investigate.

19. Therefore with these series designated by capital letters we will have this same equation :

$$\tfrac{1}{2}l2 = O + \tfrac{1}{3}P + \tfrac{1}{5}Q + \tfrac{1}{7}R + \tfrac{1}{9}S + \tfrac{1}{11}T + \text{etc.}$$

from which if the sums of the series *P, Q, R, S* etc. shall be known, thence we may obtain easily the sum of the series O sought; indeed it will be

$$O = \tfrac{1}{2}l2 - \tfrac{1}{3}P - \tfrac{1}{5}Q - \tfrac{1}{7}R - \tfrac{1}{9}S - \text{etc.}$$

20. But truly the sums of the series *P, Q, R* etc. from the ordered series, where all the odd numbers occur, we will be able to infer in the same manner, where above we have elicited that same O from the Liebnitz series

$$1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \tfrac{1}{9} - \tfrac{1}{11} + \text{etc.}$$

To this end it will be required to establish by this method the following ordered series :

$$\mathfrak{P} = 1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \frac{1}{9^3} - \frac{1}{11^3} + \frac{1}{13^3} + \frac{1}{15^3} + \text{etc.}$$
$$\mathfrak{Q} = 1 - \frac{1}{3^5} + \frac{1}{5^5} - \frac{1}{7^5} + \frac{1}{9^5} - \frac{1}{11^5} + \frac{1}{13^5} + \frac{1}{15^5} + \text{etc.}$$
$$\mathfrak{R} = 1 - \frac{1}{3^7} + \frac{1}{5^7} - \frac{1}{7^7} + \frac{1}{9^7} - \frac{1}{11^7} + \frac{1}{13^7} - \frac{1}{15^7} + \text{etc.}$$
$$\mathfrak{S} = 1 - \frac{1}{3^9} + \frac{1}{5^9} - \frac{1}{7^9} + \frac{1}{9^9} - \frac{1}{11^9} + \frac{1}{13^9} - \frac{1}{15^9} + \text{etc.}$$
$$\mathfrak{T} = 1 - \frac{1}{3^{11}} + \frac{1}{5^{11}} - \frac{1}{7^{11}} + \frac{1}{9^{11}} - \frac{1}{11^{11}} + \frac{1}{13^{11}} - \frac{1}{15^{11}} + \text{etc.}$$
etc.

But I gave the sums of all these series now some time ago by the quadrature of the circle [See Euler's *Foundations of Differential Calculus*, p.542, in this series of translations.] , clearly expressed by like powers of π , in the following manner :

$$\mathfrak{P} = \frac{1}{1\cdot 2} \cdot \frac{\pi^3}{2^4} \qquad \mathfrak{T} = \frac{50521}{1....10} \cdot \frac{\pi^{11}}{2^{12}}$$
$$\mathfrak{Q} = \frac{5}{1\cdot 2\cdot 3\cdot 4} \cdot \frac{\pi^5}{2^6} \qquad \mathfrak{U} = \frac{2702765}{1.......12} \cdot \frac{\pi^{13}}{2^{14}}$$
$$\mathfrak{R} = \frac{61}{1...6} \cdot \frac{\pi^7}{2^8} \qquad \mathfrak{V} = \frac{199360981}{1..........14} \cdot \frac{\pi^{15}}{2^{16}}$$
$$\mathfrak{S} = \frac{1385}{1.....8} \cdot \frac{\pi^9}{2^{10}} \qquad \mathfrak{W} = \frac{19391512145}{1...............16} \cdot \frac{\pi^{17}}{2^{18}}$$
etc.

21. We may set out these values in decimal fractions as far as to the sixth figure, and there will be :

	Differences
$\mathfrak{P} = 0,9689462$	0,0272116
$\mathfrak{Q} = 0,9961578$	0,0033969
$\mathfrak{R} = 0,9995547$	0,0003952
$\mathfrak{S} = 0,9999499$	0,0000448
$\mathfrak{T} = 0,9999947$	0,0000050
$\mathfrak{U} = 0,9999997$	0,0000005
etc.	etc.

22. So that now hence we may elicit the values of the letters *P, Q, R* etc., we may use the same method, by which above from the series

$$1-\tfrac{1}{3}+\tfrac{1}{5}-\tfrac{1}{7}+\tfrac{1}{9}-\tfrac{1}{11}+\text{etc.},$$

we may eliminate all the composite terms, since in place of these numbers it is agreed to write the powers of these simple numbers. Therefore we may show this operation in general for these letters. Therefore we may consider this series :

$$\mathfrak{Z}=1-\frac{1}{3^n}+\frac{1}{5^n}-\frac{1}{7^n}+\frac{1}{9^n}-\frac{1}{11^n}+\text{etc.},$$

of which the sum, as has been established above , we may designate by the letter A , so that there shall be $A=\mathfrak{Z}$, and hence we may elicit the following letters *B, C, D* etc. by the following formulas:

$$B=A+\tfrac{1}{3^n}(A-a) \text{ with there being } a=1$$

$$C=B+\tfrac{1}{5^n}(B-b) \qquad b=1-\tfrac{1}{3^n}$$

$$D=C+\tfrac{1}{7^n}(D-d) \qquad c=1-\tfrac{1}{3^n}+\tfrac{1}{5^n}$$

$$E=D+\tfrac{1}{11^n}(D-d) \qquad d=1-\tfrac{1}{3^n}+\tfrac{1}{5^n}-\tfrac{1}{7^n}$$

$$F=E-\tfrac{1}{13^n}(E-e) \qquad e=1-\tfrac{1}{3^n}+\tfrac{1}{5^n}-\tfrac{1}{7^n}+\tfrac{1}{9^n}-\tfrac{1}{11^n}$$

etc. etc.

With which values found the complements of these to unity, evidently $1-A,\ 1-B,\ 1-C,\ 1-D$ etc., will approach most readily to the value sought

$$Z=\frac{1}{3^n}-\frac{1}{5^n}+\frac{1}{7^n}+\frac{1}{11^n}-\frac{1}{13^n}-\text{etc.}$$

23. Therefore we will apply these general precepts initially to the value of the letter *P,* from which beginning there will be from the value

$$\mathfrak{P} = 0,9689462 = A,$$

and since here there is $n = 3$, we will have

$$a = 1,\ b = 0,9629630,\ c = 0,9709630,\ d = 0,9680476;$$

there will be no need for more values. Hence therefore we may deduce the following values :

$$B = A - \tfrac{1}{3^3} \cdot 0,0310538 = 0,9677961$$
$$C = B - \tfrac{1}{5^3} \cdot 0,0048331 = 0,9677574$$
$$D = C - \tfrac{1}{7^3} \cdot 0,0032056 = 0,9677481$$
$$E = D - \tfrac{1}{11^3} \cdot 0,0002995 = 0,9677479.$$

There is no need to proceed further ; hence on account of which we will have

$$P = 1 - E = 0,0322521,$$

from which we deduce now :

$$\tfrac{1}{2} l2 - \tfrac{1}{3} \cdot 0,0322521 = 0,3358229.$$

24. Now we may assume $n = 5$ and we will have :

$$A = \mathfrak{Q} = 0,9961578,$$

then truly there will be:

$$a = 1,\ b = 0,9958847,\ c = 0,9962048,\ d = 0,9961453,$$

hence therefore we will find :

$$B = A - \tfrac{1}{3^5} \cdot 0,0038422 = 0,9961420$$
$$C = B - \tfrac{1}{5^5} \cdot 0,0002573 = 0,9961419.$$

Therefore there will become

$$Q = 1 - C = 0,0038581$$

and thus

$$\tfrac{1}{2}l2-\tfrac{1}{3}P-\tfrac{1}{5}Q=0,3350513.$$

25. Now there shall be $n=7$ and $A=\mathfrak{R}=0,9995547$, then truly $a=1,\ b=0,9995428$, hence therefore there will become

$$B=A-\tfrac{1}{3^7}\cdot 0,0004453=0,9995545,$$

from which we have now :

$$R=1-B=0,0004455$$

and thus

$$\tfrac{1}{2}l2-\tfrac{1}{3}P-\tfrac{1}{5}Q-\tfrac{1}{7}R=0,3349877.$$

26. Since in this calculation there were not only $B=A$, in the following there will not even be a need for the letter B, on account of which we will have

$$S=1-\mathfrak{S}=0,0000501$$

and hence

$$\tfrac{1}{9}S=0,0000056.$$

Then truly there will be

$$T=1-\mathfrak{T}=0,0000053,$$

and hence

$$\tfrac{1}{11}T=0,0000005,$$

finally

$$U=1-\mathfrak{U}=0,0000003 \text{ and } \tfrac{1}{13}U=0,0000000.$$

Therefore with these particular values taken from the preceding there becomes

$$O=0,3349816.$$

From which it is apparent here the value hitherto to be a little greater than $\frac{1}{3}$.

27. Now therefore we are able to be sure the sum of this infinite series

$$\tfrac{1}{3}-\tfrac{1}{5}+\tfrac{1}{7}+\tfrac{1}{11}-\tfrac{1}{13}-\tfrac{1}{17}+\tfrac{1}{19}+\text{etc.}$$

to be precisely enough $=0,3349816$. Now it may be required to be investigated, whether or not this value may hold some notable ratio, either to the periphery of the circle π, or to its hyperbolic logarithm, since we have observed above the series of the reciprocals of the prime numbers

$$\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{5}+\tfrac{1}{7}+\tfrac{1}{11}+\text{etc.}$$

to express the hyperbolic logarithm of the complete harmonic series

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\text{etc.},$$

from which it can be seen this same series of prime numbers

$$\frac{1}{3}-\frac{1}{5}+\frac{1}{7}+\frac{1}{11}-\frac{1}{13}-\frac{1}{17}+\text{etc.}$$

also contains the logarithm of this same complete series

$$1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\text{etc.},$$

of which the sum is $\frac{\pi}{4}$. This in the end I may attach the hyperbolic logarithm of π, which at one time I found to be [See Ch. 7, *Intro. to Analysis*, Vol. I, in this series of translations]

$$1,14472,98858,49400,17414,34273,51353,05865.$$

Therefore it will be required to see, whether perhaps the sum found O shall be $l\pi - lN$, thus so that N shall be a simple enough number. Truly several investigations of this kind have been instituted, generally without any success.

28. But with the aid of the latter method not only have we elicited the sum of the proposed series, but also of its odd powers, which sums we may present here to be considered.

$$O=\frac{1}{3}-\frac{1}{5}+\frac{1}{7}+\frac{1}{11}-\frac{1}{13}-\frac{1}{17}+\frac{1}{19}+\text{etc.}=0,3349816$$
$$P=\frac{1}{3^3}-\frac{1}{5^3}+\frac{1}{7^3}+\frac{1}{11^3}-\frac{1}{13^3}-\frac{1}{17^3}+\frac{1}{19^3}+\text{etc.}=0,0322521$$
$$Q=\frac{1}{3^5}-\frac{1}{5^5}+\frac{1}{7^5}+\frac{1}{11^5}-\frac{1}{13^5}-\frac{1}{17^5}+\frac{1}{19^5}+\text{etc.}=0,0038602$$
$$R=\frac{1}{3^7}-\frac{1}{5^7}+\frac{1}{7^7}+\frac{1}{11^7}-\frac{1}{13^7}-\frac{1}{17^7}+\frac{1}{19^7}+\text{etc.}=0,0004455$$
$$S=\frac{1}{3^9}-\frac{1}{5^9}+\frac{1}{7^9}+\frac{1}{11^9}-\frac{1}{13^9}-\frac{1}{17^9}+\frac{1}{19^9}+\text{etc.}=0,00005010,$$
$$S=\frac{1}{3^{11}}-\frac{1}{5^{11}}+\frac{1}{7^{11}}+\frac{1}{11^{11}}-\frac{1}{13^{11}}-\frac{1}{17^{11}}+\frac{1}{19^{11}}+\text{etc.}=0,0000053$$
$$S=\frac{1}{3^{13}}-\frac{1}{5^{13}}+\frac{1}{7^{13}}+\frac{1}{11^{13}}-\frac{1}{13^{13}}-\frac{1}{17^{13}}+\frac{1}{19^{13}}+\text{etc.}=0,0000003$$

etc.

29. Indeed these same sums without doubt will deserve little attention, unless perhaps they will have been reduced to known quantities. Truly since in these series neither do the terms themselves progress according to a certain law, nor either may a certain order be observed in the signs plus or minus, this same enquiry on first being examined clearly would be seen to be impossible, on account of which this method itself, by which we have gone through the sums of these, certainly being considered to be worthy of all attention, and that therefore more, because it depends well enough on the abstruse properties of power series. Unless indeed the sums of the series

$$1-\frac{1}{3^n}+\frac{1}{5^n}-\frac{1}{7^n}+\frac{1}{9^n}-\text{etc.}$$

for the cases, for which n is an odd number, were to be known, this whole investigation would have been undertaken in vain.

DE SUMMA SERIEl EX NUMERIS PRIMIS FORMATAE

$$\frac{1}{3}-\frac{1}{5}+\frac{1}{7}-\frac{1}{11}-\frac{1}{17}+\frac{1}{19}+\frac{1}{23}-\frac{1}{29}+\frac{1}{31}-\text{etc.}$$

UBI NUMERI PRIMI FORMAE $4n-1$ HABENT SIGNUM POSITIVUM FORMAE AUTEM $4n+1$ SIGNUM NEGATIVUM

[E596]

Opuscula analytica 2, 1785, p. 240-256
[Conventui exhibita die 2. octobris 1775]

1. Cum iam Euclides demonstrasset multitudinem numerorum primorum revera esse infinitam, ego iam pridem ostendi etiam summam seriei reciprocae numerorum primorum, scilicet

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{11}+\frac{1}{17}+\frac{1}{19}+\text{etc.},$$

esse infinite magnam, atque adeo referre logarithmum summae seriei harmonicae

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\text{etc.},$$

id quod non parum mirum videbatur, cum vulgo summa seriei harmonicae ad genus quasi infimum infinitorum referri soleat. Cum autem non solum logarithmus numeri infiniti sit etiam infinitus, sed etiam logarithmi horum ipsorum logarithmorum etiamnunc sint infiniti, manifestum est dari insuper infinitos gradus inferiores infinitorum. Ita si A denotet summam seriei reciprocae numerorum primorum, etiam lA adhuc erit infinite magnus, sed ad ordinem infinitorum infinities inferiorem pertinere censendus est; tum vero etiamnunc hae formulae: llA, $lllA$, $llllA$ etc. erunt infinitae, quanquam quaelibet earum infinities sit minor quam praecedens.

2. Quoniam porro numeri primi praeter binarium quasi a natura in duas classes distinguuntur, prouti fuerint vel formae $4n+1$ vel formae $4n-1$, dum priores omnes sunt summae duorum quadratorum, posteriores vero ab hac proprietate penitus excluduntur; series reciprocae ex utraque classe formatae, scilicet:

$$\frac{1}{5}+\frac{1}{13}+\frac{1}{17}+\frac{1}{29}+\text{etc. et }\frac{1}{3}+\frac{1}{7}+\frac{1}{11}+\frac{1}{19}+\frac{1}{23}+\text{etc.},$$

ambae erunt pariter infinitae, id quod etiam de omnibus speciebus numerorum primorum est tenendum. Ita si ex numeris primis ii tantum excerpantur, qui sunt formae $100n+1$, cuiusmodi sunt 101, 401, 601, 701 etc., non solum multitudo eorum est infinita, sed etiam summa huius seriei ex illis formatae, scilicet:

$$\frac{1}{101}+\frac{1}{401}+\frac{1}{601}+\frac{1}{701}+\frac{1}{1201}+\frac{1}{1301}+\frac{1}{1601}+\frac{1}{1801}+\frac{1}{1901}+\text{etc.},$$

etiam est infinita.

3. Considererons hic autem imprimis discrimen inter numeros primos formae $4n+1$ et $4n-1$, et quia ambae series ex utroque ordine formatae sunt infinitae et quasi eiusdem ordinis, nullum est dubium, quin earum differentia habeat valorem determinatum. Hanc ob rem terminis ex forma $4n-1$ formatis tribuamus signum +, reliquis vero signum −, ut orietur ista series:

$$\frac{1}{3}-\frac{1}{5}+\frac{1}{7}-\frac{1}{11}-\frac{1}{13}-\frac{1}{17}+\frac{1}{19}+\frac{1}{23}-\frac{1}{29}+\frac{1}{31}-\text{etc.},$$

in qua nullus plane ordo in signis cernitur. Hoc tamen non obstante videamus, quomodo eius summa vero saltem proxime assignari queat. Non parum enim probabile videtur istam summam, si non fuerit rationalis vel irrationalis, saltem ad quodpiam genus notabile quantitatum transcendentium pertinere debere. Interim tamen, quoniam non solum non in signis, verum etiam multo minus in ipsis fractionibus nullus plane ordo perspicitur, primo intuitu nulla patere videtur via, qua ad eius summam pervenire liceat.

4. Quando autem contemplamur seriem notissimam LEIBNITZIANAM pro quadratura circuli datam

$$\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\text{etc.},$$

videmus in ea omnes numeros impares formae $4n+1$ habere signum +, reliquos vero formae $4n-1$ signum −, unde mutatis signis omnes termini nostrae seriei cum suis signis in hac serie LEIBNITZIANA occurrunt. Quod si ergo ex hac serie omnes numeros compositos expungamus, tandem praeter unitatem ipsa nostra series proposita, contrariis signis, remanebit. Quamobrem summam nostrae seriei obtinebimus, si ex serie LEIBNITZIANA successive omnes numeros compositos excludamus, quandoquidem omnium terminorum, qui per quamvis operationem excluduntur, summa facile assignare licebit.

5. Incipiamus igitur ab ipsa serie LEIBNITZIANA, statuendo

$$A=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\frac{1}{13}-\frac{1}{15}+\frac{1}{17}-\frac{1}{19}+\text{etc.},$$

ita ut sit $A=\frac{\pi}{4}$, ex eaque primo omnes numeros compositos per 3 divisibiles excludamus. Quem in finem hinc formemus istam seriem:

$$(A-1)\frac{1}{3}=-\frac{1}{9}+\frac{1}{15}-\frac{1}{21}+\frac{1}{27}-\frac{1}{33}+\frac{1}{39}-\text{etc.},$$

quae utique continet omnes numeros compositos per 3 divisibiles, unde si haec series ad illam addatur, omnes isti numeri compositi excludentur fietque

$$\tfrac{4}{3}A-\tfrac{1}{3}=1-\tfrac{1}{3}+\tfrac{1}{5}-\tfrac{1}{7}-\tfrac{1}{11}+\tfrac{1}{13}+\tfrac{1}{17}-\tfrac{1}{19}-\tfrac{1}{23}+\tfrac{1}{25}+\text{etc.}=B,$$

ita ut sit $B=\tfrac{4}{3}A-\tfrac{1}{3}$. In hac igitur serie, cuius summam novimus, nulli amplius numeri compositi per 3 divisibiles occurrunt, sed primus terminus compositus hic occurrens est $+\tfrac{1}{25}$.

6. Nunc igitur ex postrema hac serie omnes terminos per 5 divisibiles excludamus, quem in finem hinc formemus istam seriem:

$$(B-1+\tfrac{1}{3})\tfrac{1}{5}=\tfrac{1}{25}-\tfrac{1}{35}-\tfrac{1}{55}+\tfrac{1}{65}+\tfrac{1}{85}-\tfrac{1}{95}-\text{etc.},$$

quae omnes complectitur terminos per quinque divisibiles, qui adhuc in seriem *B* ingrediebantur, et quidem iisdem signis affectos, quare hac serie ab illa ablata remanebit haec:

$$\tfrac{4}{5}B+\tfrac{1}{5}(1-\tfrac{1}{3})=1-\tfrac{1}{3}+\tfrac{1}{5}-\tfrac{1}{7}-\tfrac{1}{11}+\tfrac{1}{13}+\tfrac{1}{17}-\tfrac{1}{19}-\tfrac{1}{23}+\text{etc.},$$

ubi iam ab initio omnes termini sunt numeri primi, et non primus, qui hic occurret ,erit $\frac{1}{49}$, sequens vero $\frac{1}{77},-\frac{1}{91}$. Ponamus autem summam huius seriei $=C$, ut sit

$$C=\tfrac{4}{5}B+\tfrac{1}{5}(1-\tfrac{1}{3}).$$

7. Nunc igitur hinc excludamus omnes terminos, qui adhuc per septem sunt divisibiles, quos complectetur sequens forma:

$$(C-1+\tfrac{1}{3}-\tfrac{1}{5})\tfrac{1}{7}=-\tfrac{1}{49}-\tfrac{1}{77}+\tfrac{1}{91}+\text{etc.},$$

quorum terminorum signa sunt contraria, quare haec series ad seriem *C* addita dabit

$$\tfrac{8}{7}C-\tfrac{1}{7}(1-\tfrac{1}{3}+\tfrac{1}{5})=1-\tfrac{1}{3}+\tfrac{1}{5}-\tfrac{1}{7}-\tfrac{1}{11}+\tfrac{1}{13}+\tfrac{1}{17}-\text{etc.},$$

in qua primus terminus compositus erit $+\frac{1}{121}$, sequentes vero $-\frac{1}{11\cdot 13},-\frac{1}{11\cdot 17}$ etc. Hanc autem seriem vocemus *D,* ut sit

$$D=\tfrac{8}{7}C-\tfrac{1}{7}(1-\tfrac{1}{3}+\tfrac{1}{5}).$$

8. Iam ex serie modo inventa *D* expungamus terminos, qui adhuc sunt per 11 divisibiles, quos complectetur ista forma:

$$(D-(1-\tfrac{1}{3}+\tfrac{1}{5}-\tfrac{1}{7})\tfrac{1}{11})=-\tfrac{1}{121}+\tfrac{1}{143}+\tfrac{1}{187}-\text{etc.},$$

qui termini in serie D contraria habent signa; quamobrem si haec series ad illam addatur, isti termini excludentur prodibitque

$$\tfrac{12}{11}D-\tfrac{1}{11}(1-\tfrac{1}{3}+\tfrac{1}{5}-\tfrac{1}{7})=1-\tfrac{1}{3}+\tfrac{1}{5}-\tfrac{1}{7}-\tfrac{1}{11}+\text{etc.},$$

in qua primus terminus non primus est $\frac{1}{169}$; istam autem seriem designemus littera E, ita ut sit

$$E=\tfrac{12}{11}D-\tfrac{1}{11}(1-\tfrac{1}{3}+\tfrac{1}{5}-\tfrac{1}{7}).$$

9. Ex hac igitur serie excludamus omnes terminos, qui adhuc insunt per 13 divisibiles, quos ergo complectetur haec forma:

$$(E-1+\tfrac{1}{3}-\tfrac{1}{5}+\tfrac{1}{7}+\tfrac{1}{11})\tfrac{1}{13}=\tfrac{1}{169}+\tfrac{1}{121}-\text{etc.},$$

hique termini eadem habent signa ac in ipsa serie E. Haec igitur series ab illa debet subtrahi, unde prodit

$$\tfrac{12}{13}E+\tfrac{1}{13}(1-\tfrac{1}{3}+\tfrac{1}{5}-\tfrac{1}{7}-\tfrac{1}{11})=1-\tfrac{1}{3}+\tfrac{1}{5}-\tfrac{1}{7}-\tfrac{1}{11}+\text{etc.},$$

ubi primus terminus non primus est $\frac{1}{289}$. Totam autem hanc seriem designemus littera F, ut sit

$$F=\tfrac{12}{13}E+\tfrac{1}{13}(1-\tfrac{1}{3}+\tfrac{1}{5}-\tfrac{1}{7}-\tfrac{1}{11}).$$

10. Quod si nunc istas operationes ulterius continuemus, dum successive hinc excludimus terminos adhuc per 17 divisibiles, tum vero per 19, per 23 etc., tandem relinquetur tantum series numerorum primorum post unitatem sequentium, quae si designetur littera Z, quam ut infinitesimam spectari oportet, erit utique

$$Z=1-\tfrac{1}{3}+\tfrac{1}{5}-\tfrac{1}{7}-\tfrac{1}{11}+\tfrac{1}{13}+\tfrac{1}{17}-\tfrac{1}{19}-\text{etc.},$$

consequenter seriei in titulo propositae summa erit $1-Z$. Ac manifestum est ad hune valorem continuo propius accedera istas formulas:

$$1-A,\ 1-B,\ 1-C,\ 1-D,\ 1-E,\ 1-F \text{ etc.}$$

11. Quemadmodum autem valores omnium harum litterarum successive ex antecedentibus colligi debeant, ex sequentibus formulis fiet manifestum:

$$B = \tfrac{4}{3}A - \tfrac{1}{3}\cdot 1$$
$$C = \tfrac{4}{5}B - \tfrac{1}{5}(1-\tfrac{1}{3})$$
$$D = \tfrac{8}{7}C - \tfrac{1}{7}(1-\tfrac{1}{3}+\tfrac{1}{5})$$
$$E = \tfrac{12}{11}D - \tfrac{1}{11}(1-\tfrac{1}{3}+\tfrac{1}{5}-\tfrac{1}{7})$$
$$F = \tfrac{12}{13}E + \tfrac{1}{13}(1-\tfrac{1}{3}+\tfrac{1}{5}-\tfrac{1}{7}-\tfrac{1}{11})$$
$$G = \tfrac{16}{17}F + \tfrac{1}{17}(1-\tfrac{1}{3}+\tfrac{1}{5}-\tfrac{1}{7}-\tfrac{1}{11}+\tfrac{1}{13})$$
$$H = \tfrac{20}{19}G - \tfrac{1}{19}(1-\tfrac{1}{3}+\tfrac{1}{5}-\tfrac{1}{7}-\tfrac{1}{11}+\tfrac{1}{13}+\tfrac{1}{17})$$
$$I = \tfrac{24}{23}H - \tfrac{1}{23}(1-\tfrac{1}{3}+\tfrac{1}{5}-\tfrac{1}{7}-\tfrac{1}{11}+\tfrac{1}{13}+\tfrac{1}{17}-\tfrac{1}{19})$$

Ubi notandum, si denominator primus fuerit formae $4n+1$, tum numeratorem primae partis fore unitate minorem, sive $4n$, alteram vero partem addi debere. Sin autem denominator primus fuerit $4n-1$, tum numeratorem primae partis fore unitate maiorem, sive $4n$, alteram partem vero hoc casu subtrahi debere.

12. Quo nunc omnes hos valores in numeris per fractiones decimales exprimamus, ante omnia notetur esse

$$A = \tfrac{\pi}{4} = 0,7853981634.$$

Pro reliquis autem litteris computentur sequentes valores:

$$1-\tfrac{1}{3} = b = 0,6666666666$$
$$1-\tfrac{1}{3}+\tfrac{1}{5} = c = 0,8666666666$$
$$1-\tfrac{1}{3}+\tfrac{1}{5}-\tfrac{1}{7} = d = 0,7238095238$$
$$1-\tfrac{1}{3}+\tfrac{1}{5}-\tfrac{1}{7}-\tfrac{1}{11} = e = 0,6329004329$$
$$1-\tfrac{1}{3}+\tfrac{1}{5}-\tfrac{1}{7}-\tfrac{1}{11}+\tfrac{1}{13} = f = 0,7098235098$$
$$G1-\tfrac{1}{3}+\tfrac{1}{5}-\tfrac{1}{7}-\tfrac{1}{11}+\tfrac{1}{13}+\tfrac{1}{17} = g = 0,7686470392$$
$$1-\tfrac{1}{3}+\tfrac{1}{5}-\tfrac{1}{7}-\tfrac{1}{11}+\tfrac{1}{13}+\tfrac{1}{17}-\tfrac{1}{19} = h = 0,7160154603$$
$$1-\tfrac{1}{3}+\tfrac{1}{5}-\tfrac{1}{7}-\tfrac{1}{11}+\tfrac{1}{13}+\tfrac{1}{17}-\tfrac{1}{19}-\tfrac{1}{23} = i = 0,6725371994,$$

in quo ordine primus terminus a unitate aequatur.

13. Pro computo autem ipsarum litterarum A, B, C, D, E etc. praestabit sequentibus uti formulis, quibus simul valores numericos harum litterarum adscribamus

$$B = A + \tfrac{1}{3}(A-a) = 0{,}713864$$
$$C = B - \tfrac{1}{5}(B-b) = 0{,}704424$$
$$D = C + \tfrac{1}{7}(C-c) = 0{,}681247$$
$$E = D + \tfrac{1}{11}(D-d) = 0{,}677377$$
$$F = E - \tfrac{1}{13}(E-e) = 0{,}673956$$
$$G = F - \tfrac{1}{17}(F-f) = 0{,}676066$$
$$H = G + \tfrac{1}{19}(G-g) = 0{,}671193$$
$$I = \mathrm{H} + \tfrac{1}{23}(H-h) = 0{,}669245$$
$$K = I - \tfrac{1}{29}(I-i) = 0{,}669358\,.$$

14. Quanquam autem calculum huc usque produximus, tamen ultra tertiam figuram decimalem de summa nostrae seriei certi esse non possumus, atque adeo in dubio relinquere cogimur, utrum ista summa aliquanto maior vel minor sit quam 0,669. Sin autem hune valorem pro vero assumamus, ipsa series proposita

$$\tfrac{1}{3} - \tfrac{1}{5} + \tfrac{1}{7} + \tfrac{1}{11} - \tfrac{1}{13} - \tfrac{1}{17} + \tfrac{1}{19} + \text{etc.}$$

summam habebit 0,331, ideoque hic valor tantillo foret minor quam $\frac{1}{3}$. Quoniam vero ablato $\frac{1}{5}$, iterum addi debeat $\frac{1}{7} + \frac{1}{11}$, quarum fractionum summa maior est quam $\frac{1}{5}$, utique fieri posset, ut verus valor superaret $\frac{1}{3}$, id quod hoc loco in dubio est relinquendum. Datur vero alia methodus multo accuratius in summam huius seriei inquirendi, quam hic evolvemus, quandoquidem operae pretium videtur veram huius seriei summam propius cognovisse.

15. Eadem methodo, qua hic ex prima serie LEIBNITZIANA successive terminos compositos expulimus, si omnes plane terminos praeter unitatem removeamus, reperiemus

$$\frac{\pi}{4} = \frac{3}{4} \cdot \frac{5}{4} \cdot \frac{7}{8} \cdot \frac{11}{12} \cdot \frac{13}{12} \cdot \frac{17}{16} \cdot \frac{19}{20} \cdot \text{etc.},$$

ubi in numeratoribus omnes numeri primi occurrunt praeter 2, denominatores vero sunt numeri pariter pares unitate vel maiores vel minores. Deinde vero si ista series reciproca quadratorum imparium:

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \frac{1}{11^2} + \frac{1}{13^2} + \text{etc.},$$

cuius summam ostendi esse $= \frac{\pi\pi}{8}$, simili modo tractetur, reperietur

$$\frac{\pi\pi}{8} = \frac{3\cdot3}{2\cdot4}\cdot\frac{5\cdot5}{4\cdot6}\cdot\frac{7\cdot7}{6\cdot8}\cdot\frac{11\cdot11}{10\cdot12}\cdot\frac{13\cdot13}{12\cdot14}\cdot\text{etc.},$$

ubi iterum in numeratoribus omnes numeri primi bis occurrunt, in denominatoribus vero iidem tam unitate aucti quam minuti. Quare si hanc expressionem per quadratum illius, quod est

$$\frac{\pi\pi}{16} = \frac{3\cdot3}{4\cdot4}\cdot\frac{5\cdot5}{4\cdot4}\cdot\frac{7\cdot7}{8\cdot8}\cdot\frac{11\cdot11}{12\cdot12}\cdot\frac{13\cdot13}{12\cdot12}\cdot\text{etc.},$$

dividamus, quotus erit

$$2 = \frac{4}{2}\cdot\frac{4}{6}\cdot\frac{8}{6}\cdot\frac{12}{10}\cdot\frac{12}{14}\cdot\text{etc.},$$

ubi omnes numeri primi tam unitate aucti quam minuti occurrunt, et numeri pariter pares in numeratore, impariter pares vero in denominatore constituuntur.

16. Postrema haec expressio igitur hoc modo exhiberi poterit:

$$2 = \frac{3+1}{3-1}\cdot\frac{5-1}{5+1}\cdot\frac{7+1}{7-1}\cdot\frac{11+1}{11-1}\cdot\frac{13-1}{13+1}\cdot\text{etc.};$$

hinc ergo logarithmis hyperbolicis sumendis habebimus:

$$l2 = l\frac{3+1}{3-1} + l\frac{5-1}{5+1} + l\frac{7+1}{7-1} + l\frac{11+1}{11-1} + l\frac{13-1}{13+1} + \text{etc.}$$

Constat autem per series infinitas esse in genere

$$\frac{1}{2}l\frac{a+1}{a-1} = \frac{1}{a} + \frac{1}{3a^3} + \frac{1}{5a^5} + \frac{1}{7a^7} + \frac{1}{9a^9} + \text{etc.}$$

hincque:

$$\frac{1}{2}l\frac{a-1}{a+1} = -\frac{1}{a} - \frac{1}{3a^3} - \frac{1}{5a^5} - \frac{1}{7a^7} - \frac{1}{9a^9} - \text{etc.}$$

Quodsi igitur harum formularum ope omnes illos logarithmos in series infinitas convertamus, nanciscemur quidem innumeras series infinitas, quas autem ad series facilius tractabiles reducere licebit.

17. Primo igitur omnium illorum logarithmorum semisses accipi oportet, et quia hic de logarithmis hyperbolicis agitur, ob

$$l2 = 0,6931471805$$

erit

$$\frac{1}{2}l2 = 0,3465735902,$$

ex altera autem parte logarithmi ita ordinentur:

$$\tfrac{1}{2}l\tfrac{3+1}{3-1} = \tfrac{1}{3}+\tfrac{1}{3\cdot3^3}+\tfrac{1}{5\cdot3^5}+\tfrac{1}{7\cdot3^7}+\tfrac{1}{9\cdot3^9}+\text{etc.}$$
$$\tfrac{1}{2}l\tfrac{5-1}{5+1} = -\tfrac{1}{5}-\tfrac{1}{3\cdot5^3}-\tfrac{1}{5\cdot5^5}-\tfrac{1}{7\cdot5^7}-\tfrac{1}{9\cdot5^9}-\text{etc.}$$
$$\tfrac{1}{2}l\tfrac{7+1}{7-1} = \tfrac{1}{7}+\tfrac{1}{3\cdot7^3}+\tfrac{1}{5\cdot7^5}+\tfrac{1}{7\cdot7^7}+\tfrac{1}{9\cdot7^9}+\text{etc.}$$
$$\tfrac{1}{2}l\tfrac{11+1}{11-1} = \tfrac{1}{11}+\tfrac{1}{3\cdot11^3}+\tfrac{1}{5\cdot11^5}+\tfrac{1}{7\cdot11^7}+\tfrac{1}{9\cdot11^9}+\text{etc.}$$
$$\tfrac{1}{2}l\tfrac{13-1}{13+1} = -\tfrac{1}{13}-\tfrac{1}{3\cdot13^3}-\tfrac{1}{5\cdot13^5}-\tfrac{1}{7\cdot13^7}-\tfrac{1}{9\cdot13^9}-\text{etc.}$$
etc.

18. Hinc iam verticaliter descendendo consideremus sequentes series pariter infinitas:

$$O=\tfrac{1}{3}-\tfrac{1}{5}+\tfrac{1}{7}+\tfrac{1}{11}-\tfrac{1}{13}-\tfrac{1}{17}+\tfrac{1}{19}+\text{etc.}$$
$$P=\tfrac{1}{3^3}-\tfrac{1}{5^3}+\tfrac{1}{7^3}+\tfrac{1}{11^3}-\tfrac{1}{13^3}-\tfrac{1}{17^3}+\tfrac{1}{19^3}+\text{etc.}$$
$$Q=\tfrac{1}{3^5}-\tfrac{1}{5^5}+\tfrac{1}{7^5}+\tfrac{1}{11^5}-\tfrac{1}{13^5}-\tfrac{1}{17^5}+\tfrac{1}{19^5}+\text{etc.}$$
$$R=\tfrac{1}{3^7}-\tfrac{1}{5^7}+\tfrac{1}{7^7}+\tfrac{1}{11^7}-\tfrac{1}{13^7}-\tfrac{1}{17^7}+\tfrac{1}{19^7}+\text{etc.}$$
$$S=\tfrac{1}{3^9}-\tfrac{1}{5^9}+\tfrac{1}{7^9}+\tfrac{1}{11^9}-\tfrac{1}{13^9}-\tfrac{1}{17^9}+\tfrac{1}{19^9}+\text{etc.}$$
etc.

Quarum serierum prima O est ea ipsa, cuius summam hic investigare nobis est propositum.

19. His igitur seriebus ita per litteras maiusculas designatis habebimus istam aequationem:

$$\tfrac{1}{2}l2=O+\tfrac{1}{3}P+\tfrac{1}{5}Q+\tfrac{1}{7}R+\tfrac{1}{9}S+\tfrac{1}{11}T+\text{etc.}$$

unde si summae serierum P, Q, R, S etc. essent cognitae, inde impetraremus facile summam seriei O quaesitam; foret enim

$$O=\tfrac{1}{2}l2-\tfrac{1}{3}P-\tfrac{1}{5}Q-\tfrac{1}{7}R-\tfrac{1}{9}S-\text{etc.}$$

20. At vero summas serierum P, Q, R etc. ex seriebus ordinis, ubi omnes numeri impares occurrunt, concludere poterimus eodem modo, quo supra seriem ipsam O ex serie LEIBNITZIANA

$$1-\tfrac{1}{3}+\tfrac{1}{5}-\tfrac{1}{7}+\tfrac{1}{9}-\tfrac{1}{11}+\text{etc.}$$

elicuimus. Hunc in finem evolvi hac methodo oportebit sequentes series ordinatas:

$$\mathfrak{P} = 1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \frac{1}{9^3} - \frac{1}{11^3} + \frac{1}{13^3} + \frac{1}{15^3} + \text{etc.}$$

$$\mathfrak{Q} = 1 - \frac{1}{3^5} + \frac{1}{5^5} - \frac{1}{7^5} + \frac{1}{9^5} - \frac{1}{11^5} + \frac{1}{13^5} + \frac{1}{15^5} + \text{etc.}$$

$$\mathfrak{R} = 1 - \frac{1}{3^7} + \frac{1}{5^7} - \frac{1}{7^7} + \frac{1}{9^7} - \frac{1}{11^7} + \frac{1}{13^7} - \frac{1}{15^7} + \text{etc.}$$

$$\mathfrak{S} = 1 - \frac{1}{3^9} + \frac{1}{5^9} - \frac{1}{7^9} + \frac{1}{9^9} - \frac{1}{11^9} + \frac{1}{13^9} - \frac{1}{15^9} + \text{etc.}$$

$$\mathfrak{T} = 1 - \frac{1}{3^{11}} + \frac{1}{5^{11}} - \frac{1}{7^{11}} + \frac{1}{9^{11}} - \frac{1}{11^{11}} + \frac{1}{13^{11}} - \frac{1}{15^{11}} + \text{etc.}$$

etc.

Harum autem omnium serierum summas iam pridem per quadraturam circuli, scilicet per similes potestates ipsius π expressas dedi, sequenti modo:

$$\mathfrak{P} = \frac{1}{1\cdot 2}\cdot\frac{\pi^3}{2^4} \qquad \mathfrak{T} = \frac{50521}{1....10}\cdot\frac{\pi^{11}}{2^{12}}$$

$$\mathfrak{Q} = \frac{5}{1\cdot 2\cdot 3\cdot 4}\cdot\frac{\pi^5}{2^6} \qquad \mathfrak{U} = \frac{2702765}{1.......12}\cdot\frac{\pi^{13}}{2^{14}}$$

$$\mathfrak{R} = \frac{61}{1...6}\cdot\frac{\pi^7}{2^8} \qquad \mathfrak{V} = \frac{199360981}{1..........14}\cdot\frac{\pi^{15}}{2^{16}}$$

$$\mathfrak{S} = \frac{1385}{1.....8}\cdot\frac{\pi^9}{2^{10}} \qquad \mathfrak{W} = \frac{19391512145}{1..............16}\cdot\frac{\pi^{17}}{2^{18}}$$

etc.

21. Hos igitur valores in fractionibus decimalibus usque ad sextam figuram evolvamus, eritque

	Differentiae
$\mathfrak{P} = 0,9689462$	0,0272116
$\mathfrak{Q} = 0,9961578$	0,0033969
$\mathfrak{R} = 0,9995547$	0,0003952
$\mathfrak{S} = 0,9999499$	0,0000448
$\mathfrak{T} = 0,9999947$	0,0000050
$\mathfrak{U} = 0,9999997$	0,0000005
etc.	etc.

22. Ut nunc hinc valores litterarum *P, Q, R* etc. eruamus, eadem methodo utamur, qua supra ex serie

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \text{etc.}$$

omnes terminos compositos exterminavimus, quandoquidem loco horum numerorum simplicium eorum potestates scribi convenit. Hanc igitur operationem in genere pro his litteris doceamus. Consideremus igitur hanc seriem:

$$\mathfrak{Z} = 1 - \frac{1}{3^n} + \frac{1}{5^n} - \frac{1}{7^n} + \frac{1}{9^n} - \frac{1}{11^n} + \text{etc.},$$

cuius summam, ut supra factum est, littera A designemus, ut sit $A = \mathfrak{Z}$, hincque sequentes litteras *B, C, D* etc. eliciamus per sequentes formulas:

$$\begin{array}{ll}
B = A + \frac{1}{3^n}(A - a) \text{ existente } a = 1 & \\
C = B + \frac{1}{5^n}(B - b) & b = 1 - \frac{1}{3^n} \\
D = C + \frac{1}{7^n}(D - d) & c = 1 - \frac{1}{3^n} + \frac{1}{5^n} \\
E = D + \frac{1}{11^n}(D - d) & d = 1 - \frac{1}{3^n} + \frac{1}{5^n} - \frac{1}{7^n} \\
F = E - \frac{1}{13^n}(E - e) & e = 1 - \frac{1}{3^n} + \frac{1}{5^n} - \frac{1}{7^n} + \frac{1}{9^n} - \frac{1}{11^n} \\
\text{etc.} & \text{etc.}
\end{array}$$

Quibus valoribus inventis eorum complementa ad unitatem, scilicet $1 - A,\ 1 - B,\ 1 - C,\ 1 - D$ etc., promptissime ad valorem quaesitum

$$Z = \frac{1}{3^n} - \frac{1}{5^n} + \frac{1}{7^n} + \frac{1}{11^n} - \frac{1}{13^n} - \text{etc.}$$

appropinquabunt.

23. Haec igitur praecepta generalia applicemus primo ad valorem litterae *P,* unde incipiendum erit a valore

$$\mathfrak{P} = 0{,}9689462 = A,$$

et quia hic est $n = 3$, habebimus

$$a = 1,\ b = 0{,}9629630,\ c = 0{,}9709630,\ d = 0{,}9680476;$$

pluribus valoribus non erit opus. Hinc igitur colligemus sequentes valores:

$$\begin{aligned}
B &= A - \frac{1}{3^3} \cdot 0{,}0310538 = 0{,}9677961 \\
C &= B - \frac{1}{5^3} \cdot 0{,}0048331 = 0{,}9677574 \\
D &= C - \frac{1}{7^3} \cdot 0{,}0032056 = 0{,}9677481 \\
E &= D - \frac{1}{11^3} \cdot 0{,}0002995 = 0{,}9677479.
\end{aligned}$$

Ulterius procedi non est opus; quamobrem hinc habebimus

$$P = 1 - E = 0{,}0322521,$$

unde iam colligimus

$$\tfrac{1}{2}l2-\tfrac{1}{3}\cdot 0,0322521=0,3358229.$$

24. Sumamus nunc $n=5$ et habebimus

$$A=\mathfrak{Q}=0,9961578,$$

tum vero erit

$$a=1,\ b=0,9958847,\ c=0,9962048,\ d=0,9961453,$$

hinc igitur reperiemus

$$B=A-\tfrac{1}{3^5}\cdot 0,0038422=0,9961420$$

$$C=B-\tfrac{1}{5^5}\cdot 0,0002573=0,9961419.$$

Erit igitur

$$Q=1-C=0,0038581$$

ideoque

$$\tfrac{1}{2}l2-\tfrac{1}{3}P-\tfrac{1}{5}Q=0,3350513.$$

25. Sit nunc $n=7$ et $A=\mathfrak{R}=0,9995547$, tum vero $a=1,\ b=0,9995428$, hinc igitur fiet

$$B=A-\tfrac{1}{3^7}\cdot 0,0004453=0,9995545,$$

unde iam habemus

$$R=1-B=0,0004455$$

ideoque

$$\tfrac{1}{2}l2-\tfrac{1}{3}P-\tfrac{1}{5}Q-\tfrac{1}{7}R=0,3349877.$$

26. Cum in hoc calculo tantum non fuerit $B=A$, in sequentibus nequidem littera B erit opus, quamobrem habebimus

$$S=1-\mathfrak{S}=0,0000501$$

hincque

$$\tfrac{1}{9}S=0,0000056.$$

Deinde vero erit

$$T=1-\mathfrak{T}=0,0000053,$$

hincque

$$\tfrac{1}{11}T=0,0000005,$$

denique

$$U = 1 - \mathfrak{U} = 0,0000003 \text{ et } \tfrac{1}{13}U = 0,0000000.$$

Particulis igitur his a praecedente valore ablatis prodit

$$O = 0,3349816 .$$

Unde patet hunc valorem adhuc aliquanto maiorem esse quam $\frac{1}{3}$.

27. Nunc igitur certi esse possumus summam seriei infinitae huius

$$\tfrac{1}{3} - \tfrac{1}{5} + \tfrac{1}{7} + \tfrac{1}{11} - \tfrac{1}{13} - \tfrac{1}{17} + \tfrac{1}{19} + \text{etc.}$$

esse satis exacte $= 0,3349816$. Investigandum iam foret, num iste valor non quampiam teneat rationem notabilem, sive ad peripheriam circuli *n,* sive ad eius logarithmum hyperbolicum, quandoquidem supra observavimus seriem reciprocam numerorum primorum

$$\tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{5} + \tfrac{1}{7} + \tfrac{1}{11} + \text{etc.}$$

exprimere logarithmum hyperbolicum seriei harmonicae completae

$$1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5} + \tfrac{1}{6} + \text{etc.},$$

unde videri potest istam seriem numerorum primorum

$$\tfrac{1}{3} - \tfrac{1}{5} + \tfrac{1}{7} + \tfrac{1}{11} - \tfrac{1}{13} - \tfrac{1}{17} + \text{etc.}$$

etiam continere logarithmum eiusdem seriei completae

$$1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \tfrac{1}{9} - \tfrac{1}{11} + \text{etc.},$$

cuius summa est $\frac{\pi}{4}$. Hunc in finem subiungam logarithmum hyperbolicum ipsius π, quem olim reperi

$$1,14472,98858,49400,17414,34273,51353,05865.$$

Videndum igitur erit, num forte sit summa inventa $O = l\pi - lN$, ita ut N sit numerus satis simplex. Verum huiusmodi investigationes plerumque sine ullo successu instituuntur.

28. Ope posterioris methodi autem non solum summam seriei propositae elicuimus, sed etiam eius potestatum imparium, quas summas hic conspectui exponamus.

$$O=\frac{1}{3}-\frac{1}{5}+\frac{1}{7}+\frac{1}{11}-\frac{1}{13}-\frac{1}{17}+\frac{1}{19}+\text{etc.}=0,3349816$$
$$P=\frac{1}{3^3}-\frac{1}{5^3}+\frac{1}{7^3}+\frac{1}{11^3}-\frac{1}{13^3}-\frac{1}{17^3}+\frac{1}{19^3}+\text{etc.}=0,0322521$$
$$Q=\frac{1}{3^5}-\frac{1}{5^5}+\frac{1}{7^5}+\frac{1}{11^5}-\frac{1}{13^5}-\frac{1}{17^5}+\frac{1}{19^5}+\text{etc.}=0,0038602$$
$$R=\frac{1}{3^7}-\frac{1}{5^7}+\frac{1}{7^7}+\frac{1}{11^7}-\frac{1}{13^7}-\frac{1}{17^7}+\frac{1}{19^7}+\text{etc.}=0,0004455$$
$$S=\frac{1}{3^9}-\frac{1}{5^9}+\frac{1}{7^9}+\frac{1}{11^9}-\frac{1}{13^9}-\frac{1}{17^9}+\frac{1}{19^9}+\text{etc.}=0,00005010,$$
$$S=\frac{1}{3^{11}}-\frac{1}{5^{11}}+\frac{1}{7^{11}}+\frac{1}{11^{11}}-\frac{1}{13^{11}}-\frac{1}{17^{11}}+\frac{1}{19^{11}}+\text{etc.}=0,0000053$$
$$S=\frac{1}{3^{13}}-\frac{1}{5^{13}}+\frac{1}{7^{13}}+\frac{1}{11^{13}}-\frac{1}{13^{13}}-\frac{1}{17^{13}}+\frac{1}{19^{13}}+\text{etc.}=0,0000003$$
etc.

29. Ipsae quidem hae summae sine dubio parum attentionis merentur, nisi forte ad quantitates cognitas reduci potuerint. Verum quia in his seriebus neque ipsi termini secundum certam legem progrediuntur, neque etiam in signis plus vel minus certus ordo observatur, ista disquisitio primo intuitu plane impossibilis videri potuisset, quamobrem ipsa methodus, qua ad earum summas pertigimus, utique omni attentione digna est censenda, idque eo magis, quod satis abstrusis serierum potestatum proprietatibus innititur. Nisi enim summae serierum

$$1-\frac{1}{3^n}+\frac{1}{5^n}-\frac{1}{7^n}+\frac{1}{9^n}-\text{etc.}$$

pro casibus, quibus n est numerus impar, fuissent cognitae, tota haec investigatio frustra fuisset suscepta.