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OBSERVATIONS REGARDING DIFFERENTIAL EQUATIONS
of the second order.

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The reduction of differential equations of the second order generally is especially involved and complicated, and usually the reduction eludes the less attentive analyst. While we are pursuing the path of synthesis, and when we rise from the first fluxion [derivative] to the higher level, since we can assume that either there is agreement with a known differential, or else nothing is to be agreed upon, then these difficulties for which something needs to be said hardy occur. This cannot be avoided, if some problem is proposed involving elements of the second degree, and one has to precede by an analytical method. There are boundless second order or differentio-differential equations that can be given, and for which there appears to be no solution, and nobody is ignorant of that : likewise, there are just as many that can be solved that are only revealed to the more acute analysts, and for which we may concede that a solution can only be found by calling on their aid. But how are these differential equations to be distinguished from each other, and by what means are they to be solved ? I think that the answer to this question is neither known nor obvious; for it is still the task of the more sublime mathematicians to examine these things at some future time, and to consider in what circumstances these expressions admit of a solution.

For example, let there be a curve constructed in which some power *m* of the abscissa *x* is set out in order along a line in the ratio of the second order differentials of the ordinates *y* and inversely as the similar differentials of the same abscissae, by which means the curve is set out by a differential equation of the second order :

$$x^m ddx = ddy :$$

I say that no curve is satisfactory among those considered to be possible solutions, when the transition is made from the first order fluxion to the second order fluxion, unless some constant value is first given in the first order differential equation; nor is it possible for such an equation to be changed in any way except by the addition of equal terms to both sides, or by the substitution of their values. Likewise, from the opposing point of view [on going from the second order to the first order], certain curves are found with a determined constant [in the first order] that satisfy the conditions of the problem, of which there is an infinite number, all satisfying the conditions of the original equation. Finally these expressions that impose the wrong kinds of conditions on us from the true ones, can legitimately be set aside, as is seen to be the case in the most profound investigations. Generally, I propose that [a theory] can be scrutinized under that mathematical criterion to which so much material is subjected, namely that it can be used in all cases; nevertheless, this is certainly hardly the case for our integral calculus.

Henceforth, all the equations of the second order differentials are reduced to forms involving differentials of only the first order, which have been reached, either with an assumed constant or not, and in which the second fluxions as with the first have been solved in some manner by quantities of finite magnitudes, provided the proposed equation for each does not depart from its own indeterminate fluents [the name given at this time to integrated quantities]. Also, that it this can be equally well said for these expressions that can be reduced [or integrated], and that have in some way been put into this form with some industry. The other first order equation formed from the remaining terms, and which will not be embraced in our progress, in some particular cases can extend the skill of analysts; for if a general canon could be found for these, he would surely be the great Apollo for me. Meanwhile, the universal equation is now presented for consideration :

$$(A)\ zdx = dy,$$

in which all the first order differential equation forms are contained ; with the magnitude of any given function of the coordinates *x* and *y* designated by the letter *z*. I cross to second order differentials, with nothing assumed constant, and the equation

$$(B)\ zddx + dzdx = ddy$$

emerges, which , while it stays in this same permanent state, cannot be integrated by any effort. For, if the form itself remains unchanged, by asking for mutually acceptable values from the expression (A), then an infinite number of formulae are produced, which demands more skill. From the simplest forms I desire an example: in place of *dx* itself is substituted the equal quantity :

$$dx = dy:z\ ,$$

and the first term of the equation (B) is multiplied by the power $z^m dx^m$, the remainder truly by the equivalent dy^m , hence the new equation results :

$$(D)\ z^{m+1}dx^m ddx + dy^{m+1}dz : z = dy^m ddy.$$

$$[For : z^m dx^m .(zddx + dzdx) = z^m dx^m .ddy;$$

giving $z^{m+1}dx^m ddx + z^m dx^{m+1}dz = z^{m+1}dx^m ddx + dy^{m+1}dz / z = dy^m .ddy.$]

Formulae of this kind can conveniently be reduced with the help of some constants, as this shall so often be the case ; generally, I describe these by some constant fluxion *dx:q*, where *q* is some magnitude given by the variables *x*, *y*, and the constants. I put

$$dx:q = dp\ [i],$$

and since *dx:q* is a constant, then it is equally the case that *dp* is a constant. Hence, in the equation *dx* = *qdp* by crossing over to the second order differentials we have

$$ddx = dqdp\ [ii].$$

[*i. e. p* is an independent variable with a constant fluxion or derivative dp.]
I decide in addition that

$$dy = udp\ [iii],$$

and with the second order differences taken according to the same hypothesis, with *dp* constant, it follows that in this case that

$$ddy = dudp\ [iv].$$

With these proposed values [*i* - *iv*] substituted in expression (D), as determined above, and with the terms found, the equation arises :

$$z^{m+1}q^m dqdp^{m+1} + u^{m+1}dz : z \times dp^{m+1} = u^m dudp^{m+1},$$

and on being divided by dp^{m+1},

$$z^{m+1}q^m dq + u^{m+1}dz : z = u^m du,$$

and on being integrated according to the common rules, with the addition of a constant *g* which is not to be omitted :

$$g + q^{m+1} : \overline{m+1} = u^{m+1} : \overline{\overline{m+1} \times z^{m+1}}\ ,$$

[or $g + q^{m+1}/(m+1) = u^{m+1}/(m+1) \times z^{m+1}$; since $d\left(\frac{u^{m+1}}{(m+1)z^{m+1}}\right) = \frac{u^m du}{z^{m+1}} - \frac{u^{m+1}dz}{z^{m+2}}$.]

which equation gives :

$$u = z \times \overline{q^{m+1} + gm + g}^{\frac{1}{m+1}}, \text{[or } u = z \times \left(q^{m+1} + gm + g\right)^{\frac{1}{m+1}}.\text{]}$$

and since *dy* = *udp* = *udx:q*, with the opportunity for a substitution being taken, the reduced equation occurs :

$$(E)\ dy = zdx \times q \times \overline{q^{m+1} + gm + g}^{\frac{1}{m+1}} \text{ [or } dy = zdx \times \left(q^{m+1} + gm + g\right)^{\frac{1}{m+1}}.\text{]}$$

By working in this manner some consequences flow.

1. If the magnitude of *z* is to be determined, the equation (E) can also be constructed by quadrature when this is possible, and the variables are separable, obviously I consider indefinite curves that correspond to our formula, and also the nature of the curve is changed on account of the change in the constant equal to *dx:q*, and for whatever the value of the constant *q*, a new equation is put in place, which can be either algebraic or transcendental.

2. Though from a different value of the magnitude *q* diverse curves draw their origin, yet it is clear for any hypothesis that a place is found for a curve for a given *q*, as thus I may say, the principal part depending on the fundamental equation : $(A)\ zdx = dy$; for if the constant *g* that we added by integration is made equal to zero, then the equation (E) is at once changed into (A). In this case, nothing that refers to a certain differential *dx:q* is accepted for the constant, since with *g* vanishing, also the quantity *q* vanishes.

3. If the equation (E) is differentiated further, the expression (D) is not restored except in two cases; either by putting *g* = 0, and by proceeding to the second differentials with no assumed constant, yet with terms multiplied by equivalent quantities, as has been done above; or by differentiation again, with the fluxion *dx:q* to be determined as before as the constant. Each, it is apparent retraces the steps of the previous analysis. The other equation (E) can be differentiated again, and with neither condition satisfied, it shows a number of formulas as high as the sky that we assume to be reduced from the expression (D).

4. With entirely the same arrangements, take the element *dy:q* for the constant; for by setting in place the method just treated, and in which for brevity I omit the deliberations, we come upon the reduced equation : $dx = dy : q \times \overline{mg + g}^{\frac{1}{m+1}}$ corresponding to our formula (D), in which it is to be noted equally, that with the constant *g* = 0 added in addition, the fundamental expression is produced : $(A)\ zdx = dy$.

5. Hence by gathering together what has been said, it can be seen, that the proposed fundamental second order differential form : (D) $z^{m+1}dx^m ddx^{m+1} + dy^{m+1}dz : z = dy^m ddy$, I would think as far as I am concerned to have satisfied the most hardened analysts, by observing how this expression was able to come about, either with no constant assumed, in which case the integral equation *zdx* = *dy* is found in its place, or by designating the fluxions *dx:q;* or *dy:q*, for constants, and then the integrations are :

$dy = zdx : q \times \overline{q^{m+1} + gm + g}^{\frac{1}{m+1}}; dx = dy : z - dy : q \times \overline{q^{m+1} + gm + g}^{\frac{1}{m+1}}$. I might note that the single equation *zdx* = *dy*, which by differentiation without the benefit of the constant, changes into equation (D), from the infinitude of others are able to be distinguished by the method explained above, which always stays the same with whatever constants are assumed, truly the rest with the constant varied are not to be used.

As we see, it remains whether in the other expressions, and in particular in the equation:

$$\text{(F) } x^m ddx = ddy$$

corresponding to the problem from the initial proposed conditions assigned that may be implemented by the change of the constant, and which differentiated again, with no constant assumed, takes the form (F) even with terms added, or subtracted, or with values put in place, which may restore the well-being of the equality. According to custom, put

$$dx = qdp \ [v] ,$$

which is on account of the constant *dp*, gives :

$$ddx = dqdp \ [vi] .$$

Again ,

$$dy = udp [vii],$$

that is :

$$ddy = dudp \ [viii],$$

and by substitution :

$$dx^m dqdp = dudp, \text{ or } dx^m dq = du,$$

and on integrating:

$$\int dx^m dq + g = u \ ;$$

but

$$dy = udx : q;$$

hence

$$dy = dx : q \times \int dx^m dq + gdx : q.$$

Here I note, since $g = 0$, and with the final equation reduced to the most simple form, clearly $dy = \int x^m dq \times dx : q,$ whatever the value of *q* in the equation, a different curve is produced, except perhaps when the exponent *m* is set equal to zero : $m = 0$, that assumption overturns the hypothesis. The same can be said for the constant decided for the fluxion *dy:q*, from which I infer, the first order differential equation will be sought in vain, since the proposed equation should be kept, and the form (F) $x^m ddx = ddy$ to be restored, without the aid of a constant; for if such an equation is given, with the supposition of some constant, something should emerge, as our analysis will yet show of the contrary.

Therefore it is agreed that the proposed problem, truly to find a curve, in which the given power of the abscissa is always in direct proportion to the second fluxion of the ordinate, and inversely as the similar fluxion of the same abscissa, can be solved, if a curve sought can be obtain with such a property for second order differences without a constant determined; and from the opposite point of view, to be satisfied by an indefinite number of curves, for if there is only one curve with no constant, then for any other a constant will be required.

It will not be inappropriate to bring forwards another example in the discussion, and the following equation is put forwards to be examined :

$$(G) \ x^m ddx = zddx + dz^2 + zzdz^2 .$$

This cannot be understood by our rule, it is seen from a first glance to be gathered from this, that each equation keeps its own functions in the variables *x* and *z*: the truth is if we put $zdz = dx$, then the new expression

$$(H)\ x^m ddx = ddy + dy^2$$

resulting from this substitution is not disputed to be nearly on a par with the above rules that set out the above solution.

1. In the first place I assign the differential dx to be a constant, and hence $ddx = 0$. Therefore with the term $x^m ddx$ vanishing, there remains the term : $-ddy = dy^2$, or $-ddy : dy = dy$, and by integrating, log. $dx:dy = y$, or $dx:dy = l^y$, hence $dx = l^y dy$, and finally $dx:x = dy$, which equation gives the common [natural] logarithm [with a base called l rather than e].

2. I set the other differential dy as a constant, in which by hypothesis, with $ddy = 0$, then $x^m ddx = dy^2$. I put $dx = sdy + cdy$, where the constant c is either positive or negative, and with the variable s to be determined. I cross over to the second differentials [in this second equation], and the equation $ddx = dsdy$ presents itself. Hence by substitution, $x^m ds = dy$; but $dy = dx : \overline{s+c}$; hence $sds + cds = x^{-m} dx$, and by integration with the omission of the useless constant g : $ss : 2 + cs = x^{-m+1} : -\overline{m+1}$, or $s + c = \sqrt{2x^{-\overline{m+1}} : -\overline{m+1} + cc}$: but $dx = \overline{s+c} \times dy = dy \times \sqrt{2x^{-\overline{m+1}} : -\overline{m+1} + cc}$; therefore $dx : \sqrt{2x^{-\overline{m+1}} : -\overline{m+1} + cc} = dy$.

3. I inquire whether with the logarithm found in no.1, taken with constant dx, the place can be taken equally with the substitution of the constant dy in no.2. By making the constant $c = 0$, I take the risk and say that possibly the quantity $\sqrt{2x^{-\overline{m+1}} : -\overline{m+1}}$ can be made equal to the magnitude x. Since hence by squaring : $2x^{-\overline{m+1}} = -\overline{m+1} \times xx$, & since it should be the same quantity in the coefficients as in the two exponents, it follows that an equality cannot be reached, except by putting $-m + 1 = 1$, in which case the value of the exponent is determined as $m = -1$.

4. In the formula

$$(H)\ x^m ddx = ddy + dy^2,$$

by limiting, as has been said, the value of the exponent to $m = -1$, then in the equation

$$x^{-1} ddx = ddy + dy^2$$

we are able to arrive at no assumed constant, the integration of this, according to this hypothesis, is the equation of the differential for the logistic : $dx:x = dy$; for by descending to the second order differential without the aid of constants, we have : $ddx : x - dx^2 : xx = ddy$, but $dx^2 : xx = dy^2$; hence $ddx : x = ddy + dy^2$.

5. For if the value of m itself is not equal to the negative quantity -1, nothing can be submitted to arrive at the expression (H), except by some fluxion as the constant may be determined.

6. Moreover , by proceeding generally, as has been done above, I repeat the equation :

$$(H)\ x^m ddx = ddy + dy^2.$$

I take for the constant element

$$dx:q = dp,$$

and equally I set up :

$$dy = udp,$$

in order that again I can obtain by differentiation :
$ddx = dqdp; ddy = dudp,$ and on dividing by *dp*,
$x^m dq = du + udy;$ or $dy = udp = udx : q;$ therefore: $x^m dq = du + uudx : q.$

7. Generally the method of separating the variables in this expression, even if the quantity is given *q* in some way by functions in which only *x* is unknown, in desperation. Nevertheless I warn, that if the exponent is made *m* = -1, it is easier for the variable *u* to be made equal to the value of the fraction *q:x*; for in place of *u* itself, with this value put in place, all the terms in the equation cancel each other out. Hence by gathering together in a subsidiary equation : $dy = adp = udx : q;$ with this value, the equation for the logarithm *dy = dx:x* is returned, for whatever constant assumed for expressing the magnitude of *dx:q*.

Finally it has been shown, at last to define our progress in the work of separating the variables under great difficulty. I have undertaken to adorn this spartan of old, and I have presented the idea in some Italian journal: but either I am deceived, or so subtle and so hard a work, upon which mainly hangs the wishes for the perfection of the infinitesimal calculus, is not to be moved forwards except by the joining together of men. Therefore in order that I may enthuse the cultivators of analysis and geometry to this inquiry of higher analysis, I propose the following problem.

In the above formula: $x^m dq = du + uudx : q,$ with the exponent *m* given as you wish, the quantity $q = x^n.$ is put in place. I ask in what ratio are the values of the other exponent *n* to be determined, in order that the separation of the variables will succeed, for the construction of the equation only by quadratures.

ANIMADVERSIONES IN AEQUATIONES
differentiales secundi gradus,
Autore Co. JACOBO RICCATO.

Reductio aequationum differentialum secundi ordinis plerumque est adeo perplexa, atque involuta, ut Analystam minus attentum frequentissime eludat. Dum syntheticae viae insistimus, & a primis fluxionibus ad altiorem gradum ascendimus, cum assumatur tanquam constans vel nota differentia, vel nulla, eae difficultates, de quibus sermo erit, vix occurrunt; quae tamen evitari nequeunt, si problema aliquod proponatur secunda elementa involvens & analytica methodo procedendum sit. Infinitas dari formulas differentio-differentiales, ad quas pervenitur, nulla adhibita constante, nemo profecto ignorat : totidem quoque exhiberi posse, ad quas pervenire non conceditur, nisi constante in subsidium vocata, acutiores non latet Analystas : at quomodo ab invicem dignosci queant, & qua ratione tractandae sint, non ita compertum neque obvium puto; cum tamen sublimioris Geometriae officium sit inspicere, quousque, & quibus in circumstantis expressiones istae solutionem admittant.

Sit ex. gr. construenda curva, in qua quaelibet abscissae dignitas se habeat directe ut secunda differentia ordinatae, & inverse ut similis differentia ejusdem abscissae, quae curva exponitur per aequationem differentialem secundi ordinis $x^m ddx = ddy$: aio nullam curvam inter possibiles quaestioni satisfacere, si a primis ad secundas fluxiones fiat transitus, absque eo quod aliqua prima differentia usurpetur pro constante, nec juvabit, salva aequalitate, aequationes ipsas quomodocunque alterare sive per additionem terminorum aequalium, sive per valorum substitutionem : at ex opposito, constante determinata, inveniuntur quidem curvae problematis conditionem implentes, sed numero infinitae, & indole differentes; utpote quae variantur ad arbitrariae constantis mutationem. Posteriores hasce expressiones, quae sub falsa specie nobis imponunt, a veris, atque legitimis secernere, videtur esse profundioris indagationis; nihilominus certum, & quantum subjecta materia patitur, generale Criterium Mathematicis examinandum propono, quod saltem usui erit in his omnibus casibus, in quibus nos calculus integralis haud describit.

Porro ad formulas primis tantum differentiis implicitias revocantur aequationes omnes differentiales secundi gradus, ad quas, sive assumta sive non assumta constante, perventum est, & in quibus secundae fluxiones cum primis, & cum finitis magnitudinibus quomodocunque miscentur, dummodo alterutra ex indeterminatis fluentibus cum suis functionibus aequationem propositam non ingrediatur; quod dicendum pariter de illis expressionibus, quae hanc formam aliqua adhibita industria redigi possunt : caeterum in reliquis, quas progressus noster non complectitur, ad aliquos casus particulares se extendere potest Analystarum solertia; at si quis canonem generalem inveniret, is profecto esset mihi magnus Apollo. Interim consideranda venit aequatio catholica $(A)\ zdx = dy,$ in qua omnes formulae differentiales primi gradus continentur; cum litera *z* designet magnitudinem utcunque datam per functiones coordinatarum x & y. Transeo ad alteriores differentias, nulla assumta constante, proditque aequatio $(B)\ zddx + dzdx = ddy,$ quae, dum in eodem statu permanet, nullo negotio integratur. Quod si ipsius forma immutetur, subrogatis valoribus ab expressione (A) mutuo acceptis, tunc infinitae formulae oriuntur, quae majus artificium postulant. A simplicioribus exemplum peto : loco ipsius *dx* substituatur quantitas aequalis *dy:z,* & primis terminus aequationis

(B) multiplicetur per dignitatem $z^m dx^m$, reliqui vero per aequivalentem dy^m, unde resultet nova aequatio $(D)\ z^{m+1}dx^m ddx + dy^{m+1}dz : z = dy^m ddy.$

Hujusmodi formulae expedite reducuntur ope alicujus constantis, quod ut fiat quantum fieri potest, generaliter designo pro constante fluxionem *dx:q*, est autem *q* magnitudo quomodocunque data per indeterminatas *x*, *y*, & constans. Pono *dx:q = dp*, & cum sit *dx:q* constans, erit pariter constans *dp.* Hinc in aequatione *dx = qdp* transeundo ad secundo differentias habebimus *ddx = dqdp*. Praeterea statuo *dy = udp*, & sumtis secundis differentiis in eadem hypothesi constantis *dp*, erit *ddy = dudp*. Subrogatis in expressione (D) valoribus ut supra determinatis, & inventis, orietur aequatio $z^{m+1}q^m dqdp^{m+1} + u^{m+1}dz : z \times dp^{m+1} = u^m dudp^{m+1}$, & dividendo per dp^{m+1}, $z^{m+1}q^m dq + u^{m+1}dz : z = u^m du$, & summando per regulas vulgares, non omissa constantis *g* additione, $g + q^{m+1} : \overline{m+1} = u^{m+1} : \overline{\overline{m+1} \times z^{m+1}}$, quae aequatio dat $u = z \times \overline{q^{m+1} + gm + g}^{\frac{z}{m+1}}$, & quia *dy = udp = udx: q*, opportuna adhibita substitutione, occurrit aequatio reducta $(E) dy = zdx \times q \times \overline{q^{m+1} + gm + g}^{\frac{z}{m+1}}$

Ex hoc operandi modo sponte fluunt nonnulla consectaria.

1. Si determinanta magnitudine *z*, aequatio (E) construatur saltem per quadraturas quando fieri potest, & indeterminatae sunt separabiles, manifestum puto curvas infinitas nostrae formulae respondere, variatur enim natura curvae, ob mutatam constantem *dx:q*, & quilibet valor quantitatis *q* novam semper aequationem localem, sive algebraicam, sive transcendentem subministrat.

2. Quanquam alterato valore magnitudinis *q* curvae diversae originem ducant, certum tamen est in quacunque hypothesi locum invenire inter ipsas curvam, ut ita loquar, principalem dependentem ab aequatione fundamentali $(A)\ zdx = dy$; nam si fiat aequalis nihilio constans g, quam addidimus integrando, statim aequatio (E) transit in aequationem (A). In hoc casu nihil refert quaenam differentia *dx:q* accepta sit pro constante, cum evanescente *g*, etiam quantitas *q* evanescat.

3. Si aequatio (E) ulterius differentietur, non restituet expressionem (D) nisi duobus in casibus; vel ponendo g = 0, & procedendo ad secundas differentias nulla constante assumta, multiplicatis tamen terminis per quantitates aequavalentes, ut supra factum est; vel iterum differentiando, determinata prius pro constante fluxione *dx:q*. Utrumque patet relegendo analyseos vestigia. Caeterum aequatio (E) rursus differentiata, neutra conditione impleta, formulam toto caelo diversam ab expressione (D) quam reducendam assumsimus, exhibet.

4. Idem omnino contingit, sumto pro constante elemento *dy:q* ; nam operationem juxta traditam methodum instituendo, quam brevitati consulens omitto, deveniemus ad aequationem reductam $dx = dy : q \times \overline{mg + g}^{\frac{z}{m+1}}$ respondentem nostrae formulae (D), in quo pariter notandum; quod facta constante superaddita *g* = 0, prodit expressio fundamentalis $(A)\ zdx = dy$;

5. Denique ex dictis colligi posse videtur, quod proposita nuda formula differentiali secundi gradus (D) $z^{m+1}dx^m ddx^{m+1} + dy^{m+1}dz : z = dy^m ddy$, putarem ex asse me satisfecisse Analystae quantumvis moroso, observando ad hanc expressionem perveniri potuisse, vel

nulla constante assumpta, quo in casu locum invenit aequatio integralis $zdx = dy$, vel designando pro constantibus fluxiones $dx : q; dy : q$, & tunc summatorias esse

$dy = zdx : q \times \overline{q^{m+1} + gm + g}^{\frac{1}{m+1}}; dx = dy : z - dy : q \times \overline{q^{m+1} + gm + g}^{\frac{1}{m+1}}$. Addenderem, aequationem unicam $zdx = dy$, quae differentiata absque constantis beneficio transit in aequationem (D) ab aliis infinitis artificio supra explicato distingui posse, quia semper eadem manet in quacunque constantis suppositione, reliquae vero variata constante mutationi sunt obnoxiae.

Superest ut videmus, utrum in aliis expressionibus, & praecipue in aequatione (F)$x^m ddx = ddy$ respondente problemati ab initio proposito assignatae conditiones adimpleantur ad mutationem constantis, & quae rursus differentiata, nulla constante assumpta formulam (F) saltem terminis additis, vel subductis, aut valoribus subrogatis, salva aequalitate restituat. Fiat igitur de more $dx = qdp$, erit propter constantem dp, $ddx = dqdp$. Sit iterum $dy = udp$, hoc est $ddy = dudp$, & substituendo $dx^m dqdp = dudp$, seu $dx^m dq = du$, & integrando $\int dx^m dq + g = u$; sed $dy = udx : q$; ergo $dy = dx : q \times \int dx^m dq + gdx : q$.

Hic noto, quod facta g = 0, & reducta ultima aequatione ad simpliciorem formam, videlicet $dy = \int x^m dq \times dx : q$, quilibet valor ipsius q aequationem, & curvam diversam subministrat, nisi fortasse poneretur exponens m = 0, quod assumtam hypothesin evertit. Idem dicendum statuta constante fluxione $dy:q$, ex quibus infero, frustra quaeri aequationem differentialem primi ordinis, quae propositum praestare queat, & formulam (F)$x^m ddx = ddy$, sine constantis auxilio, restituere; nam si daretur talis aequatio, prodere se deberet in quacunque constantis suppositione, cum tamen nostra analysis contrarium ostendat.

Constat igitur problema propositum, nempe curvam invenire, in qua data dignitas abscissae sit semper directe ut secunda fluxio ordinatae, & reciproce ut similis fluxio ejusdem abscissae, solvi posse, si curva quaesita talem proprietatem obtinere debeat, sumtis secundis differentiis nulla constante determinata, & ex opposito curvas infinitas satisfacere, si modo una, modo altera constans usurpanda sit.

Non erit abs re aliud exemplum in medium afferre, & sequentem formulam $(G)\ x^m ddx = zddx + dz^2 + zzdz^2$ examini subjicere. Hanc sub canone nostro non comprehendi, videtur primo aspectu colligi ex eo, quod aequatio utramque indeterminatam x, z cum suis functionibus contineat : verum si fiat $zdz = dx$, nova expressio $(H)\ x^m ddx = ddy + dy^2$ ex hac substitutione resultans juxta regulas supra explicatas solutionem non respuit.

1. Imprimis designo pro constante differentiam dx, unde sit $ddx = 0$. Evanescente igitur termino $x^m ddx$. Evanescente igitur termino $x^m ddx$, remanet $-ddy = dy^2$, vel $-ddy : dy = dy$, & integrando log. $dx:dy = y$. vel $dx:dy = l^y$, hoc est $dx = l^y$ dy, & tandem $dx:x = dy$, quae aequatio dat logarithmicam vulgarem.

2. Statuo tanquam constantem alteram differentiam dy, in qua hypothesi, existente $ddy = 0$, erit $x^m ddx = dy^2$. Pono $dx = sdy + cdy$, c constans est sive affirmativa sive negativa, & s variabilis determinanda. Transeo ad secundus differentias, & sese offert aequatio $ddx = dsdy$. Hinc substituendo $x^m ds = dy$, sed $dy = dx : \overline{s + c}$; igitur

$sds + cds = x^{-m}dx$, & summando omissa inutili constantis g additione

$ss:2 + cs = x^{-\overline{m+1}} : -\overline{m+1}$, seu $s + c = \sqrt{2x^{-\overline{m+1}} : -\overline{m+1} + cc}$: atqui

$dx = \overline{s+c} \times dy = dy \times \sqrt{2x^{-\overline{m+1}} : -\overline{m+1} + cc}$; igitur $dx : \sqrt{2x^{-\overline{m+1}} : -\overline{m+1} + cc} = dy$.

3. Quaero utrum logarithmica inventa num.1. adhibita constante dx, locum pariter habere possit in suppositione constanti dy num.2 usurpata. Facta constante c = 0, periculum facio an forte quantitas $\sqrt{2x^{-\overline{m+1}} : -\overline{m+1}}$ posset esse aequalis magnitudini x. Quoniam hinc inde quadrando $2x^{-\overline{m+1}} = -\overline{m+1} \times xx$, & cum eadem quantitas debeat esse, tam in coefficiente, quam in exponente binario aequalis, sequitur ad aequalitatem non perveniri, nisi ponendo $-m + 1 = 1$, quo in casu determinatur valor exponentis $m = -1$.

4. In formula $(H)\ x^m ddx = ddy + dy^2$, limitando, ut dictum est, valorem exponentis $m = -1$, tunc ad aequationem $x^{-1}ddx = ddy + dy^2$ pervenire possumus nulla assumta constante, ejusque summatoria in hac hypothesi est aequatio differentialis ad logisticam $dx:x = dy$; nam ascendendo ad secundas differentias absque constantis auxilio, habebimus $ddx : x - dx^2 : xx = ddy$, sed $dx^2 : xx = dy^2$; ergo $ddx : x = ddy + dy^2$.

5. Quod si valor ipsius m non sit aequalis quantitati negativae -1, ad expressionem (H) nullo pervenire conceditur, nisi aliqua fluxio tanquam constans determinetur.

6. Procedendo autem generaliter, ut supra factum est, repeto aequationem $(H)\ x^m ddx = ddy + dy^2$. Sumo pro constante elementum $dx:q = dp$, & statuo pariter $dy = udp$, ut obtineam rursus differentiando $ddx = dqdp; ddy = dudp$, & dividendo per dp, $x^m dq = du + udy$; sed $dy = udp = udx : q$; igitur $x^m dq = du + uudx : q$.

7. Methodus generaliter separandi variabilis in hac expressione, etiamsi quantitas q detur quocunque modo per functiones solius ignotiae x, pro desperata habenda est. Moneo tamen, quod si fiat exponens $m = -1$, simplicior indeterminatae u valor prodit aequalis fractioni $q:x$; nam loci ipsius u hoc valore subrogo, omnes termini in aequatione se mutuo destruunt. Hinc collocata in aequatione subsidiaria $dy = adp = udx : q$; hoc valore, redit aequatio ad logarithmicam $dy = dx:x$, quaecunque fuerit constans assumta per magnitudinem $dx:q$ expressa.

Denique manifestum est, nostrum operandi progressum in maximam difficultatem separationis indeterminatarum postremo definire. Hanc spartam olim exornandam suscepi, specimenque aliquod Diario Italico exhibui : sed aut ego fallor, aut negotium tam subtile, tam arduum, ex quo potissimum pendet calculi infinitorum optata perfectio, non nisi conjunctis viribus promovendum est. Ut igitur ad hanc inquisitionem profundioris analysis & Geometriae cultores excitem, sequens problema propono.

In superiori formula $x^m dq = du + uudx : q$, dato ad libitum exponente m, statuatur quantitas $q = x^n$. Peto qua ratione determinandi sint valores alterius exponentis n, ut succedat indeterminatarum separatio, & aequationis constructio per solas quadraturas.