

[1]  
**QVADRATVRAE  
CIRCULI  
LIBER PRIMUS.  
DE  
LINEARVM POTENTIIS.  
ARGVMENTVM.**

*Liber hic fere Lemmaticus est, quemadmodum & alter, qui de Circulorum variis proprietatibus tractat. Porrò quo magis materia Lectori admanum sint, omnem in tres partes dividere placuit.*

*Prima quidem maximè circa linearum proportionem versatur.*

*Secunda varias trianguli affectiones exhibet.*

*Tertia illas linearum contemplatur proprietates, quae earum potentias concernunt.*

*This book is generally concerned with a lemma [i. e. the harmonic ratio, whereby a line segment is divided internally and externally in the same ratio], as with the following one which draws out various properties of circles. Again, so that more material should be at hand to the reader, it was determined to set everything out in three parts.*

*The first part is mainly concerned with the proportions of lines.*

*The second shows the various uses of triangles.*

*The third considers these properties of lines which are concerned with powers.*

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***PARS PRIMA.***

*De variae linearum inter se proportione.*

***PROPOSITO PRIMA.***

**S**int AB, BC, CD, DE, lineae in continua proportione; sit autem media FG, inter AB, BC; & inter AB, CD; media GH: Denique inter AB. DE. media sit HI. Dico AB. FG. GH. HL. lineas esse in continua analogia

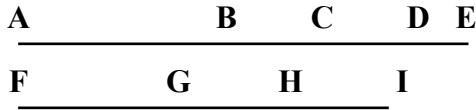
***Demonstratio.***

Quadratum AB est ad rectangulum AB.C. <sup>a</sup> ut AB. ad BC. & ABC. rectangulum ad AB.CD. rectangulum est ut BC. ad CD. id est ut AB, ad BC. Eodem modo rectangulum super AB.CD ad AB.DE rectangulum, est ut CD ad DE ; id est AB ad BC. Quadratum autem AB, ad FG. quadratum, est ut AB ad BC. Igitur quadrato AB. est ad quadratum FG, ut ABC rectangulum, ad rectangulum AB.CD. id est ad quadratum GH. Unde eandem rationem continuant quadrata AB.FG.GH.HI. ac proinde ipsae lineae; quod fuit demonstrandum.

<sup>a</sup> 1.Sexi.

**PART ONE.***Concerning the various proportions of line segments amongst themselves.***L1.§1.****PROPOSITON 1.**

**L**et AB, BC, CD, and DE be lines in continued proportion; moreover let FG be the mean proportion between AB and BC, and GH the mean proportion between AB and CD, and HI the mean between AB and DE. I say that AB, FG, GH, and HL are analogous lines in continued proportion.

**Demonstration.**

Prop.1. Fig.1

The square on AB is to the rectangle AB.BC as AB to BC, and the rectangle AB.BC is to the rectangle AB.CD as BC to CD: that is, as AB to BC. In like manner, the rectangle AB.CD is to the rectangle AB.DE as CD to DE, that is as AB to BC.

Moreover, the square on AB is to the square on FG

as AB to BC, [as FG.FG is equal to AB.BC]. Therefore the square on AB is to the square on FG, as the rectangle AB.BC is to the rectangle AB.CD, that is to the square on GH. Hence the squares on AB, FG, GH, and HI continue in the same ratio, and so on for the rest of the points on the line; which was to be shown.

**L1.§1. Prop. 1 Note:**

We use the convention that for points such as A, B, C, D, ... in sequence, the segment  $AB = a$ ,  $BC = b$ , ..., etc, using the first letter only. For the given segments in proportion, we are given that :  $a/b = b/c = c/d = d/e$ , etc., from which it follows that  $a^2/ab = a/b = ab/ac = b/c = ac/ad = c/d = ad/ae = d/e$ ; also, since  $f^2 = ab$ ,  $g^2 = ac$ ,  $h^2 = ad$ , etc.; then  $a^2/f^2 = a/b$ ;  $f^2/g^2 = b/c$ ;  $g^2/h^2 = c/d$ , etc. Thus,  $a/f = f/g = g/h$ , etc, and so these lines are also in continued proportion.

[2]

**PROPOSITO II.**

Circulum ABC contingant rectae DA, DC. & ductâ AC, ponatur altera DG occurrens in H. & circulo in G & I. Dico rectam DG. sectam esse in I & H. ut sit DG ad GH. quemadmodum est DI ad IH.

**Demonstratio.**

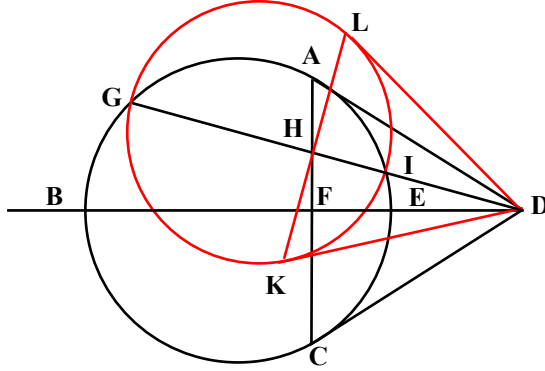
Super IG. diametro, describatur circulus GLK & per H punctum erigatur normaliter HL, vel HK, ad diametrum GI: ponaturque DL, erit itaque GHI rectangulo<sup>a</sup>, hoc est AHC, aequale rectangulum LHK, sive LH quadratum; pertinent igitur ad eundem<sup>b</sup> circulum puncta LAKC. iuncta igitur DL est aequalis tangenti DA, adeoque tangentinam LD quadratum est aequale quadrato LH; id est rectangulo GHI. id est<sup>c</sup> AHC unacum quadrato HD: id est quadratis HF, FD. unde LD quadrato rectangulum<sup>d</sup> IDG aequale; ac proinde cum IG. diameter sit circuli GLI. erit per Pappum<sup>e</sup> ut DG ad GH. ita DI ad IH. quod fuit demonstrandum. a.35.Tertii; b.ibid.; c. 34.Tertii; d. 56. Sexti; e. Lib.7.pr.155.

**L1.§1.**

**PROPOSITON 2.**

The lines DA and DC touch the circle ABC, and the line AC drawn, a line DG is drawn meeting the former line in H and the circle in G and I. I say that the line DG is divided by the points I and H so that [the ratio] DG to GH shall be as DI to IH.

**Demonstration.**



Prop.2. Fig.1

The circle GLK is described upon the diameter IG , and through the point H the line HL or HK is erected normal to the diameter GI. The line DL is put in place, and thus the rectangle GH.HI<sup>a</sup>, that is AH.HC, is equal to the rectangle LH.HK or the square on LH ; hence the points L, A K, and C pertain to lie on the circumference of another circle<sup>b</sup>. Therefore the connected line DL is equal to the tangent DA [i.e. D is the centre of the third circle, as it lies on the intersection of the two perpendicular bisectors of the chords AC and KL of this third circle.], and indeed the square on the tangent LD [from the symmetry of the diagram LD is also a tangent] is equal to LH squared, that is

[the same as] the rectangle GH.HI or AH.HC, together with the square of HD, that is also the sum of the squares on HF and FD. From which the square on LD is equal to the rectangle<sup>d</sup> ID.DG; and hence since IG is the diameter of the circle, the required ratio shall be according to Pappus<sup>e</sup> : as DG to GH, thus DI to IH. Q.e.d.

a.35.Tertii; b.ibid.; c. 34.Tertii; d. 56. Sexti; e. Lib.7.pr.155.

**L1.§1. Prop. 2 Note:**

The geometric proof makes use of well-known theorems. We present an algebraic proof as follows: Let HD = b, and GH = R + a, where GI = 2R, the diameter of the upper circle. Triangle GLI is right-angled in a semi-circle with centre O, not shown on the diagram, hence LH<sup>2</sup> = GH.HI = (R + a)(R - a) = ab, the final relation is true since OLD is also a triangle in a semi-circle : hence

$$R^2 = a(a + b), \text{ or } \frac{a+b}{R} = \frac{R}{a}; \frac{a+b}{R} \pm 1 = \frac{R}{a} \pm 1$$

$$\frac{a+b \pm R}{R} = \frac{R \pm a}{a}; \text{ hence } \frac{a+b+R}{R+a} = \frac{R}{a} = \frac{a+b-R}{R-a},$$

$$\text{or } \frac{DG}{GH} = \frac{DI}{IH}.$$

**Scholium.**

*Quoniam multoties in hoc opere recurret divisio lineae, prout praesenti propositione exposita est; operae pretium existimavi, aliquid illi peculiare nomen adiungere, quo illius cognito magnis innotescat; placuit autem nomenclaturam illi appropriare, sectionem lineae, secundum mediam et extremam rationem proportionalem; quoniam similitudinem non exiguam habet, cum ea divisione, quae in elementis vocatur ratio mediae & extremae: differt tamen ab ea, quod illa solum sit aequalitatis, hac vero de qua agimus, omnem omnino proportionem admittat; solam aequalitatis excludens.*

**Note.**

*Since often in this work there will be recourse to the division of a line, as shown with the present proposition, I have considered it a valuable aid to add some special name by which the section may become better known. Moreover, it has been decided to appropriate the nomenclature for the sections of a line following that for the ratio of mean and extreme proportions, since the ratio we use has considerable similarity to that division, which in elementary work is called the ratio of the mean and extreme. However*

our ratio is different from that ratio, which is for equality alone - indeed for the things we do here, all proportions are admitted; only equality being excluded.

[In what follows, numerous situations where this ratio arises, and even its invariance, are investigated : all of Gregory of St. Vincent's proofs of course are in the frame of Euclidean geometry. However, in some of the propositions we recognize properties from the realms of projective geometry, in which case we now talk about the invariance of the cross-ratio  $(AC/CB)/(AD/DB)$ ].

### PROPOSITO III.

Ex termino diametri circuli ABC: sumantur arcus hinc inde aequales, BC.BD. & ducta qualibet CF, quae occurrat perimetro in E, ponatur DE intersecans AB in G.  
Dico rectam FB. sectam esse in A & G. media & extrema ratione proportionali.

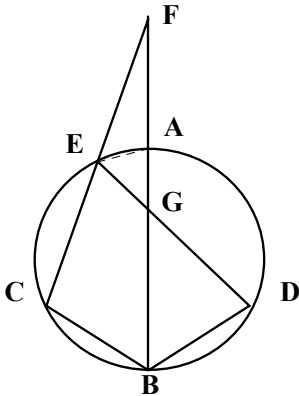
#### Demonstratio.

Demonstratum invenies libro nostro de Circulis, propos. 101. rectam BF ad FA. eandem habere rationem, quam BG.GA. hoc est totam FB. ad alterum extremorum FA, eandem rationem obtinere, quam BG alterum extremum, habet ad medium AG ; ergo secta est recta FB. in A & G. media & extrema ratione proportionali. Secundam explicationem praecedenti Scholio factam.

#### L1.§1.

### PROPOSITON 3.

From the ends of a diameter of the circle ABC: the equal arcs BC and BD are taken, and any line CF is drawn, which crosses the perimeter in E, and DE is placed cutting AB in G. I say that the line FB is cut by A and G and that the mean and extreme ratios of the lengths are in proportion.



Prop.3. Fig.1

#### Demonstration.

You will find the demonstration in our book *Concerning Circles, Prop. 101*. The lines BF to FA are in the same ratio as BG to GA. That is the whole length FB to the external line FA, maintains the same ratio as the internal line BG has to the middle line AG ; therefore the line FB cut by A & G has its points proportional in the mean and extreme ratio. Following the explanation made in the preceding Note.

**L1.§1. Prop. 3 Note:**  $EA/BD = AG/GB = EA/CB = FA/FB$ ; as triangles EAG & DBG are similar, and likewise triangles FEA & FBC. Along the same lines as Prop. 2.

[3]

### PROPOSITO IV.

Secta sit AB. in C & D. media & extrema ratione proportionali, & composita ex altera extremarum, & media, Verbi gratia CB, bifariam secetur in E.  
Dico CE quadratum, rectangulo DEA, aequale exsurgere.

#### Demonstratio.

Producatur CA. in F, ut CA, AF aequales sint inter se : quoniam est ut BA ad AC ; ita BD ad DC ; erit componendo. BA cum AC. hoc est BF. ad FA. id est AC ut BC ad DC. sed totius BF, dimidia est AE, ipsius quoque BC, dimidia est EC, per constructionem.

Igitur AE, ad AC; est ut EC ad CD, & convertendo ut AE. ad CE ita est, EC ad DE. ergo AE.CE.DE. sunt in continue ratione. Unde & quadrato a CE. rectangulum AED aequale. Quod fuit demonstrandum.

a 36.Sexti.

**L1.§1.**

**PROPOSITON 4.**

AB is divided by C & D with the mean and extreme ratios in proportion, and a line is formed by adding the extreme and the other mean length. For the sake of an example CB is bisected in E. In this case, the square CE comes to equal the rectangle DE.EA.

*Demonstration.*

CA is produced to F in order that CA and AF are equal to each other. Since: BA to AC is as BD to DC. BA is to be added with AC. That is : BF to FA (or AC) as BC to DC; but AE is half of the whole length BF; and by construction, EC is half of BC too.



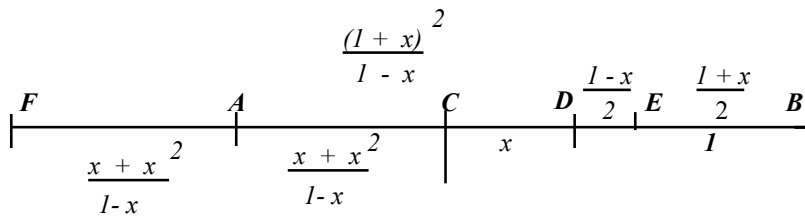
Prop.4. Fig.1

Therefore, we have AE to AC as EC to CD, & in turn, as AE to CE, thus EC to DE. Hence, AE, CE & DE are in a continued ratio. Thus the square on CE<sup>a</sup> is equal to the rectangle AE.ED. Q.e.d. a 36.Sexti.

**L1.§1. Prop. 4 Note:**

By definition, BA/AC = BD/DC, then (BA + AC)/AC = (BD + DC)/DC, or BF/AC = BC/DC. Then AE/AC = CE/DC, and AE/CE = AC/DC ; but also, (AE - AC)/AC = (CE - DC)/DC, or CE/AC = DE/DC, giving AC/DC = CE/DE = AE/CE, as required;

i. e. the lengths of the three consecutive segments AE, CE, DE are in proportion. In terms of algebra, this amounts to  $(1+x)^2/2(1-x)$ ,  $(1+x)/2$ , and  $(1-x)/2$ : thus, starting from DE, the common ratio is  $(1+x)/(1-x)$



Prop.4. Fig.2

**PROPOSITO V.**

Ponatur tres in continua ratione ED, EC, EA. & EB fiat aequalis EC , & ipsi CA. aequalis AF. dico AB. in C & D. divisam esse media & extrema ratione proportionali.

*Demonstratio.*

Quonium est ratio continuata, trium ED.EC.EA. erit quoque dividendo, DE ad DC ut EC ad CA. & componendo, ut CE ad DC sic AE ad AC. quare & BC ad DC ita BF. ad AC. hic est ad AF. & dividendo BD ad DC. ut BA ad AF. hic est , ad AC. Quod fuit demonstrandum.

**L1.§1.****PROPOSITON 5.**

Three line segments ED, EC, EA are placed in a continued ratio. EB is made equal to EC, similarly CA is equal to AF. I say that AB is divided by C & D with the extreme and mean ratios to be in proportion.

**Prop.5. Fig.1***Demonstration.*

Since the ratios of the three segments ED, EC, EA are in continued proportion. By subtraction, the ratio DE to DC as EC to CA is given too. And by addition, as CE to DC thus AE to AC. Whereby as BC to DC thus BF to AC, that is as AF, & by division BD ad DC, as BA to AF, that is, to AC. Q.e.d.

**L1.§1. Prop. 5 Note:**

This theorem is the converse of the previous one. We are given :  $EC/ED = EA/EC$  ; hence  $(EC - ED)/ED = (EA - EC)/EC$  or  $CD/ED = AC/EC$ , or  $DE/DC = EC/CA = EB/CA$  as required on inversion. Again, in a similar manner,  $CE/DC = (DC + DE)/DC = (EC + CA)/CA = AE/CA = AE/FA$ . Hence,  $CB/DC = BF/CA$ , as  $CB = 2.CE$ , and  $BF = 2.AE$  ; and by subtraction,  $(CB - DC)/DC = (BF - CA)/CA$  or  $BD/DC = AB/CA$ .

*Corollarium.*

Duo hinc consequuntur ; primum, rectangulum BDC. rectangulo ADE. esse aequale, alterum BAC. rectangulo aequale existere, EAD rectangulum. Primi demonstratio inde patet : quoniam BC. in E. secta est aequalia, & non aequalia in D. rectangulum sub inaequalibus segmentis totius BDC. unà cum <sup>a</sup> quadrato quosd sit ab intermedia DE. aequale est ei, quod à dimidia CE. describitur quadrato. Rursus cum AE sectasit in D. utcunque erit rectangulum AED. <sup>b</sup> aequale rectangulo ADE, & quadrato DE. sed iam ostensum est, AED rectangulum quadrato CE aequale. Igitur BDC rectangulum, unà cum DE quadrato, aequale est ADE rectangulo, unà cum DE quadrato. Sublato itaque communi DE quadrato, residuum erit BDC rectangulum ; rectangulo ADE aequale.

Alterum quoque sic manifestum erit. Quoniam BC in E, secta est bifariam , ipsique additur pars CA, erit rectangulum BAC, <sup>c</sup> unà cum quadrato CE, aduale quadrato AE; sed cum secetur AE in D, utcunque, erit quadratum AE, aequale duobus <sup>d</sup> rectangulis AED, & EAD. Igitur rectangulum BAC, unà cum quadrato CE, aequale est duobus rectangulis AED, EAD. sed horum alterum AED, ostendum est quadrato CE. esse <sup>e</sup> aequale. his itaque detractis, residuum manebit BAC rectangulum, rectangulo EAD aequale.

a. 5.Secundi ; b. 5.Secundi ; c. 6.Secundi ; d. 4.Secundi ; e. 4.Huius.

**Prop.5. Fig.1***Corollary.*

There are hence two consequences ; first, the rectangle BDC is equal to the rectangle ADE, the other rectangle BAC proves to be equal to the rectangle EAD. The first demonstration is apparent in this way: since BC has been cut equally by E, and not equally by D, the rectangle BDC under the unequal division of the whole segment together with the square of the intermediate length DE is equal to the square described on the half segment CE. [i.e.  $CE^2 = (CD + DE)^2 = CD^2 + 2.CD.DE + DE^2 = DE^2 + CD.(CD + 2.DE) = DE^2 + BD.DC$ ]. Again the section AE is cut in D, and the rectangle AED <sup>b</sup> is equal to the rectangle ADE and the square DE in anycase. [i. e.  $AE.ED = AD.DE + DE^2$ ]. But now it has been shown above that the rect. AED is equal to the square CE [  $AE.ED = CE^2$  by the common ratio property]. Therefore the rect. BDC,

together with the square DE, is equal to the rect. ADE, together with the square DE. Therefore by subtracting the common square DE, the rect. BDC will be left, equal to the rect. ADE .

[ i. e.  $DE^2 + BD.DC = CE^2 = AE.ED = AD.DE + DE^2$ ; hence  $BD.DC = AD.DE$  as required.]

The other consequence is made plain too [i.e. rect. BAC = rect. EAD]. Since BC is bisected in E, and the part CA is added to it, the rect. BAC<sup>c</sup>, together with the square CE will be equal to the square AE [ i. e.  $AE^2 - CE^2 = (AE - CE)(AE + CE) = BA.AC = \text{rect. BAC}$ ]; but as AE is cut in D, the square AE will be equal to the sum of the two rectangles<sup>d</sup> AED and EAD in any case [ i.e.  $AE^2 = AE.(AD + DE)$ ]. Therefore the rect. BAC, together with the square CE, is equal to the sum of the two rectangles AED and EAD. But one of these AED, has been shown to be equal to the square CE<sup>e</sup>. And thus when the square is taken away, the remainder shows that the rect. BAC equal to the rect. EAD.

[i. e.  $\text{rect. BAC} + CE^2 = AE^2 = \text{rect. EAD} + \text{rect. AED} = \text{rect. EAD} + CE^2$ , etc].

**L1.§1. Prop. 5 Corollary Note:**

This extra note is more by way of a neumonic to aid the memory than anything else.

In the first case, instead of considering the ratio BD/DC, we consider the rectangle BD.DC or BDC in the shortened version used in the text. This rectangle is found to equal the rectangle formed by the separation of the mean points A and D, and the distance of D from the mid-point of CB, i. e.  $BD.DC = AD.DE$ .

In the second case, correspondingly, the rectangle from the mean ratio is BA.AC, and this is equal to the rectangle formed from the separation of the mean points AD and the length from the other mean A to the mid-point E, i. e.  $EA.AD = BA.AC$ .

[4]

**PROPOSITO VI.**

Lineâ AB. sectâ in C & D. secundùm mediam & extremam rationem proportionalem, fiat circulus super AD. diametro, cuius centrum E, erigatur deinde BFG. normalis ad AB. ex puncto B.

Dico rectas omnes per C. ductas quae circulo in K. & L. occurrunt & FB. normali in aliquo puncto G. divisas esse secundum mediam & extremam rationem proportionalem.

*Demonstratio.*

Ex E. centro Circuli ALD, ducatur recta EH, normalis ad rectam KG; erit itaque KL. bifariam<sup>a</sup> divisa in H; & quia rectus est uterque angulorum ad H, & B, Circulus diametro GF descriptus, percurrat puncta H,<sup>b</sup> & B. Rectangulum itaque BCE,<sup>c</sup> aequale est GCH rectangulo; est quoque rectangulum BCE, rectangulo ACD aequale. per Corollarium praecedentis propositionis; huic autem aequale est d rectangulum KCL. Igitur rectangulo GCH, rectangulù KCL, aequale erit; adde ergo utriq; rectangulo quod ex CH. fit quadratù, erit denuo rectangulum KCL, unà cum quadrato CH, aequale rectangulo GCH, unà cum quadrato CH; sed rectangulo KCL, una cum quadrato CH ostenduntur est aequari rectangulum GHC, sive GCH unà cum quadrato CH id est quadratum<sup>e</sup> HL. Itaque rectangulum GHC, aequale est HL quadrato : utque CH ad HL . ita eadem HL ad HG. quo fit, ut cum sit HL aequalis HK. Ipsa GH.<sup>f</sup> in C. & L. secta sit secundum rationem extremam & mediam proportionalem, per praecedentem propositionem. Quod fuit demonstrandum.

a. 3. Tertii ; b. 31. Tertii ; c. 34. Tertii ; d. ibid. ; e. 5. Secundi; f. 5. Huius.

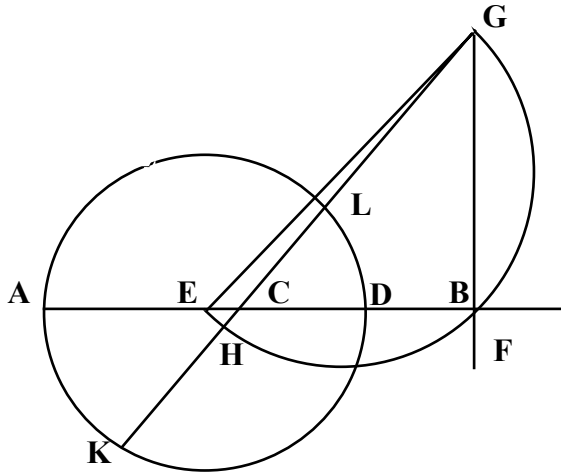
**L1.§1.**

**PROPOSITON 6.**

The line AB is cut in C and D in the ratio of mean and extreme proportions, and a circle with diameter AD is constructed with centre E. The normal BFG to AB is then erected from the point B.

I say that all the lines drawn through C which cross the circle in K and L and the normal FB in some point G, are also divided in this ratio.

[In this case, the ratio is  $AC/CD = AB/BD$ .]



Prop.6. Fig.1

*Demonstration.*

From the centre E of the circle ALD, the line EH is drawn normal to the line KG; and KL will thus be bisected<sup>a</sup> in H; and because both the angles at H and B are right, a circle with diameter GE [text has GF] can be described, that will pass through the points H and B<sup>b</sup>. And thus the rect. BCE<sup>c</sup> is equal to the rect. GCH; also, rect. BCE is equal to the rect. ACD<sup>d</sup>; therefore rect. GCH is equal to rect. KCL; therefore the square CH is added to each rectangle; it will be the new rect. KCL together with the square CH that is equal to the rect. GCH together with the square CH; but the rect. KCL<sup>f</sup> with the square CH is shown to be equal to the rect. GHC, or the rect. GCH together with the square CH, that is [both are equal] to the square<sup>e</sup> HL. And thus the rect. GHC is equal to the square HL: and thus as

CH to HL, so HL to HG; from which it shall be, as HL is equal to HK. The section GH itself<sup>f</sup> shall be divided in C and L according to MERP, by the preceding proposition. Q.e.d.

**L1.§1. Prop. 6 Note:**

Starting from  $GCH = KCL$ , we then have  $GCH + CH^2 = (GC + CH).CH = GH.HC = GHC$ ; again,  $KCL = (HL - CH)(HL + CH) = HL^2 - CH^2$ , or  $KCL + CH^2 = HL^2$ ; thus  $GH.HC = HL^2$ , and HC, HL, & HG are in continued proportion as required.

**PROPOSITO VII.**

Esto ABC trianguli basis AC divisa bifariam in D; actâque per B lineâ BE, parallelâ basi AC, agatur per D linea quaecunque EF, occurrès trianguli ABC lateribus, in G.F.

Dico EF lineam in D & G punctis extrema & media ratione proportionali esse divisam: id est esse ut EF ad FD, sic EG ad GD.

*Demonstratio.*

Quoniam EB, & AC lineae aequidistant, erit ut EB ad DC, sic EF ad DF; sed est ut EB ad AD, sic EG ad GD, igitur ut EF ad FD, sic EG ad GD. Quod erat demonstrandum. In 2. figura est FD linea in E & G divisa media & extreme proportionali.

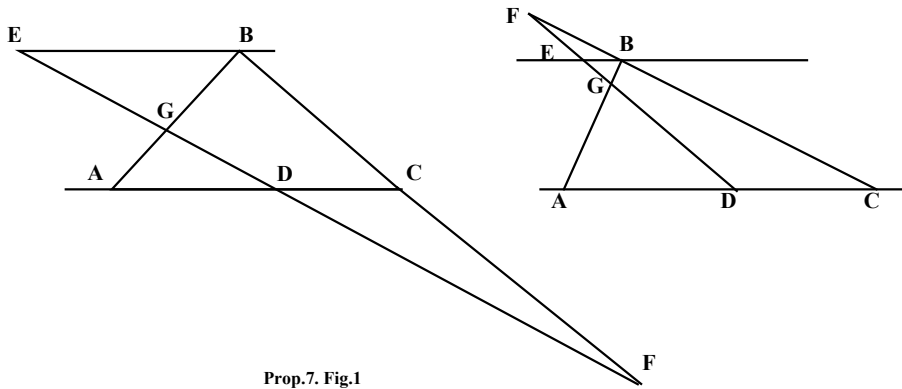


**LI.§1.**

**PROPOSITON 7.**

Let the base AC of triangle ABC be bisected in D, and a line BE is drawn through B parallel to the base AC. Some line EF is drawn through D, crossing the sides of the triangle ABC in G and F.

I say that the line EF is divided in D & G in points in the extreme & mean ratio of proportion: that is they are as EF to FD, so EG to GD.



Prop.7. Fig.1

*Demonstration.*

As the lines EB and AC are parallel, EB is to DC as EF is to DF; but as EB is to AD, thus EG is to GD, therefore as EF is to FD, so EG is to GD. Q. e. d.

In figure 2, the line FD is divided proportionally in the mean and extreme ratio by E & G.

[5]

**PROPOSITO VIII.**

Sit angulus ABC divisus bifariam rectâ in BD, ad quam erecta ex B normali BE, agatur linea quaecunque EG per punctum D.

Dico EG lineam in F & D, extrema & media ratione proportionali esse divisam.

*Demonstratio.*

Ponatur per D, linea AC, parallela rectae BE: occurrens anguli ABC lateribus, in A & C. Quoniam igitur AC linea aequidistat, rectae EB. quae normalis ponitur ad lineam BD, erunt anguli ADB. CDB recti; sunt autem per constructionem, & anguli ABD, CBD inter se aequales; & BD linea communis, igitur triangulum ABD aequale est triangulo CBD, & AD basis, aequalis basi DC : igitur per praecedentem ut EG ad GD, sic EF ad FD, Quod erat demonstrandum.

In 2. figura linea GD divisa est, secundum mediam & extremam proportionalem.

**LI.§1.**

**PROPOSITON 8.**

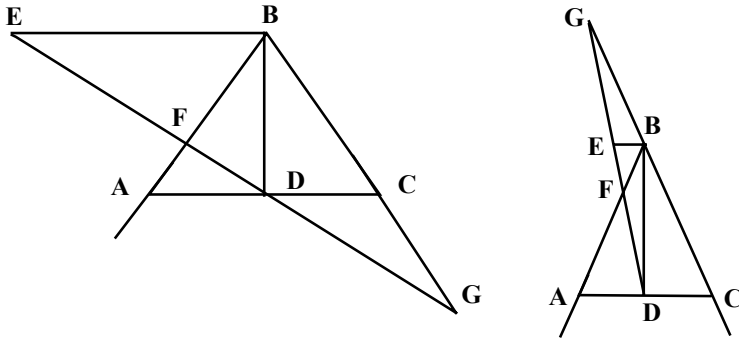
Let the line BD bisect the angle ABC, to which the normal BE has been erected from B, while some line EG is sent through the point D.

I say that the line EG is divided by F & D in the extreme & mean ratio of proportionality.

*Demonstration.*

Draw the line AC through D, parallel to the line BE: crossing the sides of the angle ABC in A & C. Since the line AC is parallel to the line EB, which has been placed normal to the line BD, the angles ADB and CDB are right; but the angles ABD and CBD are equal by construction, and BD is a common line,

therefore the triangle ABD is congruent [equal in text] to the triangle CBD, & the base AD is equal to the base DC. Therefore by the preceding as EG to GD, thus EF to FD. Q. e. d.  
 In figure 2, the line EG is divided by F & D in the extreme and mean ratio of proportionality.



Prop.8. Fig.1

**PROPOSITO IX.**

Esto ABC trianguli basis AC in D bifariam divisa, iunctisque BD, agatur per B linea BE, parallela basi AC; dein ex C, recta ducatur CE, occurrens utcunque lineae BE in E, BD rectae in F, & lateri AB in G. Dico in prima figura: lineam EC; in secunda lineam FC; in tertia rectam EG, extrema & media ratione proportionali esse divisam.

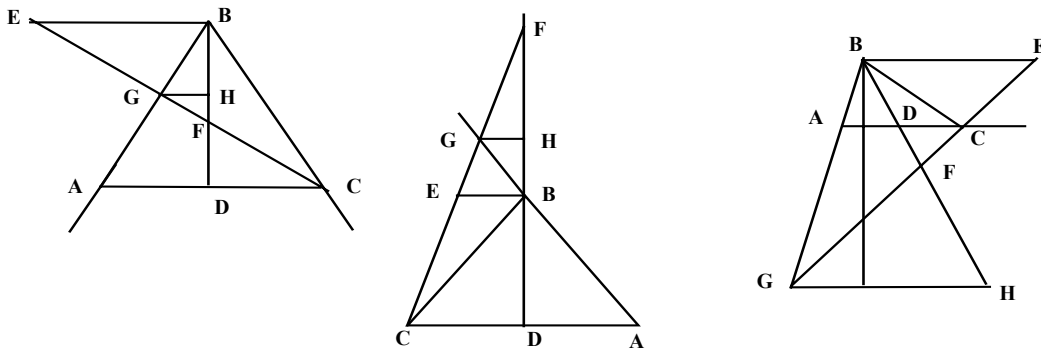
*Demonstratio.*

Ducatur GH linea, parallela basi AC: Quoniam AC, EB, GH, lineae aequidistant, erit AB ad BG ut CE ad GE; sed est ut AB ad GB, sic AD ad GH, hoc est DC ad GH, hoc est CF ad FG, igitur ut CE ad EG, sic CF ad FG. Q.e.d.

**L1.§1.**

**PROPOSITON 9.**

Let the base AC of triangle ABC be bisected in D, and with BD joined, the line BE is sent through B, parallel to the base AC; then from C, the line CE is drawn, crossing the lines BE in E, BD in F, & the side AB in G, in some manner. I say that in the first figure: the line EC, in the second figure the line FC, and in the third figure the line EG, is cut so that the mean and extreme ratios are in proportion.



Prop.9. Fig.1

*Demonstration.*

The line GH is drawn parallel to the base AC. Since the lines AC, EB, and GH are parallel, AB is to BG as CE is to GE. But as AB is to GB, thus AD to GH, that is DC to GH, that is CF to FG. Therefore as CE to EG, thus CF to FG. Q.e.d.

[6]

**PROPOSITO X.**

Demittantur ex A puncto, lineae tres AB, AC, AD, quae angulos constituent BAC, CAD rectis minores. Ponatur autem quaevis ED occurrens ductis ex A. lineis, & divisa in B, & C. extrema & media ratione proportionali esse divisam.

Dico omnes lineas ex E ductas, occurrentes rectis AB, AC, AD, ab iisdem media & extrema ratione proportionali dividi.

*Demonstratio.*

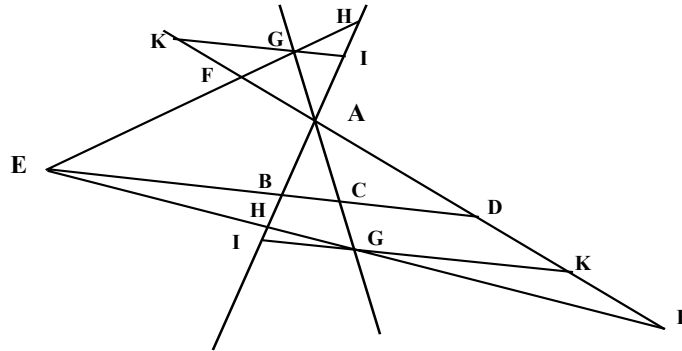
Ducatur ex E, quaevis linea EF, occurrens lineae AD in F, rectae AC in G, & AB lineae in H. dein per G, agatur linea IK, parallela rectae BD. Quoniam BD, IK lineae aequidistant; erit ut DC ad CB, sic KG ad GI; sed per hypothesim est, DC ad CB, ut DE ad EB, igitur erit ut DE ad EB, sic KG ad GI, & permutando ut DE ad KG, sic EB ad IG. est autem ut DE ad KG, sic EF ad FG, & ut EB ad IG, igitur ut EF ad FG, sic EH ad HG, & permutando ut EF ad EH, sic FG ad GH. Q. e. d.

**L1.§1.**

**PROPOSITON 10.**

Three lines AB, AC, AD are drawn from the point A, of which the constituent angles BAC and CAD are less than right angles. Moreover some line ED is drawn, crossing the lines drawn from A, and divided at B and C in the extreme & mean ratio of proportionality.

I say that all the lines drawn from E, crossing the lines AB, AC, AD, are divided in the mean & extreme proportional ratio.



Prop.10. Fig.1

*Demonstration.*

Some line EF is drawn from E, crossing the lines AD in F, AC in G, & AB in H, then through G, the line IK is sent, parallel to the line BD. Since BD and IK are parallel [equidistant in text] lines; DC is to CB, thus as KG to GI. But by hypothesis, DC is to CB, as DE to EB, therefore as DE is to EB, so KG ad GI, and by interchanging, as DE to KG, so EB to IG; but as DE to KG, so EF to FG, & as EB to IG, thus as EF to FG, so EH ad HG, & by interchanging, as EF to EH, so FG to GH. Q.e.d.

**L1.§1. Prop. 10 Note:**

DC/ CB = KG/ GI; but by hypothesis, DC/CB = DE/EB, therefore  $DE/EB = KG/ GI$ , and by interchanging,  $DE/KG = EB/IG$ ; but  $DE/KG = \underline{EF/FG}$ , &  $EB/IG = \underline{EH/HG}$ , & by interchanging the underlined terms, as EF/EH, so FG/GH, as required.

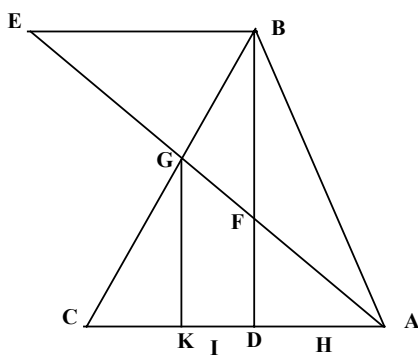
**PROPOSITO XI.**

Esto ABC trianguli basis AC, cui per B, verticum agatur aequidistans BE; oportet ex A rectam ducere AE, ut AE ad EG. datam habeat rationem maioris inaequalitatis H ad I.

*Constructio & demonstratio.*

Divisa AC bifariam D, fiat, ut H ad I sic AD ad DK, iunctisque punctis BD, erigatur ex K linea KG, parallela ipsi BD, occurrens BC lateri in G, tum ex A per G, agatur linea AE, secans BD lineam in F, & rectam EB in E; dico factum esse quod petitur. Quoniam FD GK lineae sunt parallelae erit ut AD ad DK, sic AF ad FG, sed AD est ad DK ut H ad I per constructionem igitur, ut H ad I sic AF ad FG est autem ut a AF ad FG sic AE ad EG, igitur ut H ad I sic AE ad EG. Duximus igitur ex A lineam, &c. Q. e. d.

a. IX huius.

**LI.§I.****PROPOSITON 11.**

Prop.11. Fig.1

Let the base of triangle ABC be AC, to which the parallel line BE is drawn through the vertex B; it is required to draw the line AE from A, in order that AE to EG shall give the ratio of the larger inequality H to I.

*Construction & demonstration.*

AC is divided in D so AD to DK as H to I; with the points B and D joined, the line KG is erected from K, parallel to BD itself, crossing the line BC in G; then the line AE is sent from A through G, cutting the line BD in F, & the line EB in E. I say that the construction is that sought. Because the

lines FD & GK are parallel: AD is to DK, so AF to FG, but AD to DK as H to I by construction; therefore, as H to I so AF to FG, but as AF to FG by Prop.9 above so AE to EG, therefore as H to I so AE to EG. Therefore we have drawn the line from A, &c. Q. e. d.

**PROPOSITO XII.**

Sit AC linea utcumque divisa in D, oportet illi rectam quandam CE adiacere, ut AE tota, in D & C, divisa sit media & extrema ratione proportionali.

*Constructio & demonstratio.*

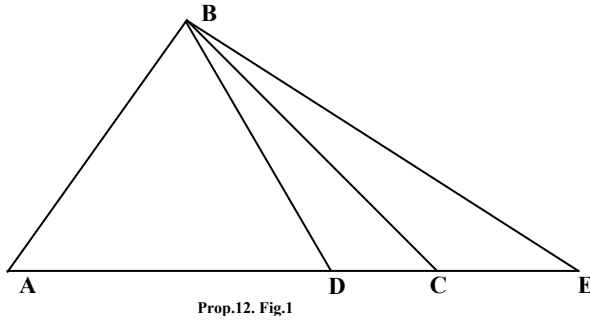
Super AC ut basi constituatur triangulum rectangulum ABC, habens ad B rectum angulum, & ex B demittatur linea BE constituens angulum EBC aequalem angulo DBC; occurrens AC lineae in E; dico factum esse quod petitur. Quoniam anguli DBC. CBE per constructionem sunt inter se aequales, & AB linea normalis ad rectam BC, <sup>a</sup> erit AD ad DC ut AE ad EC. datae igitur lineae AC adiecimus, &c. Quod erat postulatum.

a. Per 8. huius.

**LI.§I.****PROPOSITON 12.**

Let AC be a line divided somehow at D, it is required that a certain line CE be added on so that the whole line AE is divided in proportion in the mean and extreme ratio at D & C.

[7]



Prop.12. Fig.1

*Construction & demonstration.*

On AC as base a right-angled triangle ABC is constructed, having the right angle at B, & from B the line BE is drawn making the angle EBC equal to the angle DBC, and meeting the line AC in E. I say that the desired result has been achieved. Because the angles DBC & CBE are equal to each other from the construction, & the line AB is normal to the line BC, by Prop. 8 above, AD is to DC as AE ad EC. Therefore we have added to the

given line AC, &c. Which was postulated. [One may consider that the line EG containing the mean and extreme ratio in Prop. 8 has been moved to coincide with the base of the triangle]

### **PROPOSITO XIII.**

Esto ABC trianguli basis AC, in D bifariam divisa, iunctisque BD, ponatur per B linea BE parallela basi AC : tum in EB linea, punctum sumatur quodvis E, ex quo linea demittatur EC, occurrens AB lineae in G, & rectae BD in F; dein ex C recta erigatur CH, secans orthonaliter lineam AB in H. iunganturque puncta HF EH. Dico angulos EHG, FHG esse inter se aequales.

*Demonstratio.*

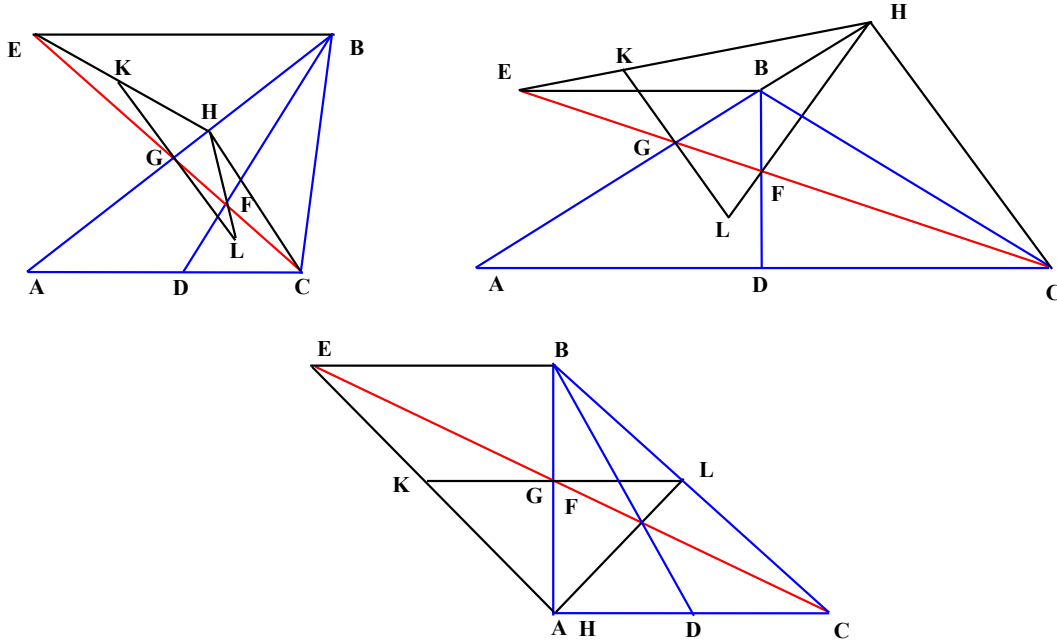
Ducatur per G linea KL, parallela rectae HC ;occurrans HE lineae in K, & FH rectae in L. Quoniam igitur HC KG, lineae sunt parallelae, erit ut HC ad KG, sic CE ad EG : sed ut CE ad EG, sic CF<sup>a</sup> est ad FG, igitur ut HC ad KG, sic CF ad FG : est autem ut CF ad FG, sic HC ad GL, igitur ut HC ad KG, sic HC est ad GL; quare KG. GL. lineae sunt inter se aequales. Rursum cum HC lineae, aequidistet linea KG, sit autem & HC normalis. ex hypothesi ad rectam AB, erit & KG linea perpendicularis ad lineam AB, adeoque anguli HGK, HGL recti ; igitur cum HG GL lineae duabus lineis HG GK sunt aequales, & anguli illis contenti recti, erunt HGK. HGL triangula inter se aequalis & similis, & anguli EHG, FHG aequalibus lineis subtensi, aequales. Q. e. d.

a.9. huius.

**L1.§1.**

**PROPOSITON 13.**

Let the base AC of triangle ABC be bisected in D, and with the line BD joined, the line BE is drawn through B parallel to the base AC : then any point E is taken in the line EB, from which the line EC is drawn, crossing the line AB line in G, and the line BD in F. From C the line CH is erected, meeting the line AB at right angles in H. Then the points HF and EH are joined. I say that the angles EHG and FHG are equal to each other.



Prop.13. Fig.1

*Demonstration.*

The line KL is drawn through G, parallel to the line HC ; crossing the line HE in K, and the line FH in L. Because the lines HC and KG are parallel, HC will be to KG, as thus CE to EG. But as CE to EG, thus CF is to FG, by Prop. 9 of this section; and therefore as HC to KG, thus CF to FG. But as CF is to FG, thus HC to GL, therefore as HC to KG, thus HC is to GL, and hence the lines KG and GL are equal in length. Again, as the line HC is parallel to the line KG, and moreover by hypothesis the line HC is normal to the line AB, and since the line KG is perpendicular to the line AB, therefore the angles HGK and HGL are right; therefore the two lines HG GL are equal to the lines HG and GK, and HGK and HGL are equal and similar triangles, and the angles EHG and FHG are subtended by equal lines. Q.e.d.

[8]

**PROPOSITO XIV.**

Esto ABC trianguli basis AC, bifariam divisa in D, iunctisque BD, agatur per B linea BE parallela rectae AC, in qua assumpto puncto quovis E, ducatur recta EC: & ex C erigatur linea CF, orthona rectae AB, occurrens EB lineae in G ; dein ponatur EF intersecans BD in I, & BC in H. Ducaturque CI, occurrens lineae EB in K.

Dico lineam EB in G & K, divisam extrema & media ratione proportionali.

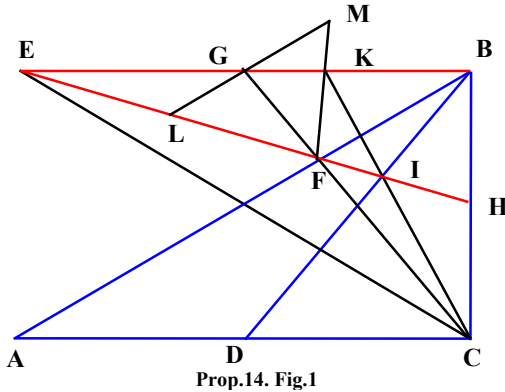
*Demonstratio.*

Agatur per G linea LM, parallela rectae AB: occurrens FK in M. Quoniam linea GC, per constructionem normalis est ad rectam AB, erunt anguli GFA, GFB recti ; est autem angulus KFB aequalis<sup>a</sup> angulo BFI, id

est angulo AFL: igitur reliquus angulus GFL aequalis est reliquo GFK ; rursum cum LM linea aequidistet lineae AB normali ad rectam GC,erunt anguli FGL, FGM recti; igitur cum tam anguli FGM, FGL, quam GFM, GFL sint inter se aequales, & FG linea communis, erunt FGM, FGL triangula, adeoque & latera LG, GM inter se aequalia. Quare ut FB ad LG, sic FB ad GM : sed est ut FB ad LG, sic BE ad EG, & ut FB ad GM, sic BK ad KG, igitur ut BE ad EG, sic BK ad KG. Quod erat demonstrandum.  
 a. Per 13. huius.

**L1.§1.**

**PROPOSITON 14.**



Let the base AC of triangle ABC be bisected in D, and with BD joined, the line BE is sent through B parallel to the line AC, in which some point E is taken, and the line EC is drawn. The line CF is erected from C, at right angles to the line AB, crossing the line EB in G, then EF is drawn intersecting BD in I, & BC in H. And CI is drawn, crossing with the line EB in K. I say the line EB is divided by G & K in the mean and extreme ratio of proportionality.

*Demonstration.*

The line LM is drawn through G parallel to the line AB, crossing FK in M. Since the line GC by construction is normal to the line AB, the angles GFA and GFB are right. Moreover, the angle KFB is equal to the angle BFI by Prop. 13, that is to the angle AFL: therefore the remaining angle GFL is equal to the remaining angle GFK ; again as the line LM is equidistet from the line AB to the normal to the line GC, the angles FGL and FGM are right; therefore as with the angles FGM and FGL, so the angles GFM and GFL shall be equal to each other, & the common line FG, the triangles FGM and FGL, and therefore the sides LG, GM are equal among themselves. Thus as FB to LG, so FB to GM : but as FB is to LG, thus BE to EG, & as FB to GM, thus BK to KG, therefore as BE to EG, thus BK to KG. Q. e. d .

**PROPOSITO XV.**

Constituant AB, BC rectum angulum, & AB maius sit latere BC. oportet AB latus producere in D, ut iunctâ DC. fiant proportionales DC, AD, & CB, AB.

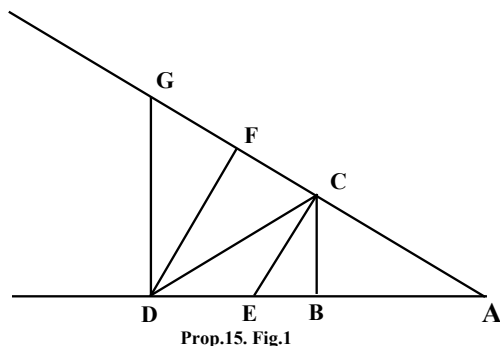
*Constructio & Demonstratio.*

Ductâ CA, demittatur ex C linea CE, normalis ipsi AC, & alia quedâ CD, que angulum DCB constituat duplum anguli CAB. dico factum esse quod petitur. Erigantur ex D lineae DF, DG;& DF quidem aequidistet ipsi EC; DG verò, rectae CB. Quoniam angulus ACE per constructionem rectus est, & CB linea normalis ipsi AE, erunt ABC. EBC triangula similia. & angulum ECB aequalia angulo CAB ; quare & angulus DCB, rectâ CE bifatium est divisus. Rursum cum DG linea. per constructionem aequidistet rectae BC; & FD linea, ipsi EC; erit angulus GDC aequalis angulo DCB, & angulus FDC, aequalis angulo ECD;adeoque. GDC angulus, rectâ FD bifariam divisus; sunt autem & anguli DFG, DFC recti (cum FD linea aequidistet ipsi EC, normali ad AC) & FD linea communis, igitur triangulum DFC, aequale triangulo DFG, & DC latus, lateri GD aequale. Quare CD ad DA ut GD ad DA ; sed est GD ad DA, ut CB ad BA, igitur ut CB ad BA sic CD ad DA. Quoad igitur AB lineam produximus, &c. Quos erat faciendum.

**L1.§1.****PROPOSITON 15.**

The lines AB and BC make a right angle, & with the side AB greater than BC. It is required that the side AB be produced to D, in order that DC joined shall make the lengths DC, AD ; CB, AB in proportion.

*Construction and Demonstration.*



With the line CA drawn, the line CE is drawn from C, normal to AC. A certain other line CD is drawn which makes the angle DCB double the angle CAB. I say that this is the sought construction.

The lines DF and DG are erected from D, and DF indeed shall be parallel [equidistant from] to EC itself, and DG truly parallel to the line CB. Since the angle ACE is right by construction, and CB is normal to AE, then ABC and EBC are similar triangles. The angle ECB is equal to the angle CAB, whereby the angle DCB is bisected by the line CE. Again, since the line DG by construction is parallel to the line BC, and

the line FD parallel to EC itself, then the angle GDC is equal to the angle DCB, and also the angle FDC is equal to the angle ECD. Hence the angle GDC is bisected by FD, and moreover the angles DFG and DFC are right (as the line FD is parallel to the line EC, normal to AC) and FD is the common line, hence triangle DFC is congruent [equal] to the triangle DFG, and the side DC is equal to the side GD. Whereby CD is to DA as GD to DA ; but GD is to DA as CB to BA, therefore as CB to BA, thus CD to DA. As long as we have produced the line AB etc. Q.e.d .

[9]

**PROPOSITO XVI.**

Sint ABC, ADC triangula, inaequalis altitudinis, super eadem base AC constituta ; oportet ducere lineam EF, parallelam base AC, ut EL ad MF datam habeat rationem TV ad XV.

*Constructio & Demonstratio.*

Constituantur per B & D lineae BG, DH parallelae basi AC; & DH quidem occurrat triangulo ABC in H ; dein AD producta, donec BG lineae occurrat in G, agatur per H, linea AHI, occurrans BG lineae in I : dividaturque BI in K, ut BI ad IK datam habeat rationem TV ad VX ; & ea K demittatur linea KA, occurrans BC lineae in L, tum per L ducatur linea EF parallela basi AC, occurrans ABC triangulo in E & L. rectae CD productae in F, & AD lineae in M.

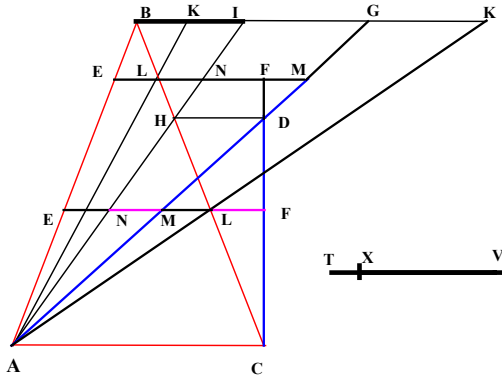
Dico EL ad MF datam habere rationem in TV ad VX. EF recta occurrat Lineae AI in N. Quoniam HAD, HCD triangula super eadem basi HD & inter eadem parallelas constituta sunt, erunt LF, MN lineae inter se aequales : demptâ igitur communis NF, vel addica ML, erunt LN, FM . lineae aequales; igitur ut EL ad LN sic EL est ad MF, sed EL est ad LN, id est BK ad KI ut TV ad VX; igitur ut TV ad VX ;sic EL est ad MF; duximus igitur lineam EF, &c. Quod erat praestandum.

Idem patet evenire si dicta duo triangula, aequales habeant bases in directû positas.

**L1.§1.****PROPOSITON 16.**

Let ABC and ADC be triangles with unequal altitudes, placed on the same base AC. It is required to draw the line EF parallel to the base AC, in order that EL to MF is in the given ratio TV to XV.





Prop.16. Fig.1

**Construction & Demonstration.**

The lines BG and DH through the vertices B and D are drawn parallel to the base AC, and indeed DH meets the triangle ABC in H. AD is then produced, thus meeting the line BG in G, and the line AHI is drawn through H to meet the line BG in I. BI is divided by K, in order that BK to KI has the given ratio TV to VX. From this point K the line KA is drawn crossing the line BC in L, then the line EF is drawn through L parallel to the base AC, cutting the triangle ABC in E and L, while the line CD is produced to F and the line AD to M.

I assert that EL to MF has the given ratio TV to VX. The line EF crosses the line AI in N. Because HAD and HCD are triangles on the same base HD, and have been placed between the same parallel lines, the lines LF and MN are equal : therefore either by taking/adding each from the common line NF or by adding each to ML, the lines LN and FM are also equal. Hence, as EL to LN thus EL to MF, but as EL is to LN, so BK to KI and TV to VX. Hence, as TV to VX, so EL is to MF; therefore we draw the line EF, &c. Which was to be presented.

The same seems to come about if the two said triangles have equal bases placed in order.

**PROPOSITO XVII.**

Sit AB linea divisa utcumque in C, oportet illi rectam addere BD, ut tota AD sit ad AB, ut AC est ad BD.

**Constructio & Demonstratio.**

Descripto super AB ut diametro, circulo AHB, erigatur ex B tangens EB, quae sit media inter AB & AC: dein ex E per F centrum circuli AHB recta ducatur BG occurrens circulo in H, addaturque lineae AB quadam BD, aequalis ipsi EH. Dico factum esse quod petitur. Quoniam HE, BD lineae sunt aequales, erunt HEG, BDA rectangula inter se aequalia; sed HEG rectangulum est aequale quadrato EB, id est ex constructione rectangulo CAB; igitur CAB, BDA rectangula sunt inter se aequalia ; sed HEG rectangulum est aequale quadrato EB, id est ex constructione tectangulo CAB; igitur CAB, BDA rectangula sunt inter se aequalia; quare AD est ad AB ut AC ad BD, datae igitur lineae AB quandam adiecimus ut, &c. Quod erat postulatum.

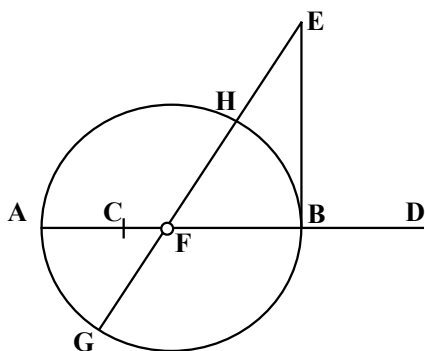
**L1.§1.**

**PROPOSITON 17.**

Let the line AB be divided at some point C. It is required to add the line BD to AB, in order that the total length AD shall be to AB, as AC is to BD.

**Construction & Demonstration.**

With the circle AHB described on AB as diameter, the tangent EB is erected from B, which shall be the mean between AB and AC [ i.e.  $EB^2$  is set equal to  $AB.AC$  or CAB]: then from E through F, the centre of the circle AHB, the line BG is drawn crossing the circle in H, and a certain line BD is added to the line AB, equal to EH. I assert that the required task has been performed.



Prop.17. Fig.1

AC/BD].

Because the lines HE and BD are equal, the rectangles HEG and BDA are equal to each other; but the rectangle HEG is equal to the square EB, which is equal to the rectangle CAB by construction. Therefore CAB and BDA are equal rectangles; but the rectangle HEG is equal to the square EB, that is from the construction for the rectangle CAB; therefore CAB and BDA are equal rectangles from which [There appears to be a repetition here of the same argument.], AD is to AB as AC to BD. Therefore to the given line AB we have added a certain line, &c. Q. e. d. [HE = BD  $\therefore$  rect. HEG = rect. BDA ; but rect. HEG = EB<sup>2</sup> = CAB (by construction)  $\therefore$  rect. CAB = rect. BDA; from which, AD/ AB =

### PROPOSITO XVIII.

Datis duabus rectis AB, CD, rectas addere, vel detrudere, in data ratione E ad F; ut compositae vel reliquae datam habeant rationem GH ad HI.

[10]

#### Constructio & Demonstratio.

1. Datae rectae AC CB sint inter se aequales. & tam ratio E ad F, quam GH ad HI, sit aequalitatis; addantur autem, vel detrahentur in ratione E ad F, lineae BK, DL, patet veritas propositionis.

2. Si fuerit AB maior quam DC; & ratio GH ad HI, maior quoque ratione AB ad CD, ratio autem E & F aequalitatis; Dico aequales addi non posse ut compositae sint in ratione data GH ad HI. Cum enim AB ponatur maior quam CD, fiat LB aequalis ipsi CD; addaturque rectae AB, quaevis BK; & si fieri possit, sit AK ad LK ut GH ad HI. erit igitur ratio AK ad LK maior ratione AB ad LB, id est ad CD; & quia ratio GH ad HI maior ponitur ratione AB ad LB, erit dividendo<sup>a</sup> quoque ratio GI ad IH maior ratione AL ad LB; sed est ut GH ad HI sic AK ad LK ex hypothesi, adeoque ut GI<sup>b</sup> ad IH, sic AL ad LK, igitur & ratio AL ad LK maior est ratione AL ad LB. Quod fieri non potest; <sup>c</sup> cum LB minor sit ipsa LK. Quare lineae aequales addi non poterunt, ut, &c, Demi vero poterunt aequales, ut reliquae sint in ratione GH ad HI: facta enim LB aequali ipsi CD, fiat ut GI ad IH, sic AL ad LM, cadet M inter L & B; ( cum AL ad LB ostensa sit minorem habere rationem, quam GI ad IH) dein rectae MB, fiat aequalis ND: Quoniam igitur per constructionem est ut GI ad IH, sic AL ad LM, erit componendo, AM ad LM, id est ad CN per constructionem ut GH ad HI. Quod erat demonstrandum.

a. 29. Quinti ; b. 17. Quinti ; c. 8. Quinti.

3. Si AB linea rursum fuerit maior recta CD, & ratio GH ad HI minor ratione AB ad CD, ratio autem E ad F aequalitatis, aequales addi poterunt ut compositae rationem habeant GH ad HI : fiat enim LB linea, aequalis rectae CD; quoniam per hypothesim, ratio AB ad LB, id est CD per construct. maior est ratione GH ad HI, erit<sup>a</sup> dividendo, ratio AL ad LB, maior ratione GI ad IH; facto igitur ut GI, ad IH, sic AL ad LN, patet LN maiorem esse linea LB; nam ratio GH ad HI minor ponitur ratione AB ad LB, id est CD, adeoque & ratio GI<sup>b</sup> ad IH, id est AL ad LN minor est ratione AL ad LB; tum CD lineae addatur quaedam DO aequalis ipsi BN: Quoniam igitur est AL ad LN, ut GI ad HI, erit componendo. AN ad LN, id est CO, ut GH ad HI: Quod erat primum. <sup>a. 29. Quinti ; b. ibid.</sup>

Iisdem positis, aequales demi non poterunt, ut reliquae datam habeant rationem GH ad HI : auferantur enim aequales KB, MD; & si fieri possit, sit ut GH ad HI sic AK ad LK, id est ad CM. cum igitur per hypothesim, ratio GH ad HU minor sit ratione AB ad LB, id est ad CD, sit autem ut GH ad HU, sic AK ad LK, id est ad CM, erit ratio AK ad LK, minor quoque ratione AB ad LB, & dividendo, a ratio AL ad LK, minor ratione AL ad LB, Quod fieri non potest cum LB linea, maior sit linea LK. quare hoc casu demi aequales lineae non poterunt ut reliquae, &c.

a. 29. Quinti ; b. 8. Quinti.

**L1.§1.**

**PROPOSITON 18.**

Lines are to be added or taken from the two given lines AB and CD, in the given ratio E to F; in order that the sum or difference shall have the given ratio GH to HI.

[10]

**L1.§1. Prop. 18 Preliminary Note:** In general, the lengths E and F are to be added/subtracted algebraically to AB and CD in order that the resulting ratio has the given value GH/HI; hence the ratio of the original lines becomes  $(a \pm e)/(c \pm f) = g/h$ ; where  $AB = a$ ,  $CD = c$ , etc. The inverse or inverted ratio F to E is also occasionally invoked, referred to as the reciprocal ratio by Gregorius. Thus a means is established for constructing a harmonic ratio from a given ratio. Gregorius now embarks on a long series of

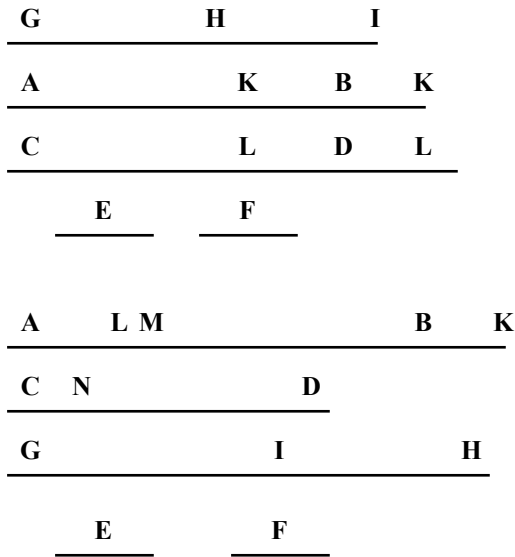
17 geometric demonstrations of different cases.

*Construction & Demonstration.*

1. The given lines AB and CD are equal to each other; and the ratio E to F is equal to GH to HI; moreover the lines BK, DL are added or taken in the ratio of E to F; [i. e.  $a = c$ ;  $ef = g/h$ .] the truth of the proposition is shown.

**L1.§1. Prop. 18 Note 1:** In the first trivial case, similar lengths are made equal :  $AB = CD$ ;  $E = F$ ;  $GH = HI$ ; AB is divided internally and externally by K;  $AB - E = AK$  or  $AB + E = AK$ ; similarly,  $CD - F = CL$ , or  $CD + F = CL$ , then  $AK/CL = GH/HI = (AB \pm E)/(CD \pm F)$ . If  $e = f$  then  $1 = g/h = e/f = (a \pm e)/(a \pm f)$ , for inner and outer division.

2. If AB is greater than DC and also the ratio GH to HI is greater than the ratio AB to CD, but the ratio E to F is still one of equality, then I assert that it is not possible to add the equal quantities E and F in order that the sums shall be in the given ratio of GH to HI.



Prop.18. Fig.i & ii

[i. e. given  $a > c$ ;  $a/c < g/h$ ;  $e = f$ , then  $(a \pm e)/(c \pm f) \neq g/h$  .]

For when AB is greater than CD, a length LB equal to CD itself is conveniently placed on the same line [The largest amount that may be taken away from AB], and some amount BK can be added to the line AB. If it is possible, AK will be to LK [i.e.  $CD + F$ ] as GH to HI. Therefore the ratio AK to LK will be greater than the ratio AB to LB (or CD). Because the ratio GH to HI is set larger than the ratio AB to LB, the ratio GI to IH is also be greater than the ratio<sup>a</sup> AL to LB

[i. e. internal division is effected by  $GH/HI - 1 > AB/LB - 1$ , giving  $GI/HI > AL/LB$ ];

but as GH is to HI thus AK to LK by hypothesis, and thus as GI<sup>b</sup> to IH, thus AL to LK on internal division, and therefore the ratio AL to LK is greater than the ratio AL to LB. Which cannot happen<sup>c</sup> when LB shall be less than LK itself: hence equal lines cannot be added, in order that, &c....

a. 29. book 5; b. 17. Book 5; c. 8. Book 5.

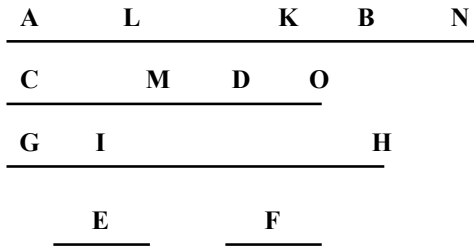
[Algebraically, the problem amounts to :  $(a + e)/(c + e) = g/h$ , where  $g/h > a/c > 1$ ; we see that  $g/h > a/c > (a + e)/(c + e) > 1$ , and so the ratio is moving away from the required ratio for  $e > 0$ .]

The equal amounts [E or F] can truly taken away, so that the remainders will be in the ratio GH to HI: indeed from the fact that LB is equal to CD itself, the ratio becomes as GI to IH, thus AL to LM, for some M falling between L and B; ( as AL to LB has been shown to have a smaller ratio than GI to IH) then the line MB [= E] shall be made equal to ND [= F]: Since indeed by construction as GI to IH, so AL is to LM, which can be added to give, AM to LM, that is equal to CN per construction, as GH to HI.

Q. e. d.

[In the subtractive mode, AB is initially divided internally by L as above :  $AB - CD = AL$ ; a new point M is placed on LB so that  $MB = E$ , & similarly,  $CD - MB = CN$ ; then  $GI/IH = (AB - E)/(CD - F) = (AB - MB)/(CD - ND) = AM/CN$  .

Algebraically:  $g/h \geq (a - e)/(c - e) > a/c > 1$ , and  $e = (gc - ah)/(g - h) > 0$ , as  $gc/h > a$ , for equality.]



Prop.18. Fig.iii

3. If the line AB is again longer than the line CD & the ratio GH to HI is smaller than the ratio AB to CD, whereas the ratio of E to F is still one of equality, equal amounts can be added in order that the terms have the composite ratio GH to HI. Indeed the line LB can be made equal to the line CD, since by hypothesis the ratio AB to LB, that is equal to CD by construction, is greater than the ratio GH to HI, and by division<sup>a</sup> the ratio AL to LB, will be greater than the ratio GI to IH

[  $AB/LB > GH/HI \therefore (AB - LB)/LB > (GH - HI)/HI$ , or  $AL/LB > GI/HI$ ; the products will therefore be as GI to

IH, so AL to LN, LN is shown to be greater than the line LB; for the ratio GH to HI is smaller than the ratio AB to LB, that is CD, & hence the ratio GI<sup>b</sup> to IH, that is AL to LN is less than the ratio AL to LB; then to the line CD a certain line DO is added equal to BN itself: therefore since AL to LN, as GI to HI, and by addition, AN will be to LN, that is CO, as GH to HI: which was the first part.<sup>a. 29. Book 5; b. ibid.</sup>

[Algebraically:  $a/c > (a + e)/(c + e) \geq g/h > 1$ ; equality holds when  $e = (ah - gc)/(g - h) > 0$ .]

With the same positions, equal amounts cannot be taken away, in order that the remainders have the given ratio GH to HI: for if the equal amounts KB and MD are taken away; and it was possible, then the ratio would be as GH to HI so AK to LK, (*i.e.* CM). When therefore by hypothesis, the ratio GH to HI is less than the ratio AB to LB (or CD), whereas it should be as GH to HI. Thus AK to LK (*i.e.* CM), the ratio will be AK to LK, which is less too than the ratio AB to LB, and by division<sup>a</sup> the ratio AL to LK is less than the ratio AL to LB. Which is not<sup>b</sup> possible when the line LB is greater than the line LK.

Wherefore in this case the equal lines cannot be taken away, in order that the remainders, &c.

a. 29. Book 5; b. 8. Book 5.

[Algebraically:  $a/c > (a - e)/(c - e) \geq g/h > 1$  has no solution for  $e > 0$ . ]

[11]

[11]

4. Si fuerint AB, CD lineae aequales, & ratio E ad F, inaequalitatis, & eadem cum ratione GH ad HI.

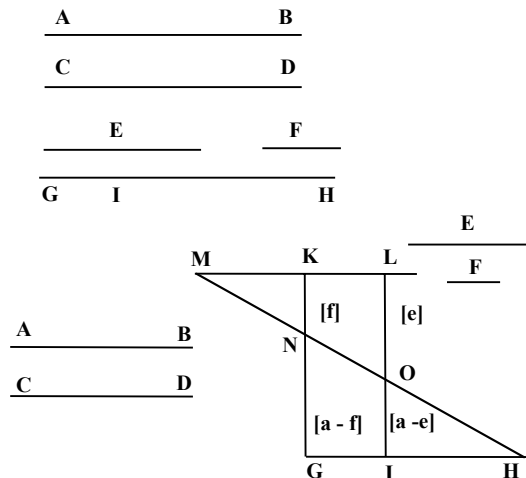
Dico addi non posse lineas in ratio E ad F. ut compositae datam habeant rationem GH ad HI. Rationi enim GH ad HI non potest addi ratio aequalitatis, quin producat ratio minor illa, quam habet GH ad HI. Ergo nec rationi aequalitatis potest addi ratio GH ad HI, quin producat ratio minor illa, quam habet GH ad HI, cum eadem sint quae resultant.

Iisdem positis, demi rectae poterunt, secundem rationem E ad F, ut residuem obtineant rationem GH ad HI reciprocem: id est, ut utriusque rationis antecedentes non sint in eadem linea: erigantur enim ex G & I parallelae GK, IL, aequales ipsis AB, CD; & ex L per K agatur linea LKM aequalis rectae GH, ducaturque linea MH occurrens GK, IL rectis in N & O. Quoniam igitur NK OL lineae sibi mutuo aequidistant, erit OL ad NK ablata ad ablatam ut LM ad MK, id est ut GH ad HI, id est ut E ad F per hypothesim; & ut GH ad IH, sic NG est ad OI, residua ad residuam: igitur lineas abstulimus secundum rationem E ad F ut reliquae datam obtineant rationem reciprocem GH ad HI.

4. If AB and CD are lines of equal length, & the ratio of E to F is one of inequality, in the same ratio as GH to HI.

I say that it is not possible to add the lines in the ratio E to F, in order that the sum has the given ratio GH to HI. For indeed a ratio equal to the ratio GH to HI cannot be reached by addition, as the ratio produced is less than GH ad HI. Therefore a ratio equal to GH to HI cannot be produced by addition, for such a ratio on being formed is less than GH is HI.

[Algebraically: It is required that:  $(a + e)/(a + f) = g/h = e/f$ , where  $e \neq f$ , which has no solution.]



Prop.18. Fig.iv

With the same lengths in place, lines can be taken away, following the ratio E ad F, in order that the ratio of the complements of GH to HI are obtained from the remainders. In order to do this, as each of the preceding ratios cannot be in the same line, the parallel lines GK and IL are erected from G and I equal to AB and CD. The line LKM is drawn from L through K equal to the line GH, and the line MH is drawn crossing the lines GK and IL in N and O. Since the lines NK and OL are parallel, OL is taken to NK as LM to MK, *i.e.* as GH ad HI, *i.e.* as E ad F by hypothesis. Thus, as GH to IH, so NG is to OI, difference to difference, therefore we have taken the lines according to the ratio E to F in order that the differences maintain the inverted ratio GH to HI given.

[*i.e.* on a line, the ratio  $(a - e)/(a - f) = g/h = ef$ , has the solution  $e = f$ , which is not allowed. but geometrically in a plane as in diagram (iv),  $GH/HI = (a - f)/(a - e)$ , where  $a - f$  is the complement of  $f$ , etc.]

5. AB, CD lineae sint rursus inter se aequales, & ratio E ad F minor ratione GH ad HI.

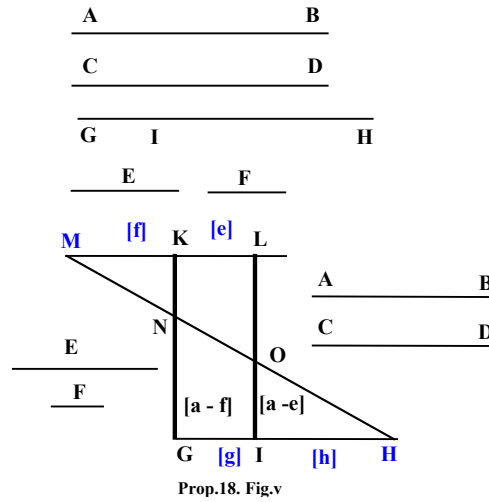
Dico addi non posse lineas secundum rationem E ad F, ut compositae rationem obtineant GH ad HI. patet: cum ratio GH ad HI ponatur minor ratione E ad F; ratio vero E ad F aucta ratione aequalitatis, fiat minor ratione E ad F.

Iisdem positis : auferri poterunt lineae, in ratione E ad F, ut reliquae rationem habeant GH ad HI, reciprocem : erigantur enim ex G & I, paralleles GK, IL, aequales ipsis AB, CD ; dein ex L per K, agatur linea LKM : factaque LM ad KM ut E ad F; ducatur recta MH occurrens GK, IL lineis in N & O: erit igitur LO ad NK, ablata ad ablatam, ut LM ad MK, id est ut E ad F; & NG ad OI, reliqua ad reliquam, ut GH ad IH.

5. Again, let AB and CD be lines equal to each other, and let the ratio E to F be less than the ratio GH to HI.

I say that it is not possible to add the lines following the ratio E to F, so that the sums have the ratio GH to HI. It is apparant that when the ratio GH to HI is compared with the ratio E to F, the given equal ratio is truly increased by the ratio of the inequality, but it is smaller than the ratio E to F. [Algebraically: It is required that:  $(a + e)/(c + f) = g/h > ef$ , where  $e > f$ , whereas it will always be less than  $ef$ .]

With the same lengths in place, lines can be taken away in the ratio E to F, in order that the remainders shall have the ratio derived from the inverted ratio AB - E to CD - F as GH to HI. For the parallel lines GH and IL are erected from G and I equal to AB and CD, then from L through K the line LKM is drawn, and the ratio LM to KM is made as E to F. The line MH is drawn crossing with the lines GK and IL in N and L: therefore LO will be to NK, difference to the difference, as LM to MK, *i. e.* as E to F, and NG to OI, remainder to the remainder, as GH to IH.



6. Sint iterum AB, CD lineae aequales, & ratio E ad F, maior ratione GH ad HI;

Dico addi posse lineas, in ratione E ad F, ut compositae habeant rationem GH ad HI. Demittantur enim ex G & I, parallelae GK, IL, datis AB, CD lineis aequales ; & ex K per L agatur parallela GH recta KM, ut KM sit ad LM, sic ut E ad F; tum ex H per M recta agatur HN; occurret illa lineis IL, GK in O & N: Cum enim ratio E ad F, id est KM ad LM, maior ponatur ratione GI ad IH; erit dividendo <sup>a</sup>, ratio KL ad LM. maior ratione GI ad IH; adeoque LM recta minor <sup>b</sup> ipsa HI, Quare & HM producta secabit lineas IL, GK in O & N: unde NK est ad LO, addita ad additam, ut KM ad LM, id est E ad F, & GN ad IO, tota ad totam, ut GH ad HI. Quod erat primum.

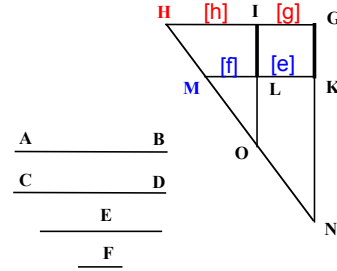
[12]

Iisdem positis : poterunt auferri lineae, in ratione E ad F, ut residuae datam habeant rationem GH ad HI, reciprocem. Erigantur enim ex G & I, parallelae GK, IL, aequales rectis AB, CD ; & ex L per K, ducatur linea LKM : ut LM sit ad KM quemadmodam est E ad F; ducaturque recta MH occurrens GK, IL lineis in N & O: patet LO ad NK, reliquam ad reliquam esse, ut LM ad MK, id est ut E ad F; & NG ad OI, reliquam ad reliquam, ut GH ad IH. Quod erat demonstratum. <sup>a</sup> 29. Quinti ; <sup>b</sup> 10. Quinti.

6. Again AB and CD shall be equal line, and the ratio E to F larger than the ratio GH to HI;

I say that it is possible to add the lines in the ratio E to F, so that the sums shall have the ratio GH to HI. For the parallel lines GK and IL are sent from the points G and I, given equal with the lines AB and CD. From K through L the line KM is drawn parallel to GH, in order that KM shall be to LM as E to F. The line HN is drawn from H through M that meets the lines IL and GK in O and N. For when the ratio E to F, *i. e.* KM to LM, is made larger than the ratio GI to IH, the ratio KL to LM will be divided<sup>a</sup> in a ratio larger than the ratio GI to IH, and thus the line LM will be less than <sup>b</sup> HI. And because the line HM produced will cut the lines IL and GK in O and N, then NK is to OL, sum to sum, as KM to LM, that is E to F, and GN to IO, total to total, as GH to HI. Which was the first case.

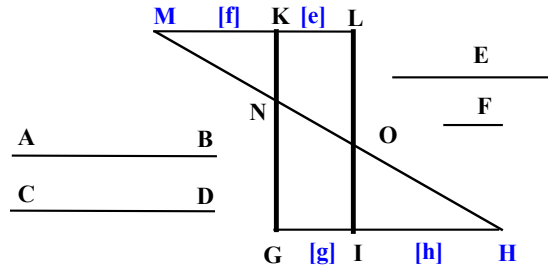
[Algebraically, we have:  $e/f > (a + e)/(a + f) = g/h$ , where  $e > f$ , which can always be satisfied.]



Prop.18. Fig.vi.a.

[12]

With the same arrangements: the lines can be taken away in the ratio E to F, in order that the remainders shall have the given inverted ratio of GH to HI. For the parallel lines GK and IL can be erected from G and I, equal to the lines AB and CD; and from L through K, the line LKM is drawn: in order that LM shall be to KM as E is to F; and the line MH is drawn crossing the lines GK and IL in N and O: it is apparent that LO to NK, which is the remainder to the remainder, shall



Prop.18. Fig.vi.b.

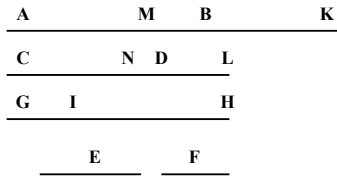
be as LM to MK, that is as E to F; and NG to OI, remainder to remainder, as GH to IH. Q. e. d.

[*i. e.*  $(a - f)/(a - e) = g/h$ ].

[12]

7. Sit AB maior recta CD, habeatque E ad F eandem ratioem quam AB ad CD, quae & eadem sit cum ratione GH ad HI. Addi poterunt lineae in ratione E ad F, ut compositae, rationem habeant GH ad HI. Addantur enim BK DL linea, secundum rationem E ad F: Quoniam igitur per hypothesin, AB est ad CD ut E ad F, id est per constructionem ut BK ad DL, erit <sup>a</sup> AK ad CL, ut AB ad CD, id est GH ad HI.

Iisdem positis: demi poterunt lineae, in ratione E ad F, ut reliquae rationem habeant GH ad HI: demantur enim BM, DN. Quoniam igitur est ut AB ad CD, sic E ad F, id est per constructionem BM ad ND, erit <sup>b</sup> AM ad CN, reliqua ad reliquam, ut AB ad CD, id est ut GH ad HI. Quod erat demonstrandum. <sup>a</sup> 19. Quinti; <sup>b</sup> *ibid.*



Prop.18. Fig.vii.

7. AB shall be longer than the line CD, and E to F shall have the same ratio as AB to CD, and which shall be the same with the ratio GH to HI. Lines can be added in the ratio E to F, in order that the sum can have the ratio GH to HI. For the lines BK and DL may be added, following the ratio E to F: Since indeed by hypothesis, AB shall be to CD as E to F, that is by construction, as BK to DL, <sup>a</sup> AK will be to CL, as AB to CD, that is GH to HI.

[For the sum, (the difference is similar):  $e/f = a/c > 1$ ; if  $g/h < a/c$  then  $e$  and  $f$  can be found such that  $a/c > (a + e)/(c + f) = g/h$ .]

With the same arrangements: the lines can be taken away in the ratio E to F, in order that the remainders shall have the given ratio GH to HI: for BM and DN may be taken away. Since therefore as AB is to CD, thus E to F, that is by construction BM to ND, will be as AM <sup>b</sup> to CN, remainder to remainder, as AB to CD, that is as GH to HI. Q. e. d. <sup>a</sup> 19. Five; <sup>b</sup> *ibid.*



Dico addi posse lineas in ratione E ad F, ut compositae rationem habeant GH ad HI reciprocam. Fiat enim ut E ad F sic KL ad ML, & MN ad KN; erigantur ex M & K parallelis, KO, MP, aequalibus ipsis AB, CD; iungantur OP, fiatque ut GH ad HI, sic OR ad PR, & PS ad OS, & PO producta occurrat MN lineae in Q, erit PS maior quam PQ. & MQ minor quam MN, ut ostendam; tum recta ducatur SL; quae secet MP OK lineas in V & T; Quoniam igitur MQ est ad KQ, ut PM ad OK, id est per constructionem ut AB ad CD, & MN ad KN, ut E ad FL sit autem ex hypothesi ratio AB ad BC, maior ratione E ad F, erit ratio MQ ad KQ, maior ratione MN ad KN, & dividendo, <sup>c</sup> ratio MK ad KQ, maior ratione MK ad KN, quare MQ linea minor est lineae MN; eadem modo ostenditur PQ recta minor recta PS. Unde KT est ad MV addita ad addita ut KL ad ML; id est E ad F, & PV ad OT tota ad totam, ut PS ad OS, id est GH ad HI. Similiter additio fiet reciproca, si recta ducatur NR occurrens OK, PM lineis in X & Z: erit enim XO ad ZP, addita ad additam, ut OR ad PR, id est per constructionem ut GH ad HI, id est ex hypothesi ut E ad F; & ZM ad XK; tota ad totam, reciprocem, ut MN ad KN, id est per constructionem ut E ad F, id est ex hypothesi ut GH ad HI.

c. 29. Quintii.

Iisdem positis: Dico addi non posse lineas in ratione E ad F, ut composita sit ad compositam, in ratione GH ad HI ordinate; id est ut utriusque rationis antecedentes sint in eadem linea: addantur enim secundum rationem E ad F, lineae BK, DL, & si fieri posset, sit AK ad CL, ut GH ad HI: erit igitur ut AK ad CL sic BK ad DL, (quia ex hypothesi ratio E ad F, eadem est cum ratione GH ad HI) adeaque & AB ad CD, ut AK ad CL, id est ut GH ad HI; quod est contra Suppositum: nam ratio AB ad CD maior ponitur, ratione GH ad HI: quare ordinata additio non continget.

Sed neque hoc casu detractio ordinata, aut reciproca continget. Detrahantur enim ordinatim lineae BM, DN, in ratione E ad F: & si fieri possit, sit AM ad CN, ut GH ad HI. Quoniam igitur AB ad CD, maiorem habet rationem, quam BM ad DN, id est E ad F, erit AM reliqua, ad reliquam CN, in maiori ratione quam AB ad CD; adeoque multo maiore quam E ad F, id est GH ad HI: quare ordinata detractio nulla fiet.

De reciproca detractio sic constabit: fiat ut E ad F sic LD ad KB; & si fieri possit, sit AK ad CL, ut GH ad HI. Cum igitur ratio AB ad CD, ex hypothesi maior sit ratione AK ad CL, id est GH ad HI, fiat AK ad CM, ut AB ad CD; eritque CN linea, <sup>b</sup> minor ipsa CL: quare cum sit ut AB ad CD, sic AK ad CM, erit quoque <sup>c</sup> KB ad ND, residua ad residuam ut AB ad CD; adeoque KB maior, quam MD, & multo maior quam LD. Quod est contra hypothesim non igitur detractio, reciproca continget.

a. 33. Quintii; b. 10. Quintii; c. 19. Quintii.

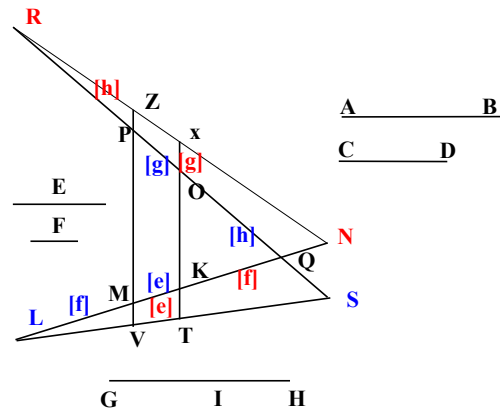
9. If AB and CD are lines of unequal lengths; and the ratio E ad F is the same as the ratio GH to HI; which is less than the ratio AB to CD, [i.e.  $a/c > g/h = e/f$ ] then:

I say that the lines can be added in the inverted ratio E ad F, in order that the sums have the ratio of GH to HI. [i.e.  $(a + kf)/(c + ke) = g/h$ ].

For by the following construction the ratios KL to ML and MN to KN can be made equal to E to F [See Fig. ix a]. From M and K the parallel lines MP and KO are erected equal to AB and CD, the

points O P are joined, and OR to PR is made in the ratio GH to HI, and also as PS to OS [i.e.  $OR/PR = g/h = PS/OS$ ]. Now PO produced crosses the line MN in Q, and PS is greater than PQ. Also MQ is less than MN as shown; then the line SL is drawn; which cuts the lines MP and OK in V and T.

The result is demonstrated as follows: Since MQ is to KQ as PM is to OK, therefore as AB is to CD from the construction, & MN to KN, as E to F; but the ratio AB to BC is larger than the ratio E to F by hypothesis, and the ratio MQ to KQ is larger than the ratio MN to KN [i.e.  $MQ/KQ = PM/OK = AB/CD = a/c > MN/KN = e/f$ ]; and by division of the line<sup>c</sup>, the ratio MK to KQ is larger than the ratio MK to KN, whereby the line MQ is shorter than the line MN [i.e.  $MQ/KQ - 1 > MN/KN - 1$ , or  $MK/KQ > MK/KN$ ]. In the same manner it can be shown that the line PQ is shorter than the line PS. From which KT is to MV, sum to sum, as KL to ML: that is, E to F; and PV to OT, total to total, as PS to OS, that is GH to HI. [i.e.  $KT/MV = KL/ML = e/f$ ; and  $PV/OT = (a + kf)/(c + ke) = PS/OS = g/h$ , for some  $k > 0$ .] <sup>c. 29. Five.</sup>

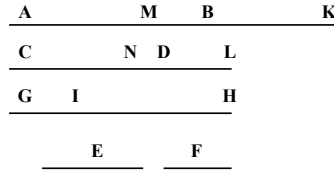


Prop.18. Fig.ix a.



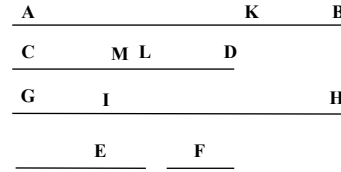
Similarly the inverted ratio can be formed from addition: if the line NR is drawn crossing the lines OK and PM in X and Z ; for now XO is to ZP, sum to sum , as OR to PR, which is by construction as GH to HI, which is by hypothesis as E to F; & ZM to XK; total to total, with the inverted ratio, as MN to KN, which is by construction as E to F, that is by hypothesis as GH to HI. [*i. e.*  $XO/ZP = OR/PR = g/h = e/f$ ; &  $ZM/XK = (a + kf)/(c + ke) = g/h$ .]

With the same ratios in place [*i. e.*  $a/c > g/h = e/f$ ]: I say that it is not possible to add the lines in the ratio E ad F, in order that the ratio [in the form] sum to sum shall be accordingly GH to HI ; both of the preceding ratios should be in the same line: for the lines BK and DL are added following the ratio E to F, and if it were possible to make the ratio AK to CL as GH to HI: therefore AK to CL would be thus as BK to DL, (since by hypothesis the ratio E to F is the same as the ratio GH to HI) and so AB to CD, as AK to CL, as GH to HI; which is contrary to the supposition : for the ratio AB to CD is put larger than the ratio GH to HI: whereby the ordered addition will not happen. [*i. e.*  $(a + ke)/(c+kf) \neq e/f$ ]



Prop.18. Fig.ix b.

But neither subtraction with the ordered or inverted ratio can give the desired ratio either. Indeed, the lines BM and DN being taken away in the order E ad F, if it were possible, would become AM to CN as GH ad HI. Since therefore AB to CD, has a greater ration than BM to DN, that is E to F, the remainder AM will be, to the remainder CN, in a greater ratio than AB ad CD; and thus in a much greater ratio than E to F, that is GH to HI: whereby no ordered subtraction will be possible.



Prop.18. Fig.ix c.

Concerning the inverted subtraction it will be agreed thus : E to F will become thus as LD to KB; and if it were to become possible, AK to CL would be thus as GH to HI. Therefore when the ratio AB to CD, shall be greater than the ratio AK to CL from hypothesis, that is GH to HI, AK to CM shall become as AB to CD; and the line CN<sup>b</sup> shall be less than the line CL : whereby when AB thus shall be to CD as AK to CM, and <sup>c</sup> KB to ND also, remainder to remainder as AB to CD; and hence KB will be greater than MD, & much greater than LD. Which is contrary to the hypothesis and therefore subtraction with the inverted ratio cannot happen.

a. 33. Five; b. 10. Five; c. 19. Five.

[14]

10. Sint AB, CD lineas inaequales ; & ratio E ad F eodem cum ratione GH ad HI ; quae maior sit ratione AB ad CD.  
 Dico additionem, neque ordinatam, neque reciprocam, fieri posse. Addantur enim ordinatim in ratione E ad F, lineae BL, DM, & si fieri possit, sit AL ad CM, ut GH ad HI. Quoniam est BL ad DM, ut AL ad CM, igitur erit quoque <sup>a</sup> ratio AB ad CD, eadem cum BL ad DM, hoc est E ad F, vel GH ad HI quodcontra suppositum; ponitur enim ratio AB ad CD minor ratione E ad F, sed neque reciproca additio continget : fiat enim ut E ad F, sic DL ad BK, addita ad additam, & si fieri possit, sit AK ad CL ut GH ad HI. Fiatque; ut GH ad HI, sic AB ad CM <sup>b</sup> erit CM minor quam CD, quia ratio AB ad CD minor est ratione GH ad HI, id est AB ad CM. igitur cum dit ut AK ad CL, id est GH ad HI, sic AB ad CM, erit <sup>c</sup> BK ad ML, residuum ad residuum ut AB ad CM ; adeaque; BK maior ipsa ML, & multo maior quam LD. Quod est contra hypothesin: quare nec additio reciproca continget.  
 Iisdem positis, neque ordinata detractio continget : detrahantur enim in ratione E ad F, lineae KB LD, & si fieri possit sit AK ad CL, ut GH ad HI. Quoniam igitur ratio AB ad CD, ex hypothesi minor est ratione E ad F, id est KB ad LB, ablatae ad ablatam, erit <sup>d</sup> ratio AK ad CL, reliquae ad reliquam, minor ratione AB ad CD, adeoque multo minor ratione KB ad LD, id est GH ad HI; quod est contra suppositum: quare detractio ordinata non continget.  
 Continget vero detractio reciproca hoc modo. Erigantur duae parallelae KM, LN, aequales ipsis AB CD ;iunctisque punctis MN, KL fiat ut E ad F, sic MO ad NO, & NP ad MP, item ut GH ad HI, sic KQ ad LQ, & LR ad KR: cadent R & P intra concursum linearum OP.QR., ut ostendam: quare rectae ducantur RO, PQ; & PQ quidem secet lineas KM, NL in S & T. RO vero lineam PQ in V, & rectas MK, LN in Z & X convenient OP. RQ lineae in β. Quoniam ratio AB ad CD, id est per constructionem MK ad NL, id est Nβ ad Mβ, minor est ratione E ad F, id est per constructionem PN ad MP, erit dividendo, <sup>e</sup> quoque ratio NM ad

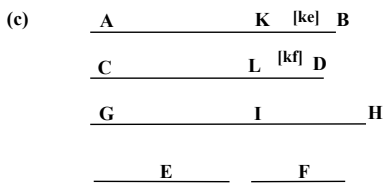
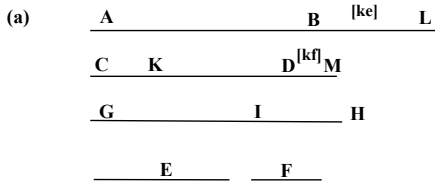
Mβ, minor ratione NM ad MP; adeoque MP linea<sup>f</sup> minor quam Mβ: eadem modo ostenditur KR linea minor quam Kβ : quare iunctae RO, PQ secabunt lineas MK, NL ut dictum est : unde NT est ad MS, ablata ad ablatam, ut NP ad MP, id est E ad F, & SK est ad TL, reliqua ad reliquam, ut SQ ad TQ, id eat GH ad HI : similiter erit MZ ad NX, ablata ad ablatam, ut MO ad NO;

[15]

id est E ad F, & XL ad ZK, reliqua ad reliquam, ut LR ad KR, id est GH ad HI.

a Ibid. ; b 10. Quintii. ; c 19. Quintii. ; d 33. Quintii. ; e 29. Quintii. ; f 10. Quintii.

10. Let AB and CD be lines of unequal lengths. Also, the ratio E to F shall be the same as ratio GH to HI, which shall be larger than the ratio AB to CD [*i.e.*  $a/c < g/h = e/f$ ].



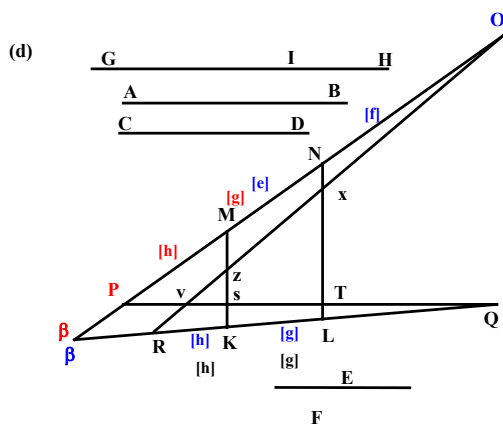
Prop.18. Fig.x.

given, hence  $g/h = AB/CM$ . Therefore, according to what is being said, as AK to CL, *i. e.* GH to HI, thus AB to CM will be as<sup>c</sup> BK to ML, remainder to remainder as AB to CM. Hence BK is itself larger than ML, and much larger than LD. Which contradicts the hypothesis: whereby addition of reciprocals cannot be used in this case.

[*i. e.* we suppose  $(a + kf)/(c+ke) = e/f$ . In this case a length CM is assumed that insures the equality of the ratio  $g/h = e/f = AB/CM$  ; but in this case the remaining ratio BK/ML must also have the value  $e/f$  or  $AB/CM$  and so  $BK > ML > DL$ , which is not so as  $DL > BK$  as  $e > f$  is given].

With the same ratios, neither will ordinary subtraction give equality of the ratios:

[*i.e.* again  $a/c < g/h = e/f$  and we show that  $(a - ke)/(c - kf) \neq e/f$ ; see Fig. (x; c) ; The method of the additive case could be used. ]: for if the lines KB and LD are taken in the ratio E to F from AB and CD, and it is assumed that the resulting ratio AK to CL is as GH to HI. Because the ratio AB to CD by hypothesis is less than the ratio E to F, that is KB to LB, then the ratio of the remainder to the remainder will give the



Prop.18. Fig.x .

I assert that neither ordinary nor inverted addition of the E to F ratio can be used to give an equal ratio between AB to CD and GH to HI. [*i.e.*  $(a + ke)/(c+kf) \neq g/h$  ].

For the regular addition of the lines BL and DM in the ratio E to F, if it were possible, would give the ratio AL to CM equal to the required ratio GH to HI [See Fig. (x; a)]. But since BL to DM is in the same ratio as AL to CM, then AB to CD also is as BL to DM, or GH to HI : which is against the supposition that the ratio AB to CD is less than the ratio E to F.

[If  $(a + ke)/(c+kf) = g/h = e/f$ , then  $a/c = e/f$  also, contrary to the original hypothesis.]

Neither will equality occur if the inverteds are added [*i.e.* again  $a/c < g/h = e/f$  and we show that  $(a + kf)/(c+ke) \neq e/f$ ; see Fig. (x; b) ;]:

For as E to F, thus DL to BK, with the ratio of AB to CD augmented by addition. If it were possible, then AK to CL shall be as GH to HI. However, in truth as GH to HI, so AB to some CM<sup>b</sup> where CM is less than CD, because the ratio AB to CD is less than the ratio GH to HI [for  $a/c < g/h$  is

ratio AK to CL, which is less than the ratio AB to CD [as a larger quantity is taken from the numerator than the denominator], and thus much less than the ratio KB to LD, that is GH to HI; which is contrary to the supposition: whereby ordinary subtraction cannot be used.

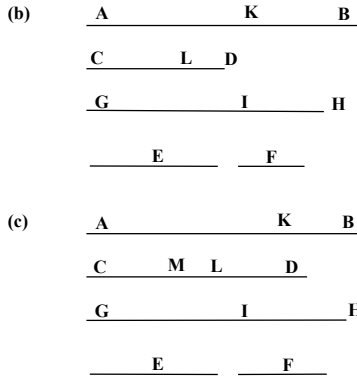
Reciprocal subtraction gives the required ratio using the following construction:

Two parallel lines KM and LN are erected equal to CD and AB, and the points MN and KL are joined.



ratio GH to HI. Now, PO crosses the line LM in Q, and OR is a longer line than OQ, and LQ smaller than LM : as shown in Fig. (xi;a). Whereby the line MR drawn from M through R, crossing the lines OK and PL in S and T. RN is joined, and it cuts the same lines in V and X.

Demonstration: Because the ratio LM to KM is equal from the construction to the ratio E to F which is less than the ratio LQ to KQ, that is LP to KO, that is AB to CD, then by division the ratio LK to KM is less than the ratio LK to KQ [i. e. given  $g/h < a/b$ ; then  $(LM/KM - 1) < (LQ/KQ - 1)$ ; hence  $KL/KM < KL/KQ$ ], and hence the line  $LM^b$  is longer than LK [should this be  $KM > KQ$ ?] : in the same manner it can be shown that the line OR is longer than the line OQ. Hence LT to KS by ordinary addition to addition is as LM to KM, that is as E to F. Also, TP is to SO, sum to sum, as PR to OR, that is as GH to HI. Hence we may add the lines together in the ordinary way to give the required ratio, etc.



[i. e.  $g/h = PT/SO$  (from  $\Delta PRT$ ) =  $(PL + LT)/(OK + KS)$  (from  $\Delta$ 's  $MLT$  &  $PQL$ ) =  $(a + ke)/(b + kf)$ ].

The reciprocal lines can indeed be added in the ratio E to F, etc. as I show thus. VK is to LX, sum to sum, as KN to LN, that is by the construction as E to F; and XP is to OV total to total, as PR to OR; that is as GH to HI; therefore, &c.

[i. e.  $g/h = PR/OR$  (from  $\Delta PRT$ ) =  $(PL + LX)/(OK + KV)$  (from  $\Delta$ 's  $PQR$  &  $NKV$ ) =  $(a + kf)/(b + ke)$ ].

With these ratios set as before : I declare that subtraction either of the regular or of the inverted ratio to give the required ratio is not possible. Indeed the lines KB and LD [see Fig. xi;b] may be taken away in the ordinary manner in the ratio E to F: and if it were possible, AK to CL would be as GH to HI. Because the ratio AB to CD is larger than the ratio E to F, i. e. KB to LD, the ratio AK to CL, remainder to remainder, will be larger than the

Prop.18. Fig.xi.

ratio AB to  $CD^c$ , which is contrary to the hypothesis, when the ratio AK to CL i. e. GH to HI, is set less than the ratio AB to CD, whereby ordinary subtraction cannot be considered.

[i. e.  $a/c > e/f$  then  $AK/CL = (a - ke)/(c - kf) > a/c$ , as a larger number is taken from the denominator than from the numerator, all quantities remaining positive: but  $g/h < a/c$ , hence the equality cannot hold.]

The taking of reciprocals will be considered next thus, [see Fig. xi;d]. The inverted ratios LD and KB are taken in the ratio E to F: and if this were possible, the ratio AK to CL would thus be as GH to HI. Hence AB to CD would thus be as AK to CM, where CM is less than CL, as the ratio AB to CD, (i. e. AK to CM), is set larger than the ratio GH to HI, which is AK to CL. Therefore when AB to CD is thus as AK to CM, taken to taken, KB to  $MD^d$  is, remainder to remainder, as AK to CM : and the line KB will be greater than the line MD: and thus much greater than the line LD. Which is contrary to the hypothesis: whereby subtraction of the reciprocals cannot be considered.

[i. e.  $a/c > e/f$  and  $AK/CL = (a - kf)/(c - ke) = g/h$ , by assumption. Now, if  $AK/CL = g/h < a/c$ , then  $a/c = AB/CD = AK/CM$ , where  $CL > CM$ . In this case,  $(AB - KB)/(CD - MD) = AK/CM$ , and  $KB > MD$  as a greater length must be taken from the numerator than the denominator to give equality: but this cannot be the case, as  $F < E$ . Thus, the original supposition is false.]

12. Sint AB CD lineae inaequales, & ratio E ad F minor ratione AB ad CD, maior vero ratione GH ad HI.

Dico neque ordinatim, nec reciproce lineas posse detrahi, in ratione E ad F, ut reliquae obtineant rationem GH ad HI: dematur enim reciproce in ratione E ad F,

[16]

rectae LD KB, & si fieri possit, sit AK ad CL ut GH ad HI, fiatque ut AB ad CD, sic AK ad CM, erit CM minor ipsa CL, quia ratio AB ad CD, id est AK ad CM, maior est ratione GH ad HI, id est AK ad CL, cum igitur sit ut AB ad CD, sic AK ad CM, erit & KB ad MD, residuum ad residuum, ut AK ad CM : adeoque KB maior ipsa MD, & multo maior recta LD. Quod est contra hypothesim : quare detractio reciproca non continget. Sed neque ordinata detractio, casu hoc continget.

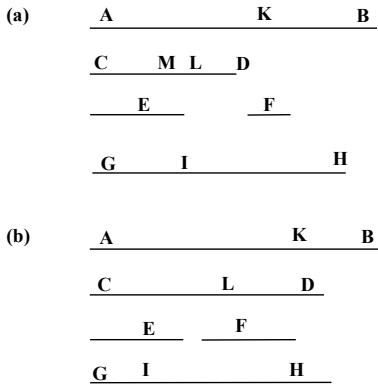
Detrahantur enim in ratione E ad F, lineae KB LD; & si fieri possit, sit AK ad CL, ut GH ad HI cum igitur ratio AB ad CD, maior ponatur ratione E ad F, id est KB ad LD, erit & ratio AK ad CL, a maior ratione AB ad CD; adeoque multo maior ratione GH ad HI, id est AK ad CL. Quod est contra hypothesim. Igitur neque ordinata detractio fieri potest.

Iisdem positus, neque additio ordinata continget. Addantur enim in ratione E ad F, lineae BL, DM: & si fieri possit, sit AL ad CM ut GH ad HI. Quoniam igitur ratio AL ad CM, id est GH ad HI, minor ponitur

ratione E ad F, id est BL ad DM, erit <sup>b</sup> ratio AB ad CD, minor ratione AL ad CM, id est GH ad HI. Quod est contra hypothesim ; quare nec ordinata additio continget.

Iisdem positis, additio reciproca fieri poterit: fiat enim ut E ad F, sic KL ad ML, & MN ad KN, erectisque ex M & K parallelis KO, MP, quae rectis AB, CD sint aequales, iungantur OP, & fiat OQ ad PQ, item PR ad OR, ut GH ad HI, occurratque PO linea, rectae MN in S; patet ex saepe dictis, SO lineam minorum esse recta OR, & SK minorem ipsa KN ; quia ratio AB ad CD, id est MP ad OK, id est PS ad OS, maior est ratione GH ad HI, id est PR ad OR; item MS ad KS, maior ratione MN ad KN, id est E ad F. Quare iuncta RL secabit rectas, OK PM, in T & V. Unde KT est ad MV, addita ad additam, ut KL ad ML, id est E ad F, & PV ad OT, composita ad compositam, ut PR ad OR, id est per constructionem GH ad HI.

Addi iterum poterunt lineae in ratione E ad F, ut OK minor utriusque rationis antecedentes habeat : ducatur enim in eadem figura, ex Q per L recta; occurret illa lineis PV OT in Z, & X; quia ratio OQ ad PQ, id est GH ad HI, minor ponitur ratione KL ad LM, id est E ad F; nam si LQ linea non concurreret cum PV, OT lineis, sed illis aequidistaret, esset ratio KL ad ML. eadem cum ratione OQ ad PQ. Quod est contra hypothesim ; occurret igitur LQ linea, rectis PV, OT in Z & X. Unde KX est ad MZ, addita ad additam, ut KL ad ML. id est E ad F, & OX ad PZ, ut OQ ad PQ. id est GH ad HI.



Prop.18. Fig. xii.

a 33: Quintii ; b *ibid.*

12. AB and CD are two unequal lines with  $AB > CD$ . The ratio E to F is less than the ratio AB to CD, but greater than the ratio GH ad HI.

[i. e.  $g/h < e/f < a/c$ .]

I declare that the lines cannot be taken in either the ordinary or the inverted ratio E to F, in order that the remainders obtained will be in the ratio GH to HI.

The lines LD and KB shall indeed be taken reciprocally in the ratio E to F,

[16]

and if it were possible, AK to CL shall be as GH to HI, and AB to CD thus becomes as AK to CM [On being 'scaled down', but still in the ratio  $a/c$ ]. CM will be less than CL itself, as the ratio AB to CD (i. e. AK to CM) is greater than the ratio GH to HI [for  $a/c > g/h$  by the hypothesis], that is AK to CL.

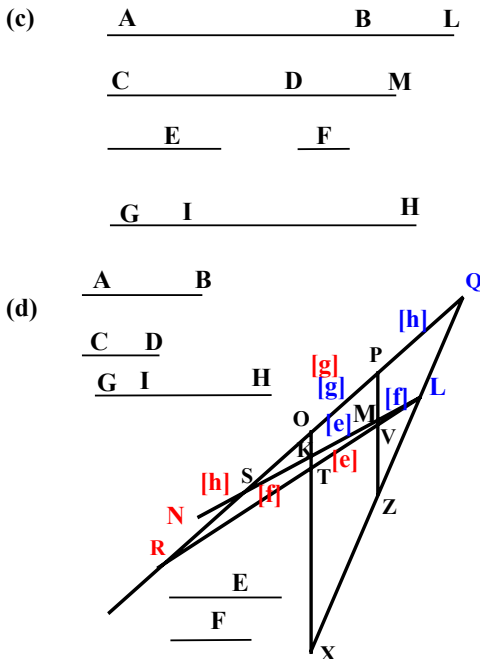
Whereby AB to CD shall be thus as AK to CM, and KB to MD shall be, remainder to remainder, as AK to CM [on being similarly scaled down, but still in the ratio  $a/c$ ]: and thus KB is larger than MD itself, and even larger than LD [i. e.  $F > E$ ]. Which is contrary to the hypothesis, whereby subtraction of the

reciprocals is not tenable.

But neither is the result true for ordinary subtraction, with this case now touched on.

The lines KB and LD are to be taken in the ratio E to F, and if it were possible, the remainders AK to CL should be as GH to HI. When the ratio AB to CD is set larger than the ratio E to F, (i. e. KB to LD), then the ratio AK to CL<sup>a</sup> is larger than the ratio AB to CD ; and thus much larger than the ratio GH to HI, i.e. AK to CL. Which is contrary to the hypothesis. Therefore ordinary subtraction of the ratios can be effected either to produce the given ratio. [i. e. if  $a/c > e/f$ ; we wish to show that  $(a - e)/(c - f) > a/c$ . Algebraically, assume this is true, then  $a(1 - e/a)/c(1 - f/c) > a/c$ , giving  $1 - e/a > 1 - f/c$ ,  $e/a < f/c$ , and  $e/f < a/c$ , which is true. Note: The  $k$  in the  $e/f$  ratio will be taken as understood henceforth.]

With the same ratios in place [i. e.  $g/h < e/f < a/c$ ], ordinary addition cannot be used to give the required ratio [Fig.xii;c]. Indeed the lines BL and DM may be added in the given ratio E ad F, and if it were possible, AL to CM would be as GH to HI. Since the ratio AL to



Prop.18. Fig.xii.

CM (*i. e.* GH to HI) is therefore made less than the ratio E to F, (*i.e.* as BL is to DM), the ratio<sup>b</sup> AB to CD is less than the ratio AL to CM, (GH to HI). Which is contrary to the hypothesis ; whereby ordinary addition will not produce the required ratio.

With the same ratios in place , inverted addition does give the required ratio [Fig.xii;d]. Construction:

Indeed KL to ML is made in the same ratio as E to F, and also MN to KN. The parallel lines MP and KO are erected from K and M equal in length to the lines AB and CD. The points O and P are joined, and both OQ to PQ and PR to OR are made in the ratio GH to HI, and the lines PO and MN cross at S.

Demonstration: It is apparent from what has often been said, that OS is less than OR, and KS is less than KN. Since the ratio AB to CD (*i. e.* MP to KO, and also PS to OS), is greater than the ratio GH to HI, *i. e.* PR to OR, and likewise MS to KS is greater than the ratio MN to KN, *i. e.* E to F. Whereby the line RL will cut the lines OK and PM in T and V. Then KT is to MV, the ratios to be added, are as KL to ML, *i. e.* E to F, and PV to OT, the sum to the sum, is as PR to OR, that is by construction GH to HI.

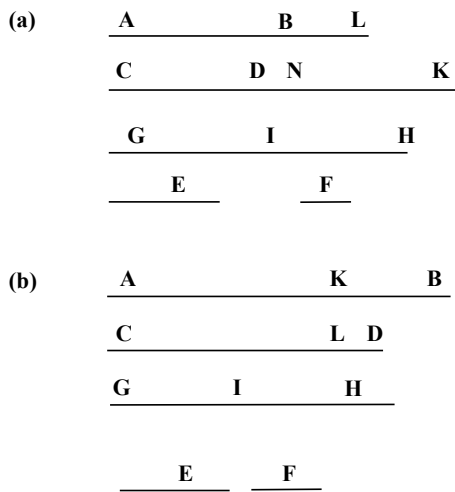
Again the lines can be added directly in the ratio E to F, as OK is the smaller of the two lines in the preceding ratio. Indeed in the same figure, a line can be drawn from Q through L that crosses the lines PV and OT in Z and X. Since the ratio OQ to PQ is as GH to HI, which is made less than the ratio KL to ML, (E to F); for if the line LQ does not intersect with the lines PV and OT, but should be parallel to them, the ratio KL to ML will be the same as the ratio OQ ad PQ, which is contrary to the hypothesis. Hence the line LQ cuts the lines PV and OT in Z and X; then KX is to MZ, the ratio to be added, shall be as KL to ML, *i. e.* E to F, and the required ratio OX to PZ, as OQ to PQ, *i. e.* GH to HI.

[17]

13. Sint AB CD lineae inaequales; ratio vero GH ad HI , minor ratione E ad F ; quae maior sit ratione AB ad CD. fieri non poterit additio reciproca. addantur enim DK, BL in ratione E ad F, & si fieri possit sit AL ad CK ut GH ad HI : dein fiat ut AB ad CD, ita BL ad DN; erit DN linea minor recta DK, cum ipsa BL minor ponatur recta DK, adeoque tota CN minor, CK. igitur cumsit ut AB ad CD, sic BL ad DN, & componendo , ut AB ad CD, sic AL ad CN, erit ratio AL ad CN, minor ratione GH ad HI, id est per constructionem ratione AL ad CK. quid fieri nin potest; cum CN linea minor sit recta CK : quare additio reciproca non continget.

Neque etiam ordinata detractio fieri poterit : demantur enim lineae KB, LD in ratione E ad F, & si fieri possit , sit AK ad CL, ut GH ad HI; Quonim igitur ratio AB ad CD, minor ponitur ratione KB ad LD, sive E ad F, erit ratio AK ad CL, id est GH ad HI, <sup>a</sup> minor ratione AB ad CD, quod est contra Suppositum: quare ordinata detractio non continget.

Detrahi tamen poterunt lineae, in ratione E ad F, ut reliquae reciprocè habeant rationem GH ad HI. Fiat enim ut E ad F, sic KL ad ML, & MN ad KN; erectisque ex K & M parallelis KO, MP, quae rectis AB, CD, sint aequales iungantur OP; & fiat ut GH ad HI, sic OQ ad PQ, & PR ad OR : ostendetur ut in casu decimo huius propositionis puncta N & R, esse intra concursum linearum LN, QR, quare ductae RL, NQ secabunt lineas OK, PM : & N1 quidem illas secet in ST, recta vero RL eisdem occurrat in V & X. patet TM esse ad SK, ablatam ad ablatam , ut MN ad KN, id est per constructionem ut E ad F, & SO esse ad PT, residuam ad residuam, ut IQ ad PQ, id est GH ad HL.



Prop.18. Fig.xiii.

Addi etiam poterunt ordinatim lineae in ratione E ad F, ut compositae rationem habeant GH ad HI, agatur enim ex R per N linea RN, occurret illa rectis IK, PM: quid saepius ostensum est, cum ratio XR ad VR, id est PR ad OR, id est per constructionem GH ad HI, minor ponatur ratione E ad F, id est MN ad KN. occurrat igitur in Z & α. erit Mα ad KZ, addita ad additam , ut MN ad KN , id est E ad F, & Tα ad OK, composita ad compositam, ut PR ad OR, id est GH ad HI.

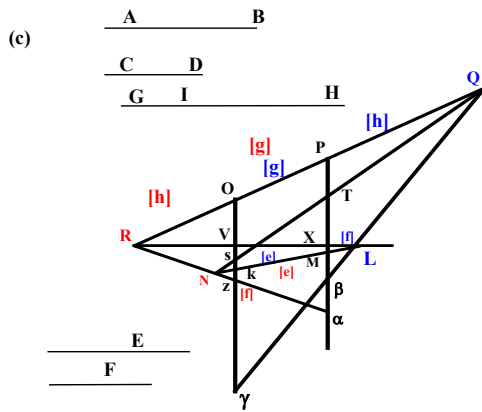
Rursum addi possunt lineae ut minor OK, utriusque rationis, habeat antecedentes; ducatur enim ex Q per L recta, quae cum OK, PM lineis conveniet ut ostesum saepius; quia ratio OQ, ad PQ minor est ratione KL ad ML : conveniat igitur in β & γ erit Kγ ad Mβ, addita ad additum, ut KL ad ML, id est per constructionem ut E ad F, & Oγ ad Pβ, tota ad totam, ut OQ ad PQ, id est GH ad HI. <sup>a</sup>. 33. Quinii

13. AB and CD are lines of unequal length, while the ratio of GH to HI is truly less than the ratio E to F, which in turn is larger than the ratio AB to CD. [i. e.  $a/c < g/h < e/f$  and  $a > c$ . ]

It is not possible to add lines in the inverted ratio.

Indeed let DK and BL be added in the ratio E to F in order that AL to CK is in the ratio GH to HI, if it shall be possible [See Fig.xiii;a]. Now, as AB is to CD, thus BL is to DN, where DN is a line less than DK as BL is itself established as a line smaller than DK, and therefore the total CN is less than CK [i. e.  $BL/DN = a/c$ ]. Therefore as AB to CD, thus BL is to DN; and by adding the ratios, as AB to CD, thus AL to CN. [For the numerators and denominators of like ratios can be added and equality maintained.] The ratio AL to CN is smaller than the ratio GH to HI [For  $a/c < g/h$ ], that is by construction set equal to the ratio AL to CK. which cannot be the case, as CN is a smaller line than CK : whereby the required ratio cannot be formed by the addition of the reciprocals.

Neither indeed will regular subtraction of the ratio be possible : Let the lines KB and LD are taken in the ratio E to F from AB and CD, and if possible, the ratio AK to CL is as GH to HI [Fig.xiii;b]. Because the ratio AB to CD is put less than the ratio KB to LD, or E to F, [For  $a/c < e/f$ ], the ratio AK to CL, that is by assumption equal to the ratio GH to HI<sup>a</sup>, shall be less in turn than the ratio AB to CD, which is contrary to the supposition: whereby ordinary subtraction cannot be used to produce the required ratio.



Prop.18. Fig.xiii.

However the lines can be taken in the ratio E to F, so that the remainders are reciprocally in the ratio GH to HI [See Fig.xiii;c].

Construction: Let KL to ML and MN to KN indeed be made in the ratio E to F. The parallel lines KO and MP are erected from K and M, which are equal to the lines AB and CD, the points O and P are joined; and the points Q and R are chosen so that the ratios OQ to PQ and PR to OR are made in the ratio GH to HI : it is made clear in the tenth case of these propositions that the points N and R, lie on the joins of the lines LN and

QR, whereby the lines drawn RL and NQ cut the lines OK and PM : and NQ indeed cuts these lines in S and T, the line RL cuts the same in V and X. It is apparent that TM is to SK, taken to taken, as MN is to KN, that is by construction as E is to F, and SO is to PT, remainder to remainder, as OQ to PQ, that is GH to HL. [Thus,  $(PM - XM)/(OK - VK) = (c - e)/(a - f) = PX/OV = RP/RO = g/h$ . ]

The lines can also be added in the ordinary ratio E to F, in order that the sums are in the ratio GH to HI. [Recall that  $a/c < g/h < e/f$  and  $a > c$ . ] Indeed the line RN can be sent from R through N, that crosses the lines OK and PM: which more often has been shown, when the ratio XR to VR, that is PR to OR, that is by construction GH to HI, which is put less than the ratio E to F, that is MN to KN. The line RN crosses these lines in Z and  $\alpha$ . The ratio  $M\alpha$  to KZ, sum to sum, shall be as MN to KN, that is E to F, and  $P\alpha$  [text has  $T\alpha$ ] to OK, sum to sum, as PR to OR, that is GH to HI.

$$[\text{Thus, } (PM + M\alpha)/(OK + KZ) = (a + e)/(c + f) = P\alpha/OZ = PR/OR = g/h. ]$$

Again the lines can be added, as happens with the smaller of both ratios preceding. Indeed a line is drawn from Q through L, which meets the lines OK and PM as has often been shown. Because the ratio OQ to PQ is less than the ratio KL to ML, the line QL meets the lines PM and OK in  $\beta$  and  $\gamma$ . The ratio  $K\gamma$  to  $M\beta$ , sum to sum, shall be as KL to ML, that is by construction as E to F, and  $O\gamma$  to  $P\beta$ , total to total, shall be as OQ to PQ, that is GH to HI.

$$[\text{Thus, } (OK + K\gamma)/(PM + M\beta) = (c + e)/(a + f) = O\gamma/P\beta = OQ/PQ = g/h. ]$$

[18]

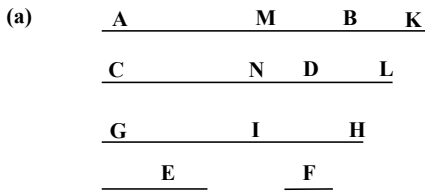
14. Sint iterum rectae AB, CD inaequales, ratio vero E ad F maior ratione AB ad CD; quae eadem sit, cum ratione GH ad HI: neque addi, neque demi ordinatim poterunt lineae, in ratione E ad F, ut compositae vel reliquae, rationem habeant GH ad HI. addantur enim ordinatim lineae BK, DL, in ratione E ad F; & si fieri possit, sit AK ad CL, ut GH ad HI. Quoniam igitur AK est ad CL, ut AB ad CD, erit<sup>a</sup> reliqua BK ad DL

reliquam, id est E ad F, ut AK ad CL, id est ex hypothesi GH ad HI. Quod est contra Suppositu. Quare ordinatim addi non poterunt lineae, &c. Neque demi possunt lineae in ratione E ad F ut residuae rationem obtineant GH ad HI, auferantur enim lineae MB, ND in ratione E ad F, & si fieri possit, sit ut GH ad HI, sic AM ad CN, cum igitur sit ut AB ad CD, sic GH ad HI, id est per constructionem AM ad CN, erit <sup>b</sup> MB reliqua ad reliquam ND, ut AB ad CD ; Quod est contra hypothesin. Quare neque ordinatim demi poterunt lineae, etc.

Iisdem positis; neque additio reciproca fieri poterit addantur enim lineae DK, BL, secundum rationem E ad F, & si fieri possit, sit AL ad CK ut GH ad HI. erit igitur AL ad CK, ut AB ad CD, & BL reliqua ad reliquam DK, ut AB ad CD : quare BL maior quam DK. Quod est contra hypothesin. Unde additio reciproca fieri non poterit.

Poterunt ramen detrahi reciprocem lineae in ratione E ad F, ut reliquem rationem obtineant GH ad HI, fiat enim KL ad ML, ut GH ad HI, erectisque ex K & M parallelis KO MN, quae aequales sint lineis AB, CD, ducatur recta ON; occurret illa lineae KL in L; quia OL est ad NL, ut OK ad NM, id est ut AB ad CD, id est GH ad HI, id est per constructionem ut KL ad ML : dein OP facta aequali ipsi NL fiat ut E ad F, sic KQ ad MQ, ducaturque recta QP : occurret illa lineis OK, NM, quia ratio OL ad NL, id est GH ad HI, id est ratio KL ad ML, minor ponitur ratione E ad F, id est KQ ad MQ:adeoque MQ linea <sup>c</sup> minor recta ML :secet igitur PQ linea, rectas OK, NM in V & X : patet VM esse ad XK ablatam ad ablatam, ut KQ ad MQ, id est E ad F : & NV esse ad OX , residuum ad residuum, ut NP ad Opid est GH ad HI.

Addi etiam poterunt lineae in ratione E ad F, ut minor utriusque rationis habeat antecedentes. Facta enim KR aequali ipsi MQ, ducatur recta PR : occurret illa lineis OK, NM in S & T: quia ratio MR ad MR, id est E ad F, maior ponitur ratione GH ad HI, id est NP ad OP. Unde patet MT esse ad KS, additam ad additam ut MR ad KR, id est E ad F, & NT esse ad OS, compositam ad compositam, ut NP ad OP, id est GH ad HI. <sup>a</sup>

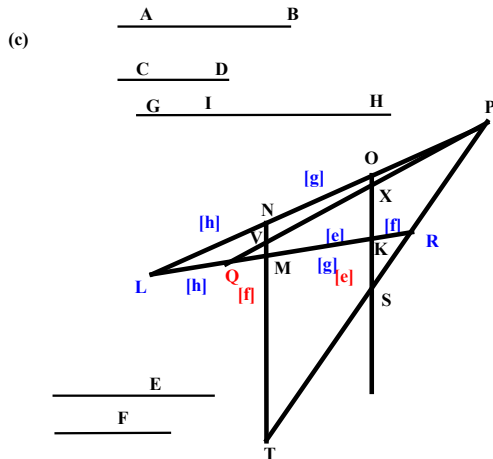
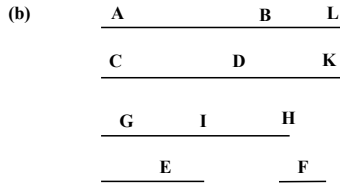


Prop.18. Fig.xiv.

14. AB and CD are again lines of unequal length, with the ratio of E to F truly greater than the ratio of AB to CD, which is the same as the ratio GH to HI: the lines cannot be added or taken in the regular manner in the ratio E to F, in order that the sum or difference has the ratio GH ad HI.

[i. e.  $g/h = a/c < e/f$  and  $a > c$ .]

Indeed the lines BK, DL can be added in the ratio E ad F; and if it were possible, AK to CL will be as GH to HI [See Fig.xiv;a]. Since AK is therefore to CL as AB to CD, the remainder<sup>a</sup> BK to the remainder DL, that is E to F, shall be as AK to CL. that is by hypothesis as



Prop.18. Fig.xiv.

GH to HI. This is contrary to the supposition. Whereby regular addition of the lines cannot be performed, etc.

Neither can the lines be taken in the ratio E to F in order that the remaining ratio GH to HI is obtaind, Indeed the lines MB and ND can be taken in the ratio E ad F, and if it were to be the case, GH is to HI as AM is thus to CN. Since AB is to CD, so GH is to HI, that is by construction AM to CN, hence the remainder<sup>b</sup> MB to the remainder ND, as AB to CD: which is contrary to the hypothesis. Whereby ordinary subtraction of the lines cannot be effected, etc.

With the same lines in position, neither by addition of the reciprocals can the ratio be made to work. [See Fig.xiv;b]. Indeed the lines DK and BL can be added, following the ratio E to F, and if it were to be possible, AL to CK will be as GH to HI. Therefore AL to CK will be as AB to CD, and the remainder BL will be to the remainder DK, as AB to CD : Whereby BL is larger than DK. Which is contrary to the hypothesis. Hence inverted addition is not possible.



However the reciprocal of the lines may be taken in the the ratio E to F, in order that the resulting ratio gives the ratio GH to HI.

[Recall:  $g/h = a/c < e/f$  and  $a > c$ .]

Construction: [See Fig.xiv;c.] Indeed, KL to ML, shall be made in the ratio GH to HI, and the parallel lines KO and MN erected from K & M, which shall be equal to the lines AB and CD, and the line ON is drawn. That line cuts the line KL in L; because OL is to NL as OK to NM, that is as AB to CD, it is as GH to HI, that is by construction as KL to ML. Then OP is made equal to NL itself; and as E to F, thus KQ to MQ. The line QP is drawn.

Demonstration. Because the ratio OL to NL, that is equal to the ratios GH to HI and KL to ML, is put less than the ratio E ad F, that is KQ to MQ. Indeed the line MQ<sup>c</sup> is less than the line ML: therefore the line PQ cuts the lines OK and NM in V and X. It is apparent that VM is to XK, in the taken to taken ratio, as KQ to MQ, that is as E to F: amd NV to OX, remainder to remainder, as NP to OP. That is as GH to HI.

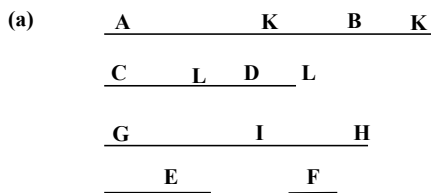
Also the lines can be added in the ratio E to F, as happens with the smaller of both ratios preceding. Indeed KR is made equal to MQ, and the line PR is drawn and that line cuts the lines OK and NM in S and T. Because the ratio MR to MR, that is E to F, is put larger than the ratio GH to HI, that is NP to OP. Hence it is apparent that MT is to KS, in the ratio of sum to sum, as MR is to KR that is E to F, and NT to be to OS, sum to sum, as NP to OP, that is GH to HI.

15. Sit iterum ratio AB ad CD, eadem cum ratione GH ad HI, quae maior sit ratione E ad F ; dico ordinatim addi, nec demi posse lineas, in ratione E ad F, ut compositae, vel reliquae, rationem habeant GH ad HI; addantur enim lineae BK, DL in ratione E ad F, & si fieri possit, sit AK ad CL, ut GH ad HI : erit igitur ut AK ad CL, sic AB ad CD ; unde & BK ad DL, id est per constructionem E ad F, ut AK ad CL, id est GH ad HI: Quod est contra hypothesim : igitur ordinata additio non continget.

Eodem modo detrahantur ordinatim BK, LD in ratione E ad F; & si fieri possit sit AK ad CL, ut GH ad HI; erit igitur AK ad CL, ut AB ad CD, quare & BK ad LD, id est E ad F, ut AB ad CD, quod est contra hypothesim, igitur, etc.

[19]

Neque etiam addi, vel demi reciprocem poterunt lineae in ratione E ad F, ut residuae vel compositae rationem habeant GH ad HL. addantur enim DL, BK in ratione E ad F, & si fieri possit sit AK ad CL, ut GH ad HI : erit igitur ut AK ad CL, sic AB ad CD. quare & BK ad DL, est ut AB ad CD, ideoque BK maior ipsa DL. Quod est contra hypothesim: ergo, &c. eodem modo ostenditur detractionem reciprocam fieri non posse.



**Prop.18. Fig.xv.**

15. Again the ratio AB to CD shall be the same as the ratio GH to HI, which shall be larger than the ratio E to F ; I say that the lines cannot be added or subtracted in the regular manner in the ratio E ad F, in order that the sum or difference shall be in the ratio GH to HI.

[i. e.  $g/h = a/c > e/f$  and  $a > c$ .]

Indeed the lines BK and DL can be added in the ratio E to F, and we may assume that AK to CL is in the ratio GH to HI. [See Fig.xv;a.] Therefore as AK to CL, thus AB to CD.

Hence BK to DL, that is by construction E to F, is thus as AK to CL, that is GH to HI. Which is contrary to the hypothesis : therefore the required ratio cannot be effected by regular addition.

In the same way the ratio BK to LD may be taken in the ratio E to F; and we may assume that AK to CL is in the ratio GH to HI. Therefore AK to CL is as AB to CD, whereby BK to LD, that is E to F, shall be as AB to CD, which is contrary to the hypothesis, therefore, etc,

The inverted or inverted ratio can neither be added nor taken in the ratio E to F, in order that the sum or difference gives the ratio GH to HL. Indeed the ratio DL to BK can be added in the ratio E to F, and if this were possible, then AK to CL would be as GH to HI. Hence, as AK to CL, thus AB to CD. Whereby BK to DL is then as AB to CD, and therefore BK shall be greater than DL, which is contrary to the hypothesis: therefore, etc. It is shown in the same way that subtraction of the reciprocals is not possible.<sup>a 33. Quinti.</sup>

16. Sit ratio E ad F eadem cum ratione AB ad CD, minor autem ratione GH ad HI : dico ordinatim addi nec demi posse lineas in ratione E ad F, sic ut compositae vel reliquae , rationem habeant GH ad HI, addantur enim BK, DL in ratione E ad F, & si fieri possit sit AK ad CL, ut GH ad HI : erit igitur ratio AK ad CL, id



ad reliquam, ut AB ad CD; quod est contra hypothesim: cum ratio AK ad CL, id est per constructionem GH ad HI, minor ponatur ratione AB ad CD. Quare ordinata detractio non continget.

Eodem modo probatur ordinatam additionem fieri non posse.

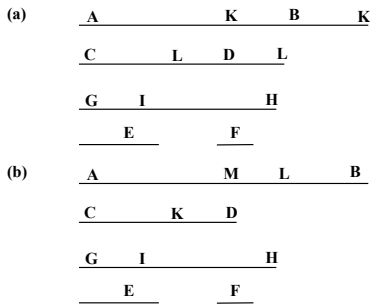
Sed neque detractio reciproca continget: demantur enim in ratione E ad F, lineae KD, LB; & si fieri possit, sit ut GH ad HI, sic AL ad CK : fiat dein ut AB ad CD, sic MB ad KD, erit MB maior, quam KD, adeoque multo maior recta LB. Igitur cum sit AB ad CD, sic MB ad KD, erit & AM ad CK, ut AB ad CD, sed ratio AM<sup>a</sup> ad CK, minor est ratione AL ad CK, id est GH ad HI: Quod est contra hypothesim. Quare nec reciprocam lineae detrahi poterunt in ratione E ad F, &c.

Poterit tamen fieri additio reciproca, &c. Fiat enim ut E ad F, sic KL ad ML, erectisque ex K & M parallelis KN, MO, quae AB, CD lineis sint aequales; ducatur recta ON, fiatque OP ad NP, item NR ad OR, ut GH ad HI : occurratque OP lines rectae KM in Q ; erit NQ linea minor recta NP. quia ratio OP ad NP, id est GH ad HI, minor est ratione OQ ad NQ, id est OM ad NK, sive AB ad CD: quare & iuncta PL secabit lineas NK, OM, in S & T. Unde KS est ad MT, addita ad additam, ut KL ad ML, id est E ad F, & OT ad NS, ut OP ad NP, id est GH ad HI.

Additi etiam sic poterunt lineae in ratione E ad F, ut NK minor utriusque rationis habeat antecedentes, ducatur enim recta RL: occurret illa lineis OM, NK in V & X, quia ratio KL ad ML, id est E ad F maior ponitur ratione NR ad RO. Unde erit KX ad MV, addita ad additam, ut KL ad ML, id est E ad F, & NX, ad OV, composita ad compositam, ut NR ad RO, id est GH ad HI. <sup>a 3. Quinti.</sup>

*Scholion.*

Possent in hac propositione plures casus determinari, & aliqui etiam, quibus poopositio absolui non potest ; sed ne taedium Lectori adferam, consultò abstinui; satis esse ducens, viam ad reliquas determinationes Geometriae studioso aperuisse.



Prop.18. Fig.xvii.

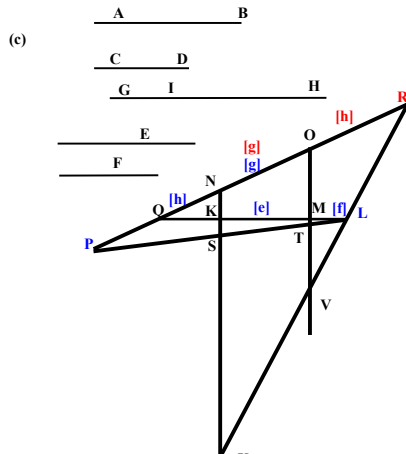
17. Again the ratio E to F shall be unequal, and the same as the ratio AB to CD, which shall be larger than the ratio GH ad HI. I say that the lines in the ratio E to F cannot be added or taken in the regular manner, in order that the difference or sum shall be in the ratio GH ad HI. [i. e.  $a/c = ef > gh$  and  $a > c$ .]

Indeed the lines BK and LD can be taken away in the ratio E to F, [See Fig.xvii;a], and if it were to be possible, AK to CL is as GH to HI. Therefore, since AB to CD is in the same ratio as E to F, this is KB to LD in the 'take to take' ratio. Thus AK to CL, remainder to remainder, shall be as AB to CD, which is contrary to the hypothesis: since the ratio AK to CL is by construction GH to HI, put smaller than the ratio AB to CD. Whereby regular

subtraction cannot be effected.

In the same way ordinary addition cannot be approved.

Inverse subtraction does not hold either. Indeed the lines KD and LB can be taken in the ratio E to F [See Fig.xvii;b], and if it were possible, AL to CK would thus be as GH to HI. As AB to CD, MB to KD is



Prop.18. Fig.xvii.

then made in the same ratio, and MB is larger than KD, and thus much larger than LB. Therefore as AB to CD, so thus is MB to KD, and AM to CK, will be as AB to CD, but the ratio AM<sup>a</sup> to CK is smaller than the ratio AL to CK, that is GH to HI. Which is contrary to the hypothesis. Whereby lines cannot be taken in the inverse ratio E ad F, etc.

However inverse addition can be performed, [See Fig.xvii;c]. Thus the ratio KL to ML can be made as E to F, and the parallel lines KN and MO erected from K and M which shall be equal to AB and CD. The line ON is drawn, and the ratio OP to NP, likewise NR to OR, shall be made as GH to HI. The line KM cuts the line OP in Q ; then NQ is shorter than NP, because the ratio OP to NP (that is GH to HI), is smaller than the ratio OQ to NQ (that is OM to NK), or AB to CD.

Whereby the joined line PL cuts the lines NK and OM in S and

T. Hence KS is to MT, 'sum to sum', as KL to ML (that is E to F), and OT is to NS, as OP to NP (that is GH to HI).

[i. e.  $(a + f)/(b + e) = g/h.$  ]

The lines can also be added in the regular ratio E to F, as NK is the smaller part of the foregoing ratio. The line RL is drawn that cuts the lines OM and NK in V and X, and the ratio KL to ML (that is E to F) is put greater than the ratio NR to RO. Hence KX to MV, added to added, shall be as KL ad ML (that is E to F), and NX to OV, sum to sum, as NR to RO, (that is GH to HI). [i. e.  $(a + e)/(b + f) = g/h.$  ]

*Note.*

Many cases are determined in this proposition, and indeed others for which the proposition could not be solved ; but to spare the reader the tedium , I have purposely withheld proceeding further; to be satisfied to show the way, and to leave the rest of the determinations for the eager student of geometry.