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# QVADRATVRÆ
# CIRCULI
## LIBER SECUNDUS.
## DE
## PROGRESSIONIBUS GEOMETRICIS.
## AD LECTOREM.

Praesens liber, quem de progressionibus Geometricis inscribimus, omnino necessarius est ad sternendam viam, quam inimus circulo ad quadratum reducendo: non ita tamen hoc velim intelligas, ut omnes omnino propositiones, quae in toto eius decursu reperiuntur, ad eum finem requiri credas; sed quod sine usu huius libri, quoad partes maxime principales, difficulter ad scopum pervenire quis possit: exigebat autem libri argumentum, ad doctrinae formam vel leviter saltem concinnari, & cognatis materiis exornari, ne faetum imperfectum ederemus. Idem de sequentibus libris iudicium derre dignabere.

## TO THE READER.

The present book, in which we write about geometric progressions, is all the more necessary as we establish in it the path we must follow to undertake the reduction of the circle to a square. I would not have you believe that all of the propositions presented in this discourse are needed to reach that end; however, without the use of this book, especially as far as the main parts are concerned, it would be more difficult for anyone to be able to reach that goal. Thus the main subject matter of the book is prepared according to the basic principles of geometry lest we should publish an abortive attempt, while other parts are dealt with more lightly and furnished with related material. The same applies to the rest of the books in this work that are deemed worthy to be brought forward for your judgement.

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## ARGVMENTVM.

*Plurimi de hoc argumento Speculationes non tantum Theorematicas, verum etiam Problematicas conscripserunt : sed omnes, prout ex libris huc usque conscriptis cognoscere potui, Arithmeticas materias concernantes, in medium attulerent : quas authoribus suis omnino intactas relinquo; mei enim instituti est progressiones tractare Geometricas, non Arithmeticas : & per illas cognoscere quantitatum omnis speciei magnitudines, sine illae in lineis, sine superficiebus, vel etiam solidis exhiberi debeant. Occasionem huic considerationi subministrarunt nonnulla, cum in Archimede, tum in Euclide loca, quae iubent in constructione Geometrica, auferri (verbi gratia) ab aliqua quantitate dimidium , & huius iterum dimidii dimidium, & pro clausula adfertur,* & hoc semper fiat*. Titillavit me haec particula, & coegit morosiore cogitatione circa haec versari : tandem post longas vexas intellectus, ea quae mihi inciderent, tibi benigne Lector communico, ut quae huic materiae desunt supplere digneris. Illam etenim* *solum direxi ad cognoscendas quantitates, quas instituto meo necessarias esse arbitrarus sum: communi enim Geometria, quam a veteribus*

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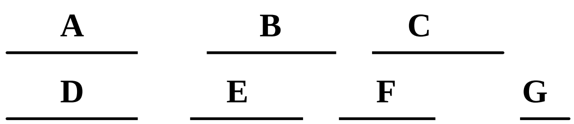
*accipimus, non existimavi posse quempiam viam sibi sternere, ad problemata solvenda, quae a Mathematicis per tot saecula desiderantur: unde novas artes & methodos novas indicavi excogitandas, quae supplerent Geometriae antiquae defectus; dignaberis itaque benigne Lector, boni haec consulere, & cum adverteris , multa hic esse quae incudi reddi debeant, utpote male tornata, memineris velim : ita scientias in orbem esse ingressas; primo mutilas & inconcinnas, quas multorum tandem manus ad perfectum nitorem reduxerunt; facili etenim negotio inventis quidpiam addi potest, plura quippe vident oculi, quam oculus; qui domini sui iussu certis terminis se continere cogitur.*

*In quatuor partes partimur hunc tractatum, quo distinctius procedamus, & captui tyronum magis inserviamus.*

*Prima inserviet contemplationi progressionum inchoatarum , sine necdum terminatarum, quod terminus progressionum nondum in considerationem adducatur. quid autem sit Progressio Geometrica, quis eius terminus, & his similia, patebit ex sequentibus. Sed tribus verbis conabimur praesentis partis nomenclaturam explicare.*




*Datur quaevis ratio A ad B ;& petatur tertia proportionalis ad has duas quantitates, exhibeaturque tertia C : continuatio trium horum terminorum dicetur esse progressionis interminatae : eo quod possit ulterius procidi in eadem serie : nam si dentur tres quantitates D, E, F, & addatur quarta G , quae continuet eadem rationem D ad E, progressio terminorum D,E,F,G, ulterius est producta, quam sit progressio terminorum A, B, C: cum haec consistat in duabus rationibus; illa vero in tribus. Porro hac methodo procedendo, continuo auctis rationibus; augetur progressio ; manet interim semper interminata. Haec igitur pars primae non perget ulterius, & sistet in sola consideratione Progressionis huius, quam vocamus interminata, ad distinctionem alterius, quae tota exhaurietur, ac proinde terminabitur seipsa. variis igitur proprietatibus inchoatae progressionis in medium allatis, subsequitur.*

*Secunda pars, quae tota versabitur circa progressionem terminatam, absolutam, sine exhaustam; atque hoc universim in quocunque genere quantitatis : indagando scilicet cuiuscunque rationis, si in infinitum censeatur continuata, magnitudinem , seu quantitatem. neque velim quis in animum inducat, nos materiam ingredi, quae placitis philosophorum contradicat :imo ostendemus luce clarius, hac nostra methodo dissolui etiam gravissimas difficultates, quibus in Gymnasiis & Philosophorum Licaeis solent in materia quantitatis invicem esse molesti. Quod ut exemplo uno manifestius explicetur, subeat memoria argumenti, quod Achilles Zenonis nominatur, quo omnem motum eliminare ex orbe se posse contendebat: omnibusque studebat persuadere, falli oculos, dum quid loco moveri arbitrarentur; argumento ad id formato duorum, quae moverentur; Achillis scilicet velocissime currentis, & testudinis tardissime reptantis*


*Ponatur , inquiebat ille, Achilles cursor pernicissimus, ex A puncto, testudinem pergentem per semitam BC tardissimo motu, velle assequi suo curfsu : Quo tempore Achilles tendit ex A in B, mota est testudo ad aliquod spatium, perveniens in D, igitur necdum Achille assecutus est testudinem : iterum quo tempore*

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*ex B Achilles currit, ut assequatur testudinem existentem in D, mota est testudo perveniens ad E punctum ; igitur nondum assecutus est testudinem, atque hoc in infinitum continget ; igitur nunquam assequetur Achilles testudinem. Argumentum hoc Zenonis facillime expedietur per ea, quae secunda huius libri parte adferemus ; nam ex doctrina illius partis, non solum manifestum fiet, Achillem perventurum ad testudinem , sed ipsum punctum assignabitur in quo apprehensio testudinis futura est.*

*Tertia Pars huius libri versabitur circa progressones terminatas, planorum praesertim similium, quae commodiora sunt ut ad doctrinae seriem reducantur : ut si datis duobus quadratis in ratione maioris inaequalitatis, quis requirat superficiem exhiberi, quae aequalis existat magnitudini, quam tota illa series quadratorum produceret, orta ex progressione rationis primi quadrati ad secundum, & huius secundi ad* *tertium , & sic in infinitum : atque ut idipsum familiarius exponam, ponatur quis equum generosissimum velle a quopiam mercari, qui vulgi opinione mille aureis aestimatur: cumque mille aurei eidem non sint ad manum, hac sponsione contrahit, se post mensem centum aureos ei donaturum ; secundum vero mensem quinquaginta. post tertium vero vigintiquinque, atque ita deinceps : post singulos menses, perpetuo censu dimidium dimidii pretii pendat eius, quid postremo mense creditori persolverit: tandem pertaesus molestiarum, conetur pacisci cum eodem, offerendo ducentos aureos pro residuo, ut ab illo censu sese expediat, quis horum duorum tali contractu decipitur; emptorne an venditor? huius & similium quaestionum solutiones, doctrina tertiae partis huius libri, liquidissime experiet : & assignabit, quae post singulos menses summa sit residue debiti; imo & si quis postularet nosse qualis summa residua esset futura, post centesimum, imo millesimum mensem, Geometrica certitudine ex praesentis Partis scientiae cognoscet.*

*Quarta Pars totam doctrinam prioribus partibus explicatam corporibus solidisque applicabit : spero huius tractatus argumentum non fore illis ingratum, qui communi Geometria exculti sunt; admiranda enim Theoremata huius libri notitiae eruuntur, & Problemata ( quae communi Geometria difficili methodo solvuntur ) praxi commodissima experiuntur : sine adminiculo enim huius libri nulla esset huius operis lucubrationum certitudo, quae maxima ex parte huic fulcimento innixa est.*

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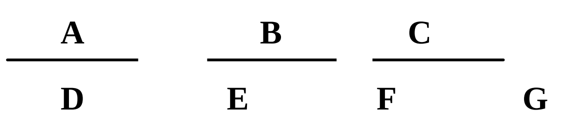
## CONTENTS.

*Most of the contents section for this book is concerned with speculations rather than the theorems or even problems that have been written down in it. All the material concerning arithmetical progressions, as taken from the books of others up to this point, can in the main be brought forwards : and which I leave with their authors entirely undisturbed. I have undertaken to investigate geometric progressions rather than arithmetical ones. Through these progressions I have found how to measure the sizes of all kinds of quantities* [*i.e. areas, volumes, and the like*] *that should be made known to everyone, whether the quantities are associated with lines, surfaces , or volumes. The opportunity to consider several problems of this kind has been provided to us both by Archimedes and Euclid, in which place instructions are given for geometric constructions in which, for example, half is to be taken from some quantity, and half of the half again, and so on, until some conclusion is produced, and this shall always be the case. This has always been a point of great interest for me, and finally I have brought to fruition many difficult thoughts that I have turned over in my mind concerning these things for a long period of time. Finally, gentle reader, I share with you these thoughts that befell me, in order that what is lacking in this material you may deem worthy to supply. My one and only concern has been directed towards understanding the quantities that I have considered necessary: indeed from the common geometry that we have accepted from the ancients, I have not been able to establish any way to solve the problems which mathematicians through the ages have desired to solve. From what I have said, new skills and methods need to be thought out which can be applied to the shortcomings of the geometry of the ancients. Thus, kind reader, you will consider it worthy to deliberate on these good things, and with your attention turned to them, there is much here badly turned out that needs to be returned and reshaped, as I would have you remember. Thus the knowledge of the world progresses: at first new knowledge is awkward and not properly formed, and which finally is brought to a state of perfection by the work of the hands of many; and indeed it is a relatively easy task to add something to such discoveries, for many eyes of course see possible additions that have escaped the eye of the master who thought through his own judgement to have contained the work within certain boundries.*

*We have divided this tract into four parts, by which means we may procede in more definate directions, and to bring to task this terrible master we wish to serve.*

*The first part is concerned with considering the start of progressions, without yet thinking about the end, as the end of a progression has not yet been brought forward for examination. Exactly what the geometic progression shall be, and what its end shall be, will be made clear in the following sections. But with three terms only we try to explain the nomenclature of the present part. Some ratio A to B is given, and a third*



proportion to these two quantites is sought, and then the third proportion C can be written down. A continuation beyond these three terms we will call the start of an unbounded progression, in which extra terms can fall into position beyond the third term in the same series. For if the three quantities D, E, and F are given, and a fourth is added continuing in the same ratio as D to E, then a progression of the terms D, E, F, and G has been produced. This goes further than the progression of the three terms A, B, C, first considered which could terminate after the two ratios, and this other which could indeed finish after three ratios. We can again continue with the preceding method, whereby the progression is augmented successively by terms in the given ratio, and meanwhile the progression remains unterminated. Thus the first progresson we examined does not procede further beyond three terms and stopped, and we are to distinguish that kind of progression from this other sort called non-terminating: the difference of the first being that it is totally finished and has thus terminated itself. Hence in what follows , a variety of starting conditions for progressions are examined along general lines.

The second part of the book is completely given over to these geometric progressions that end generally in some kind of final quantity, but with terms added indefinitely, through an investigation performed by means of some ratio of course, in the case where the final magnitude or quantity is considered with an indefinite number of terms added on. Now I do not wish to give you the idea that we are advancing material which is contrary to the pleas of the philosophers: and indeed we will show here quite clearly that by using our method even the gravest difficulties are resolved. Indeed in the grammar schools, and in turn in the schools of the philosophers as well, students are in the habit of experiencing much difficulty with such quantitative matters.

We will set forth our method planely by means of an example that goes straight to the history of the matter with an argument by Zeno, concerning Achilles and the tortoise. Zeno contended that all motion could be banished from the earth, and strove to convince everone by deceiving the eye, while the motion of a body from some place was being observed. The argument had the form of two bodies that could be set in motion: The athlete Achilles could of course run the fastest, while a tortoise crawled the slowest.


Achilles, said Zeno, the most nimble of runners, is set to run from point A, while the tortoise is set to go along the path from B to C by the slowest motion. Achilles desires to overtake the tortoise by running : but in the time taken for Achilles to go from A to B, the tortoise has moved some distance to arrive at D. Therefore Achilles has not yet caught up with the tortoise : again, by the time Achilles has ran from B to the place D where the tortoise was, so that he might catch up with it, the tortoise has moved further to arrive at some point E; and hence Achilles has not yet caught up with the tortoise, and this process can be continued indefinitely, and Achilles can never catch up with the tortoise [as he is involved in an infinite sequence of catch-up steps of ever diminishing length.] Zeno's argument is readily dispensed with through the work we present in the second part of this book. For from what is taught here, not only do we show that Achilles will overtake the tortoise, but also the point will be found where this occurs.

The third part of this book is based around geometric progressions that terminate, especially those associated with like planes, which are more suitable for reduction to a series by the method : for if two squares are given in the ratio of the larger inequality, from which it is required to show a surface, to which size it may prove to be equal, as the whole series of squares may produce, that has come from the progression of the ratio of the first square to the second, of the second to the third, and so on indefinitely.

In order that I may explain this in more familiar terms that may be put as follows : consider someone wishing to buy from somewhere a horse of the highest pedigree, which common opinion values at one thousand gold pieces. Now, since a thousand gold pieces are not at hand, this solemn contract is entered, that after a month the buyer will give the vendor one hundred gold pieces, after the second month he will give him fifty gold pieces, after the third month twentyfive, and so on henceforth. After several months, constantly assessing half of the half of the amount he should pay of this, which he will pay to the vendor at the end of the month. Finally wearied of the troubles, he tries to make a deal with the seller, by offering two hundred gold pieces for the remainder, in order that he might be freed from the bargain. Which of the two is cheated by the deal, the buyer or the seller? The third part of this book will demonstrate most clearly the solution of this problem and similar questions, and will assign what sum will be the remainder of the debt after several months. Indeed, someone might want to know also what the remaining sum should be after a *hundred or even a thousand months. The knowledge will be known with certainty from the geometry of the present part.*

*The fourth part will applied all the methods set fourth in the previous parts to finding the volumes of solid bodies. I hope the subject of this tract will not be received in an unpleasant manner by those who have cultivated common geometry. Indeed the notions contained in the wonderful theorems of this book are to be brought to light, and (which are hard to solve by the methods of common geometry) can be proven with suitable practise. Indeed, without the help of this book there would be nothing of certainty in this laborious work, which has leaned on its support to a large extent.*

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# *DEFINITIO PRIMA.*

*Geometricam seriem voco quantitatem finitam, divisam secundum continuationem cuiuscunque rationis datae.*

*Explicatio.*

Quamvis sensus, quem verba indicant obscurus non videatur, nihilominus mentem meam circa definitionem praesentem censui apponendam: ponatur itaque linea quaevis AB, divise in C, secundum quamcumque proportionem; & fiat quem admodum est tota AB ad CB, ita CB ad DB, & denuo ut CB ad DB, ita

**A C D E F G B**

DB ad EB, & hoc semper fiat. Oritur in hac ratione procedendi duplex consideratio; prima, rationis AB ad CB, & CB ad DB, & ulterius DB ad EB, atque ita deinceps : altera rationis AC ad CD, & CD ad DE, iterum DE ad EF, & ita consequenter ; & licet hae considerationes videantur diversae, in unum tamen scopum collimant: nam cum ratio AB ad CB, & CB ad DB, eadem sit cum ratione AC ad CD, & CD ad DE, atque ita ulterius, si fiat rectae AC aequalis HI & haec HI, **H K L M I**
dividatur in punctis K, L, M, secundum rationem AB ad CB, & CB ad DB, &c. tunc apparebit ratio, qua duae istae considerationes in unam coalescunt : si quis igitur petat, quid velim intelligi nomine seriei? respondeo me nomine serie, totam illam continuationem linearum intelligere, quarum prima est AC, secunda CD, ita ut ratio AC ad CD, eadem sit cum illa quam obtinet CD ad DE, atque ita deinceps terminatam eodem termino, quo terminatur ratio AB ad CB, &c. Et quia in definitione, particula adiuncta est *finitam* ; hinc hoc loco explicare cogor seriem per rationes AB ad CB, & CB ad DB, & ita consequentur, cum necdum demonstratum sit quo puncto lineae A C D terminus existat seriei rationis AC ad CD. nam plusquam notum est, apud philosophos nunquam perveniri posse ad terminum continuationis, per partes proportionales AC ad CD, & CD ad DE, cum residua semper remaneat quantitas dividenda, secundum easdem rationes ; quare terminus, hoc tenore progrediendi acquiri nequit : sed si quis procedat iuxta considerationem continuationis AB ad CB, & CB ad DB, & sic in infinitum, semper includit terminum, ad quem per continuationem rationis AC ad CD, perveniri nequit ; seriem itaque voco, quantitatem finitam AB, divisam secundum continuationem rationis AC ad CD, & CD ad DE. quae eadem semper existit cum ea quae sit continuando rationes AB ad CB. Quotiescunque igitur occurret mentionem fieri continuationis alicuius seriei AC ad CD, & CD ad DE; iubeat animo continuationem hanc finitam esse, & totam illam terminorum infinitorum collectionem, alibi terminatam esse : quae collectio series alicuius rationis Geometricae nuncupatur.

# *FIRST DEFINITION.*

*A geometrc series can be formed from a fixed length by the process of continued division according to some given ratio.*

*Explanation.*

Although I am of the opinion that the wording of the definition is not vague or obscure [note: as the translator, I have taken some liberties with the phrasing of this definition, to make it more accessible to the modern reader, as the original is a little obscure], I have it in mind nevertheless to add something to the present definition. Thus some line AB is put in place, which is divided at the point C, according to some given proportion; and the division certainly gives the ratio of the whole length AB to CB, thus again the remainder is divided as CB to DB, and as CB to DB, thus


DB to EB, and in this manner the ratio of subdivision always continues. Now, two series arise from the proceding ratios which are to be considered. The first series comes from the ratios AB to CB, CB to DB, DB to EB, and thus henceforth for further terms, as we have just seen. The other series comes from the ratio AC to CD, CD to DE, DE to EF, and so on for those that follow. Although these series are considered to be different, they do however aim [*i.e.* converge in modern terms] towards a single point: for with the ratio AB to CB, and CB to DB, the same situation shall be true for the ratio AC to CD, and CD to DE, and so on for further terms beyond.


For the line AC can be made equal in lenth to HI, and this line HI is divided by the points K, L, and M following the ratio AB to CB, CB to DB, etc. Then it will become apparent that the ratio formed from these two series merges into one : if someone so desired, by what name should the series be known ? I myself reply with a name for the series, that it must be understood to be the whole continuation of the lines, the first of which is AC, the second CD, thus in the ratio AC to CD, that will be the same as obtained from the ratio CD to DE, and so on for the rest of the terms with the same limit as that with which the ratio AB to CB, etc finishes [Here we use the word *limit* in its original sense, as a barrier or boundary].

Now, a small part is added to the definition, that is, *the end* of the series. Hence I have to give an explation about this place to which a series runs together, defined by the ratios AB to CB, CB to DB, and so on for the rest of the terms, since it has yet to be shown which point on the line ACD the end of the series formed by the ratio AC to CD shall prove to be. For, furthermore, it has been noted by the philosophers, that it is not possible for one to come to the end of the continuation of terms formed by the proportional parts AC to CD, CD to DE, etc, since there is always an amount left over from the division, according to the same ratios ; whereby the terminal point can never be attained by maintaining this course: yet if one could proceed to the consideration of the nearby continuations of AB to CB, CB to DB, and so on indefinitely, then the boundary would always be reached, but which one can never reach by a finite continuation of the ratios AC to CD.

Thus I give the name series to a finite quantity AB, divided according to a continuation of the ratios AC to CD, CD ad DE, etc, which shall always be the same as that formed by the continuation of the ratios AB to CB. However many times this occurs, there has to be a mention made of that other series of continued proportionals AC to CD, CD to DE, etc. One can consider that there is a limit to this continuation of ratios, and that for the total collection of an infinte number of terms, the limit is elsewhere [*i.e.* it is not one of the ratios] : any such collection of geometrical ratios is called a series.

## *DEFINITIO SECUNDA.*

**P**rogressio Geometrica est quotcunque terminorum secundum eandem rationem continuatio.

*Explicatio.*

**E**st itaque omnis progressio pars seriei , cum ut explicatum est, omnis series sit continuatio alicuius rationis eo usque producta, donec amplius protrahi nequeat, modo prius explicato; Progressio vero prout differt a serie, proprie interminata est ; ac proinde eo usque pars : ubi tamen inveneris in contextu sequentium, agi de tota progressione rationis alicuius continuata, de serie agi memineris, neque enim haec

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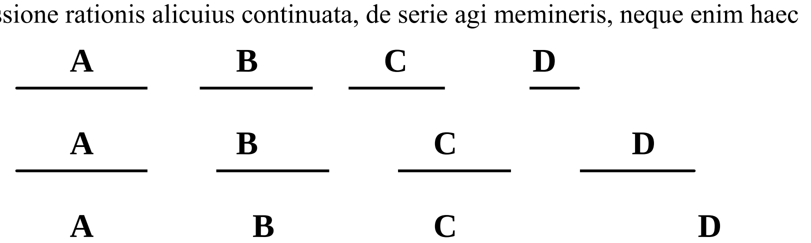
magis scrupolose observari volo, quam a Geometris omnibus scribentur huius continuatae proportionis, vel rationis; quarum licet a



altero in duobus terminis consistare, altera ......ribus, nihilominus pro arbitraque scribentis, passim ......ntur. Sint igitur haec praemissa, si non exactarum definitio cum loco, ...... rhetoricis explicationis supplemento. Datis itaque A & B :vel ratio A ad B est maioris inaequalitatis, vel prorsus aequalitatis, vel minoris inaequalitatis: quod si fiant continuae proportiones A B C D huiusmodi continuatio progressio vocetur, quotcunque tandem termini existant.

Progressio vero Geometrica iam explicata duplex est; alia continua, alia discreta.Continua est cum omnes termini rationem connectentes, habent rationem antecedentis, & consequentis; ut in secumatque rationum explicatarum, si fuerit quemadmodum A ad B, ita B ad C; & quemadmodum B ad C, ita C ad D, atque ita in quovis tandem numero terminorum; huiusmodi progressionem continuam voco.

Discreta progressio est similium rationum secundum aliquam continuationem positio, ut consequentes non fiant antecedentes. exempli causa; si fiat quem quemadmodum A est ad B, ita C ad D, & B fuerit minor quam A, vel C, talem progressionem in quodlibet tandem terminis constituta sit, voco discretam; etiam in his terminis 1, 2. 5, 10. 3, 6. 8, 16. Ubi discreta ratio valde interrupta est, quia est continuatio similium rationum.

**A** **B** **C** **D**

## *SECOND DEFINITION.*

**A** geometric progression is a number of terms which are formed with the same continuing ratio.

*Explanation.*

**T**hus it is the case that the progression is part of the whole series: as has been explained, the whole series is a continuation of some ratio as far as that can be produced, and which cannot be extented further along the lines of the previous explanation. The progression is truly different from the series, which properly is unending, and the progression is thus part of this series : and you will however arrive at the progression in the following context that is taken from some whole series of continued ratios as you may recall, and indeed that I do not wish to observe more closely, since everything concerning the continued proportions or ratios can be written down from geometry, whereby from two given terms it is permissable to imply the others [ the text is very difficult to read here], nevertheless in order to judge the writings [?], everywhere ....Therefore, with these premises, if the exact definition cannot be put in place, we can supplement with other rhetorical [?] definitions. Therefore with A and B given : either the ratio A to B is a greater inequality [i.e. A > B, corresponding to a decreasing progression], or they are equal [i.e. A = B], or it is a lesser inequality [i.e. A < B, corresponding to an increasing progression]: in which case they produce the proportions A B C D, in what is termed a continuous progression of this kind, until a certain number of terms shall arise.

**A** **B** **C** **D**

**A** **B** **C** **D**

**A** **B** **C** **D**

Indeed the geometic progression has been set forth in a two-fold manner : the one continued and the other discrete. The continued one has all the terms connecting the ratio, the preceding terms have the same ratio as the subsequent, set forth according to the form of the ratio. If it were as A to B, thus B to C; and as B to C, thus C to D, ans thus in some final number of terms ; I call a progression of this kind a contining progression.

A discrete progession is composed of like ratios with the

**A** **B** **C** **D**

position following some continuation, as a subsequent term is not made from the antecedent. For example, if the terms are made as A is to B, thus C to D, and B were less than A or C, such a progression that can then be set up with any terms I call discrete; now also with these terms 1, 2. 5, 10. 3, 6. 8, 16, where the individual ratio has been greatly interrupted, as it is a continuance of the same ratio.

## *DEFINITIO TERTIA.*

**T**erminus progressionis est seriei finis, ad quem nulla progressio pertinget, licet in infinitum continuetur; sed quovis intervallo dato propius ad eum accedere poterit.

*Explicatio.*

**A C D E F G B**

Ponatur rect AB, divisa in C D E F G. ut continuent eandem rationem AB, CB, DB, EB, FB, G. Cum sit eadem ratio AC ad CD, & CD as DE, atque ita deinceps, cum ea, quae reperitur inter lineas AB, CB, DB, &c. similis quoque erit eiusdem rationis progressio AC ad CD, cum progressione AB, ad CB. terminus autem rationis AC ad CD, dicitur punctum B, sive illum intrinsecum velis, sive extrinsecum, per me licet; nam de re nobis est hic quaestio, non de verbo : ad quod punctum nulla progressio pertingere valet, cum omnis progressio interminata pars seriei existat : nihilominus tamen, poterit ad illum progressio per continuationem magis ac magis acccedere, ita ut vicinior ultimus terminus progressionis interminatae existat ipsi termino seriei, quam sit distantia quaecunque proposita.

An autem talis detur terminus progressionis, & quo pacto investigati debeat, libro secundo huius tractatos disceptabitur, illud interim insinuatum hoc loco desidero, huiusmodi terminum solum reperiri in iis progressionibus, quem proportionem maioris inaequalitaiis continuant, nam earum progressionum quae vel terminis semper aequalibus constant, vel certe terminis minoris inaequalitatis, nullum assignari posse terminum, manifestum nimis est, quam ut explicatione indigeat; quandoquidem dictae progressiones, si continuentur, magnitudinem quavis data maiorem exhibere natae sint. Itaque secundo libro, tertim et quarto, in quibus progressiones iam terminatae

**[56]**

considerantur, nullae proportiones praeter eas, quae maioris erunt inaequalitatitis : in primo vero, quoniam illis a termina adhuc abstrahitur, etiam progressiones aequalitatis, & minoris aequalitatis contemplabimur. Terminus igitur progressionis talis est, quaemadmodum explicuimus, cum scilicet aggregatum, sive summa terminorum progressionis, quantumvis continuatae numquam excedit quandam magnitudinem; excedit vero omne minus llla magnitudine , atque ita posset etiam dici productum sive quantitas totius, datae progressionis, & magnitudo illa aequalis dicetur toti progressioni dati ; hoc est omnibus terminis proportionalibus simul sumptis. Idem igitur quoad rem erit, sive terminum quaerere progressionis , huc magnitudinem toti progressini parem , sive ipsammet integram seriem progressionis exhibere. Si quid praeteria pertinebit ad terminoram explanationem, in decursis operis exponetur.

## *THIRD DEFINITION.*

**T**he limit [terminus] of the progression is the end of the series, a point which no progression will reach even if it is allowed to continue indefinitely; but for any given interval around the end point, terms in the progression will be able to come nearer.

*Explanation.*

**A C D E F G B**

The line AB is put in place, divided in the points C D E F G, in order that they continue in the same ratio AB, CB, DB, EB, FB, GB. The ratio AC to CD, and CD to DE, etc, and thus henceforth, shall be the same as which is found between the lines AB, CB, DB, etc, and the ratio of the progression AC to CD will be similar too will be the the progression with the progression AB to CB. But the limit [terminus] of the ratio AC to CD is said to be the point B, either inside as you may wish, or outside, permitted by me; for concerning this thing this is the question for us, but not concerned with words : what is the point that no progression is able to reach, with the whole infinite part of the progression set out ? Yet nevertheless, more and more terms can be added to that progression by continuation, and thus as the final term of the progression gets closer to the limit of the infinite series itself, that shall be the proposed distance whatever.

However, just so many terms of a progression may be given, and the arrangement by which such a progression ought to be investigated will be discussed in the second book of this tract. Meanwhile, a number of these terms have insinuated themselves into the place I desire. A limit of this kind is only to be found in these progressions that continue with the larger inequality. For those progressions where the terms are either always equal or constant, or indeed for terms with the smaller inequality, nothing can be assigned to the limit, as is quite clear by way of any explanation needed, since the said progressions, if they are to continue, show some given magnitude become larger. Thus with the second, third, and fourth books, in which terminating progressions are now considered, there are no proportions except these from greater inequalities. In the first book, indeed, since for these so far only parts of the progressions separate from the limit will be contemplated, as with also with progression of equal or smaller ratios. The limit of a progression is therefore such, as we have explained, with a known collection or sum of terms of the progression, although continued to any will never exceed a certain amount. Truly it will exceed all magnitudes less than that amount, ans thus it may also be said to be the product or size of the whole given progression. This magnitude is said to be the total of the given progression ; that is , the sum of all the proportional terms taken together. Therefore it amounts to the same thing: either the limit of the required progression, here equal to the magnitude of the whole progression , or the sum of the whole series of the progression itself is shown. If there is anything else besides pertaining to the explanation of the limit, then it will be set out in the works as we proceed.

**[57]**

## *PROGRESSIONUM GEOMETRICARUM*
# *PARS PRIMA.*

### *Progressiones considerat indeterminatas.*

## *PROPOSITIO PRIMA.*

Continue proportionalium differentiae sunt in continua analogia eiusdem rationis suorum integrorum.


**Prop. 1. Fig. 1.**

Sunt magnitudines continuae proportionales AG, BG, CG, DG, EG, FG. ostendendum est AB, BC, CD, DE, EF differentias in continua esse analogia suorum integrorum, & contrae si AB, BC, CD, &c, fuerint continuae, & FG addatur ut AB, BG, AB, BC sint proportionales;

Dico & AG, BG, CG, &c. esse in continua analogia.

### *Demonstratio.*

Quoniam AG est ad BG, ut BG ad CG : erit dividendo ut AB ad BG, sic BC ad CG :& permutando AB ad BC, ut BG ad CG, sive ut AG ad BG. Similiter BG est ad CG, ut CG ad DG : & dividendo BC ad CG ut CD ad DG, & pernutando BC est ad CD, ut CG ad DG, id est ut BG ad CG. id est ut AG ad BG. atqui erat ut AG ad BG, sic AB ad BC, ergo BC est ad CD ut AB ad BC. Continuae sunt igitur proportionales AB, BC, CD, & quidem in analogia suorum integrorum AG, BG, simili ratione ostendam reliquas cum his tribus esse continuas: quod erat prinum.

Sint iam AB, BC, CD, &c. continuae, quibus addita sit FG, sic ut AG sit ad BG ut AB ad BC; Dico omnes AG, BG, CG, DG, &c. esse in continua analogia differentiarum. Quia ut AB ad BC, sic AG ad BG, ergo permutando & dividendo, & rursum permutando ut AB ad BC, hoc est ex datis ut AG ad BG, sic BG ad CG. quoniam autem iam BC est ad CD, ut AG ad BG, hoc est ex datis ut AB ad BC, hoc est rursum ex datis ut BC ad CD, eodem plane discursu ostendemus BG, CG, DG esse continuas, &c. quidem in ratione BC ad CD, hoc est AB ad BC; Quare cum etiam AG, BG, CD sint in eadem ratione continuae, omnes quatuor AG, BG, CG, DG erunt in ratione AB ad BC continuae. similiter ostendemus & reliquas cum iisdem esse continuas. Patet igitur veritas propositionis.

### *GEOMETRIC PROGRESSIONS.*

# *PART ONE.*

### *Indeteminate progressions are considered.*

## L2.§1. *PROPOSITION 1.*

The differences of continued proportions are in the same continued ratio as their whole lengths.


**Prop. 1. Fig. 1.**

The magnitudes of the lines AG, BG, CG, DG, EG, FG are continued proportionals. It is to be shown that the differences AB, BC, CD, DE, EF are in the continued ratio of their whole lengths, and likewise if AB, BC, CD, etc, are continued proportions, and FG is added in order that AG, BG, AB, BC are proportionals, then I say that AG, BG, CG, &c. are in the continued ratio.

### *Demonstration.*

Since AG is to BG, as BG to CG : by division it becomes as AB to BG, thus BC to CG: and on interchanging the terms, as AB to BC, so BG to CG, or as AG to BG. In the same way, as BG is to CG, so CG is to DG : and by division, BC to CG as CD to DG, and on interchanging the terms, BC is to CD, as CG to DG, that is as BG ad CG, which is as AG to BG. Yet as AG is to BG, thus AB to BC, hence BC is to CD as AB to BC. Therefore AB, BC, and CD are continued proportionals, and indeed in the ratio of their whole lengths AG to BG, and I can show that the rest are in continued proportion with the same ratio as these three: which is the first part.

Now AB, BC, CD, etc. are continued proportions, to which FG is added, thus as AG is to BG so AB to BC. I say that all the lengths AG, BG, CG, DG, etc. are in the continued ratio of the differences. Because as AB to BC, thus AG to BG, therefore by interchanging terms and dividing, and again by interchanging terms as AB to BC, that is from the given ratios as AG to BG, thus BG to CG. But now since BC is to CD, thus AG is to BG, that is from the given AB to BC as BC to CD. By the same clear discourse we can show that BG, CG, and DG are in continous proportion, etc. Indeed in the same ratio BC to CD as AB to BC.

Whereby as also AG, BG, and CG are in the same continued ratio, all four AG, BG, CG, and DG are in the continued ratio AB to BC. Similarly we can show the rest to be in the same continuous ratio with these. Therefore the truth of the proposition is apparent.

## L2.§1. Prop. 1 Note:

AG/BG = BG/ CG : hence AG/BG - 1 = BG/ CG - 1 or AB/BG = BC/CG: and AB/BC = BG/CG = AG/BG. In the same way, as BG/CG = CG/ DG : so BC/CG = CD/DG, and BC/CD = CG/DG = BG/CG = AG/BG. As AG/BG = AB/BC, hence BC/CD = AB/BC. Therefore AB, BC, and CD are continued proportionals, and indeed in same ratio as AG to BG.

Now AB/BC = BC/ CD = .... DE/EF = EF/FG are continued proportions, to which FG is added. Hence, if we assume AG/BG = AB/BC, then all the lengths AG, BG, CG, DG, etc. are in the continued ratio of the differences. Since AB/BC = AG/BG, then AG/AB - 1 = BG/BC - 1 giving BG/AB = CG/BC and AB/BC = BG/CG = AG/BG. Again from BC/CD = BG/CG we can show that BG, CG, and DG are in continous proportion, and hence all four AG, BG, CG, and DG are in the continued ratio AB to BC.
Note also that we could have started an addition process from the right-hand end and arrived finally with the initial assumption that AG/BG = AB/BC.

**[58]**

## *PROPOSITIO II.*

Si quatuor fuerint quantitates in continuata analogia, erunt aggregata ex prima & secunda, ex secunda & tertia, ex tertia & quarta, in continua proportione.


**Prop. 2. Fig. 1.**

### *Demonstratio.*

Sunto quatuor in continua analogia AB, BC, CD, DE. Dico etiam AC, BD, CE, esse continue proportionales. cum enim ut AB ad BC, ita BC ad CD, erit [a] utraque antecedens AC, ad utramque consequentem BD, ut AB ad BC, una antecedens , ad unam consequentium : simili modo quia BC ad CD, est ut CD ad DE, erit utraque antecedens BD, ad CE utramque consequenem, ut BC ad CD, id est ut AB ad BC, hoc est ut AC ad BD. Sunt igitur AC, BD, CE in continua analogia. quod fuit demonstrandum.
*a 11 Quinti*.

## *L2.§1.* *PROPOSITION 2.*

If there are four quantities in continued proportion, then the sums of the first and the second, of the second and the third, and of the third and the fourth terms are also continuously in proportion.

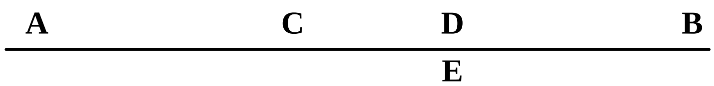
### *Demonstration.*

Let the four terms in continued proportion be AB, BC, CD, and DE. I say that the terms AC, BD, and CE are also in continued proportion. For indeed as AB is to BC, thus BC is to CD, [a] both the preceeding terms AC, to both the following terms BD, shall be as AB to BC, the one preceeding term, to the one following term : in the same manner as BC to CD is as CD to DE, both the preceeding terms BD, shall be to both the following terms as CE as BC to CD, that is as AB to BC, that is as AC ad BD. Hence AC, BD and CE are continued ratios. q.f.d.
*a 11 Quinti*.
[From AB/BC = BC/CD it follows that AB/BC + 1 = BC/CD + 1, that is AC/BC = BD/CD, etc.]

## *PROPOSITIO III.*

Sit AB divisa in C & D ut ratio AB ad AC duplicata sit eius, quam habet BD ad DC. Dico AC, AD, AB tres esse in continua ratione.

***Demonstratio.***



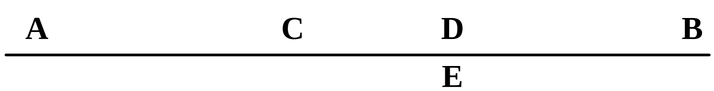

**Prop. 3. Fig. 1.**

Ponatur AE, media inter AB, & AC; igitur tres erunt continae quantitates AC, AE, AB quare per primam huius erit ratio AB ad AE, eadem cum ratione BE ad EC. sed ratio AB ad AC, duplicata est rationis AB ad AE, igitur & ratio AB ad AC duplicata est rationis BE ad EC. quare divisa est CB in E, ut divisa est eadem CB in D : ac proinde punctum D, unum idemque; est cum puncto E. Unde cum sint tres continuae proportionales AC, AE, AB, ex constructione, erunt quoque in continuata ratione AC, AD, AB. Q.E.D.

## *L1.§1.* *PROPOSITION 3.*

Let AB be divided in C and D in order that the ratio AB to AC is equal to the square of that which BD has to DC. I say that AC, AD and AB are three terms in a continued ratio.

***Demonstration.***




**Prop. 3. Fig. 1.**

Put the mean AE between AB and AC; then the three quantities AC, AE, and AB are in continued proportion, whereby from the first PROPOSITION of this book, the ratio AB to AE is the same as the ratio BE to EC. But the ratio AB ad AC is equal to the square of the ratio AB ad AE, and therefore the ratio AB to AC is equal to the square of the ration BE ad EC. Whereby CB has been divided in E, as the same CB has been divided in D ; and hence the point D, is one and the same as the point E. Thus with three continued proportionals AC, AE, AB, from the construction, are also in the continued ratio AC, AD, AB. Q.E.D.

[AC/AE = AE/AB from the definition of the mean; Now, from the first proposition, {or from AC/AE + 1 = AE/AB + 1, that is EC/AE = EB/AB}, AB/ AE = BE/EC; but AB/AC = (AB/AE).(AE/AC) = $(BE/EC)^2$ ; and AB/AC = $(BD/DC)^2$ is given; hence CB is divided by E in the same way that it is divided by D. Hence D and E co-incide. This result is thus far easier to obtain by regarding AB, AD, and AC as a geometric progression, than by using means of ratios, etc, which is the point of this proposition.]

## *PROPOSITIO IV.*

Quod si AB, AC,AD, sint quantitates proportionales:
Dico AD ad AB, rationem eius habere duplicatam, quam habere DC ad CB.

***Demonstration.***


**Prop. 4. Fig. 1.**

Ratio enim AD ad AB, duplicata est rationis AD ad AC, sit per primam huius ut AD ad AC, ita quoque est DC ad CB, igitur ratio AD ad AB duplicata est rationis DC ad CB. Q.E.D.

## *L2.§1. PROPOSITION 4.*

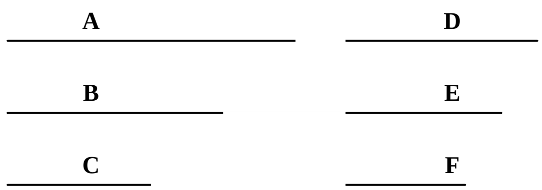
For if AB, AC, AD are proportional quantities, then I say that the ratio AD to AB is equal to the square of the ratio DC to CB.

*Demonstration.*

Indeed the ratio AD to AB is equal to the square of the ratio AD to AC, which is by the first PROPOSITION of this book, as AD to AC, thus too is DC to CB. Hence the ratio AD to AB is the square of the ratio DC to CB. Q.E.D.

## *PROPOSITIO V.*

Sint tres continuae proportionales A, B, C, sit autem ratio A ad B, triplicata eius quam habet D ad E : ratio quoque B ad C, triplicara rationis E ad F. Dico D, E, F quantitates in continua analogia.




**Prop. 5. Fig. 1.**

*Demonstratio.*

Cum ratio A ad B, sit eadem cum ratione B ad C; igitur etiam ratio B ad C triplicata est rationis D ad E. est autem & ratio B ad C ex suppositione triplicata eius quam habet E ad F; igitur ratio D ad E, est eadem cum ratione E ad F. Sunt igitur in continuata proportione D, E, F. Quod demonstrandum fuit.

## *L2.§1. PROPOSITION 5.*

Let there be three continued proportionals A, B, and C. Moreover, the ratio A to B is the cube of the ratio D to E : and also, the ratio B to C is the cube of the ratio E to F. I say that the quantities D, E, and F are in a continued ratio.

*Demonstration.*

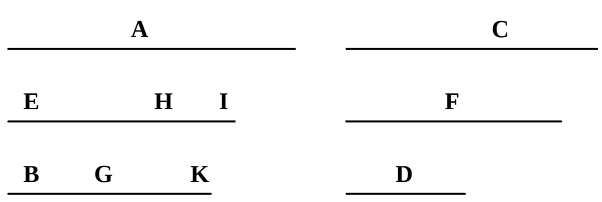
As the ratio A to B, shall be the same as the B to C; therefore the same ratio B to C is the cube of the ratio D to E. Moreover the ratio B to C from supposition is the cube of the ratio E to F; therefore the ratio D to E is the same as the ratio E to F. Therefore D, E, F are in a continued proportion . Quod demonstrandum fuit.

**[59]**

## *PROPOSITIO VI.*

Sit A ad BK in minore ratione, quam C ad D.

Dico A ad EI mediam proportionalem inter A & BK, esse in minori ratione, quam sit C ad F mediam inter C & D.




**Prop. 6. Fig. 1.**

*Demonstratio.*

Fiat enim A ad BG in ratione C ad D, erit [a] BG minor quam BK. Et fiat inter A & BG media EH, erit quoque EH minor quam EI. Rursam quia A ad BG, duplicatam habet rationem A ad EH, & C ad D eamdem ex constructione habet rationem, quam A ad BG; habebit etiam C ad D rationem duplicatam eius, quam habet A ad EH : sed & C ad D duplicatam habet eius, quam C ad F; igitur ratio C ad F, eadem est cum ratione A ad EH: sed EI ostensa est maior quam EH, igitur A ad EI [b] minorem habet rationem, quam A ad EH, id est quam C ad F. Quod erat ostendendum.

*a 10 Quinti; b 8 Quinti.*

*Corollarium.*

Simili modo demonstrabitur si plures mediae inter primam & secundam, & inter tertiam & quartam ponantur, fueritque in prioribus maior proportio vel minor primae ad secundam , quam in posterioribus tertiae ad quartam; fore etiam in prioribus primae ad primam mediarum, vel secundam, aliamque quamcumque maiorem vel minorem proportionem, quam teriae ad similem ordine mediam inter posteriores.

## *L2.§1. PROPOSITION 6.*

Let the ratio A to BK be less than the ratio C to D.

I say that A to EI, the mean proportion between A and BK, is less than the ratio C to F, the mean proportional between C and D.

*Demonstration.*

Indeed let A to BG be made in the ratio C to D, and [a] BG will be less than BK. And let the mean between A and BG be EH, which will also be less than EI. Again, since A to BG is the square of the ratio A to EH, and C to D by construction has the same ratio as A to BG; also C to D will have the same square ratio of that which A has to EH : but C to D has the square of that, which C has to F. Therefore the ratio C to F is the same as the ratio A to EH. But EI has been shown to be larger than EH, therefore A to EI [b] has the lesser ratio than A to EH, that is C to F. Which had to be shown.

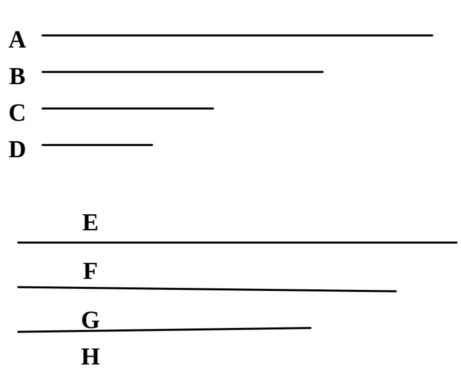
*a 10 Quinti; b 8 Quinti.*

[$A/EH = EH/BG$ ; $A/BG = A/EH \,.\, EH/BG = (A/EH)^2 = C/D = (C/F)^2$ ; hence $A/EH = C/F$. But $EI > EH$, hence $A/EI < A/EH = C/F$.]

*Corollory.*

In the same way it will be shown that if more means are places between the first and the second terms, and between the third and the fourth terms, then the first to second terms will be in the initial greater or less proportion as in the final third to fourth terms; indeed the ratio in the initial first term to the first or second mean, or some other larger or smaller proportion, than the ratio of the third term to a similar mean between the terms of the second ratio.

## *PROPOSITIO VII.*




**Prop. 7. Fig. 1.**

Sint duo ordines continues proportionalium A, B, C, D, & E, F, G, H: & maior sit ratio A ad B, quam E ad F.

Dico maiorem quoque esse rationem A ad C tertiam, vel D quartam, quam E ad G tertiam, vel H quarta.

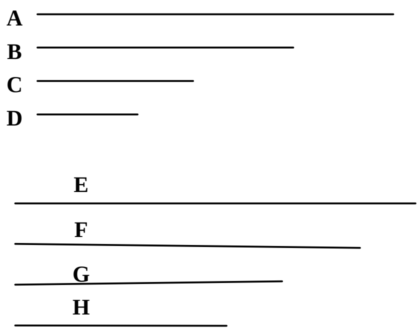
*Demonstratio.*

Cum enim sit eadem ratio A ad B, quae B ad C, & C ad D ; Similiter F ad G ratio, & G ad H eadem, quae est E ad F, sitque A ad B maior ratio, quam E ad F : erit etiam tam B ad C quam C ad D maior ratio, quam F ad G, aut G ad H : & proinde erit A ad c C maior ratio, quam E ad G : & similiter B ad D maior ratio, quam F ad H; & A ad D maior quam E ad H : quae erant demonstranda.

## ***L2.§1. PROPOSITION 7.***

Let there be two continued orders of proportionals A, B, C, D, and E, F, G, H: and the ratio of A to B shall be greater than the ratio of E to F.

I say that the ratio of A to the third proportion C, or to fourth proportion D is also larger than the ratio of E to the third proportion G or the fourth H.




**Prop. 7. Fig. 1.**

*Demonstration.*

Since indeed the ratio of A to B is the same as the ratio of B to C and also of C to D; similarly the ratio F to G , and the ratio G to H, is the same as the ratio E to F. The ratio of A to B is greater than the ratio of E to F: also the ratio of B to C and C to D is greater than the corresponding ratios F to G and G to H : and hence A to C [c] is greater than the ratio E to G : and similaly the ratio B to D is greater than the ratio F to H; and A to D is greater than E to H : which were to be shown.

**[60]**

## ***PROPOSITIO VIII.***

Sint tres magnitudines AC, AB, AF; & FE aequalis sit CB.

Dico si AC minima trium, & CB minor differentia, & BE, utriusque differentiae, differentia sint continue proportionales, ipsas quoque magnitudines AC, AB, AF esse in continua ratione.


**Prop. 8. Fig. 1.**

*Demonstratio.*

Rectangulum FAC [a] aequatur rectangulo FCA una cum quadrato CA, rectangulum autem FCA aequatur [b] rectangulo ACB, & rectangulo AC BE (id est [c] quadrato CB, cum AC, CB, BE ponantur continuae) ac praeterea rectangulo AC EF; id est ACB, quia aequales sunt CB, EF : igitur rectangulum FAC aequatur quadratis AC, CB, & rectangulo ACB bis; hoc est quadrato [d] AB; sunt [e] igitur tres proportiones magnitudines AC, AB, AF. Quod erat demonstrandum. *a 3. secundi; b 3 Secundi; c 17 Sexti; d 4 secundi; e 17 Sexti.*

*Aliter.*

Ut BC ad CA, sic FB ad BC, id est EF : igitur componendo ut BA ad CA, sic BF ad EF, hoc est BF ad BC ergo permutando & componendo FA ad BA, ut BA ad CA. Quod erat demonstrandum.

*Corollarium.*


**Prop. 8. Fig. 2.**

Si autem AC, CB, EF fuerunt proportionales, & BE aequalis CB minori differentiae, erunt rursum AC, AB, AF continuae proportionales. Demontratio eadem est, quae propositionis iam positae.

## *L2.§1.* *PROPOSITION 8.*

AC, AB, and AF are three magnitudes ; and FE is equal to CB.

I say that if AC the smallest of the three, and both CB the least difference, and the other difference BE, are successive proportionals of differences, then the magnitudes themselves AC, AB, AF are also in a continued ratio.

*Demonstration.*

The rectangle FA.AC [a] is equal to the rectangle FC.CA together with the square CA; but the rectangle FC.CA is equal to [b] the rectangle AC.CB and the rectangle AC.BE (that is [c] to the square CB, since AC, CB, and BE are placed in continued proportion) and to the rectangle AC.EF in addition, that is AC.CB, since CB and EF are equal. Therefore FA.AC is equal to the squares AC and CB together with twice the rectangle AC.CB; that is to the square [d] AB. The three magnitudes AC, AB and AF are therefore [e] in proportion. Q.E.D. *a 3. secundi; b 3 Secundi; c 17 Sexti; d 4 secundi; e 17 Sexti.*

[FA.AC = FC.CA + $AC^2$ = AC.CB + AC.BE + AC.CB + A $C^2$ = $CB^2$ + $CA^2$ + 2.AC.CB = $(AC + CB)^2$ = $AC^2$ . Hence AC, AB, and AF are in continued proportion.]

*Otherwise.*

As BC is to CA, thus EB is to BC, that is EF : therefore by addition, as BA to CA, thus BF to EF, that is BF to BC. Hence by interchanging and additiom, FA is to BA, as BA is to CA. Q.E.D.

[BC/CA = EB/BC, giving BA/CA = EC/BC = BF/BC of BF/BA = CB/AC giving AF/AB = AB/AC.]

*Corollary.*

But if AC, CB, and EF are proportionals, and BE is equal to the smallest of the differences CB, again AC, AB, and AF will be continued proportionals. The demonstration is the same as that for the proposition just considered.

[$CB^2$ = AC.EF; BE = CB; consider AC.AF = AC.(AC + CB + BE + EF) = $AC^2$ + 2.AC.BC + $BC^2$ = $AB^2$.]

## *PROPOSITIO IX.*

Sint in continua analogia AB, AC, AD, & minori differentiae aequalis sit ED.

Dico minimam AB, & minorem differentiam BC, deinde & CE, utriusque differentiae differentiam, esse proportionales.


**Prop.9. Fig. 1.**

*Demonstratio.*

Rectangulum DAB [f] aequatur rectangulo DBA, & quadrato AB : rectangulum autem DBA & aequatur rectangulis ABC, & AB ED, id est rursus rectangulo ABC ( sunt enim ED, BC lineae aequales) & rectanguloAB CE: ergo rectangulum DAB, aequatur quadrato AB, & rectangulo ABC bis, & praeterea rectangulo ABCE; quadratum vero AC, aequatur quadrato AB, rectangulo ABC bis, & quadrato BC. Atque cum AB, AC, AD ponantur continuae proportionales, erit rectangulum [h] DAB aequale quadrato AC ; ergo quadratum AB, rectangulum ABC bis, & rectangulum ABCE simul sumpta, aequantur quadrato AB, rectangulo ABC bis, & quadrato BC simul sumptis. Itaque demptis communibus quadrato AB, & rectangulo ABC bis, aequalia remanent quadratum BC, & rectangulum ABCE; quare AB [g], BC, CE, sunt tres continuae proportionales. Q.e.d. *f 3. Secundi; g 1. Secundi; h 17. sexti; i ibidem.*

**[61]**

*Aliter.*

Quoniam DA, CA, BA ponuntur proportionales, ergo per primam huius DC ad CB est ut CA ad BA : quare (cum aequales sint DE, BC ex hypothesi) ut CD ad ED, sic CA ad CA. ergo dividendo CE ad ED, hoc est, CB, ut CB ad BA. Quod erat demonstrandum.

*Corollarium.*

**A B C E D**

**Prop.9. Fig. 2.**

Quod si positis continue proportionalibus AB, AC, AD, sumatur CE aequalis minori differentiae BC. erut AB, BC, ED in continua analogia : quod demonstrabitur prorsus eadem modo quo propositio iam posita.

## *L2.§1.* *PROPOSITION 9.*

AB, AC, AD are [increasing] continued ratios, and ED is set equal to the smallest of the differences BC. I say that AB, the smallest difference BC, and thus CE, the difference of the other difference, are in proportion.

*Demonstration.*

The rectangle DA.AB [f] is equal to the sum of the rectangle DB.BA and the square AB : but the rectangle DB.BA is equal to the sum of the rectangles AB.BC, AB.ED, that is again equal to the rectangle AB.BC ( for the lines ED and BC are equal) and the rectangle AB.CE: therefore the rectangle DA,AB is equal to the sum of the square AB and twice the rectangle AB.CC bis, and to the rectangle AB.CE besides; the square AC is truly equal to the sum of the square AB with twice the rectangle AB.BC and the square BC. And with AB, AC, AD placed in continued proportion, the rectangle [h] DA.AB is equal to the square AC ; therefore the sum of the square AB, twice the rectangle AB.BC and the rectangle AB.CE , is equal to the sum of the square AB, twice the rectangle AB.BC , and the square BC. Therefore with common square AB and twice the rectangle AB.BC taken away, there remains the square AB equal to the rectangle AB.CE; whereby AB [g], BC, CE, are three continued proportionals. Q.e.d.

*f 3. Secundi; g 1. Secundi; h 17. sexti; i ibidem*.

[DA.AB = DB.BA + $AB^2$ ; DB.BA = AB.BC + AB.ED ( = AB.BC as ED = BC ) + AB.CE. Therefore DA.AB = $AB^2$ + 2.AB.BC + AB.CE ; $AC^2 = AB^2$ + 2.AB.BC + $BC^2$ ; also, $AC^2$ = DA.AB from continued proportions. Therefore, $AB^2$ + 2.AB.BC + AB.CE = $AC^2 = AB^2$ + 2.AB.BC + $BC^2$ ; hence AB.CE = $BC^2$, and thus AB, BC, and CE are in continued proportion as required.
In modern terms, if AB = $a$, AC = $ar$, and AD = $ar^2$, then ED = BC = $a(r - 1)$,
CE = $ar^2 + a(1 - r) - ar = a(r - 1)^2$; hence $a$, $a(r - 1)$, $a(r - 1)^2$ are terms in the continued ratio $(r - 1)$ ]

*Otherwise.*

Since DA, CA, BA are placed as proportionals, it follows by the first proposition of this book, that DC to CB is as CA to BA : whereby (as DE is equal to BC by hypothesis) as CD to ED, thus CA to BA. Therefore on dividing, CE is to ED, that is CB, as CB to BA. Q.e.d.

[DA/CA = CA/BA giving CD/CA = CB/BA = DE/BA or CD/DE = CA/BA, giving CE/DE = BC/BA or CE/BC = BC/BA as required.]

*Corollory.*

**A B C E D**

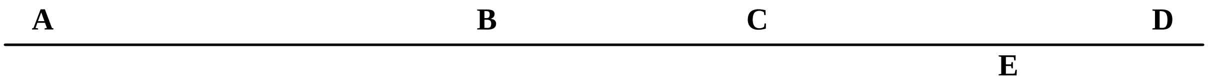
**Prop.9. Fig. 2.**

For if you take AB, AC, AD for the continued proportions, CE is taken equal to the smallest difference BC. AB, BC, ED are then in a continued ratio : which in short will be explained in the same way as the proposition now in place.

## *PROPOSITIO X.*

Sint tres magnitudines AB, AC, AD, in continua analogia; ponantur autem maxima trium AD, & maior differentia CD, dein & tertia quaepiam ED, continuae proportionales.

Dico BC, CE, aequales esse lineas.




**Prop.10. Fig. 1.**

*Demonstratio.*

Cum tres ponantur continuae AB, AC, AD, erit DC ad [a] CB, ut DA ad CA: quare [b] rectangulum DCA rectangulo DA BC, aequale est. Similiter cum ponantur continuae AD, CD, ED, est AC ad CE [c] ut AD ad CD : ergo rectangulum idem ACD aequatur etiam rectangulo ADCE rectangula. rectangula ergo ADBC, AD, CE inter se aequantur ; unde BC, [d] CE aequales sunt. quod erat demonstrandum.

*a 1. Huius; b 16 Sexti; c 1. Huius; d 1. Sexti.*

*Aliter.*

Quandoquidem ponantur continuae AB, AC, AD, ergo per primam huius ut DA ad CA, sic DC ad CB. Iterum quoniam ponuntur continuae AD, CD, ED : per primam huius ut AD ad CD, sic AC ad CE ; & permutando ut AD ad AC, sic CD ad CE. Sed ut AD ad AC, sic ostendi esse CD ad BC; ergo CD est ad CE, ut CD est ad BC. aequales igitur sunt BC, CE. quod erat demonstrandum.

*Corollarium.*


**Prop.10. Fig. 2.**

Simili plane modo, si positis continuis AB, AC, AD, etiam AD trium maxima. CD maior differentia, & tertia quaepiam CE fuerint continuae proportionales.Dico BC, ED aequales esse. Demonstratio eadem estque propositionis decimae.

## *L2.§1.* *PROPOSITION 10.*

The three magnitudes AB, AC, and AD are in a continued ratio; moreover the largest of the three AD, the larger difference CD, and some other third length ED are put in place in order that these three lengths are continued proportionals.

I say that in this case the lines BC and CE are equal in length.

*Demonstration.*

Since the three lengths AB, AC, and AD are put in a continued ratio, DC is to [a] CB as DA to CA: quare [b] rectangulum DCA rectangulo DA BC, aequale est. Similiter cum ponantur continuae AD, CD, ED, est AC ad CE [c] ut AD ad CD : ergo rectangulum idem ACD aequatur etiam rectangulo ADCE rectangula. rectangula ergo ADBC, AD, CE inter se aequantur ; unde BC, [d] CE aequales sunt. quod erat demonstrandum.

*a 1. Huius; b 16 Sexti; c 1. Huius; d 1. Sexti.*

[AC/AB = AD/AC, hence BC/AB = CD/AC = BC/AC = CD/AD; whereby DC.CA = DA.BC. Similarly, CD/AD = ED/CD, hence AC/AD = CE/CD and AC.CD = AD.CE. Hence AD.CE = AD.BC, and so CE = BC.

In modern terms again, the ratios are AD = $ar^2$, CD = $ar(r - 1)$, and ED = $a(r - 1)^2$ .]

*Otherwise.*

Since the continued ratios AB, AC, AD are put in place, therefore by the first proposition of this book, DA is to CA, thus as DC is to CB. Again, since the lengths AD, CD, ED are placed in continued proportion : by the first prop. as AD is to CD, thus AC is to CE ; and by interchanging, as AD to AC, so CD to CE. But as AD is to AC, thus I have shown that CD is to BC; hence CD is to CE, as CD is to BC. Therefore BC and CE are equal. Q.E.D.

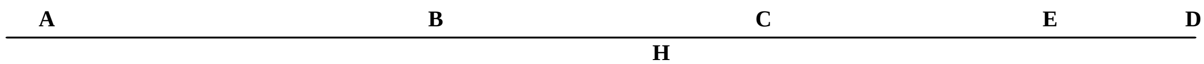
*Corollary.*


**Prop.10. Fig. 2.**

Clearly in the same way, if AB, AC and AD are placed in continuous proportion. Again, AD the greatest of the three, CD the greater difference, and the third some length CE are continued proportionals. I say that BC and ED are equal. The demonstration is the same as for proposition ten.

## *PROPOSITIO XI.*

Sint AB, AC, AD in continua analogia, & minori differentiae BC aequalis sit CE. Dico ED, CD, AD in continua quoque analogia esse.




**Prop.11. Fig. 1.**

**[62]**

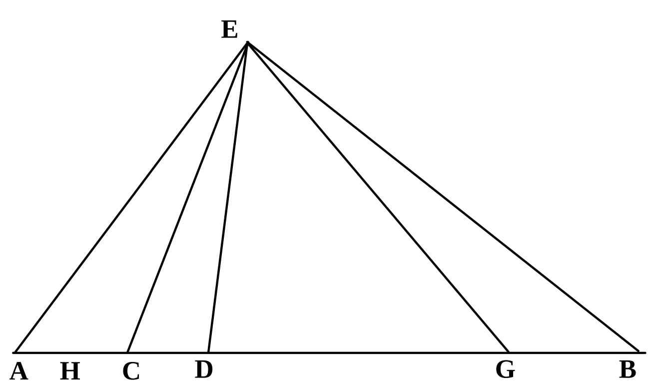
*demonstratio.*

Rectangulum ADE aequatur [a] rectangulo AED cum quadrato ED : rectangulum autem AED, aequatur rectangulo [b] DEC, & DECB, hoc est rursus rectangulo DEC (cum CE, CB sint aequales) & rectangulo DE BA. Sed per corollarium nonae huius, AB, BC, ED sunt continuae. ergo rectangulum ED AB aequale est quadrato BC, hoc est quadrato EC. rectangulum igitur AED aequatur rectangulo DEC bis, & quadrato EC, quare rectangulum ADE aequatur rectangulo DEC bis, & quadratis EC, ED, hoc est [c] quadeato CD. Unde AD, CD, ED sunt tres continuae proportionales. quod erat demonstrandum.

*a 3. Secundi; b 1. Secundi; c 4 Secundi.*

*Aliter.*

Quoniam DA, CA, BA ponuntur continuae, erit per primam huius DA ad CA, sicut DC ad CB, id est CE : quia aequales sunt ex datis BC, CE : Itaque dividendo ut DC ad CA, sic DE ad EC : & envertendo ut AC ad CD, ita CE ad ED, & componendo ut AD ad CD, sic CD ad ED. Q.e.d.




**Prop.11. Fig. 2.**

*Corollarium.*

Ex hac propositione educitur hoc Theorema. Sint AB, CB, DB proportionales, erigatur autem ad angulum quemcunque recta BE, aequalis ipsi CB, iunganturque puncta AE, CE, DE.Dico AEC, CED angulos esse aequales: & si AEC, CED anguli fuerint aequales, & CB linea aequetur rectae BE. dico AB, CB, DB esse proportionales : si vero AEC, CED anguli aequentur, & AB, CB, DB sint proportionales; dico CB, EB lineas esse aequales.

*Demonstrationem.*

Quoniam ponitur ut AB ad CB, hoc est BE, sic BE ad DB, & angulum ABE sit communis, erit DEB [d] triangulum, AEB triangulo simile, adeoque angulus DEB angulo EAB aequalis. Rursum cum CB, EB latera aequentur, erunt anguli CEB, ECB aequales; est autem angulus ECB, aequalis [e] duobus angulis aequalis EAC, AEC. Igitur angulus CEB aequatur duobus angulis AEC, EAC : sed DEB aequalis ostensus est angulo EAC; demptis igitur aequalibus EAC, DEB, manent AEC, CED reliqui aequales.

Sint iam anguli AEC, CED, & CB, BE lineae aequales, dico AB, CB, DB esse continuas. Quoniam angulus ECB hoc est CED aequetur angulis EAC, [f] ABC, & CED ponatur aequalis ipsi AEC, erit DEB reliquus reliquo EAC aequalis: est autem angulus ABE communis: ergo tertius tertio aequatur, adeoque & AEB, DEB triangula similia: unde ut AB ad EB hoc est CB, ita EB sive CB ad DB. Quod erat secundum.

Iam vero dint AEC, CED anguli aequales, & AB, CB, DB tres continuae proportionales : dico CB, EB lineas aequari : si enim non sint aequales, sit CB maior, fiatque CG ipse BE aequalis . erunt igitur per secundam partem huius propos. AG, CG, DG continuae proportionales. unde si fiat HC aequalis ipso CD, erunt tam & AG, AC, AH; quam AB, AC, AH continuae proportionales; igitur tam AGAH rectangulum, quam ABAH rectangulum, aequale quadrato AC mediae communi : quare AGAH, ABAH rectangula aequantur inter se. Unde ut AH ad AH h sic AB ad AG; sed AH sunt aequales, ergo AG,AB rectae aequantur, & G punctum idem est quod punctum B: & CB, CG. id est BE lineae aequales, eodem modo ostenditur CB non esse minorem ipsa BE.

## *L2.§1.* *PROPOSITION 11.*

Let AB, AC, and AD be in a continued ratio, and the smallest difference BC is equal to CE. I say that ED, CD, and AD are in a continued ratio.

*Demonstration.*

The rectangle AD.DE is equal to the sum of the rectangle [a]AE.ED and the square ED : but rectangle AE.ED is equal to the sum of the rectangle [b] DE.EC and DE.CB, which is equal to the rectangle DE.EC (as CE and CB are equal) and also with rectangle DE.BA. But by the corollary of Prop. nine of this book, AB, BC, ED are continued proportions. Therefore rectangle DE.BA is equal to the square BC, that is also equal to the square EC. Therefore rectangle AE.ED is equal to the sum of twice the rectangle DE.EC and the square EC, whereby the rectangle AD.DE is equal to the sum of twice the rectangle DE.EC together with the squares EC and ED, that is to the square [c] CD. Hence AD, CD, ED are three continued proportionals. Q.e.d.

*a 3. Secundi; b 1. Secundi; c 4 Secundi.*

[AD.DE = AE.ED + $ED^2$ ; also, AE.ED = DE.EC + DE.CB (= DE.EC) + DE.BA. But AB, BC, and ED are in continued proportion, hence, $BC^2$ = DE.BA= $EC^2$; hence AD.DE = 2.DE.EC + $EC^2$ + $ED^2$ = $CD^2$ as required.]

*Otherwise.*

As DA, CA, and BA are in continued proportion, by the first proposition of this book, it follows that : DA is to CA, thus DC is to CB, *i.e.* CE, as BC and CE are given as equal to each other. Thus by division, as DC to CA, thus DE to EC : and on inverting, as AC to CD, thus CE to ED, and on addition, as AD to CD, thus CD to ED. Q.e.d.

[CA/DA = BA/CA giving DC/DA = CB/CA or DC/CB = DA/CA = DC/EC; and again, DC/CA = DE/EC, or AC/CD = CE/DE and AD/CD = CD/DE as required.]

*Corollary.*

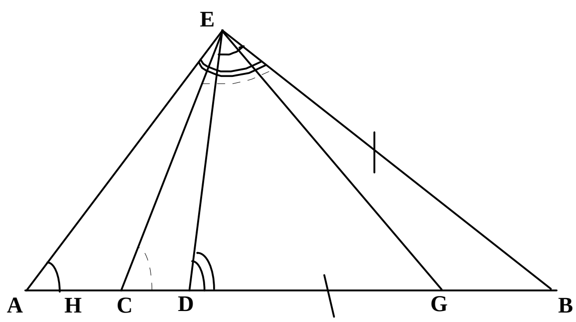
The following theorem can be established from this proposition. Let AB, CB, and DB be proportionals, and moreover a line BE equal to CB is erected at some angle. The points AE, CE, and DE are joined. I say that the angles AEC and CED are equal: and if the angles AEC and CED are equal, then the line CB is equal to the line BE.

I say that the lines AB, CB, DB are indeed proportionals if the angles AEC and CED are equal, and if AB, CB, DB are proportionals, then CB, EB are lines of equal length.

*Demonstration.*

Since the ratio AB to CB, or BE, is thus as BE to DB, and the angle ABE is common, then triangle DEB [d] is similar to triangle AEB, and hence the angle DEB is equal to the angle EAB. Again since the sides CB and EB are equal, then the angles CEB and ECB are equal. But the angle ECB is equal to the sum of the two angles EAC and AEC. Therefore angle CEB is equal to the sum of the angles AEC and EAC. But angle DEB has been shown to be equal [e] to the angle EAC ; therefore with the equal angles EAC and DEB taken away, there remains the equal angles AEC and CED. [Thus, $\angle$ CED = $\angle$ EDB - $\angle$ ECB = $\angle$ AEB - $\angle$ CEB = $\angle$ AEC.]

Now with the angles AEC and CED equal, and the lines CB and BE equal, I can say that the lines AB, CB, and DB are in continued proportion. Since the angle ECB, or CEB, is equal to the sum of the angles EAC and AEC [f], and the angle CED is set equal to the angle AEC, then the remaining angle DEB is equal to the angle EAC : but the angle ABE is common: therefore the third angle to angle is equal , and thus the triangles AEB and DEB are similar: thus as AB to EB or CB, thus EB or CB to DB. Which establishes the second part.



**Prop.11. Fig. 3.**

Now indeed the angles AEC and CED are set equal, and the lines AB, CB, and DB are three continued proportionals : in this case I say that the lines CB and EB are equal. For indeed if they are not equal, then let CB be the greater, and BE is set equal to CG. Therefore by the second part of this proposition, it follows that AG, CG, and DG are continued proportionals. Thus if HC is set equal to CD, as AG, AC, and AH, so AB, AC, AH are continued proportionals. Therefore, both rectangles AG.AH and AB.AH are equal to the square AC of the common mean. Whereby the rectangles AG.AH and AB.AH are equal to each other. Thus as AH to AH [h] so AB to AG; but the AH's are equal, therefore the lines AG and AB are equal, and the point G is the same as the point B: and the lines CB, CG or BE are equal. In the same way it can be shown that CB cannot be less than BE.

**[63]**

## *PROPOSITIO XII.*

Dentur tres lineae in continua ratione AB, BC, CD, ita ut BC sit maior AC, & fiat rectae AC aequalis BE.

Dico AB, BE, ED esse proportionales.

**Prop.12. Fig. 1.**

*Demonstratio.*

Cum ponantur in continua analogia AB, BC, CD; igitur erit ut BC [a] ad CD ita AC, hoc est ex constructione BE, ad BD. Ergo dividendo ut BD ad DC, ita ED ad DB; & invertendo uti CD ad BD, ita se habet BD ad DE. Quare [b] BD quadratum aequatur EDC rectangulo; quadratum autem [c] BE aequale est quadratis BD, DE, & rectangulo EDB bis sumpto : atque rectangulum ABED aequatur iisdem : nam rectangulum ABED, aequatur EDC [d] rectangulo (hoc est , ut ostendi, quadrato BD) & rectangulo EDAC ( hoc est rectangulo BED), id est rursum[e] quadrato ED cum rectangulo EDB, & rectangulo insuper EDB, rectangulum itaque ABED, dum iisdem aequale sit, aequabitur BE quadrato: patet igitur AB, [f] BE, BD esse continua analogia. quod demonstrandum fuit.

a. 1. huius; b 17. Sexti; c 4. Secundii; d 1 Secundi; e 3. Secundi; f 17. Sexti.

## *L2.§1. PROPOSITION 12.*

The three lines AB, BC, and CD are given in a continued ratio, with BC greater than AC, and AC equal to BE.

I say that the lines AB, BE, and ED are in proportion.

*Demonstration.*

As AB, BC, and CD are put in a continued ratio, then as BC[a] is to CD thus as AC, or BE by construction, is to BD. Hence on dividing, as BD to DC, thus ED to DB; and on inverting, as CD to BD, thus BD to DE. Whereby [b] the square BD is equal to the rectangle ED.DC ; but the square [c]BE is equal to the sum of the squares BD and DE, together with twice the rectangle ED.DB : and the rectangle AB.ED is equal to the same : for the rectangle AB.ED is equal to the sum of the rectangle ED.DC [d] (that is, as shown, to the square BD) together with the rectangle ED.AC ( that is to the rectangle BE.ED, that is again equal to the square [e] ED with the rectangle ED.DB), and with the above rectangle ED.DB. Thus the rectangle AB.BD, is equal to the same, is equal to the square BE: therefore it is apparent that AB, [f] BE, BD are in continued proportion. Which was to be shown.

a. 1. huius; b 17. Sexti; c 4. Secundii; d 1 Secundi; e 3. Secundi; f 17. Sexti.

[AB/BC = BC/CD is given; hence AC/BC = BD/CD or BC/CD = AC/BD = BE/BD; again, BC/CD - 1 = BE/BD - 1 or BD/CD = ED/BD and CD/BD = BD/ED on inverting. Hence, $BD^2$ = ED.DC; also, $BE^2 = BD^2 + DE^2$ + 2.BD.DE; again, AB.ED = ED.DC (or $BD^2$) + ED.AC ( or BE.ED = $ED^2$ + ED.DB) + ED.DB : Thus $BE^2$ = AB.ED as required.]

## *PROPOSITIO XIII.*

Ponantur AB, BC, CD, tres lineae in continua proportione, & fiat BC aequalis DE. Dico AE, EC, CD, esse in continuae analogia.

*Demonstratio.*

Quandoquidem ponuntur esse continuae proportionales lineae AB, BC, CD, etiam erunt continuae proportionales AB, DE, CD, & BC mediae aequalis DE erit quadratum DE rectangulo ABCD aequale, quadratum autem [g] CE aequatur CD, DE quadratis, & rectangulo CDE bis sumpto : quibus etiam aequatur rectangulum AECD nam rectangulum AECD aequatur ECD rectangulo (hoc est quadrato [h] CD cum rectangulo CDE) & rectangulo DCB. hoc est ( quia ex constructioneBC, DE aequantur) iterum rectangulo CDE & rectangulo insuper ABCD, ( id est, ut modo ostendimus, quadrato DE) quare rectangulum AECD cum iisdem aequale sit, quadrato CE aequare necesse est : adeoque CD, CE, EA esse in continua analogia. Q. e. d. *g 4. Secundi; h 3. Secundi.*


**Prop.13. Fig. 1.**

## *L2.§1.* *PROPOSITION 13.*

The three lines AB, BC, and CD are placed in continued proportion, and DE is made equal to BC.

I say that the three lines AE, EC, and CD are also in continued proportion.

*Demonstration.*

Since the lines AB, BC, and CD are placed in continued proportion, the lines AB, DE, and CD are also in continued proportion, as the means BC and DE are set equal. The square DE is equal to the rectangle AB.CD, but the square [g] CE is equal to the sum of the squares CD and DE , together with twice the rectangle CD.DE: which is also equal to the rectangle AE.CD. For the rectangle AE.CD is equal to the sum of the rectangle EC.CD (that is, to the square [h] CD with the rectangle CD.DE) together with the rectangle DC.CB, that is (as BC is equal to DE by construction) again equal to the sum of the rectangle CD.DE, and the above rectangle AB.CD, ( or, as we have shown to the square DE); whereby the rectangle AE.CD is equal to the same, is necessarily equal to the square CE: thus CD, CE, and EA are in continued proportion. Q. e. d.

*g 4. Secundi; h 3. Secundi.*

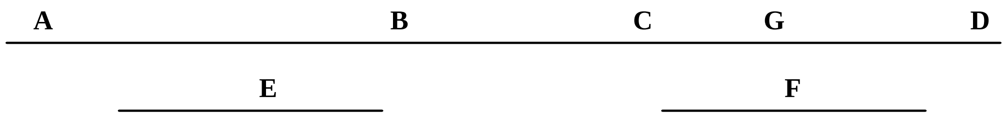
[$DE^2$ = AB.CD; also $CE^2 = CD^2 + DE^2$ + 2.CD.DE , which is also equal to AE.CD as AE.CD = EC.CD ( *i.e.* $CD^2$ + CD.DE) + DC.CB (or DC.DE) + AB.CD (or $DE^2$): for, on adding the terms in brackets, we have the same sum for $CE^2$. Hence, $CE^2$ = AE.CD, and so AE, CE, and CD are also in continued proportion.]

## *PROPOSITIO XIV.*

Sint AB, AC, AD, in continua analogia ; & fiat ut AB ad BC, ita BC ad E lineam; & ut AD ad DC, sic DC fiat ad F.

Dico E & F esse inter se aequales.

*Demonstratio.*




**Prop.14. Fig. 1.**

Quoniam supponuntur continuae AB, AC, AD, si fiat DG aequalis BC, erunt continuae [i] AB, BC, CG, ergo cum ex datis etiam fiat in continua ratione,AB, BC, & E ,

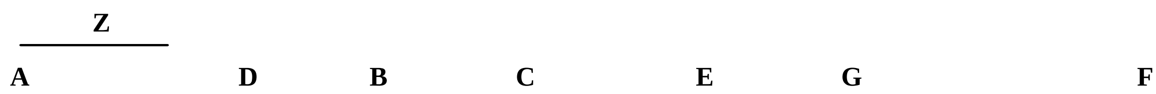
**[64]**

erit linea E aequalis CG : deinde quai continuae sunt AB, AC, AD, si fiat BC aequalis GD, erunt quoque in continua analogia a CG, DC, DA : ponuntur autem continua: AD, DC & F ; erunt igitur continuae F, DC, AD; est igitur F aequalis CG : unde F & E quoque sunt aequales, utpote ipsi CG aequales. Quod erat propositum demonstrare.

i 9. Huiua; a. 11. huius.

## ***Lemma*.**

Dati sint duo ordines trium quantitatum CA, BA, DA & CF, EF, GF; & sit ut CA ad DA, sic CF ad GF, sitque item ut BA ad DA, sic CF ad EF.
Dico etiam esse CA ad BA, ut EF est ad GF.




**Prop.14. Fig. 2.**

*Demonstratio.*

Si enim non est CA ad BA, ut EF ad GF, erit ergo aliqua maior ver minor, quam CA, nempe Z ad BA, ut EF est ad GF; quoniam ergo ut CF prima ad EF secundam, ita in altero ordine BA secunda est ad DA tertiam, & ut EF secunda ad GF tertiam, ita in ordine altero Z prima, est ad BA secundam , ergo ex aequo in proportione perturbata, ut CF ad GF, ita Z [b] ad DA: atque ex hypothesi CF ad GF, ergo ut CA ad DA, sic Z (maior vel minor quam CA) est ad DA, quod est [c] absurdum: non est igitur CA ad BA, in alia pooportione, quam EF ad GF. *b 23. Quinti; c 8 Quinti.*

## ***L2.§1. PROPOSITION 14.***

Let AB, AC and AD be in continued proportion ; and the ratio AB to BC is made as BC to the line E ; and as AD to DC, so DC to F.
I say that E and F are equal to each other.

*Demonstration.*

Since the lines AB, AC, and AD are considered to be in continued proportion , if DG is made equal to BC, then AB, BC, CG are continued proportions [i], therefore from that given, AB, BC, and E are also in a continued ratio. The line E is thus equal to CG : hence as the lines AB, AC, AD are in continued proportion, if BC is made equal to GD, CG, DC, DA are also in continued proportion [a] : but AD, DC and F are placed in continued proportion; therefore F, DC, AD are in continued proportion ; therefore F is equal to CG : hence F and E are equal too, as CG itself is equal to E. Which was the PROPOSITION to be shown.

i 9. Huiua; a. 11. huius.

### ***Lemma.***

Two orders of three quantities CA, BA, DA and CF, EF, GF are given as follows: as CA is to DA, thus CF is to GF, and also as BA is to DA, so CF is to EF.
I say that CA is to BA, as EF is to GF.

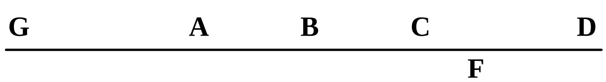

**Prop.14. Fig. 2.**

*Demonstration.*

For indeed, if the ratio CA to BA is not the same as EF to GF, then it will be either a larger or smaller ratio, namely Z to BA, rather than CA to BA, that is the same as EF to GF. Therefore, since the first term CF is to the second term EF, thus in the other order the second term BA is to the third term DA, and as the second term EF is to the third term GF, so in the other order the first term Z is to the second term BA, therefore from the equality in the disturbed proportion, as CF to GF, thus Z [b] to DA: and from the hypothesis, CF is to GF, thus as CA is to DA, thus Z (greater or less than CA) is to DA, which is [c] absurd: therefore the ratio CA to BA is not in any other proportion than EF ad GF. *b 23. Quinti; c 8 Quinti.*
[If CA/DA = CF/GF and BA/DA = CF/EF then CA/BA = CA/DA . DA/BA = CF/GF. EF/CF = EF/GF as required.]

## ***PROPOSITIO XV.***

Sint proportionales AB, AC, AD; & linea BD composita ex utraque differentia BC, CD, dividatur bifarium in F, ac mediae proportionali AC, aequalis sit producta AG.
Dico FC, FB, FG in continuata esse proportione.




**Prop.15. Fig. 1.**

*Demonstratio.*

Cum enim GC bisecta sit in A, & BD in F, erit CG ad AG, ut BD ad FD. praeterea cum AB, AC, AD ponantur continuae, erit per primam huius BC ad CD, ut AB ad AC, id est AG; & componendo ut BG ad AG, ita BD ad CD: sed ante ostenderamus esse ut CG ad AG, ita BD ad FD. Quare per lemma praecedens etiam erit ut CG ad BG, ita CD ad BF, hoc est FD; & dividendo CB ad BG, ut CF ad FD, hoc est FB; Itaque invertendo, permutando, componendo erit ut GF ad BF, ita BF ad CF; ergo GF, BF, CF sunt proportiones. Quod erat demonstrandum.

*Corollarium.*


**Prop.15. Fig. 2.**

Ex hac propositione hoc Theorema deducimus. Sint tres proportionales AB, BC,CD, divisaque sit AB bifariam in E. Dico AE, EC, & compositam ex AE & lineis BC, CD bis sumptis, esse proportionales, & si AE, EC, & composita ex AE & BC, CD lineis bis sumptis, sint continuae; dico AB,BC, CD esse in continua analogia.

**[65]**

*Demonstratio.*

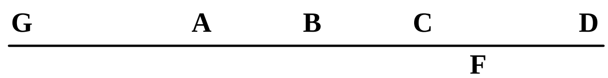
Producatur CD in F, ut DF linea aequetur BD compositae : BA vero producatur in G, ut GA, BC aequentur. Quoniam GA, BC lineae sunt aequales, erunt [a] DC, DB, DG continuae proportionales. Rursum cum GA aequetur rectae BC, & AE ipsi EB sit aequalis, erit GC in E bifariam quoque divisa: est autem DF aequalis ipsi BD per constructionem ; ergo EB, EC, & [b] EF, hoc est AE, EC, & composita ex AE & BC, CD bis sumptis, sunt continae proportionales. quod erat primum.

2. Quoniam EB, EC, EF sunt continuae, & EC lineae aequatur EG, & BF linea ex construc. sit in D bifariam divisa, erunt DC, DB, [c] DG continuae : quare & AB, BC, CD tres erunt [d] continuae proportionales. quod erat demonstrandum. *a. 8 huius; b 15 Huius; c Ibid; d 9 huius.*

## *L2.§1.* *PROPOSITION 15.*

AB, AC, and AD are proportionals; and the line BD, which is the sum of both the differences BC and CD, is bisected in F. The line AG produced is equal to the mean proportional AC.

I say that FC, FB, and FG are in continued proportion.




**Prop.15. Fig. 1.**

*Demonstration.*

Since GC is bisected in A, and BD in F, CG is to AG, as BD is to FD. Furthermore, as AB, AC, and AD are placed in continued proportion, by the first proposition of this book, BC to CD, is as AB and AC, or AG; and by adding: BG to AG, is thus as BD to CD; but we have shown before that CG is to AG, thus BD is to FD. Whereby by the preceding lemma, also as CG to BG, thus CD to BF, or FD. On division, CB is to BG, as CF to FD, or FB. Thus on inverting, exchanging, and addition, GF is to BF, thus as BF is to CF; hence GF, BF, and CF are proportions. Q.e.d .

[CG/AG = BD/FD; from AB/AC = AC/AD it follows that BC/CD = AB/AC = AB/AG. On adding, this gives BG/AG = BD/CD; also, CG/AG = BD/FD; hence by the previous lemma, or from BG/AG . AG/CG = BD/CD . FD/BD, we have BG/CG = FD/CD or CG/BG = CD/FD on inverting. Hence, CB/BG = CF/FD = CF/FB, from which BG/CB = FB/CF, and BG/FB = CB/CF, giving GF/FB = FB/CF as required.

Using algebra, if AB = $a$, AC = $ar$, and AD = $ar^2$, then BD = $a(r^2 - 1)$ and BF = $a(r^2 - 1)/2$; FC = BF - BC = $a(r - 1)^2/2$; and FG = FA + AG = $a(1 + r)^2/2$ .
Hence, $FB^2 = a^2(r^2 - 1)^2/4 = a(r + 1)^2/2 \,.\, a(r - 1)^2/2$ = FG.FC.]

*Corollarium.*


**Prop.15. Fig. 2.**

We can deduce these theorems from this proposition:

1. Let AB, BC, and CD be three proportionals, and AB divided in two equal parts by E. I say that AE, EC, and the sum of AE with twice the sum of the lines BC and CD are in continued proportion.

2. If AE, EC, and the sum of AE and twice the sum of the lines BC and CD are in continued proportion, then I say that AB, BC and CD are in a contined ratio.

*Demonstratio.*

CD is produced to F, in order that the line DF is equal to the lines making BD : BA truly is produced to, in order that GA and BC are equal. Since the lines GA and BC are equal, [a] DC, DB, and DG are continued proportionals. Again as the line GA is equal to the line BC, and AE is equal to EB, GC is likewise divided in two by E : furthermore, DF is equal to BD from the construction ; hence EB,EC, and [b] EF , that is AE, EC, and the sum of AE and twice BC and CD, are continued proportionals. Which establishes the first theorem. *a. 8 huius; b 15 Huius; c Ibid; d 9 huius.*

[$BC^2$ = AB.CD, or BC/AB = CD/BC or AB/BC = BC/CD, then GB/BC = BD/CD, or GB/BD = BC/CD, giving GD/BD = BD/CD or $BD^2$ = GD.CD; i. e. DC, DB, and DG are continued proportionals. Also, as GA = BC and AE = EB, then GE = EC. Again, AE/BC = BC/2.CD giving EC/BC = (BC + 2.CD)/2.CD, hence AE/BC = EC/(BC + 2.CD) or BC/AE = (BC + 2.CD)/EC giving EC/AE = (AE + 2.BC + 2.CD)/EC, as required.]

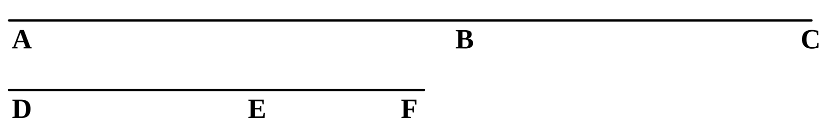
2. Since EB, EC and EF are continued proportionals, and the line EC is equal to the line EG, and the line BF from the construction is cut in two equal parts by D, then the lines DC, DB, and [c] DG are in continued proportion : whereby AB, BC and CD are [d] three continued proportionals. Q.e.d.

[The converse of 1. AE + 2.BC + 2.CD = AE + BF = EF and AE = EB, hence EC/EB = EF/EC from 1; BC/EB = CF/EC;
EC/EB = CF/BC; hence BC/EB = AB/BC (take) and AC/EB = BF/BC (add); hence AC/AB = BD/BC and BC/AB = CD/BC (take) as required.]

## PROPOSITIO XVI.

Si sit prima ad secundam, ita tertia ad quartam, erit ut quarta [prima in text] cum tertia ad tertiam, ita omnes quatuor ad primam cum tertia; vel ut prima cum secunda ad secundam, ita omnes quatuor ad secundam cum quarta.

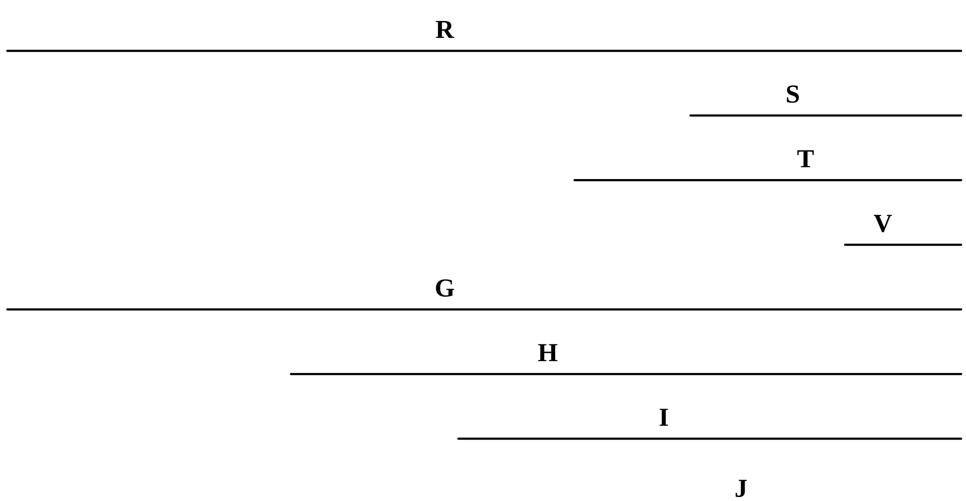
***Demonstratio.***




**Prop.16. Fig. 1.**

Esto AB ad BC, ut DE ad EF, dico esse AC ad AB, ut aggregatum ex AC & DF ad aggregatum ex AB, DE. Cum enim sit AB ad BC , ut DE ad EF, erit invertendo & componendo AC ad AB, ut DF ad DE: qua propter erit etiam veraque antecedens AC, DF, [e] ad utramque consequentem AB, DE, ut una antecedens AC ad AB unam consequentem. quod erat propositum. *e 11 Quinti.*

*Corollarium.*




**Prop.16. Fig. 2.**

Si igitur ex quatuor proportionalibus fiat una aequalis omnibus quatuor, verbi gratia R: & fiat altera S aequalis primae, ac tertiae, & tandem recta T exhibeatur aequalis primae & secundae : denique si ponatur V aequalis primae, erit R ad S, ut T ad V : similiter si omnibus quatuor fiat aequalis G, & secundae cum quartam aequalis H, & prima cum secunda aequalis I, & secundae aequalis K, erit ut G ad H, sic I ad K, res haec a Pappo

**[66]**

lib. tertio propositione decimaseptima in tribus continue proportionalibus proposita fuit; quae cum univeralis sit, censuimus etiam hoc ipsum verbo indicandum ad ampliorem usum.

## *L2.§1.* *PROPOSITION 16.*

If the first term is to the second term, thus as the third is to the fourth, then the fourth and the third is to the third, thus as all four is to the first and the third; or as the first and the second to the second, thus as all four to the second and the fourth.

### *Demonstration.*

Let AB be to BC as DE to EF, I say that AC to AB is as the sum of AC and DF to the sum of AB and DE. For indeed as AB to BC, thus DE to EF, and on inverting and adding, AC is to AB thus as DF is to DE: which is indeed the sum of the first terms AC and DF, [e] to the other subsequent terms AB and DE, as the one preceeding term AC to the one subsequent term AB. quod erat propositum. *e 11 Quinti.*

[The first part : AB/BC = DE/EF, then BC/AB = EF/DE and AC/AB = DF/DE giving AC/DF = AB/DE and (AC + DF)/DF = (AB + DE)/DE; hence (AC + DF)/(AB + DE) = DF/DE = AC/AB.
For the second part : we are to show that (AC + DF)/(BC + EF) = AC/BC. From AB/BC = DE/EF we have AC/BC = DF/EF and BC/AC = EF/DF or DF/AC = EF/BC and (AC + DF)/AC = (BC + EF)/BC, giving the required result.]

*Corollary.*

If therefore from four terms in proportion, there is one term equal to the sum of all the terms, for example R: and another S is equal to the first and the third, and the final term the line T is shown equal to the sum of the first and the second, then if V is put equal to the first term: R is to S, as T is to V.

[For (AC + DF)/(AB + DE) = AC/AB becomes R/S = T/V.]

In the same way, if all four terms are set equal to G, the second with the fourth equal to H, the first and second equal to I, and the second equal to K, then G is to H as I is to K.

[For (AC + DF)/(BC + EF) = AC/BC becomes G/H = I/K.]

This result comes from Pappus, Book III, Prop. 17, where the original theorem is proposed for three continued proportionals. We consider that it is of more general use as the example indicates.

## *PROPOSITIO XVII.*

Sit A aequalis B, & C aequalis D; omnibus autem A, B, C, D, ponatur aequalis E, duabus vero B, D, ponatur aequalis F: similiter duabus C, D , aequalis G; & rectae D, aequalis H.

Dico rationem E ad F, & G ad H esse duplam. Quod si rectis E, F, G, H, aequalis ponatur I, & duabus F & H, aequalis statuatur K, & lineis G, H, aequalis ponatur L, denique residuae H, fiat M aequalis;

Dico rationem L ad K, & L ad M esse triplam.

***Demonstratio.***

A cum B, dupla est ipsius B, & C cum D, ipsius D dupla est: ergo etiam A, B, C, D simul sumptae , duarum B, D, simul sumptoram duplae sunt. Sed lineis A, B, C, D aequalis est E, & lineis B, D, aequalis F, ergo E dupla est F : similiter C, D, simul sumptae, hoc est G, ipsius D, id est H, sunt duplae. Quod erat primum. Secundam partem ita expedimus. E continet rectam A bis, & C bis; ipsa vero F continet A semel, & C semel : ipsa tandem G continet C bis; & H aequalis est C; igitur omnes E, F, G, H, , hoc est recta L continet tertio, & rectam C sexies : recta vero K, continet ex hypothesi F, & H; sed F continet B & D, hoc est ipsam A semel , & semel rectam C, igitur si addatur H, hoc est C, recta K erit semel A, & C bis sumptata. sed A sumpta semel, cum C bis, est tertia pars A lineae ter sumptae, & lineae C sexies sumptae: ratio ergo I ad K tripla est. Eodem modo ostendemus quod L, sit tripla lineae M, nam L aequalis est G, H; ipsa vero G continet bis C lineam, cui si adiungatur H, ipsi C aequalis, erit G cum H, hoc est L, tripla rectae C, hoc est H, hoc est M; patet ergo veritas propositione exposita.

*Scholium.*

*Pappui lib. 3 propositine 18, tribus continuis quantitatibus applicuit hanc materiam, quae scilicet eandem continuant rationem; quia vero adventi non solum continuis rationibus convenire hanc proportionalitas proprietatem, verbum etiam discretissimus, modo rationes similes assumantur: hinc opera pretium duxi, hoc insinuare. Notatum autem dignum existimo, si quis ulterius*

**[67]**

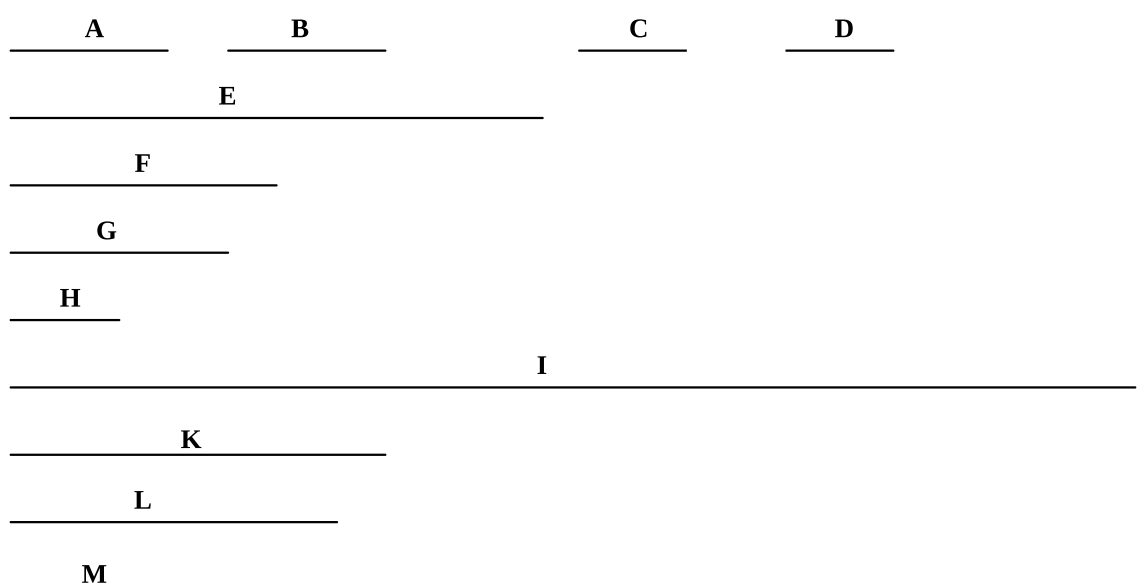
*proportiones ita convertat, quemadmodum hac propositione secimus, in infinitum repariet plures ac plures subdivisiones rationum, sola praxis in propositione posita observatione.*

## *L2.§1.* *PROPOSITION 17.*

A is equal to B, & C is equal to D; moreover E is set equal to the sum of A, B, C, and D, while F is the sum of the two B and D : similarly G is the sum of the two C and D, and the line D equals H.

I say that the ratio E to F and the ratio G to H is two.

While if I is put equal to the sum of the lines E, F, G, H, and K is set equal to the two lines F and H, and L is placed equal to the lines G and H, and then H set equal to the remaining line M. In this case, I say that the ratio L to K and the ratio L to M is three.




**Prop.17. Fig. 1.**

***Demonstration.***

The sum of A and B is twice B, and the sum of C and D is twice D: therefore also, the sum of A, B, C, and D is twice the sum of B and D. But E is equal to the sum of the lines A, B, C, and D, and F is equal to the sum of the lines B and D, therefore E is twice F : similarly G is the sum of C and D, which is twice D or H. Which shows the first part to be true.

We second part we thus expedite as follows. The line E is composed of the line A twice and the line C twice; while F is composed of A and C once: finally G is composed of the line C twice; and H is equal to C. Therefore the sum of the lines E, F, G and H, that is the line L, contains the line A three times and the line C six times. The line K by hypothesis is composed of the lines F and H; but F is composed of B and D, that is A itself once and the line C once; therefore if H or C is added, then the line K is composed of A once and twice C. But A taken once with C taken twice is the third part of the sum of A taken three times and of C taken six times. Therefore the ratio L to K is three. In the same way we can show that L is three time the line M, for L is equal to the sum of G and H; and G is composed of twice the line C, to which if H or C is added, then L or the sum of G and H, is three times the line C or H or M. Hence it is apparent that the truth of the proposition has been shown.

[Using small letters and algebra, we have $a + b = 2b$; and $c + d = 2d$; hence, $a + b + c + d = 2b + 2d$. Now, $e = a + b + c + d$ , and $f = b + d$; hence $e = 2f$, and $g = c + d = 2d$ or $2h$. (first part)
Again, $e = 2a + 2c$, and $f = a + c$; $g = 2c$ and $h = c$. Hence, $e + f + g + h = l = 3a + 6c$; $k = f + h$, but $f = b + d$ ; hence $k = a + 2c = 1/3.(3a + 6c)$, hence $l/k = 3$. Similarly $l = 3m$, for $l = g + h$. Again, $g = 2c$, and $l = g + h = 3c = 3h = 3m$. (second part)].

*Scholium.*

*This material is an application of Proposition 18, Book 3 of Pappus for three continued quantities, which of course give the same continuing ratio. Because indeed not only is this a proportionality property to be found for continuous ratios considered together, but also for the most separate numbers; moreover similar kinds of ratios are assumed :and thus in this way I have exdeavoured to iintroduce the value of the work. Moreover I can judge their worth by observation, for if further proportions are thus inverted, in this way we can follow more and more subdivisions of the ratio, and these can be produced indefinitely by this proposition , put into the proposition for practise in observing the outcome.*

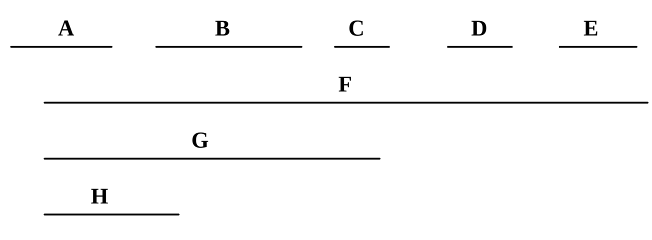
## *PROPOSITIO XVIII.*

Sint quaecumque & quotcumque magnitudines A, B, C, D, E ; ponaturque F omnibus (praeter ultimam) bis sumptis, ac ultimae E semel sumptae aequalis; G vero aequalis omnibus simul sumptis, denique H aequalis ultimae E.

Dico F, G, H arithmeticam analogiam continuare.

***Demonstratio.***

Linea F, id est A, B, C, D, bis sumptae, una cum E semel, excedit recta G, id est A, B, C, D, E semel sumptas, excedit A, B, C, D, semel sumptis. Sed eodem excessu excedit linea G, ipsam H, quae aequalis ponitur rectae E, igitur F, G, H lineae arithmeticam continuant proportionem. Quod erat demonstrandum.




**Prop.18. Fig. 1.**

*Scholium.*

*Quare miram videri non debuit Frederico Commandino, Pappum, cum lib. 3 ex Geometrica proportione, analogias, ac meditatas eruit, Arithmeticas neglexisse; quando quidem non ex Geometrica tantum sed quacumque quantitatum serie producatur; quod Pappum in caeterum sagacissimum latere potuisse vix credi potest; vel certe opere misset Commandinum dum Pappi defectum, ut vocat, eodem libro propositione*

*19. supplere mititur, continuitatem Geometricam assignarem ex qua, cum qua plane ratione Arithmeticam analogiam deduxisset, non ideo caeteris non proproportionalibus quantitatibus commune esse, posset demonstrari : si quidem a problematici nitaris splendore alienum videtur, certum & determinatum quid in construectionem adhibere, cum obviam quodlibet fuerit sufficiens. Sed haec ingratiam antiquaratis dicta sint, quam venerari omnes deberent: illius enim saeculi virorum labores, & ingeniorum partis haec usque non vidi a recentioribus adequata.*

## *L2.§1.* *PROPOSITION 18.*

Let A, B, C, D, E be some number of unspecified magnitudes ; and F is put equal to twice the sum of all of them except the last one term E, which is taken only once. G is taken as the sum of all the terms taken together, and then H is equal to the final term E. I say that F, G, and H continue in an arithmetic ratio.

*Demonstration.*

The line F, that is the sum of twice A, B, C, and D together with E taken once, exceeds the line G, that is the sum of A, B, C, D, and E, that in turn exceeds the sum of A, B, C, and D. But the line G exceeds the line H or E by the same amout. Hence the amounts F, G, H are line in continued arithmetical proportion. Q.e.d. [For 2G = F + H, and the common difference is A + B + C + D.]

*Scholium.*

*Does it not appear astonishing for Frederico Commandino [*1509 - 1575*: an Italian savant who translated several of the works of the ancient Greek masters, including the Collections of Pappus, published in 1565] to disregard arithmetic ratios, when Pappus in book 3 on geometrical proportion has considered the topic so carefully ?* [*Print hard to read in this section.*] *Indeed some quantity of interest can be produced not so much from geometry, but rather from a series; which fact Pappus is able to obscure from Commandino in a most able manner that can scarcely be believed. In this regard at least, the work published by Commandino is so poor regarding Pappus. For he invokes in his published work, according to PROPOSITION 19 of the same book, a continuitation of the geometrical assignation from which, planely the arithmetical analogy for the ratio may be deduced; however instead he asserts that for the rest there is no common proportionality that can be demonstrated. For indeed it seems that if you should struggle with the wonderful problems of others, then you should be able to overcome any difficulties by adhering to sure and tried methods of proof. This ingratitude to the ancients must be mentioned, as they should all be held in veneration:indeed the labours of the men of that age, and especially these with a share of ingenuity, have not seen an equal up to the present time.*

## *PROPOSITIO XIX.*

Datis duobus progressionibus terminorum in continua ratione Arithmetica A, B, C, D, E, F; ex utriusque seriei terminis A, D, B, E, C, F, in unum constatis constituntur tertia series G, H, I.

Dico etiam G, H, I esse in continua Arithmetica analogia.

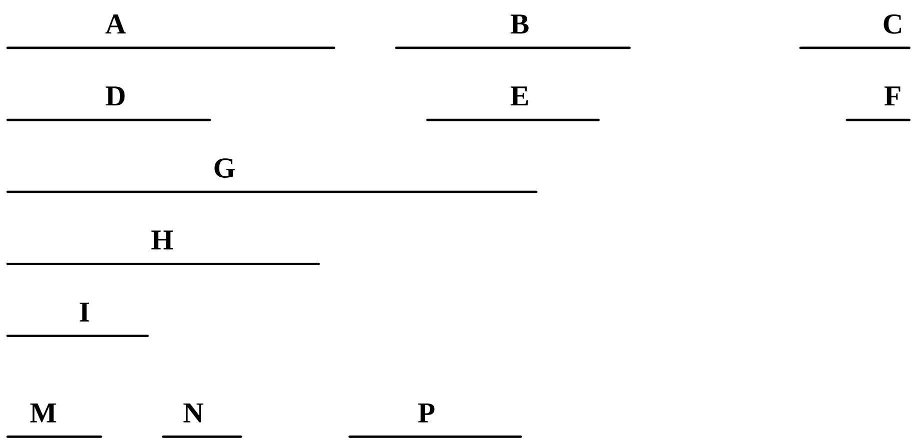
***Demonstratio.***

Terminorum seriei A, B, C, mutuus excessus sit M; N. vero excessus alterium M, autem & N sibi additi, faciant P : quoniam igitur A superat B excessu M, & D superat E excessu N, A, & D simul sumpti, hoc est G, superabunt B & E simul sumptos, hoc est H, excessibus M & N simul sumptis, hoc est excessu P : similiter ostendemus H superare I excessit P. sunt igitur G, H, I in Arithmerica analogia. Quod erat demonstrandum.

## *L2.§1. PROPOSITION 19.*

A, B, C; & D, E, F are two given continued arithmetic progressions. From both series of terms A, D, B, E, C, F, it is agreed to set up the third series G, H, I.
I say that the terms G, H, and I are also in a continued arithmetic ratio.

***Demonstratio.***




**Prop.19. Fig. 1.**

The mutual excess of the terms of the series A, B, & C is M; while N is the excess of the other series, and moreover the sum of N and M is P. Therefore as A is greater than B by M, and D is greater than E by N, then the sum of A and D, or G, is greater than the sum of B and E, or H, by the sum of the excesses M and N, that is the excess P : similarly we can show that H is greater than I by the excess P. Hence G, H, and I are in arithmetic progression. Q. e. d.

## *PROPOSITIO XX.*

Esto AB ad BC, ut BD ad DE; & fiat ipsi BC aequalis EF, ubicumque tandem cadat punctum E.
Dico esse ut AB ad AC, sic AD ad AF, & BD ad BE.

***Demonstratio.***

Quoniam est ut AB ad BC, ita BD ad DE, erit componendo, convertendo ut AB ad AC, ita BD ad BE: ulterius, cum sint EF, BC aequales, erit tota AF aequalis quatuor proportionalibus AB, BC, BD, DE : & AD aequalis primae & tertiae, uti & AC primae ac secundae : quare ut a AC ad AB, ita AF ad AD: & invertendo ut AB ad AC, ita AD ad AF, & ut ante ostendimus, ita BD ad DE. Quod erat demonstrandum. *a 16 Huius.*

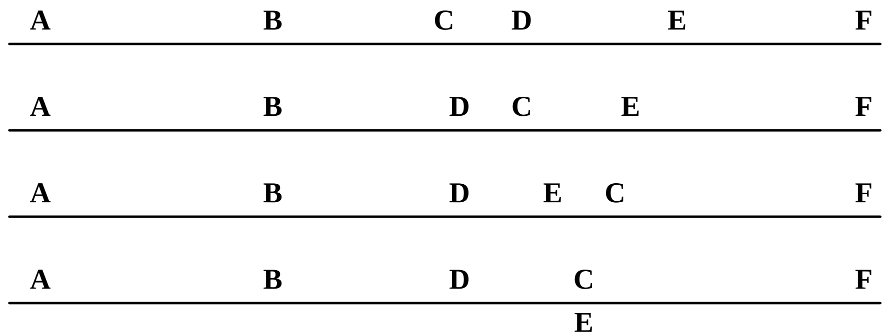
## *L2.§1. PROPOSITION 20.*

The ratio AB to BC shall be as BD to DE; and EF is made equal to BC, and finally the position of the point E is variable .

I say that as AB is to AC, thus AD to AF, and BD to BE.

***Demonstration.***

Since AB is to BC, thus as BD is to DE, by adding and rearranging as AB to AC, thus as BD to BE: further, as EF is equal to BC, the whole length AF is equal to the sum of the four ratios AB, BC, BD, and DE : AD is equal to the sum of the first and the third, as AC is equal to the first and the second. Whereby as [a] AC is to AB, thus AF is to AD: & invertenso ut AB ad AC, ita AD ad AF, & ut ante ostendimus, ita BD ad DE. Quod erat demonstrandum. *a 16 Huius.*
[BC/AB = DE/BD, giving AC/AB = BE/BD on adding, and AB/AC = BD/BE on inverting. Again, the




**Prop.20. Fig. 1.**

sum of the four ratios is AB + BC + BD + DE = AB + BD + EF + DE = AF; while AD = AB + BD and AC = AB + BC. From EF/AB = DE/BD we have EF/DE = AB/BD, giving DF/DE = AD/BD and DF/AD = DE/BD gives AF/AD = BE/BD = AC/AB and on inverting, AB/AC = AD/AF = BD/BE as required.]

## *PROPOSITIO XXI.*

Datae sint quatuor proportionales, minina AB, secunda AC, tertia AD, quarta AE.

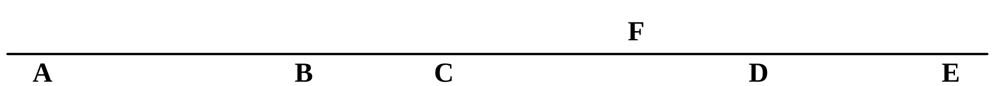
Dico primo DB differentiam primae & tertiae, minorem esse EC, differentia secundae & quartae.

**[69]**

Secundo si BD minori auferatur aequalis EF, ex CE maiori , ut AB ad BD, vel ut AC ad CE, sic BC differentiam primae & secundae, fore ad CF differentiam differentiarum BD & CE.

***Demonstratio.***

Cum ex hypothesesi AB sit ad AC, ut AD ad AE: igitur permutando ut AB ad AD, sic AC ad AE; & dividendo ut AB ad BD, sic AC ad CE; igitur permutando ut AB ad AC, sic BD ad CE; atque AB minor est quam AC, igitur & BD minor est quam CE. Quod erat primum. deinde ex discursu iam facto, ut AB ad AC,




**Prop.21. Fig. 1.**

sic est BD ad CE; quare cum EF ex hypothesi sit aequlis ipsi BD. erit etiam AB ad AC, ut EF ad CE; ac dividendo ut AB ad BC, sic EF ad FC, & permutando ut AB ad EF, id est ut AB ad BD, sic BC ad CF: atque etiam ut AB ad BD, sic est AC ad CE, (ut ante ostendi) ergo ut AB ad BD, vel AC ad BD, vel AC ad CE, sic est BC ad CF. Quod erat demonstrandum.

## *L2.§1.* *PROPOSITION 21.*

There are four proportionals given, AB is the least, AC the second, AD the third, and AE the fourth.

I say in the first place, that the difference DB between the first and the third, is less than the difference EC between the second and the fourth.

In the second place, if the smaller difference BD is taken from the greater difference CE, equal to EF, as AB to BD, or as AC to CE, thus the difference BC between the first and second, will be as CF, the difference of the differences BD and CE.

### *Demonstration.*

For by hypothesis, AB is to AC, as AD to AE: therefore on interchanging, as AB to AD, thus AC to AE; and on division, as AB to BD, thus AC to CE; therefore on interchanging, as AB to AC, thus BD to CE; and AB is less than AC, and hence BD is less than CE. Which establishes the first part of the proposition.

Further, from what has been established, as AB is to AC, thus BD is to CE; whereby as EF by hypothesis is equal to BD, then also AB is to AC, as EF is to CE; and on division, as AB to BC, thus EF to FC, and on interchanging, as AB to EF, or AB to BD, thus BC to CF: and also as AB to BD, thus AC is to CE, (as shown before) hence as AB to BD, or AC to CE, thus BC is to CF. Q. e. d.

[$AB/AC = AD/AE$, giving $AB/AD = AC/AE$ and on inverting, $AD/AB = AE/AC$ and on taking, $BD/AB = CE/AC$ or $AB/BD = AC/CE$ and finally, $AB/AC = BD/CE < 1$.
Again, from $AB/AC = BD/CE = FE/CE$, and on inverting, $AC/AB = CE/FE$, and on taking, $BC/AB = CF/FE$ giving $AB/BC = FE/CF$ and on interchanging, as $AB/EF = BC/CF = AB/BD$; also, as $AB/BD = AC/CE$ from above, thus, $AB/BD = AC/CE = BC/CF$. ]

## *PROPOSITIO XXII.*

Datae sint quatuor proportionales, minina AB, secunda AC, tertia AD, quarta AE.

Dico primo, differentiam primae & secundae BC, minorem esse differentia DE, tertiae & quartae:

Secundo si ex ED maiori tollatur EF, aequalis minori BC, fore ut AD ad DE, sic BD differentiam primae & tertia, ad DF differentiam differentiarum EC, DE.

### *Demonstratio.*

Cum sit AB ad AC, ut AD ad AE: igitur invertendo, dividendo rursumque invertendo, ut AB ad BC, sic AD ad DE: Itaque permutendo ut AB ad AD, sic BC ad DE. sed AB minor est AD, ergo & BC minor est quam DE. Quod erat primum.


**Prop.22. Fig. 1.**

Deinde cum modo ostensum sit, esse AB ad AD , ut BC est ad DE; & cum FE ponatur aequalis BC, etiam AB est ad AD, ut EF ad DF. Itaque invertendo, ac convertendo ut AD ad BD, sic DE ad FE, ac deinum permutando ut AD ad DE, sic BD ad DF. Atqui etiam ut AD ad DE, sic est AB ad BC. (ut ante ostendi) ergo ut AD ad DE, sic est AB ad BC , sic est BD ad DF. Quod erat demonstrandum.

## *L2.§1.* *PROPOSITION 22.*

There are four proportionals given, AB is the least, AC the second, AD the third, and AE the fourth.

I say in the first place, that the difference BC between the first and the second is less than the difference DE between the third and the fourth.

In the second place, if an amount EF equal to the lesser difference BC is taken from the greater difference ED, then the ratio AD to DE thus will be as the difference BD between the first and third proportionals, to DF, the difference of the differences EC and DE.

***Demonstration.***

For as AB is to AC, so AD is to AE: therefore on inverting, dividing, and again inverting, as AB to BC, thus AD to DE; and on interchanging, as AB to AD, thus BC to DE; but AB is less than AD, and hence BC is less than DE, as AB to AC. Which establishes the first part of the proposition.

Further, in this way that it can be shown that AB is to AD as BC is to DE, and with FE put equal to BC, also AB is to AD as EF is to DF. For on inverting and converting, as AD is to BD, thus DE is to FD, and then on interchanging, as AD to DE, thus BD to DF. But also, as AD is to DE thus AB is to BC, (as shown before) hence as AD to DE, thus AB to BC, thus BD is to DF. Q. e. d.

[AB/AC = AD/AE, giving AC/AB = AE/AD , BC/AB = DE/AD , and AB/AD = BC/DE < 1 as required. Again, from AB/AD = BC/DE = FE/DE, and on inverting, AD/BD = DE/FD, and on interchanging, AD/DE = BD/DF. Also, AD/DE = AB/BC, giving AD/DE = AB/BC = BD/DF as required. ]

## *PROPOSITIO XXIII.*

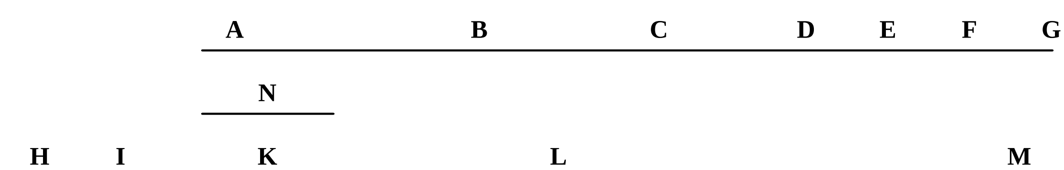
Sint continui proportionalium processus, AB, BC, CD, DE, EF: assumta vero quavis alia N, fiat ut AB ad N, sic N ad HI: utque BC ad N, sic N fiat ad IK, & ut CD ad N, sic N ad KL, & sic deinceps.

Dico HI, IK, KL, &c. esse in continua analogia.

**[70]**

***Demonstratio.***

Ex hypothesi lineae AB, N, HI, sunt continuae proportionales; item BC, N, IK; ergo tam rectangulum ABHI quam rectangulum BCIK [a] aequatur quadrato N. ac proinde aequalia sunt inter se rectangula; ergo ut AB ad [b] BC, sic IK ad HI, similiter rectangula BCIK, & CDKL, aequantur inter se, quia eidem quadrato N




**Prop.23. Fig. 1.**

aequalia sunt; igitur ut BC ad CD, sic KL ad KI : atqui ex datis BC est ad CD, ut AB ad BC. hoc est, ut ostendi, sicut IK ad HI, ergo KL est ad IK, ut IK ad HI: continuae sunt proportionales igitur HI, IK, KL. Quod erat demonstrandum. *a. 17 Sexti; b. 16 Sexti.*

## *L2.§1.* *PROPOSITION 23.*

AB, BC, CD, DE, and EF is a procession of continued proportionals : assume that there is some other length N, constructed so that as AB is to N, thus N is to HI; and as BC is to N, thus N is made in the ratio IK, and as CD to N, thus N to KL, and thus henceforth.

I say that HI, IK, KL, etc are in a continued ratio.

***Demonstration.***

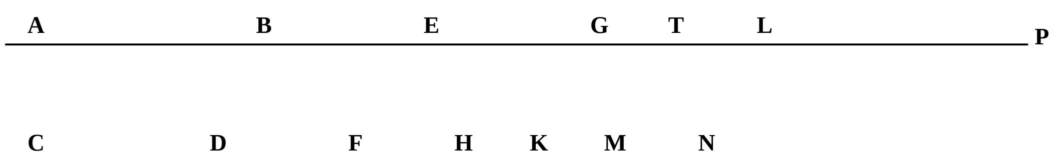
By hypothesis, the lines AB, N, and HI are in continued proportion; likewise for BC, N, and IK. Hence the rectangles AB.HI and BC.IK are equal to the square N [a]. Hence the rectangles are equal to each other; hence as AB is to BC, thus IK is to HI; similarly the rectangles BC.IK and CD.KL are equal to each other as both are equal to the same square N. Thus BC is to CD, thus KL is to KI; but as BC to CD is given as equal to AB to BC, as shown, thus IK to HI, as KL is to IK, as IK to HI. Hence HI, IK, KL are continued proportions. Q. e. d.

## *PROPOSITIO XXIV.*

Si continue proportionalium rectangulorum bases sint in continua analogia: erunt & altitudines in continuata proportione.

***Demonstratio.***

Sint AB, BE, EG, GI, IL, insuper & rectangula AB.CD, BE.DF, EG.FH &c.in continua analogia . Dico etiam CD,DF, FH, &c. esse continue proportionales. Rectanguli enim AB.CD proportio ad rectangulum BE.DF, [c] componitur ex rationibus AB ad BE, & CD ad DF : item rectanguli BE.DF proportio ad EG.FH




**Prop.24. Fig. 1.**

rectangulum , componitur ex rationibus BE ad EG, & DF ad FH: quare cum ex hypothesi rectangulorum AB.CD, BE.DF, EG.FH, aequales sive eaedem proportionales sint, erunt quoque rationes AB ad BE, & CD ad DF, simul sumptae aequales, sive eaedem cum rationibus BE ad EG, & DF ad FH simul sumptis . Sunt autem ex hypothesi etiam aequales rationes AB ad BE, & BE ad EG; Quare si ab aequalibus rationibus, nempe a composita ex rationibus AB ad BE, & CD ad DF; itemque a composita ex rationibus BE ad EG, & DF ad FH, auferas aequales rationes, AB ad BE, & BE ad EG : patet reliquas proportiones CD ad DF, & DF ad FH fore aequales : ut constat ex definitione compositionis proportionum. Continue sunt igitur proportionales CD, DF, FH. Simili discursu etiam reliquae HK, KM, &c. cum praecedentibus eandem continuabunt ratione. Constat ergo propositium erat demonstrare. c*. 15 Sexti.*

*Corollarium.*

Eodem genere discursus, si rectangula ABCD, BEDN, EGFN, &c. itemquebases AB, BE, EG, &c. sint in continua analogia, demonstrabimuseorum altitudines CN, DN, FN, &c. continuam quoque seruare analogiam.

## *L2.§1.* *PROPOSITION 24.*

If the bases of rectangles in continued proportion are in continued proportion, then the altutudes are also in continued proportion.

***Demonstration.***

The bases AB, BE, EG, GI, IL as well as the rectangles ABCD, BEDF, EGFH , etc, are in continued proportion. I say that the altitudes CD, DF, FH, etc, are also in continued proportion. For the proportion of the rectangle ABCD to the rectangle BEDF, [c] is composed from the ratios AB to BE, and CD to DF : likewise the proportion of the rectangle BEDF to the rectangle EGFH is composed from the ratios BE to EG and DF to FH. Wereby by hypothesis, if the rectangles ABCD, BEDF, EGFH, are all in the same proportion, then the ratios AB to BE and CD to DF are equal for the ratios taken together, as likewise they are for the ratios BE to EG and DF to FH taken together. But by hypothesis, the ratios AB to BE, and BE to EG are also equal; whereby if from the equal ratios, truly composed from the the ratios AB to BE, and CD to DF, and likewise from the rectangles composed from ratios BE to EG, and DF to FH, you take away the equal ratios, AB to BE, and BE to EG : it is apparent that the proportions left, CD to DF, and DF to FH will be equal : in agreement with the definition of proportion. Therefore CD, DF, and FH are continued proportions. By a similar discussion the remaining terms HK, KM, etc also are in the same continued ratio with the preceeding. This is in agreement with the proposition that was to be shown. c*. 15 Sexti.*

*Corollary.*

From the same discussion, if the rectangles AB.CN, BE.DN, EG.FN, etc and likewise the bases AB, BE, EG, etc. are in continued proportion, then we can show that the altitudes CN, DN, FN, etc of the same are also in continued proportion.

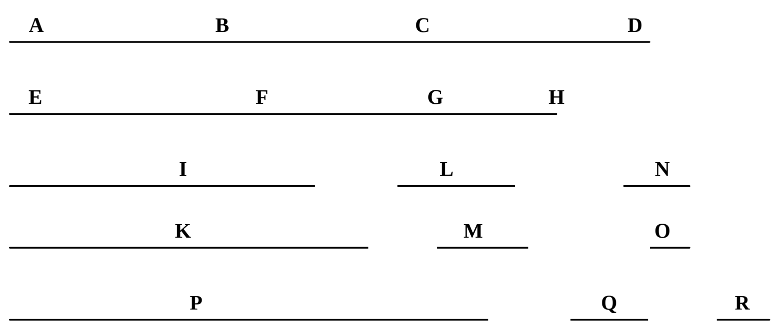
## *PROPOSITIO XXV.*

Ponatur denuo AB, BC, CD proportionales, uti etiam EF, FG, GH, & ratio AB ad EF, continuetur quomodocunque in I, K, P, & similiter ratio BC ad FG producatur in L, M, Q: & ratio CD ad GH, pergat in N, O, R, &c.

Dico etiam I, L, N, & K, M, O, item P, Q, R, continuare suam rationem.

**[71]**

***Demonstratio.***




**Prop.25. Fig. 1.**

Quadratum enim EF est aequale rectangulo sub AB & I; & quadratum FG, aequale est rectangulo sub BC & L : uti etiam quadratum GH, rectangulo sub CD & N contento : igitur ut quadrata EF, FG, GH inter se sunt , ita etiam rectangula sub lateribus AB & I, sub BC & L, item sub CD, & N, contenta: sed quadrata sunt in continuata analogia, (cum latera super quibus fiunt ex hypothesi sint in continua analogia) igitur etiam rectangula, sub lateribus AB & I, BC & L, CD & N, sunt continue proportionalia : cum autem ipsorum bases ponantur in continuata ratione AB, BC, CD; etiam I, L, N, altutudines erunt in continuata analogia per praecedentem. Eodem modo quia ex hypothesi EF, FG, GH, & ex demonstratis modo I, L, N, sunt continuae; itemque ex hypothesi EF, I, K : & FG, L, M; item GH, N, O, continuam servant analogiam : demonstrabimus K, M, O esse continuas. similiter quoque procedetur in lineis P, Q, R, atque ita in infinitum. constat ergo veritas proportionis. *a. 27 Sexti.*

## *L2.§1.* *PROPOSITION 25.*

The proportionals AB, BC, and CD are set in place anew, as also are the proportionals EF, FG, and GH, and the ratio of AB to EF is set to continue in some manner in successive proportionals I, K, and P. Similarly the ratio BC to FG proceeds as to the proportionals L, M, Q; and the ratio CD to GH goes on as N, O, R, etc.

I say that I, L, N, and K, M, O, and likewise P, Q, and R are also continued proportionals in their own ratio.

***Demonstration.***

The square EF is indeed equal to the rectangle under AB and I; and the square FG is equal to the rectangle under BC and L; as also the square GH is equal to the rectangle contained under CD and N. Thus as the squares EF, FG, and GH are self-contained , so also are the rectangles contained under the sides AB and I, BC and L, and likewise CD and N. But the squares are in a continued ratio ( since the sides of the squares upon which they are formed are in a continued ratio by hypothesis). Hence the rectangles under the sides AB and I, BC and L, and CD and N, are also in continued proportion: but since the bases themselves are placed in the continued ratio AB, BC, and CD; thus the altitudes I, L, and N also are in a continued ratio from the preceeding proposition.

By the same method, whereby if by hypothesis EF, FG, GH are in continued proportion, then I, L, N are shown to be in continued proportion; it then can be shown in the same manner that EF, I, K are in a continued ratio: as are FG, L, M; and likewise GH, N, O. We can then show that K, M, O are in a continued ratio, and similarly the method can be applied to the lines P, Q, R, and thus indefinitely, hence in agreement with the truth of the proposition. *a 17 Sexti.*

[$EF^2$ = AB.I; $FG^2$ = BC.L; $GH^2$ = CD.N; as EF, FG, GH are in continued proportion, as are AB, BC, and CD, it follows that I, L, N are also in continued proportion. We can then set $I^2$ = EF.K; $L^2$ = FG.L; $N^2$ = GH.O and so establish that K, M, O are in a continued ratio, etc.]

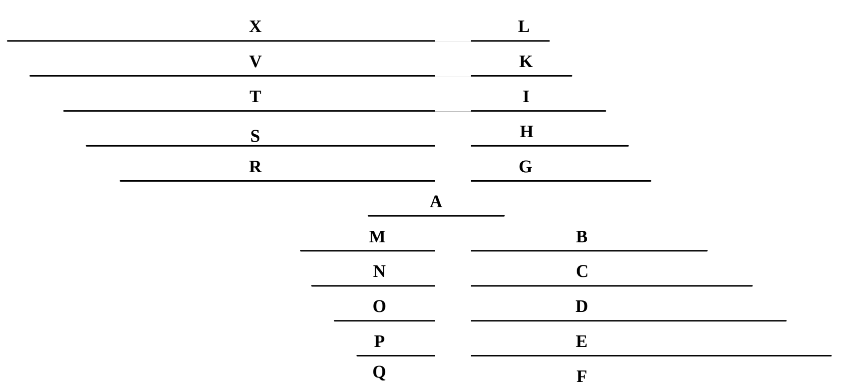
## *PROPOSITIO XXVI.*

Sint series continuae proportionalium, habentes primum terminum A, communem; A, B, C, D, E, F, & A, M, N, O, P, Q; continuata autem serie utriusque , fiat ut B ad A, ita A ad G, &c. & ut M ad A, ita A ad R, sic ut omnes F, E, D, C, B, A, G, H, I, K, L: item omnes Q, P, O, N, M, A, R, S, T, V, X sint in continuata analogia.

Dico esse B ad M, ut R ad G, & C ad N, ut S ad H, & D ad O, ut T ad I, &c.

***Demonstratio.***

Cum B, A, G sint tres continuae, item M, A, R, erit tam [b] rectangulum BG, quam rectangulum MR, quadrato A, ideoque & inter se aequalia. Quare ut B ad M, sic [c] R ad G: similiter quoniam C, B, A, G, H, sunt continuae, ita ut mediam A,




Prop.26. Fig. 1.

aequalis utrimque proportionalium numerus cingat: patet ex elementis A esse mediam proportionalem, inter C & H : rectangulum igitur HC aequatur quadrato A; eodem modo rectangulum SN quadrato A aequale erit: ergo inter se aequantur rectangula HC, SN. Quare [d] ut C ad N, sic S ad H, simili discursu erit ut D ad O, sic T ad I, & c sic de caeteris. Quod erat demonstrandum. *b. 17 Sexti; c. 16 Sexti.*

**[72]**

## *L2.§1.* *PROPOSITION 26.*

There are continued series of proportionals having the first term A in common : A, B, C, D, E, F; and A, M, N, O, P, Q. Moreover, a further series in continued proportion is constructed according to B to A, thus A to G, etc.; and as M to A, thus A to R, in order that all of F, E, D, C, B, A, G, H, I, K, L, and likewise Q, P, O, N, M, A, R, S, T, V, X are in continued proportion.

I say that as B is to M, so R to G, and as C to N, to S to H, as as D to O, so T to I, etc.

***Demonstration.***

Since B, A, and G are three terms in continued proportion, as well as M, A, and R, it follows [b] that the rectangles B.G and M.R are equal to each other and to the square A. Whereby as B to M, thus [c] R to G : similarly, since C, B, A, G, H are in continued proportion, in order that the mean A is surrounded by an equal term from both sides of the proportion, it is apparent from this principle that A is the mean proportional between C and H. Therefore the rectangle H.C is equal to the square A, and in the same way the rectangle S.N is equal to the square A. Hence the rectangles H.C and S.N are equal to each other. Whereby [d] as C is to N, thus S to H, and by a similar discourse, as D is to O, thus T is to I, and so on for the rest. Q.e.d. *b. 17 Sexti; c. 16 Sexti.*

**[72]**

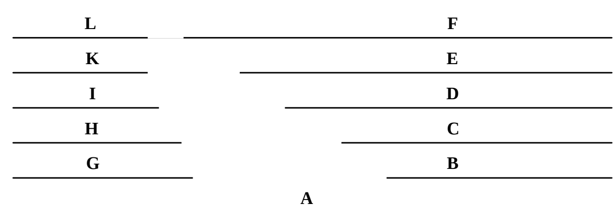
## *PROPOSITIO XXVII.*

Sint duo ordines continuae proportionalium eandem habentes primum, A, B, C, D, E, F, & A, G, H, I, K, L.

Dico proportionem H ad C, esse duplicatum proportionis G ad B, & rationem I ad D, rationis G ad B esse triplicatam : & rationem K ad E, quadruplicatam : L vero ad F quintuplicatam eiusdem rationis G ad B: & sic in infinitum.

*Demonstrat hanc Euclid.lib.14.pr.28 de quatuor continue proportionalibus. Nos eandem de quotcunque, & alia plane methodo demonstribimus.*

***Demonstratio.***




Prop.27. Fig. 1.

Quoniam tam A, G, H quam A, B, C, sunt continuae proportionales, rectangulum HA, quadrato G , & rectangulum CA, quadrato [a] B aequales erit. unde rectangulum HA, ad rectangulum CA, est ut quadratum G, ad quadratum B: itaque cum quadrata G, B, sint [b] in duplicata ratione basium G ad B, erit & rectangulorum proportio duplicata, rationis G ad B : sed ratio rectanguli HA, ad rectangulum CA, est ratio [c] H ad C; ergo proportio H ad C; ergo proportio H ad C est duplicata rationis G ad B. Deinde quoniam tam G, H, I , quam B, C, D , sunt continuae, erit rectangulum IG, quadrato H, & rectangulum DB, quadrato C aequale, itaque ratio rectangulorum IG, DB, hoc est ratio [d] composita ex proportionibus laterum I ad D, & G ad B, aequalis est rationi quadrati H ad C: atqui ratio quadratorum H, C , est quadruplicata rationis G, B; ( est enim ratio quadratorum H, C duplicata rationis H ad C, quae ostensa modo est duplicata rationis G ad B:) ergo proportio composita ex rationibus I ad D, & G ad B, est quadruplicata rationis G ad B. ex quo patet rationem I ad D solam, esse triplicatam rationis G ad B. Simili discursu demonstrabimus rationem K ad E, esse quadruplicatam; & rationem L ad F, quintuplicatam rationis G ad B. constat ergo veritas propositionis. *a. 17 Sexti; b. 20 Sexti; c. 1 Sexti; d. 23 Sexti.*

## *L2.§1.* *PROPOSITION 27.*

There are two orders of continued proportionals having the same first term : A, B, C, D, E, F; and A, G, H, I, K, L.

I say that the proportion of H to C is double the proportion of G to B, and the ratio I to D to be the triple of the ratio G to B: and the ratio K to E, the quadruple; L to F the quintuple of the ratio G to B, and thus indefinitely.

*This is shown by Euclid in book 14, prop. 28 for four continued proportionals. We show the same for any number of proportionals by another clear method*.

### *Demonstration.*

Since A, G, and H as well as A, B, C are in continued proportion, the rectangle H.A is equal to the square G, and likewise the rectangle A.C is equal to the square [a] B. Thus, the rectangle H.A is to the rectangle C.A, as the square G is to the square B. Thus as the squares G and B are in the ratio of the squares of the bases G and B, the ratio of the rectangles will be the square of the ratio G to B. But the ratio of the rectangles H.A and C.A is the ratio [c] H to C: hence the ratio H to C is the duplicate [i.e. square] of the ratio G to B. Hence, since G, H, and I, as well as B, C, D are in continued proportion, the rectangle I.G is equal to the square H, and the rectangle D.B is equal to the square C. Thus the ratio of the rectangles I.G to D.B , that is composed from the proportions of the sides I to D and G to BM, is equal to the ratio of the squares H and C. But the ratio of the squares H and C is the quadruple of the ratio G to B; (the ratio of the squares of H to C is indeed the duplicate of the ratio H to C, which has been shown to be the duplicate of ratio G to B); hence the ratio composed from the ratios I to D and G to B is the quadruple of the ratio G to B. From which it is apparent that the ratio I to D alone is the triplicate of the ratio G to B. By a similar argument we can show that the ratio K to E is the quadruple, and the ratio L to F the quintuple of the ratio G to B, in agreement with the truth of the proposition. *a. 17 Sexti; b. 20 Sexti; c. 1 Sexti; d. 23 Sexti.*

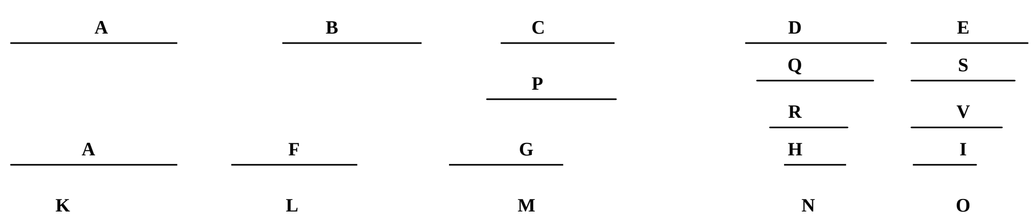
[$G^2$ = H.A; $B^2$ = C.A; hence $G^2/B^2$ = H/C. Again, $H^2$ = I.G, and $C^2$ = D.B, I.G/D.B = I/D.G/B = $H^2/C^2$ = $G^4/B^4$, from which it follows that I/D = $G^3/B^3$. Again, K.H = $I^2$ and E.C = $D^2$ ; hence, K/E.H/C = $I^2/D^2$ = $G^6/B^6$ ; hence, K/E = $G^4/B^4$ and L/F = $G^5/B^5$, etc. Note: the reader should bear in mind that the terms duplicate, triplicate, etc, refer to squares, cubes, etc, of ratios.]

## *PROPOSITIO XXVIII.*

Sint duo ordines continuarum A, B, C, D, E; & A, F, G, H, I eandem nacti primam A : ponatur autem tertius ordo continue proportionalium, K, L, M, N, O, secundo ordini similis; ita tamen ut A & K, sint inaequales. Deinde inter tertias C & G, ponatur media P; & inter quartas D & H. ponantur duae mediae Q & R; tandem inter quintas ponantur tres mediae S, T, V, & ita deinceps.

Dico esse ut L ad B, ita M ad P, & N ad R, ut O ad V, &c.

### *Demonstratio.*




**Prop.28. Fig. 1.**

[73]

Quoniam utriusque sereri, primus terminum idem est A, ergo per praecedentem G est ad C, in duplicata ratione F ad B; sed G etiam est ad C, in duplicata ratione G ad P, igitur G est ad P, ut F ad B: sed ex supposito F ad G est ut L ad M; igitur alternando M est ad G, ut L ad F. est igitur ex aequo M ad P, ut L ad B. Simili modo per praecedentem H ad D, triplicatam, habit ratonem, F ad B; sed etiam H ad D, habet triplicatam eius , quam habet H ad R; ergo ut F ad B, sic H ad R: deine est F ad H, ut L ad N, unde alternando, & ex aequo N ad R ut L ad B. Eadem prorsus ratione demonstrabitur esse O ad V, ut est L ad B. Quare patet veritas propositionis.

Hinc etiam patet B, P, R, V esse continuas cum L, M, N, O, sint continuae, exquidem in eadem analogia in qua sint lineae A, F, G, H, I, ut patet.

. *a. 17 Sexti; b. 20 Sexti; c. 1 Sexti; d. 23 Sexti.*

## *L2.§1.* *PROPOSITION 28.*

There are two series of continued proportionals obtained from the same first term A; A, B, C, D, E; and A, F, G, H, I. A third series of continued proportinals K, L, M, N, O, is put in place, similar to the second series but with A not equal to K. Subsequently, a mean P is placed between the third terms C and G ; and two means Q and R is placed between the fourth terms D and H ; finally, three means S, T, and V are placed between the fifth terms, and henceforth.

I say that as L is to B, thus M is to P, and N is to R, as O to V, etc.

***Demonstration.***

Since the first term of the second series is also A, it follows by the preceeding proposition that G is to C in the duplicate [or square] ratio F to B. However, G is to C in the square ratio G to P, and therefore G is to P as F is to B. Moreover, from supposition, F is to G as L is to M; therefore alternatively, M is to G as L is to F. Therefore on equating, M is to P, as L is to B. In the same way, by the preceeding theorem, H is to D, in the triplicate ratio F to B; but also H to D is in the triplicate ratio of that which H has to R; therefore as F to B, thus H to R: then as F to H, so L to N, from which otherwise, and on equating, N is to R as L is to B. In short for the same ratio it can be shown that O is to V as L is to B. Whereby the truth of the proposition is apparent.

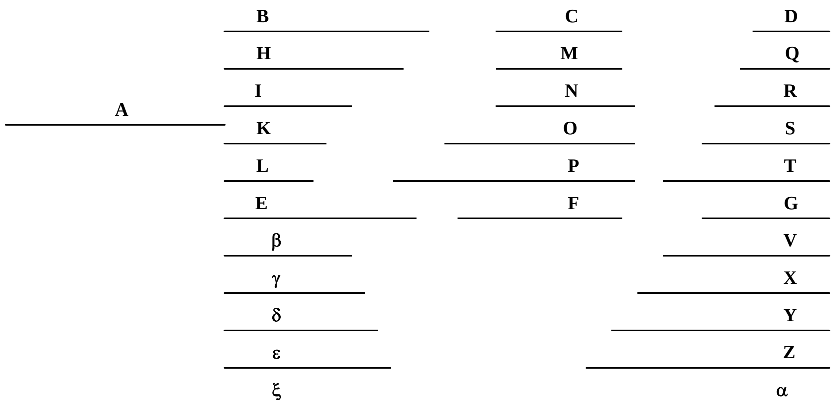
Hence, it can also be seen that B, P, R, V are in continued proportion with L, M, N, O; and these indeed are apparent to be in the same ratio as the lines A, F, G, H, I.
[$G/C = F^2/B^2$. Again, $G^2/P^2 = G/C$ ; hence, $G/P = F/B$. But $F/G = L/M$ , or $M/G = L/F$ and from $G/P.M/G = F/B.L/F$, it follows that $M/P = L/B$.
Again, $H/D = F^3/B^3$, but in this case from the three means we have $H/R = R/Q = Q/D$ or $H/D = H^3/R^3$, from which $H/R = F/B$ or $F/H = B/R$; also, $F/H = L/N$, hence $N/R = L/B$. Again, $O/V = L/B$, etc. (Thus, the power associated with a mean between like terms in a series of proportions can take the place of the power associated with the first ratio, for any term in a series of proportions).]

## *PROPOSITIO XXIX.*

Ponatur duae series continuarum A, B, C, D ; & A, E, F, G, communem habentes primam A : & inter secundas B, E, sint quotvis mediae H, I, K, L, totidem quo inter tertias interponantur; M, N, O, P, Similiter inter quartas D & G ponantur mediae Q, R, S, & T.

Dico A, H, M, Q; item A, I, N, R; & A, K, O, S, & A, L, P, T esse in continua analogia.

***Demonstratio.***




**Prop.29. Fig. 1.**

Ponantur enim inter A & M, media proportionalis V ; & inter A & N media X; similiter Y, Z, α mediae ponantur inter A, O, & A, P, & A, F. Quoniam ergo A, B, C, & A, V, M sunt continuae proportionales, erit [a] ratio C ad M, duplicata eius, quam habet B as V. uterius cum A, V, M & A, X, N etiam sint continuae proportionales, erit [b] M ad N, duplicata eius quam habet V ad X : eodem pacto ostenditur, rationes N ad O, & O ad P, & P ad F, esse duplicatas rationum X ad Y, & Y ad Z, & Z ad α : Quare cum C, M , N, O, P, F ponantur esse continuae, etiam B, V, X , Y, Z , α patet esse continuas. Est igitur ratio B ad α, quintuplicata rationis B ad V, & quia tam α, quam E, mediae sunt inter A & F, inter se aequales erunt ideoque & ratio B ad α, & ratio B ad E, eadem est; ergo ratio B ad E, quintuplicata est rationis B ad V, quia autem B, H, I, K, L, E ponantur continuae, etiam ratio B ad E, est quintuplicata rationis B ad H; est igitur B ad V, ut B ad H, unde aequales sunt H & V. Similiter ostenditur I & X, K & Y, L & Z aequales esse. Quare cum ex constructione A, V, M; A X N; A Y O; A, Z, P, sint continuae; etiam A, H, M; A, I, N; A, K, O; A, L, P continuae erunt. Non alia ratione ostendemus, etiam ipsas A, H, M, Q; item A, I, N, R, &c. esse in continuata analogia;

**[74]**

(Si nempe ipsis A, H, M inveniamus quartam β, & ipsis A, I, N quartam , γ, & ipsis A, K, O quartam δ, & ipsis A, L, P, ε, denique ipsis A, E, F quartam ξ) demonstrabimus esse ut prius ipsas β, γ, δ, ε, ξ, ipsis Q, R, S, T, G aequales; ac proinde omnes quatuor A, H, M, Q; A, I, N, R; &c. esse continuas. Quod erat demonstrandum.

*a. 27 huius; b. ibid.*

## *L2.§1.* *PROPOSITION 29.*

There are two series of continued proportionals A, B, C, D ; and A, E, F, G ; put in place having the same first term A. Between the second terms B and E certain means H, I, K, L are inserted; and the same number of means are inserted between the third terms C and F: M, N, O, P . Similarly, the means Q, R, S, T are inserted between the fourth terms D and G.

I say that the series A, H, M, Q; as well as A, I, N, R; A, K, O, S ; and A, L, P, T are in continued ratios.

***Demonstration.***

For the mean V of the proportionals A and M can be placed between A and M ; and the mean X between A and N ; similarly the means Y, Z, and α are placed between A, O; A, P; and A, F. Hence, as A, B, C and A, V, M are continued proportionals, the ratio [a] C to M is the duplicate [i. e. square] of the ratio B to V. Further, as A, V, M and A, X, N also are continued proportionals, [b] M to N, is the duplicate of V to X : in the same manner it can be shown that the ratios N to O, O to P, and P to F, are the duplicate ratios of X to Y, Y to Z, & Z to α. Whereby as C, M , N, O, P, F are placed in a continued ratio [the 5th order geometric mean between C and F], so also B, V, X , Y, Z , $\alpha$ have been shown to be continued proportionals [the 2nd order geometric mean between A and the series just mentioned]. Therefore the ratio B to $\alpha$ is the quintuplicate [i.e. B/V raised to the 5th power] of the ratio B to V; and as for $\alpha$, so also for E, which is the mean of A and F, and the ratio B to $\alpha$ is the same as the ratio B to E: hence E and $\alpha$ are equal to each other; therefore the ratio B to E is the quintuplicate of the rato B to V. Since in addition, B, H, I, K, L, E are placed in continued proportion, the ratio B to E also is the quintuplicate of the ratio B to H; therefore B to V is as B to H, and hence H and V are equal. In the same manner it can be shown that I and X, K and Y, and L and Z are equal. Whereby as A, V, M; A, X, N; A, Y, O; A, Z, P are in continued proportin by construction , also A, H, M; A, I, N; A, K, O; A, L, P are in continued proportion. We do not show others further, but A, H, M, Q; and likewise A, I, N, R, &c. are continued ratios. (For A, H, M themselves we can find the fourth mean $\beta$, and likewise for A, I, N, $\gamma$; A, K, O, $\delta$; A, L, P, $\varepsilon$; A, E, F, $\xi$) . We can show that as before $\beta$, $\gamma$, $\delta$, $\varepsilon$, $\xi$, are equal to Q, R, S, T, G; and hence all four A, H, M, Q; A, I, N, R; etc. are in continued proportion. Q.e.d.

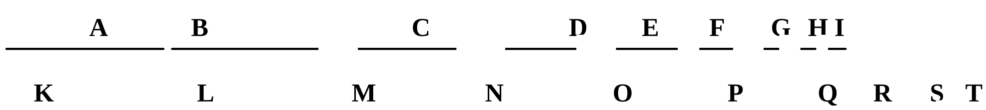
[We have the continued proportionals A, B, C; A, V, M; A, X, N; A, Y, O; A, Z, P; A, $\alpha$, F. Now, $C/M = B^2/V^2$; and similarly, $M/N = V^2/X^2$; as $N/O = X^2/Y^2$, $O/P = Y^2/Z^2$, $P/F = Z^2/\alpha^2$. Hence as C, M , N, O, P, F are in continued proportion, (as they are the successive 5th order mean proportionals between C and F), so also B, V, X , Y, Z , $\alpha$ are in a continued ratio, and $B/\alpha = B^5/V^5$; $B/\alpha = B/E$ giving $\alpha = E$. Again, in the same way, $B/E = B^5/V^5$ ; as B, H, I, I, L, E are in a continued ratio, $B/E = B^5/H^5$: $B/V = B/H$ and $V = H$. In the same way, $I = X$; $K = Y$; $L = Z$. Hence, as A, V, M; A, X, N; A, Y, O; A, Z, P are in continued proportin by construction , also A, H, M; A, I, N; A, K, O; A, L, P are in continued proportion, etc.]

## *PROPOSITIO XXX.*

Dentur binae series continuae proportionalium in diversis rationibus: A, B, C, D, &c; K, L, M, N, &c. ita tamen ut A, K, L, B sint etiam continuae proportionales.

Dico omnes A, K, L, B, N, O : C, Q, R, D, & sic deinceps, (omisso in serie K L tertio quoque termino M, P, S) esse in continua analogia.

***Demonstratio.***




**Prop.30. Fig. 1.**

Quia A, K, L, B ponantur continue, ergo ut K ad L, sic L ad B; sed etiam est ex hypothesi ut K ad L, sic L ad M; ergo L ad B, & M eadem habet rationem; aequales [a] igitur sunt B & M. ideoque [b] rationes B ad C, M ad C aequales sunt. Quoniam autem A, K, L, B sunt quatuor continuae proportionales, erit ratio A ad B, id est ex hypothesi ratio B ad C, id est ex demonstratione ratio M ad C, triplicatata rationis A ad K: sed ratio A ad K, ex hypothesi est ratio K ad L, id est ratio M ad N : ergo ratio M ad C, triplicata est rationis M ad N : est autem ratio M ad P, triplicata rationis M ad N, (sunt enim M, N, O, P continuae) igitur ut M ad C, sic M ad P: unde & aequales sunt C & P. quare cum loco M, in serie statuatur illi aequalis B, & loco P, illi aequalis C. erunt A, K, L, B, N, O, C continuae proportionales. similiter ostendemus seriem hanc per terminos Q, R, D, &c. continuati in infinitum. Quod erat demonstrandum.

*a. 9 Quinti; b. 7 Eius.*

*Corollarium.*

Hinc sequitur : rationem A ad B, triplicatam esse rationis K ad L : & B ad C, triplicatam ipsius L ad M : item C ad D, triplicatam ipsius M ad N : & sic de ceteries, nam ratio A ad B continuatur, estque illa triplicata rationis K ad L, quam per reliquos terminos continuatur.

## *L2.§1.* *PROPOSITION 30.*

Two series of continued proportionals are given with different ratios: A, B, C, D, etc; K, L, M, N, etc. that also however have the series A, K, L, B, etc in continued proportion.

I say that all the terms A, K, L, B, N, O : C, Q, R, D, and thus henceforth, (with the terms K L and also the terms M, P, S disregarded in the third series) are in a continued ratio.

### *Demonstration.*

Since A, K, L, B are placed in a continued ratio, thus K is to L, as L is to B; but also from hypothesis as K to L, thus L ad M. Hence L to B and L to M are both equal to the same ratio K to L; and therefore [a] B and M are equal, and likewise the ratios [b] B to C and M to C are equal. As A, K, L, B are four continued proportionals, the ratio A to B, (that is by hypothesis equal to the ratio B to C, that has been shown to be equal to the ratio M to C), is the triplicate of the ratio A to K: but the ratio A to K, from hypothesis is equal to the ratio K to L, that is the ratio M to N : hence the ratio M (or B) to C is the triplicate of the ratio M to N. But the ratio M to P is the triplicate of the ratio M to N, (for M, N, O, and P are in a continued ratio): therefore as M is to C, thus M is to P: and thus C and P are equal. Whereby B can be set up equal to and in place of M in the second series, and C is set in place of P. The terms A, K, L, B, N, O, C are hence continued proportionals. Thus in the same way we can show that the series can be continued by the terms Q, R, D, etc., indefinitely. Q.e.d.

*a. 9 Quinti; b. 7 Eius.*

*Corollary.*

Hence it follows that the ratio A to B is the triplicate of the ratio K to L : and B to C is the triplicate of L to M itself : likewise C to D is the triplicate of M to N : and so on for the rest, for the ratio A to B on being continued, is the triplicate of the ratio K to L for the remainder of the terms in the progression.

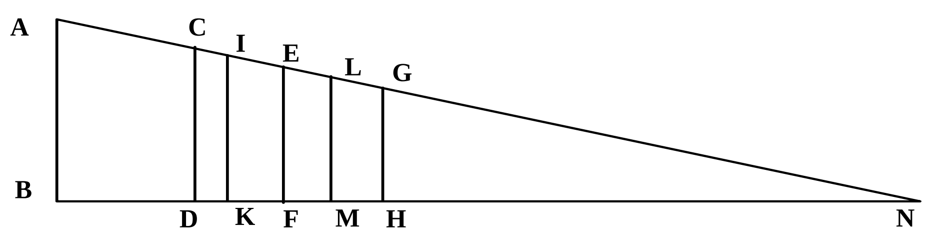
[For K and L are the cubic mean proportonals of A/B, etc. Thus, in the second series, B, C, D, ... can replace the terms which are multiles of three, i.e. M, P, S, ...]

## *PROPOSITIO XXXI.*

In triangulo quovis ABN quatuor ponantur parallelae AB, CD, EF, GH in continua analogia : ac inter AB, GH, media sit IK, inter EF vero & GH, media sit LM.

Dico rationem AB ad IK triplicatam esse rationis LM ad GH.

### *Demonstratio.*




**Prop.31. Fig. 1.**

Ratio AB ad GH, duplicatum est ex hypothesi, rationis AB ad IK : & ratio EF ad GH, duplicatam rationis LM ad GH; ergo cum iterum, ex hypothesi ratio AB ad GH triplicata sit rationis EF ad GH. erit ratio AB ad

**[75]**

IK,quae dimidiata est rationis AB ad GH; triplicata rationis LM ad GH; quae dimidiata est rationis EF ad GH. Quod erat demonstrandum.

## *L2.§1.* *PROPOSITION 31.*

In some triangle ABN four parallel lines AB, CD, EF, GH are set in a continued ratio. The mean between AB and GH is IK, and the mean between EF and GH is LM.

I say that the ratio AB to IK is the triplicate of the ratio LM to GH.

***Demonstration.***

The ratio AB to GH by hypothesis is the duplicate [square] ratio AB to IK : and the ratio EF to GH is the duplicate of the ratio LM to GH. Hence in the same way from hypothesis, the ratio AB to GH is the triplicate [cube] of the ratio EF to GH, and the ratio AB to IK, which is square root of the ratio AB to GH, is the triplicate of the ratio LM to GH, which is square root of the ratio EF to GH. Q.e.d.

## *PROPOSITIO XXXII.*

Quantitas ex quotvis continuae proportionalibus composita, ad aliam ex pari numero terminolum eiusdem seriei productae constatam, multiplicem rationem habet proportionalis primae ad secundam, ex quot terminis alterutra quantitatum componitur.

***Demonstratio.***


**Prop.32. Fig. 1.**

Ponatur series rationis alicuius, constituere quantitatem AB, & series ulterius producta conficiat quantitatem BC ; hoc tamen pacto ut utraque pari numero constet terminorum continue proportionalium eiusdem seriei productae; verbi gratia si numeret AB quantitas terminos octo continue proportionales, & BC totidem eiusdem seriei producae; dico AB ad BC, octuplicatam habere rationem uniformiter eius qua AD ad DE. Cum enim tota serieis rationis AD ad DE, pergat uniformiter & continue ex supposito usque ad C, & sint tot termini eiusdem seriei in AB, quot suntin BC, exempli causa in singulis octo; habebit ergo prima AD ad BF, octuplicatam rationem eius, quam habet AD a ad DE : similiter ratio DE ad FG, octuplicata est rationis DE ad EH, id est rationis AD ad DE : quare cum rationes AD ad BF, & DE ad FG, octuplicatae sint eiusdem rationis AD ad DE, erit ut AD ad BF, sic DE ad FG : eodem modo erit ut DE ad FG, sic EH ad GI, & sic de ceteris. Quare omnes [b] antecedentes, id est octo termini qui constituunt AB, se habent ad omnes consequentes, hoc est ad octo terminos qui constituunt BC, ut una antecedens AD, ad unam consequentem BF. Quare cum AD ad BF, rationem habeat octuplicatam rationis AD ad DE. habebit quoque AB ad BC octuplicatam eius, quam habet AD ad DE. Quod erat demonstrandum.

*a Defin. 5 Sexti ; b 11 Quinti.*

## *L2.§1.* *PROPOSITION 32.*

A quantity [of terms in continued proportion] is composed from some continued proportionals, and is to have certain terms in common with that series, for which the quantity is a multiple of the ratio of the first to the second proportional, from which another of the terms of the new series is composed.

***Demonstration.***

A series is formed from some ratio, initially to give the AB, and the series is produced further to give the term BC ; this term corresponds to another term of the same series produced in continued proportion. For example, if eight terms in continued proportion are to be enumerated in the initial length AB, and in BC an equivalent number of terms are enumerated for the whole series produced ; I say that AB to BC is in the eight-fold ratio of AD to DE uniformly set our. For indeed the whole of the series of ratios AD to DE appear uniformly and continue from supposition as far as C, and there is the same number of terms of the series in AB, as there are in BC, which is eight in each for the example. Hence, the first ratio AD to BF is eight-fold the ratio of AD [a] to DE : and similarly the ratio DE to FG, is the eight-fold of the ratio DE to EH, that is also the ratio of AD to DE. Whereby as the ratios AD ad BF, and DE ad FG, are the eight-fold of the same ratios AD to DE, as AD to BF, thus DE is to FG : in the same manner, DE is to FG thus as EH is to GI, and so on for the rest. Whereby all the eight terms in the preceeding interval that constitute [b] AB, are held with all the following eight terms that constitute the interval BC, in order that one preceeding term AD to a following term BF has the same ratio. Whereby AD to BF has the eight-fold ratio of AD to DE. Also, AB to BC is the eight-fold of that which AD has to DE. Q.e.d.

*a Defin. 5 Sexti ; b 11 Quinti.*

[This result can easily be found using algebra: e.g. Let AD = $a$, DE = $ar$, EG = $ar^2$, etc.; then AB = $a(1 + r + r^2 \ldots. + r^7)$; BC = $a(r^8 + \ldots. + r^{15})$; giving BC/AB = $r^8$, etc. Gregorius's geometric proof seems a little intuitive.]

## *PROPOSITIO XXXIII.*

Inter tres continue proportionales AD, BD, CD, mediae sint GD, ED : rursum inter AD, GD, media sit FD, & inter BD, ED, media sit HD: & hoc semper fiat:

Dico esse ut AB ad BC, sic AG ad BE, & AF ad BH, &c.


**Prop.33. Fig. 1.**

### *Demonstratio.*

Cum inter tres continuas AD, BD, CD, mediae sint GD, ED; erunt AD, GD, BD, ED, CD, (ut patet ex elementis) omnes continuae proportiones: proindeque etiam [c] AG, GB, BE, EC erunt continuae proportionales: ergo AG ad GB, ut BE ad EC : & componendo ac per conversionem rationis AB ad AG, ut BC ad BF. Igitur alternando AB ad BC ut AG ad BE: Deinde quoniam AD, GD, BD, ED, sunt continuae, AD est ad GD, ut BD ad ED : Quare si inter AD, GD mediae sit FD, & inter BD, ED, media HD, patet ex elementis esse AD ad FD, ut BD ad FD, atqui cum AD, FD, GD, sint continuae, AF est ad FG, ut AD ad FD. per 1. huius. Et cum BD, HD, ED sint continuae etiam BH est ad HE, ut BD ad HD, hoc est (sicut iam ostendi) ut AD ad FD, ergo AF est ad FG, ut BH ad HE. Quare componendo ac per conversionem

[76]

rationis ut AG est ad AF, sic BE ad BH: & permutando ut AG ad BE, hoc est (sicut ostendi) ut AB ad BC, sic AF ad BH. Constat ergo propositionis veritas.

## *L2.§1.* *PROPOSITION 33.*

Between three continued proportionals AD, BD, and CD, the means are GD and ED. Again, between AD and GD, the mean is FD, and between BD and ED the mean is HD; and this shall always be the case.

I say that as AB is to BC, thus AG is to BE, and AF is to BH, etc.

### *Demonstration.*

Since between the three continued proportions AD, BD, CD, the means are GD, ED, then all the terms AD, GD, BD, ED, CD, (as is apparent from elementary considerations) are continued proportions: and hence also [c] AG, GB, BE, EC are continued proportionals: thus AG is to GB as BE is to EC : and on addition and conversion of the ratio, AB is to AG as BC is to BF. Therefore on re-arranging, AB is to BC as AG is to BE. Hence, since AD, GD, BD, ED, are in a continued ratio, AD is to GD, as BD is to ED : Whereby if the mean between AD and GD is FD, and between BD and ED, the mean is HD, then it is apparent from elementary considerations that AD is to FD, as BD is to FD, because as AD, FD, GD, are in a continued ratio, AF is to FG, as AD is to FD. by Prop. 1. of this book. And since BD, HD, ED are in continued proportion, also BH is to HE, as BD is to HD, that is (as has been shown) as AD is to FD, thus AF is to FG, as BH is to HE. Whereby by adding and by the converse of the ratio, as AG is to AF, thus BE is to BH: and on permuting, as AG is to BE, that is (thus as shown) as AB to BC, thus AF to BH. Thus the truth of the proposition is agreed upon.

## *PROPOSITIO XXXIV.*

Sint in continua analogia AB, CB, DB, & inter AB, CB media sit EB; inter CB vero & DB sit media FB; deinde inter EB, CB, ac CB, FB mediae sint GB, HB : Denique inter GB, CB, & CB, HB sint mediae IB, KB, & sic deinceps.

Dico rationem AC ad CD, duplicatam esse rationis EC ad CF; quadruplicatam autem rationis GC ad CH, & octuplicatam rationis IC ad CK : atque ita in infinitum.


**Prop.34. Fig. 1.**

***Demonstratio.***

Cum AB, CB, DB sint continuae proportionales, erit AC ad CD [a] ratio eadem cum ratione AB ad CB: item quia continuae sunt AB, EB, CB, erit ratio AE ad EC eadem cum ratione AB ad EB : & quia inter tres continuas AB, CB, DB mediae sunt EB, FB, patet ex elementis AB, EB, CB, FB, DB omne esse continue proportionales: quare ratio AB ad CB, id est ut iam ostendi, ratio AC ad CD, duplicata est rationis AB ad EB; id est ut ostendi, rationis AE ad EC, quia autem continuae proportionales sunt AB, EB, CB, FB, etiam erunt AE, EC, [b] CF continuae; ergo ut AE ad EC, sic EC ad CF : ergo ratio AC ad CD, etiam duplicata est rationis FC ad CF; quod erat primum. Eodem discurrendi modo demonstrabimus, rationem EC ad CF, duplicatam esse rationis GC ad CH : adeoque rationem AC ad CD, quadruplicatam esse rationis GC ad CH; denique ostendemus etiam simili ratiocinatione, rationem GC ad CH, duplicatam esse rationis IC ad CK. Unde manifestum est rationem AC ad CD, eiusdem octuplicatam esse. Quae erant demonstranda.

*a ibid; b 1. Huius.*

## *L2.§1.* *PROPOSITION 34.*

Let AB, CB, and DB be in a continued ratio, and let the mean between AB and CB be EB; and likewise between CB and DB the mean is FB; between EB, CB, and CB, FB the means are GB and HB : Further, between GB, CB, and CB, HB the means are IB, KB, and so on.

I say that the ratio AC to CD is the duplicate [square] of the ratio EC to CF; the four-fold [ fourth power] of the ratio GC to CH, and the eight-fold [eighth power] of the ratio IC to CK : and so on indefinitely.

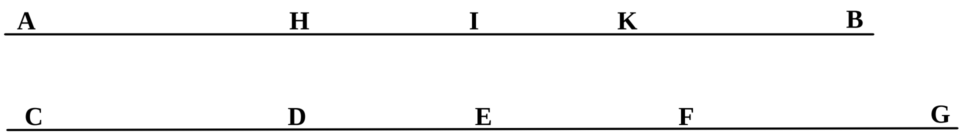
***Demonstration.***

Since AB, CB, and DB are continued proportionals, the ratio AC to CD [a] is the same as the ratio AB to CB: furthermore since AB, EB, and CB are continued proportionals, the ratio AE to EC is the same as the ratio AB to EB : and since between the three continued proportionals AB, CB, and DB the means are EB and FB, it is apparent from elementary considerations that AB, EB, CB, FB, DB are all continued proportionales: whereby the ratio AB to CB, that is as thus shown, the ratio AC to CD, is the duplicate of the ratio AB ad EB; or as shown, of the ratio AE to EC; but as AB, EB, CB, and FB are continued proportionals, AE, EC, and [b] CF are also continued proportionals; hence as AE is to EC, thus EC is to CF : hence the ratio AC to CD is indeed the duplicate of the ratio FC ad CF; which establishes the first part of the proposition. We can show by the same kind of discussion that the ratio EC to CF is the duplicate of the ratio GC to CH: and thus the ratio AC to CD is the four-fold of the ratio GC to CH; and then we can also

show by similar reasoning that the ratio GC to CH is the duplicate of the ratio IC to CK. From which it can be showm that the ratio AC to CD is the eight-fold ratio of the same. Q.e.d.
*a ibid; b 1. Huius.*

## *PROPOSITIO XXXV.*

Datam lineam secare in quotvis continue proportionales, secundem datam rationem.




**Prop.35. Fig. 1.**

***Constructio & Demonstratio.***

Data sit linea AB dividenda sicut postulat progressio, in quatuor continuas, secundum rationem datam CD ad DE. Continuatur toties ratio CD ad DE, quot continuas numero desiderat in linea data AB, nempe CD, DE, EF, FG : tum divide lineam c AB, sicut divisa est linea CG in punctis H, I, K. Dico AB, esse divisam prout exigit propositio : patet demonstratio ex constructio.
*c 10. Sexti.*

## *L2.§1.* *PROPOSITION 35.*

To cut a given line in some number of continued proportionals, following a given ratio.

***Construction & Demonstration.***

The progression demands that the given line AB is to be divided in four continued ratios following the given ratio CD to DE. The ratio CD to DE is to be continued in the given line AB as many times as the desired number, namely CD, DE, EF, FG : then divide the line [c] AB as the line CG has been divided, in the points H, I, and K. I say that AB has been divided as required by the proposition, as is apparent from the construction.
*c 10. Sexti.*

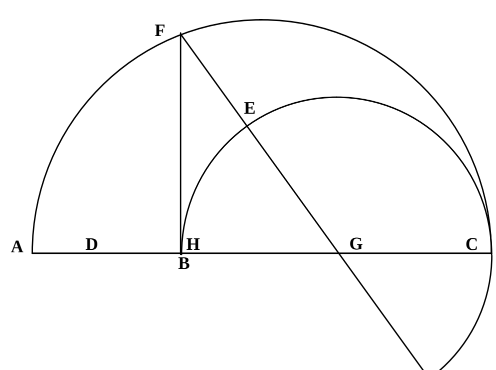
## *PROPOSITIO XXXVI.*

Datarum linearum alteram ita secare, ut partes lineae sectae, cum insecta, sint in continua analogia.

[77]

***Constructio & Demonstratio.***

Datae sint duae lineae AB & BC, quarum alteram scilicet AB, oporteat dividere in D puncto, ut AD, DB, BC sint in continua analogia. pro constructione , super BC diametro describatur circulus, item super linea AC, composita ex duabus datis AB & BC, qui sint BEC & AFC; & ex puncto B erigatur perpendicularis BF, ad rectam AC; ducaturque ex F puncto, linea FI. per centrum minoris circuli G, fiat denique rectae GF, aequalis GD. Dico factum esse quod imperatum fuit : quoniam quadrato FB iam aequatur rectangulum ABC, quam [a] EFI, hoc est BDC, erunt BDC, ABC, rectangula aequalia inter se : sed ABC rectangulo aequantur



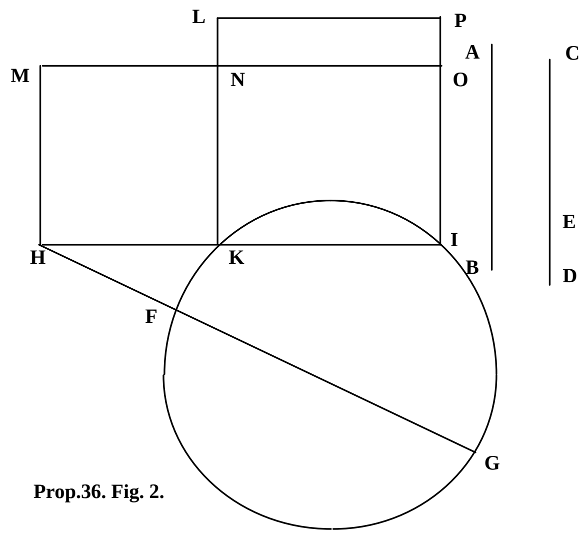

**Prop.36. Fig. 1.**

rectangula ADBC, DBC; & BDC rectangulo aequatur rectanglum DBC, una cum quadrato DB; dempto igitur communi rectangulo DBC, manet DB quadrato, aequale rectangulum ADBC. igitur AD,DB, BC sunt continuae.

*Aliter.*

Datae sint AB, CD, quarum una CD, ita secanda sit in E, ut AB, CE, ED sint in analogia continua.

***Constructio & Demonstratio.***



**Prop.36. Fig. 2.**

Assumpta FG linea maiori quam AB, fiat super FG tanquam diametro circulus FIG: deinde (quod ex elementis facile perficitur) rectangulum GHF aequale fiat rectangulo ABCD; & ex puncto H ita ducatur HKI, ut KI aequalis sit AB (quod ab aliis factum, & nos libro nostro de circulis alia atque alia methodo praestabimus) denique ex K erecta normali KL, aequali ipsi CD, praescio datur ex KL (ostendam enim esse maiorem) linea KN, aequalis ipsi HK. Dico factum quod petebatur.

Nam rectangulum [b] IHK aequatur rectangulo GHF, id est ex constructione rectangulo ABCD, id est rursum ex constructione IKL, ergo ut IH ad KI, ita est reciproce Kl ad KH : atqui IH maior est quam IK, ergo & LK, quam KH maior erit, quod assumptum fuerat in constructione: perficiantur iam rectangula IHK, IKL, auctis per N & L parallelis ad HI, & normalibus HM, IOP. Quoniam igitur rectangula IM, IL, aequalia sunt, ablato communi KO, reliqua OL, KM aequalia erunt. Quare cum KM (ut ex condtructione patet) sit quadratum, erunt NO, KN, LN tres continuae. Atqui NO est KI, hoc est AB: & KL est CD. Factum igitur est quod petebatur. *a 35 Tertii; b 36 Sexti.*

[78]

*Scholium.*

*Hanc eandem propositionem nonnullis alii & proposuerant. & feliciter solverunt. Uti ? A. P.Clausius Magister meam (amicus per plures annos familiaris & domesticus auditor fui) Peletarius?, & Guido Ubaidius?, sed exercitis causa, etiam meo eam Matre ? expedire volui, ut alienis plumis me exornare vello videar : nosquam cuius propositionem statui meis lucubrationibus interserere, quam diverse discursis demonstratione meam non fecero; vel authoris nomen in publicum non protulere. Rogo benigne Lector huius res ,memor esse velis, dum in posterum simile occurret.*

## *L2.§1.* *PROPOSITION 36.*

To cut one of two given lines into two parts in order thus that the cut parts of the line together with the other section are in a continued ratio.

***Construction & Demonstration.***

Let AB and BC be two given lines, it is required to divide one of these, such as AB, by the point D in order that AD, DB, and BC are in a continued ratio. By way of construction, a circle BEC is described with diameter BC, and likewise a circle AFC is described with diameter AC, where AC is the sum of AB and BC. From the point B a perpendicular BF is erected to the line AC; and from the point F the line FI is drawn through the centre of the smaller circle G, and thus the lines GF and GD are made equal. I say that the required task has been performed : For the square FB thus is equal to the rectangle AB.BC, [a] since the rectangles EF.FI, BD.DC [as EF = DB and FI = DC] and AB.BC are equal to each other. But the sum of the rectangles AD.BC and DB.BC is equal to the rectangle AB.BC, and the rectangle BD.DC is equal to the sum of the rectangle DB.BC and the square DB; therefore by removing the common rectangle DB.BC, the rectangle AD.BC remains equal to the square DB. Therefore AD, DB, and BC are in continued proportion.

*An alternative method.*

Given the lines AB and CD, of which one CD is to be cut thus in E in order that AB, CE, and ED are in continued proportion.

***Construction & Demonstration.***

It is assumed that the line FG is longer than the line AB, and the circle FIG is constructed with FG as diameter : hence (and which can easily be established from first principles ) the rectangle GH.HF is made equal to the rectangle AB.CD; and thus from the point H the line HKI is drawn, in order that KI is equal to AB (which has been done by the other method, and from our book on the circle and the other method we shall succeed one way or another) and hence from K a normal is erected KL, equal to CD, and from what is already known, the line KN from KL (which I will show to be the greater) is equal to HK. I say that what was demanded has been done.

For the rectangle [b] IH.HK is equal to the rectangle GHF, which is from the construction equal to the rectangle AB.CD, that is again from the construction equal to IK.KL, thus as IH is to KI, thus reciprocally as Kl to KH : but IH is greater than IK, and therefore LK is greater than KH, which was assumed in the construction: thus the rectangles IH.HK and IK.KL can be established, by raising parallel lines to HI through N and L, and with the normals HM and IOP. Hence, as the rectangles IM and IL are equal, with the common rectangle KO taken away, the remainders OL and KM are equal. Whereby as KM is a square(as is apparent from the condtruction) , the lines NO, KN, and LN are three continued proportionals. But NO is equal to KI, or AB: and KL is equal to CD. Hence that which was demanded has been done.

*a 35 Tertii; b 36 Sexti.*

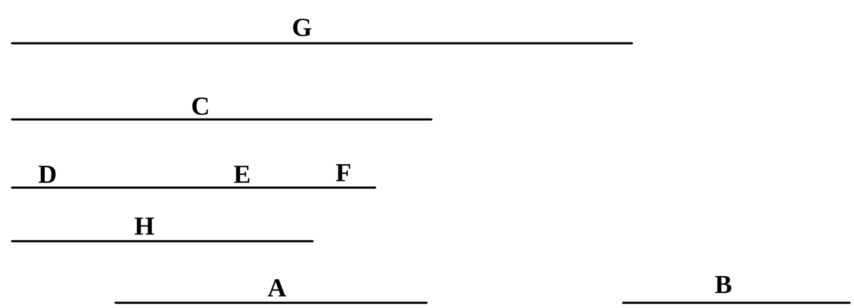
*Scholium.*

*Thus the same proposition and some others have been proposed and happily solved by me. This problem was posed to me by my teacher R. P.Clavius (with whom I was a close family friend and listener for many years), as an exercise, so that I would have a more rounded education : indeed I have not stated the propositions composed from my lucubrations elsewhere, as I do not wish to make my demonstrations from different discourses; or my name would not be brought forward in public. I ask kind reader, that you may remember this, as later the same kind of thing will occur. [Translation paraphrased in parts due to poor quality of text on microfiche*

# *PROPOSITIO XXXVII.*

Datarum duarum rectarum alteram ita secate, ut rectangulum sub insecta, & parte lineae sectae, ad residum lineae quadratum, datam habeat rationem.

***Constructio & Demonstratio.***



**Prop.37. Fig. 1.**

Data sit ratio A ad B, deinde datae lineae sint C & DF, postulat propositio, fecari rectam DF in E, ut rectangulum sub C & EF, ad quadratum DE, eam rationem habeat, quam data B ad datam A. Fiat C ad G, ut B ad A, & per praecedentem secetur DF in puncto E, ut sint tres continuae proportionales lineae, G, DE, EF. Dico factum quod petebatur. fiat enim ut B ad A, sic H ad DE: erit ergo, ut G ad C, sic DE ad H: & permutando ut G ad DE, ita C ad H ; sed ut G ad DE, ita DE ad EF ex constructione; ergo ut DE ad EF, ita C ad H : rectangulum igitur super lineis C & EF, aequale est [a] rectangulum sub lineis DE & H constructum, porro rectangulum sub lineis DE & H consturctum, ad quadratum DE, eam habet proportionem quam lineae ipsae, scilicet quam habet [b] H ad DE: ergo & rectangulum C & EF, est ad quadratum DE, ut H ad DE: id est ut B ad A : igitur datarum rectarum alteram, &c. Quod fuit praestandum.

*a 26?. Sexti. ; b 2 Sexti.*

## *L2.§1.* *PROPOSITION 37.*

To divide one of two given lines thus, in order that the rectangle formed from the one line and with a part of the line divided by the intersection, may have a given ratio to the square of the rest of the line divided.

***Construction & Demonstration.***

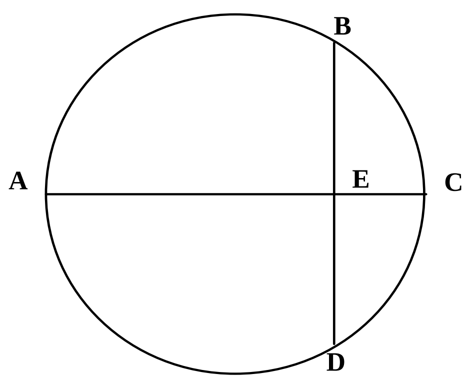
The ratio A to B is given, as well as the lines C and DF, and according to the proposition, the line DF is to be diveded in E, in order that the ratio of the rectangle under C and EF to the square DE has that given ratio A to B. The ratio C to G is made as B to A, and by the preceding proposition, DF is divided by the point E in order that the three lines C, DE, and EF are in continued proportion. I say that what is required has been done. For the ratio H to DE is made as B is to A,: hence as G is to C, thus DE is to H. On re-arranging, we have as G to DE, thus as C to H; but as G is to DE is thus as DE to EF by construction; hence as DE is to EF, thus C is to H. Hence the rectangle on the lines C and EF is equal to [a] the rectangle formed under the lines DE and H; again the ratio of the rectangle constructed under the lines DE and H to the square DE, has the proportion [b] as lines themselves that H has to DE. Thus the rectangle formed from C and EF, is to the square DE as H is to DE, or B to A. Therefore the proposition follows as required.

*a 26?. Sexti. ; b 2 Sexti.*

## *PROPOSITIO XXXVIII.*

Data media trium in continua ratione existentium, & aggregato trium linearum primam & tertiam constituentium, exhibere primam & ultimam.

***Constructio & Demonstratio.***




**Prop.38. Fig. 1.**

Super aggregato tanquam diametro, circulus describatur ABC; in quo recta accommodetur BD, quae aequalis sit duplae datae mediae: quod fieri posse eo fiat ex datis & ex constructione : dein agatur per centrum linea AC, secans orthogonaliter in E, lineam BD. Dico factum esse quod petitur, est enim AC linea aequalis aggregato, & BE ([c] dimidia ipsius BD) datae mediae aequalis : quare cum AC orthogona sit ad BE, erit AEC [d] rectangulum aequale quadrato BE: igitur data media trium continue proportonalium, illarumque aggregato; exhibuimus primam & tertiam. Quod erat faciendum.

*c 3. Tertii. ; d 33 Tertii.*

[79]

*Lemma.*

Si fuerint tres BA, DA, CA in continua analogia, rectangulum super maxima AB, & excessu CD secundae supra tertiam, non erit rectus quadrato dimediae ipsius AB.

***Demonstratio.***


**Prop.38. Fig. 2.**

Quoniam enim BA, DA, CA sunt continuae proportionales, erit ut BA [a] ad DA, sic BD ad DC. Quare rectangula [b] BACD & BDA aequantur. Sed rectangulum BDA, non est maius quadrato dimidiae AB [c] ut patet. ergo neque rectangulum BACD maius erit quadrato dimidiae AB. Quod erat demonstrandum.
*a 7 Huius; b 16 Sexti; c 5 Secundi.*

## *L2.§1. PROPOSITION 38.*

Given the mean arising from three lines in a continued ratio, and given the sum of first and third of the three lines put in place, to show the first and the last of the lines.

***Construction & Demonstration.***

The circle ABC is described on the diameter taken as the sum of the first and last terms, in which the line BD is established which is equal to twice the given mean: which can be done from the given sum by construction. The line AC is drawn passing through the centre, and cutting the line BD at right angles in E. I say that what was required has been accomplished. For indeed the line AC is equal to the sum of the terms, and BE (equal to half of BD [c]) is equal to the given mean, whereby as AC is perpendicular to BE, the rectangle AE.EC [d] is equal to the square BE: therefore given the mean of three lengths in continued proportion, and the sum of the first and last of these, we have shown the first and the third. *c 3. Tertii. ; d 33 Tertii..*

[If *a, b,* and *c* are the lengths of three lines in continued proportion, with the mean satisfying $b^2 = ac$. The sum of the first and third proportionals is $a + c$, which is set equal to the diameter AC; while BE is set equal to the mean *b*. From geometry, $BE^2 = AE.EC$, or $b^2 = a.c$. ]

*Lemma.*

If BA, DA, and CA are three lines in continued proportion, the rectangle with the greatest area on AB, and in excess of the second CD on the third, is not larger than half the square of AB itself.

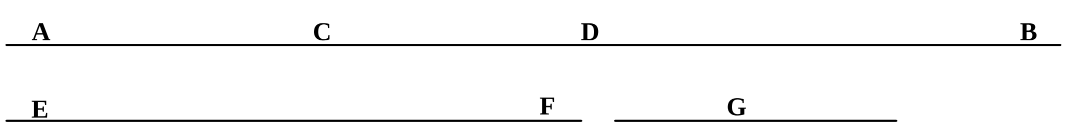
***Demonstration.***

For since BA, DA and CA are continued proportionals, BA [a] will be to DA, thus as BD to DC. Whereby the rectangles [b] BA.CD & BD.DA are equal. But the rectangle BD.DA is not greater than the square of the half of AB [c] as is apparent from the proposition. Therefore also the rectangle BA.CD cannot be greater than the square of half AB. Q.e.d. *a 7 Huius; b 16 Sexti; c 5 Secundi.*

## *PROPOSITIO XXXIX.*

Data maxima trium continuarum, & excessu quo media superat minimam, exhibere mediam & minimam.

***Constructio & Demonstratio.***




**Prop.39. Fig. 1.**

Data sit AB linea maxima trium proportionalium, & G equalia excessui quo media superat minimam : oporteat igitur invenire mediam & minimam. fiat rectangulo ABG aequale quadratum EF: quoniam ergo per lemma praecedentes quadratum EF, id est rectangulum ABG, non maius est quadrato dimidiae AB, patet ita [d] fecari posse AB, ut rectangulum sub partibus, aequale sit quadrato EF. Itaque dividatur recta AB, in puncto D, ut rectangulum ADB aequale sit quadrato EF. Dico punctum D esse quod problema soluit : fiat enim rectae G, aequalis linea DC : erit itaque rectangulum ABCD aequale rectangulo ADB: cum utrumque aequale sit quadrato EF; ergo ut AB ad AD [e] , ita est BD ad DC: si iam a rectangulo ABCD, auferis rectangulum BDC, remanebit rectangulum ADC; si vero a rectangulo ADB, auferas item BDC, reliquum erit rectangulum ACDB: atqui tota ABCD, ADB sunt aequalia , itaque oblato communi, erunt & reliqua rectangula ADC, ACDB aequalia inter se; igitur ut BD ad DC, id est [f] (sicut iam ostendi) ut BA ad DA, sic DA ad CA. Sunt igitur tres in continua analogia AB, AD, AC; & CD aequalis G, est excessus, quo media AD, excedit minimum AC. Factum igitur est quod requirabatur.

*d 5 Secundi; e 16. Sexti ; f 14 Sexti.*

## *L2.§1.* *PROPOSITION 39.*

From the given maximum of three lines in continued proportion, and the excess of the mean over the mininum, to find the mean and the minimum.

### *Construction & Demonstration.*

The maximum AB is given of the three lines in proportion, and G is equal to the excess of the mean over the mininum: it is required to find the mean and the minimum. The rectangle AB.G is set equal to the square EF: therefore, according to the previous proposition, the square EF, or the rectangle AB.G, is not greater than the square of half of AB, it is thus apparent that AB can be divided in order that the rectangle under the sections is equal to the square EF. Thus the line AB is divided in some point D in order that the rectangle AD.DB is equal to the square EF. I say that the point D is the solution to the problem. For the line G is made equal to the line DC: and thus the rectangle AB.CD is equal to the rectangle AD.DB, as each is equal to the square EF; therefore as AB is to AD[e], so BD is to DC. If now from rectangle AB.CD you take away the rectangle BD.DC then the rectangle AD.DC remains; if indeed from rectangle AD.DB likewise BD.DC is taken away, then the remainder is the rectangle AC.DB: but the whole amounts AB.CD and AD.DB are equal, and hence on taking away the common amount, the remaining rectangles AD.DC and AC.DB are equal to each other. Therefore as BD is to DC, or [f] as BA is to DA (as now shown), thus DA to CA. The three lines AB, AD, and AC are hence in continued proportion, and CD equal to G, is the excess by which the mean AD exceeds the minimum AC. Hence what was required has been done.

*d 5 Secundi; e 16. Sexti ; f 14 Sexti.*

[Essentially the argument is the converse of AC/AD = AD/AB giving CD/AD = DC/AB and AD.DB = $EF^2$ = AB.CD.]

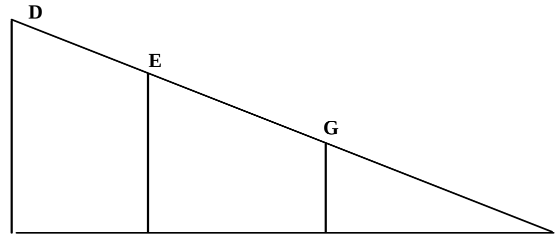
## *PROPOSITIO XL.*

Datis duobus excessibus, trium magnitudinum in continua analogia existentium, exhibere tres continuas.

### *Constructio & Demonstratio.*

Dati sint excessui AC, CB, qui ponantur in directum, & erigantur AD, CE parallelae, quae inter se eandem rationem servent quam AC ad CB; & ducta per puncta D & E recta DE, conveniat cum AB producta in puncto quodam F. Dico factum quod postulatur :

**[80]**



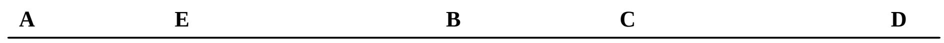

**Prop.40. Fig. 1.**

nam AD ad CE, eandem habeat rationem, quam linea AF ad CF. Sed quam rationem habet DA ad CE, eandem per constructione

habet AC ad CB; igitur quam rationem habet tota AF ad totam CF, eandem habet AC ablata ad CB ablatam. Ergo ut AF tota ad CF totam ac quoque [a] CF reliqua ad reliquam BF. Quare rectae AB adiuncta est linea BF, quae factiat AF, CF, BF in continua proportione; quod petebatur.
*a 19 Quinii.*

*Aliter.*

Cum huic constructio solis lineis conveniae subiungamus aliam, quae in omni genere quantitatis locum habeat:




**Prop. 40. Fig.2.**

Dati igitur sint excessus AB : BC : fiat AE differentia datorum excessuum, ipsisque AE, AB inveniatur tertia proportionalis continua AD : Dico AD absuluere problemata. Cum enim ex constructione AE sit AB, ut AB ad AD, erit dividenfo AE ad EB, ut AB ad BD; & componendo AB erit ad BE, hoc est BC ut AD ad BD; & permutando AB ad AD, ut BC ad BD; & dividendo AB ad BD ut BC ad CD; componendo igitur AD, BD, CD sunt continuae. Fecimus ergo quod petebatur. *a 19 Quinti.*

## *L2.§1. PROPOSITION 40.*

With the two differences given of three magnitudes present in continued proportion, to show the three continued proportions.

### *Construction & Demonstration.*

For the differences AC and CB are given, which are placed along a line, and the parallel lines AD and CE are erected which maintain the same ratio AC to CB between themselves; and the line DE is drawn through the points D and B and meets the line AB produced in some point F. I say that what was required has been done: for AD to CE has the same ratio as AF to CF. But DA to CE is in the same ratio as AC to CB by construction; therefore as the whole has the ratio AF to CF, AC taken to CB taken has the same ratio. Hence as the total AF to the total CF also the ratio of the remainder CF to the remainder BF. Whereby the line BF is added on to the line AB, which makes AF, CF, and BF in continued proportion, as sought.
*a 19 Quinii.*
[From CE/AD = CF/AF and CE/AD = CB/AC we have AF/CF = AC/CB or AF/AC = CF/CB, giving CF/AC = BF/CB or CB/AC = BF/CF = AF/CF as required.]

*In another way.*

As to this construction involving a single line, we can add another line for which there is a place for all kinds of quantities.

The ratio of the differences AB : BC is therefore given, and AE is the difference of the given differences. From AE and AB themselves the third of the continued proportionals AD can be found. I say that AD solves the problem. As indeed from construction, AE is to AB as AB is to AD; by subtraction, AE is to EB , as AB is to BD ; and by addition, AB is to BE, or BC , as AD is to BD; and on interchanging, AB is to AD as BC is to BD; and on subtraction, AB is to BD as BC is to CD; on addition, AD, BD, and CD are therefore in continued proportion. Thus, we have accomplished what was sought.

[For, if $a$, $b$, and $c$ are three numbers in continued proportion with $a > b > c > 0$, then $a/b = b/c$ , and $(a - b)/b = (b - c)/c$ or $d_1/b = d_2/c$ i.e. $d_1/d_2 = b/c$; then again, $(d_1 - d_2)/d_2 = (b - c)/c = d_2/c = d_1/a$. In this case, AB = $d_1$, BC = $d_2$, and AE = $d_1 - d_2$ : hence AE/BC = BC/AD, giving AD; or, if you wish, $d_2/d_1 = b/a$; from which AE/AB = AB/AD as in the *aliter*, again giving AD. It then follows that EB/AB = BD/AD giving BD, while AB/AD = EB/BD = BC/BD, as EB = AB - AE = AB - AB + BC; etc.]

## *PROPOSITIO XLI.*


**Prop. 41. Fig.1.**

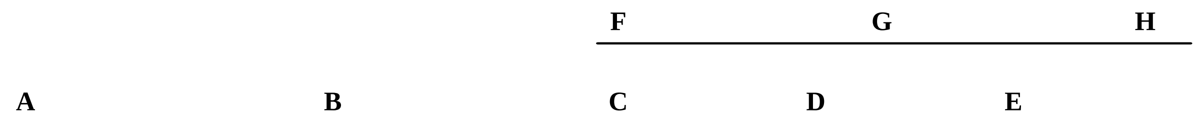
Datam magnitudinem AE semel sectam in C, proportione maioris inaequalitatis, ita in duobus aliis punctis B & D subdividere, ut quatuor partes AB, BC, CD, DE sint in continua proportiones; quae dimidiata sit rationis AC ad CE, ex prima divisione ortae.

***Constructio & Demonstratio.***

Per quadragesimam huius addatur EF, ut AF, CF, EF sint proportionales; tum inter AF, CF, media ponatur BF : inter CF quoque & EF, media DF. Dico factum quod petebatur. cum enim, inter tres continuas AF, CF, EF, mediae sint BF, DF, paret ex elementis omnes AF, BF, CF, DF, EF, esse continuae proportionales. Quare etiam AB, BC, CD, DE sunt b continuae, & quidem in ratione AF ad BF, quae ex constructione dimidiata est rationis AF ad CF, hoc est AC ad CE. Factum igitur est quod petabatur.

*b 7 Huius; c Ibid.*

*Aliter.*




**Prop. 41. Fig.2.**

Inter AC, CE inventam mediam FH ita divisae, ut FG sit ad GH, ut AC ad FH, seu FH ad CE : fiantque CB, CD ipsis FG, GH aequales. Dico factumquod petitur. Cum enim sit AC ad FH, ut FG ad GH, id est BC ad CD, erit quoque AC ad BD (quae ex constructione aequalis est FH) ut BC ad CD: & permutando AC ad BC ut BD ad CD: & dividendo AB ad BC, ut BC ad CD. Sunt igitur AB, BC, CD tres continuae proportionales in ratione BC ad CD. Similiter cum FH sit ad CE, ex constructione ut FG ad GH, erit quoque BD ad CE, ut BC ad CD, ac permutando convertendo BD ad CD, ut CB ad DB: Ideoque dividendo BC est ad CD, ut CD ad DB. Sunt igitur tres continuae BC, CD, DE in ratione BC ad CD : ac proinde omnes quatuor sunt continuae proportionales in ratione BC ad CD, id est FG ad GH, id est AC ad FH, quam ex

**[81]**

constructione dimidiata est rationis AC ad CE; fecimus ergo quod fuerat propositum.

*Corollarium.*

Ex hoc problemate licebit praxim desumere, non solum subdividendi duas in quatuor continuas, sed etiam in sex continuas; imo quotvis datas in duplo plures continuas, & quidem in ratione dimidiatis eius, in qua ipsae existunt.

## *L2.§1.* *PROPOSITION 41.*

Given the magnitude AE cut once in C in a proportion greater than one, which is thus to be subdivided by two other points B and D, in order that the four parts AB, BC, CD, and DE are in continued proportion, in a ratio which is the root of the ratio AC to CE arising from the first subdivision.

***Construction & Demonstration.***

By the 40th proposition of this book, EF is added in order that AF, CF, and EF are proportionals; then the mean BF is placed between AF and CF, and the mean DF is placed between CF and EF. I say that what was required has been done. For indeed , between the three continued proportionals AF, CF, and EF, the means are BF and DF, and it is apparent from the fundamentals that all of AF, BF, CF, DF, and EF are continued proportions. Whereby also AB, BC, CD, and DE are continued proportionals [b] , and indeed these are in the ratio AF to BF, which by construction is the square root of the ratio AF to CF, or AC to CE. Thus what was required has been done.

*b 7 Huius; c Ibid.*

*In Another Way.*

The mean FH is found between AC and CE, thus as AC is to FH, so FH to CE : and thus by division, AC is to FH as FG is to GH and CB, CD are made equal to FG, GH themselves. I say that what was sought has been done. As indeed AC is to FH as FG to GH, or BC to CD; also AC to BD ( which by construction is equal to FH) is as BC to CD : and by rearranging, AC is to BC as BD to CD. On subtracting, AB is to BC as BC is to CD. Therefore there are three proportionals AB, BC, CD in the ratio BC to CD. Similarly, as FH is to CE, by construction as FG is to GH, also BD is to CE as BC is to CD [For: BD/CE = BC/CD; BD/BC = CE/CD; CD/BC = DE/CD; BD/BC . BC/CD = CE/CD . CD/DE or BD/CD = CE/DE]; and on rearranging and multiplying, as BD is to CD, thus CE is to DE. Thus on subtracting, BC is to CD, as CD is to DE. There are hence three continued proportions in the ratio BC to CD : and thus all four are continued proportionals in the ratio BC to CD, or FG to GH, or AC to FH, which by construction is the square root of the ratio AC to CE; we have therefore done what was proposed.

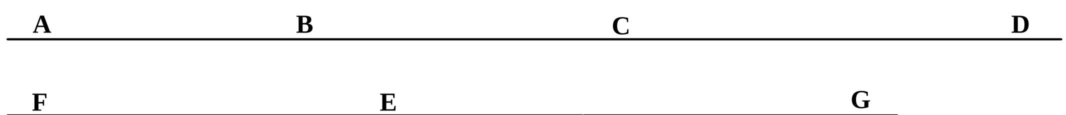
*Corollory.*

From this problem you can get some practise, not only in the subdivision by two into four continued proportions, but also into six; and indeed for whatever number into twice as many continued proportions, and they are in the ratio of the square root of that ratio in which these themselves are present.

## *PROPOSITIO XLII.*

Sint AB, BC, CD in continua analogia; deinde secetur quaepiam linea FG in E, ut FE, ad EG, eadem habeat rationem, quam AB ad BC.

Dico rectangulum BCFG, aequale esse duobus ABEG & EFCD rectangulis.




**Prop.42. Fig. 1.**

***Constructio & Demonstratio.***

Cum enim sit ut AB ad BC, ita FE ad EG, rectangulum BCEF, [a] aequale est rectangulo ABEG. Similiter, quia ut BC ad CD, ita FE est ad EG, erit etiam rectangulum BCEG aequale rectangulo CDFE. Cum igitur BCGF, BCEG rectangula, id est [b] rectangulum BCFG, aequalia sunt duobus ABEG, CDFE rectangulis. Quod erat ostendendum. *a 16 Secundii; b 2 Secundii ?*

## *L2.§1. PROPOSITION 42.*

AB, BC, and CD are in a continued ratio; then some line FG is cut in E, in order that FE to EG has the same ratio as AB to BC.

I say that the rectangle BC.FG is equal to the sum of the two rectangles ABEG and EFCD.

***Constructio & Demonstratio.***

Since indeed as AB is to BC as thus FE to EG, then the rectangle BC.EF, [a] is equal to the rectangle AB.EG. Similarly, since BC is to CD, thus as FE is to EG, also the rectangle BC.EG is equal to the rectangle CD.FE. Therefore as the sum of the rectangles BC.EF and BC.EG , that is the [b] rectangle BC.FG, is equal to the sum of the two rectangles AB.EG and CD.FE. Which it was required to show. *a 16 Secundii; b 2 Secundii ?*

## *PROPOSITIO XLIII.*

Si fuerint quotvis continuae proportionales AB, CB, DB, EB, &c. Dico rectangula ABEF, CBDE, DBCD, EBAC esse inter se aequalia. Hoc est rectangula sub lineis seriei AB, CB, DB, &c. & sub residuis EF, DE, CD, AC esse inter se aequalia; modo retrograde coniungantur.


**Prop.43. Fig. 1.**

***Demonstratio.***

Quonium est ut AB ad [c] CB, sic DE ad EF, rectangulum ABEF aequatur [d] rectangulo CBDE. Similiter, quia ut CB ad DB, ita CD est ad DE, erit rectangulum CBDE aequale rectangulo DBCD. rursus quoniam DB est [e] ad EB ut AC ad CD, rectangulum DA/BCD? aequalia est ..... EB..AC: aequalia sunt igitur omnis inter se. Quod erat demonstrandum. *c 16 Huius; d 16 Sexti ; e 1 Huius.*

## *L2.§1. PROPOSITION 43.*

AB, CB, DB, EB, etc. are some lines in continued proportion. I say that the rectangles AB.EF, CB.DE, DB.CD, and EB.AC are equal to each other. That is the rectangles under the lines of the series AB, CB, DB, &c. and under the differences EF, DE, CD, AC are equal to each other; joined together in a backward manner.

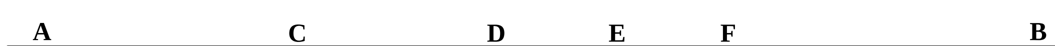
***Constructio & Demonstratio.***

Since indeed as AB is to [c] CB, thus DE ad EF, then the rectangle AB.EF is equal to the [d] rectangle CB.DE. Similarly, because CB is to DB, thus as CD is to DE, then the rectangle CB.DE is equal to the rectangle DB.CD. Again, since DB is to [e] EB as AC is to CD, then the rectangle DB.CD is equal to the restangle EB.AC: all the rectangles are therefore equal to each other. Q.e.d. *c 16 Huius; d 16 Sexti ; e 1 Huius.*

## *PROPOSITIO XLIV.*

Iisdem positis:
Dico rectanguli EBAC, DBCD, CBDE esse inter se aequalia.

***Demonstratio.***




**Prop.44. Fig. 1.**

Cum enim ut CB ad DB, ita CD[f] sit ad DE ; igitur rectangulum CBDE aequale est[g] rectangulo DBCD. Sed rersum ut DB ad EB, sic AC ad[h] CD, quare etiam rectangulum DBCD, aequale erit rectangulo EBAC; ostensum autem fuit rectangulum CBDE, esse rectangulo DBCD aequale, quare & haec duo EBAC & CBDE, proindeque omnia tria inter se erunt aequalia. Quod erat demonstrandum. *f ibid; g 16 Sexti ; h 1 Huius.*

**[82]**

Pari ratione si ponatur ulterius produci series progressionis, ita ut AB, CB, DB, EB, FB, &c. ponatur continuae proportionales, demonstrari poterit quatuor rectangula FBAC, EBCD, DBDE, & CBEF inter sese aequalia esse.

## *L2.§1.* *PROPOSITION 44.*

With the same points in place:

I say that the rectangles EB.AC, DB.CD, and CB.DE are equal to each other.

***Demonstratio.***

Since indeed as CB is to DB, so CD[f] is to DE ; and therefore the rectangle CB.DE is equal[g] to the rectangle DB.CD. But again, as DB is to EB, thus AC is to[h] CD, whereby also the rectangle DB.CD is equal to the rectangle EB.AC; but is has been shown that the rectangle CB.DE is equal to the rectangle DB.CD, and whereby these two EB.AC and CB.DE are equal, and hence all three are equal to each other. Q.e.d. *f ibid; g 16 Sexti ; h 1 Huius.*

If the series of the progression is produced further in the same ratio, thus in order that AB, CB, DB, EB, FB, &c. are continued proportionals, it can be show that the four rectangles FB.AC, EB.CD, DB.DE, & CB.EF are equal to each other.

## *PROPOSITIO XLV.*


**Prop.45. Fig. 1.**

Sint AB, CB, DB, EB in continua analogia, &c.

Dico quinque rectangula inter se aequalia esse; quorum primum est illud, quod sub AB, BC tanquam una linea, & sub DE continetur : alterum, quod describitur a CB, & DB tanquam una, ac recta CD; tertium, quod a DB & EB tanquan una, & recta AC conficitur: quartum, quod ab EC, & CB; denique illud quod ab AD, DB describitur.

***Demonstratio.***

Rectangulum enim ABDE aequale est[a] rectangulo CBCD; & rectangulum CBDE aequale est[b] rectangulo DBCD : duo igitur rectangula ABDE, & CBDE, id est rectangulum sub ABCB tanquam una, & D... aequale sunt duobus CBCD & DBCD, id est contento sub CBDB tanquam una, & sub recta CD: eodem modo ostendam sub DBEB & AC contentum, aequari prioribus; de reliquis quoque idem simili discursu demonstrabitur. Constat ergo veritas propositionis. *a 43 Huius; b Ibid.*

.

## *L2.§1.* *PROPOSITION 45.*

The lines AB, CB, DB, EB are in continued proportion, etc.

I say that there are five rectangles equal to each other, the first of which is that which is contained under the sum of AB and BC regarded as one line, and DE; the second is described from the sum of CB and DB as one line, and the line CD; the third is made from the sum of DB and EB as one line, and the line AC; the fourth from EC and CB; and then that which is desctibed from AD and DB.

***Demonstratio.***

Indeed the rectangle AB.DE is equal to the [a] rectangle CB.CD; and the rectangle CB.DE is equal to the [b] rectangle DB.CD : therefore the two rectangles AB.DE, and CB.DE, that is the rectangle AB.CB as one line, and DE is equal to the two CB.CD and DB.CD, that is contained by CBDB as one line, and the line CD: in the same manner I can show that the rectangle contained by DB.EB and AC is equal to the previous rectangles; and concerning the remaining rectangles a similar argument can be shown. Hence the truth of the proposition can be agreed upon. *a 43 Huius; b Ibid.*

## *PROPOSITIO XLVI.*


**Prop.46. Fig. 1.**

Sint continuae proportionales AB, CB, DB, EB, &c.

Dico rectangula ABCE, CBAD, & duo simul sumpta ACB, ACDB , denique trium aggregatum , scilicet quadrato CE, rectanguli ACE, & rectanguli CEB, esse inter se aequalia.

***Demonstratio.***

Quoniam est ut AC ad CD, ita CD ad DE, erit componendo & alternando, ut AD ad CE, sic CD ad DE: sed ut CD ad DE, ita est AB [c] ad CB : igitur ut AD ad CE, sic est AB ad CB, & rectangulum [d] ABCE aequalia erit rectangulo CBAD. Insuper quia est ut AB ad CB, ita AC ad CD, erit rectangulum ABCD, aequale rectangulo AC,CB. Quia vero ex aequo etiam est ut AC ad DE, ita AB ad DB, erit quoque rectangulum ACDB, aequale [e] sunt rectangulo ABCE: rectangulum igitur ABCE, aequale etiam est duobus ACB & AC DB. Deinde quia recta AB secta est in C & E, erit [f] ABCE aequale tribus CE in CA, & CE in CE ducto, (hoc est quadrato CE) & eidem CE in EB ducto: aequalia sunt igitur inter se. Quod fuerat demonstrandum. *c 1 Huius; d 16 Sexti; e 1 Secundi; f 1 Secundi.*

.

## *L2.§1.* *PROPOSITION 46.*

The lines AB, CB, DB, EB, &c. are continued proportionals

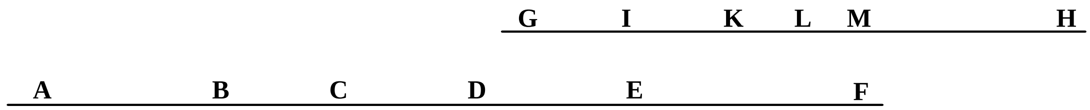
I say that the rectangles AB.CE, CB.AD, as well as the sum of the two rectangles AC.CB and AC.DB, and again the sum of the three rectangles AC.CE, AC.CE & CE.EB, are all equal to each other.

***Demonstration.***

Since AC is to CD thus as CD is to DE then by adding and rearranging, as AD is to CE, thus as CD is to DE: but as CD is to DE, thus AB [c] is to CB : therefore as AD is to CE, thus AB is to CB, & the rectangle [d] AB.CE is equal to the rectangle CB.AD. In addition, since AB is to CB, thus as AC is to CD, then rectangle AB.CD is equal to rectangle AC.CB. Whereby also from equality, AC is to DE, thus as AB is to DB, and the rectangle AC.DB too is equal to the [e] to the rectangle AB.CE: therefore the rectangle AB.CE is also equal to the sum of the two AC.CB & AC.DB. Hence as the line AB is divided in C & E, [f] AB.CE is equal to the sum of the three CE by CA, CE by CE, (or the square CE) & CE by EB: these sums of rectangles are therefore equal to each other. Q.f.d. *c 1 Huius; d 16 Sexti; e 1 Secundi; f 1 Secundi.*

[From Prop. 1 of this book, as AC/CD = CD/DE; then AD/CD = CE/DE giving AD/CE = CD/DE ; again, AB/CB = CD/DE = AD/CE : hence <u>rect. AB.CE = rect. CB.AD</u>. Again, as AB/CB = AC/CD, then rect. AB.CD = rect. AC.CB. Again, as AC/DE = AB/DB, then rect. AC.DB = rect. AB.DE; but rect. AB.DE + rect. AB.CD = rect. AB.CE; hence also, rect. AB.CE = rect. AC.CB + rect. AC.DB (the under-lined pairs). Hence, as AB = AC + CE + EB, then rect. CE.AB = rect. CE.AC + rect. CE.CE + rect. CE.EB = rect. AC.CB + rect. AC.DB . The rectangles under-lined are those sought.]

## *PROPOSITIO XLVII.*




**Prop.47. Fig. 1.**

Sint in continuae analogia minoris inaequalitatis AB, AC, AD, AE, AF, &c. & totidem aliae maioris inaequalitatis eiusdem seriei, GH, IH, KH, LH, MH, &c.

**[83]**

Dico rectangula ABGH, ACIH, item ADKH, AELH, AFMH, &c. esse omnia inter se aequalia.

***Demonstratio.***

Ex datis ut AB ad AC, sic IH ad GH, ergo rectangulum [a] ABGH aequatur rectangulo ACIH. rursum ex datis ut AC ad AD, sic KH ad IH; ergo rectangulum ACIH [b] rectangulo ADKH aequale erit. Simili discursu reliquorum aequalitatem ostendimus. est igitur quod fuerat demonstrandum. *a 16 Sexti; b Ibid.*

***Corollarium.***

Quoniam per primam huius positis continue proportionalibus AB, AC, AD, AE, AF, itemque GH, IH, KH, LH, MH, earum differentiae FE, ED, DC, &c. GH, IK, KL, &c. sunt etiam in ratione continua, manifeste patet eadem discurrendi methodo demonstrati rectangula FELM, DEKL, CDIK, BCGI, quoque inter se aequalia esse.

## *L2.§1. PROPOSITION 47.*

Let AB, AC, AD, AE, AF, etc. be a series of lesser inequalities in continued proportion [i.e. from right to left], and GH, IH, KH, LH, MH, etc. the whole of another series of greater inequalities [i.e. from left to right] of the same series.

I say that the rectangles AB.GH, AC.IH, likewise AD.KH, AE.LH, AF.MH, etc. are all equal to each other.

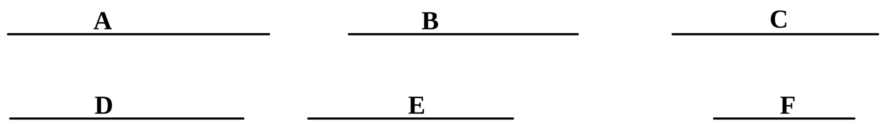
***Demonstration.***

From the given ratio, as AB is to AC, thus IH is to GH, and therefore the rectangle [a] AB.GH is equal to the rectangle AC.IH. Again, as AC is to AD, thus KH is to IH; hence the rectangle AC.IH [b] is equal to the rectangle AD.KH. We can demonstrate the rest of the equalities by a similar argument. This is what had to be shown. *a 16 Sexti; b Ibid.*

***Corollary.***

Since from the positions of the continued proportions from the first part of this proposition: AB, AC, AD, AE, AF; and in the same manner GH, IH, KH, LH, MH, then the differences FE, ED, DC, etc , and GI, IK, KL, etc of these, are also in a continued ratio. It is clearly obvious from the same kind of discussion that the rectangles FE.LM, DE.KL, CD.IK, BC.GI, can also be shown to be equal to each other.

## *PROPOSITIO XLVIII.*




**Prop.48. Fig. 1.**

Sint duae series quotcumque continuarum eiusdem rationis, A, B, C, G, H; D, E, F, I, K: sit autem rectangulum AB, aequale DE rectangulo:

Dico etiam rectangula BC, EF, CG, FI, GH, IK; & sic deinceps aequalia esse.

***Demonstratio.***

Proportio rectanguli AB, ad rectangulum DE, componitur [c] ex rationibus A ad D , & B ad E; sed rectanguli BC ad rectangulum EF, proportio quoque componitur ex rationibus B ad E, & C ad F, hoc est A ad D (nam cum ex datis & ex aequo sit A ad C, ut D ad F, erit permutando A ad D, ut C ad F) ergo ex iisdem rationibus componuntur proportiones rectangulorum AB, DE: BC, EF, adeoque eadem sunt. quare cum ratio rectangulorum AB, DE ponatur aequalitatis, rectangulorum quoque BC, EF aequalitatis; eodem modo reliqua reliquis ostendentur aequalia. Quod erat demonstrandum. *c 23 Sexti.*

## *L2.§1.* *PROPOSITION 48.*

A, B, C, G, H; D, E, F, I, K are any two series in continued proportion with the same ratio: however, the rectangle A.B is equal to the rectangle D.E:

I say that the rectangles B.C, E.F, C.G, F.I, G.H, I.K and so on are also equal to each other.

***Demonstration.***

The proportion of the rectangle A.B to the rectangle D.E, is composed [c] from the ratios A to D , & B to E; but the proportion of rectangle B.C to rectangle E.F, is also composed from the ratios B to E, & C to F, that is A to D (for from what is given and from the equality A is to C thus as D is to F, then on rearranging, A is to D as C is to F) hence the proportions of the rectangles A.B, D.E: B.C, E.F are composed from the same ratios. Whereby as the ratio of the rectangles A.B and D.E is put equal to one, the rectangles B.C and E.F are also equal to each other; and in the same manner the rest of what remains can be shown to be equal. Q. e.d. *c 23 Sexti. a 16 Sexti; b Ibid.*

## *PROPOSITIO XLIX.*


**Prop.49. Fig. 1.**

Si rectangula AB, BC, CG,GH; & DE, EF, FI, IK singula singulis sint aequalia:

Dico esse A ad C, ut D ad F, & B ad G, ut E ad I, & sic deinceps.

**_Demonstratio._**

Cum rectangulum AB, aequale sit rectangulo DE, & rectangulum BC, rectangulo EF; ergo ut rectangulum AB ad DE, sic BC ad EF: & permutando

**[84]**

ut AB ad BC, sic DE ad EF aequi rectangulum AB ad BC, est ut A ad C, & DE ad EF est ut D ad F; ergo A ad C, ut D ad F. Eodem discursu erit B ad G, ut E ad I. Quod erat demonstrandum.

## *L2.§1. PROPOSITION 49.*

If the individual rectangles in the two series AB, BC, CG,GH; & DE, EF, FI, IK are equal to each other term by term:

I say that A is to C as D is to F, & B is to G, as E is to I, & thus henceforth.

**_Demonstration._**

Since rect. A.B is equal to rect. D.E, & rect. B.C is equal to rect. E.F; it follows that as A.B is to D.E, thus B.C is to E.F: & on interchanging, as A.B is to B.C, thus D.E is to E.F; and the ratio A.B to B.C is equal to the ratio A to C, & D.E to E.F is equal to D to F; hence A is to C, as D is to F. By a similar argument, B is to G, as E is to I. Q.e.d.

## *PROPOSITIO L.*

Si rectangula AB, BC, CG,GH, &c. rectangulis DE, EF, FI, IK singula singulis sint aequalia:

Dico utramque laterum seriem A, B, C, G, H; & D, E, F, I, K, si ponantur esse continuae proportionales, esse quoque continuas eiusdem rationis.


**Prop.50. Fig. 1.**

**_Demonstratio._**

Per praecedentem est A ad C, ut D ad F: quia autem tam A, B, C, quam D, E, F, sunt continuae proportionales ex datis, erit tam ratio A ad C, (id est D ad F), rationis A ad B duplicata; quam ratio D ad F (id est A ad C) duplicata sit rationis D ad E. Quare ut A ad B, sic D ad E; & B ad C, ut E ad F, &c. sunt igitur A, B, C, G, H, & D, E, F, I, K continuae eiusdem proportionis. Quod erat demonstrandum.

## *L2.§1. PROPOSITION 50.*

If the rectangles A.B, B.C, C.G, G.H, &c. and the rectangles D.E, E.F, F.I, I.K are equal to each other term by term:

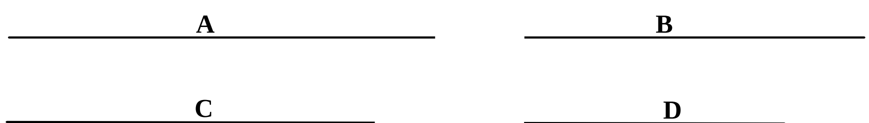
I say that each term of the series A, B, C, G, H; & D, E, F, I, K, if they are placed in continued proportion, are in the same ratio.

**_Demonstration._**

By the preceding proposition, A is to C as D is to F: but since A, B, C, and D, E, F, are given as continued proportionals, so the ratio A to C, (or D to F) is the square of the ratio A to B ; just as the ratio D to F (or A to C) is the square of the ratio D to E. Whereby as A is to B, thus D is to E are in the same ratio; & so B is to C as E is to F, &c. Therefore A, B, C, G, H, & D, E, F, I, K are continued in the same proportion. Q.e.d.

## *PROPOSITIO LI.*

Si prima A ad secundam B, eamdem habeat rationem, quam tertia C ad quartam D: Dico tria rectangula ex hisce facta, esse in continuata proportione; nempe rectangula AB, BC, & CD.




**Prop.51. Fig. 1.**

***Demonstratio.***

Rectangulum AB, est ad rectangulum BC, ut A ad C. sed etiam rectangulum BC, ad [a] CD, (ob eandem rationem) est ut B ad D. cum igitur sit ratio A ad C eadem cum ratione B ad D; cum quoque rationem habet rectangulum AB, ad BC, quam habet BC ad CD rectangulum : sunt igitur in continuate rationis, prout erat demonstrandum. *a 1 Sexti.*

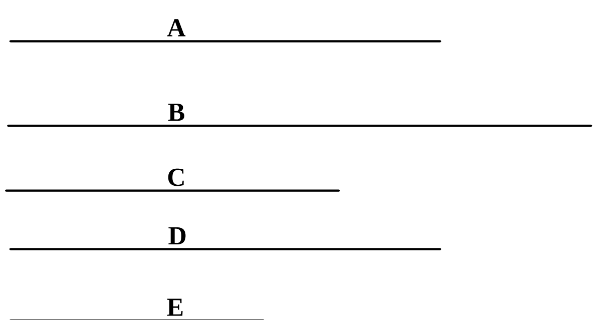
## *L2.§1.* *PROPOSITION 51.*

If the first A has the same ratio to the second B, as the third C to the fourth B: I say that the three rectangles made from these, namely AB, BC, & CD, are in a continued proportion.

***Demonstration.***

The rect. AB is to rect. BC, as A is to C; but also the rect. BC is to the rect. [a] CD, (on account of the same ratio) is as B to D. Therefore since the ratio A to C is the same as the ratio B to D; and since too the rect. AB has the same ratio to the rect. BC , as the rect.BC has to the rect.CD: hence the three rectangles are in continued proportion, as was to be shown. *a 1 Sexti.*

## *PROPOSITIO LII.*




**Prop.52. Fig. 1.**

Si prima A ad secundam B eandem habeat rationem, quam tertia C ad quartam D, fiatque ut prima A ad tertiam C, ita tertia C ad quintam E Dico rectangula ex his lineis constituta, esse in continuata proportione; nempe rectangula AB, BC, CD, & DE.

***Demonstratio.***

Rectangulum AB ad rectangulum BC, habeat eandem rationem quam A ad C; sed ratio A ad C,

**[85]**

eadem est , cum ratione C ad E ex constructione, igitur ratio rectanguli AB, ad BC, eadem est cum ratione C ad E; sed quoniam ut A est ad B, ita C ad D, erit permutando A ad C, ut B ad D: ergo ratio rectanguli AB ad BC, est ratio B ad D: sed rectangulum BC ad CD, etiam est [a] ut B ad D ; ergo ratio AB rectanguli ad rectangulum BC, eadem est cum ratione rectanguli BC , ad CD rectangulum. est autem ratio B ad D , hoc est A ad C, eadem quae est C ad E; unde etiam ratio rectanguli CD ad DE rectangulum, eadem est cum ratione rectanguli BC, ad CD rectangulum; quocirca in continua analogia sunt rectanguli AB, BC, CD, DE, cum sunt in ratione A ad C. Constat igitur veritas propositionis. *a Ibid.*

## *L2.§1.* *PROPOSITION 52.*

If the first A to the second B has the same ratio, as the third C to the fourth D, and the first A is made to the third C, thus as the third C to the fifth E :

I say that the rectangles constituted from these lines, surely the rectangles AB, BC, CD, & DE are in continued proportion.

***Demonstration.***

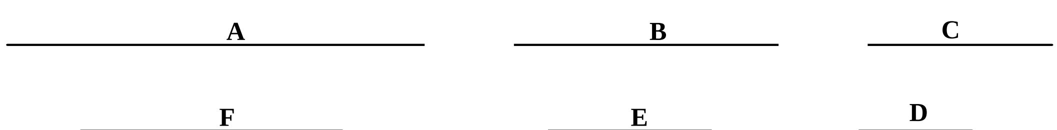
Rect. AB to rect. BC, has the same ratio as A to C; but ratio A to C is the same as the ratio C to E from the construction, therefore the ratio of rect. AB to rect. BC is the same as the ratio C to E; but since A is to B as C is to D, it becomes on interchanging, A to C as B to D: hence the ratio of rect.AB to rect.BC is in the ratio B to D: but rect.BC to rect.CD, is also as [a] B to D ; hence the ratio of rect. AB to rect.BC is the same as the ratio of rect.BC to rect.CD. But the ratio B to D, or A to C, is the same as that which C has to E; hence also the ratio of rect.CD to rect.DE, is the same as the ratio of the rect.BC to the rect.CD ; wherefore the rectangles AB, BC, CD, DE are in continued proportion, as they are in the ratio A to C. Therefore the truth of the proposition is agreed. *a Ibid.*

[AB/BC = A/C = C/E; also, A/B = C/D, giving A/C = AB/BC = B/D = BC/CD = CD/DE as required.]

## *PROPOSITIO LIII.*

Sint A, B, C, tres in continua ratione. Sintque D, E, F, in eadem vel diversa continuata ratione:

Dico rectangula CD, CE, CF; item BD, BE, BF: item AD, AE, AF esse in continuata analogia.




**Prop.53. Fig. 1.**

***Demonstratio.***

Rectangulum CD, est ad rectangulum CE, ut D [b] linea est ad E : & CE rectangulum est ad rectangulum CF, ut E ad F : sed D, E, F ex hypothesi sunt continuae proportionales; ergo & rectangula CD, CE, CF sunt in continua analogia. Eodem modo probantur BD, BE, BF rectangula, item AD, AE, AF esse continue proportionalia. Quod erat demonstrandum. *b 1 Sexti.*

## *L2.§1.* *PROPOSITION 53.*

A, B, and C are three quantities in a continued ratio; while D, E, F are three other quantities in the same or in another ratio.

I say that the rectangles CD, CE, CF; likewise BD, BE, BF: and likewise AD, AE, AF are in continued ratios.

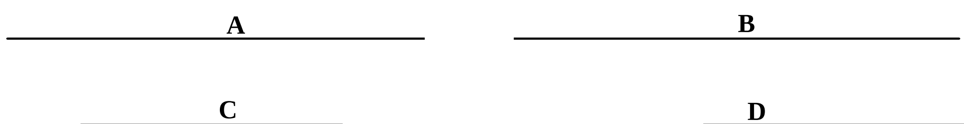
***Demonstration.***

Rect. CD is to rect. CE, as the line D [b] is to the line E : & rect. CE is to rect. CF, as E is to F : but D, E, F from hypothesis are continued proportionals; & hence the rectangles CD, CE, CF are in a continued ratio. In the same manner it can be agreed that the rectangles BD, BE, BF ; and likewise the rectangles AD, AE, AF are continued proportionals, which was to be shown. *b 1 Sexti.*

## *PROPOSITIO LIV.*

Sit A prima ad B secundam, ut C tertia ad D quartam.

Dico quadratum sub prima, rectangulum sub secunda & tertia, & quadratum quartae, in continua esse analogia.




**Prop.54. Fig. 1.**

***Demonstratio.***

Ratio quadrati A ad rectangulum CB, [c] componitur ex ratione A ad B (hoc est C ad D) : & ex ratione A ad C. Sed ratio rectanguli BC ad D quadratum, composita est ex iisdem rationibus; nam ratio rectanguli BC ad D quadratum, componiter ex ratione B ad D, hoc est A ad C, & ex ratione C ad D, hoc est A ad B; ergo sunt continuae quantitates, quadratum A, rectangulum BC, & quadratum D. Quod erat demonstrandum.
*c 23 Sexti.*

## *L2.§1.* *PROPOSITION 54.*

Let the first A be to the second B, as the third C to the fourth D.

I say that the square under the first, the rectangle under the second and the third, and the square under the fourth are in continued proportion.

***Demonstration.***

The ratio of the square A to the rectangle CB, [c] is composed from the ratio of A to B (that is C to D): & from the ratio A to C. But the ratio of the rectangle BC to the square D, is composed from the same ratios; for the ratio of the rectangle BC to the square D, is composed from the ratio B to D, that is A to C, & from the ratio C to D, that is A to B; hence the square A, the rect. BC, & the square D are continued quantities: which was to be shown.
*c 23 Sexti.*

## *PROPOSITIO LV.*

Sint tres lineae AB, BC, CD in continua analogia.

Dico AB quadratum primae, rectangulum ABC, sub prima & secunda, quadratum BC sub secunda; rectangulum BCD sub secunda & tertia; CD quadratum tertia, esse in serie eiusdem rationis AB ad BC.


**Prop.55. Fig. 1.**

**[86]**

***Demonstratio.***

Ratio quadrati A ad rectangulum CB, [c] componitur ex ratione A ad B (hoc est C ad D) : & ex ratione A ad C. Sed ratio rectanguli BC ad D quadratum, composita est ex iisdem rationibus; nam ratio rectanguli BC ad D quadratum, componiter ex ratione B ad D, hoc est A ad C, & ex ratione C ad D, hoc est A ad B; ergo sunt continuae quantitates, quadratum A, rectangulum BC, & quadratum D. Quod erat demonstrandum.

*c 23 Sexti.*

## *L2.§1.* *PROPOSITION 55.*

Let the first A be to the second B, as the third C to the fourth D.

I say that the square under the first, the rectangle under the second and the third, and the square under the fourth are in continued proportion.

***Demonstration.***

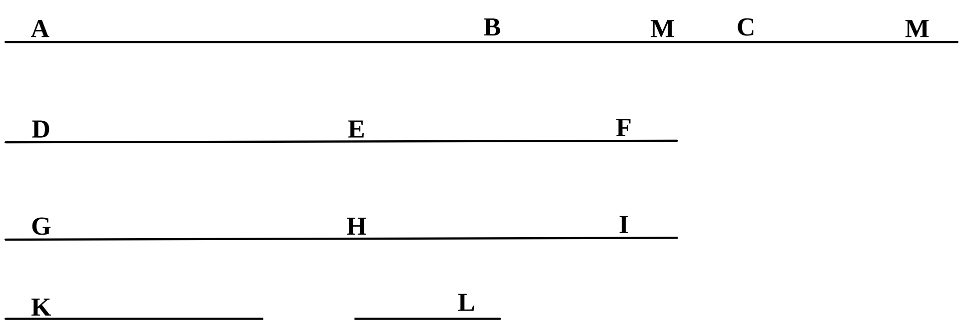
The ratio of the square A to the rectangle CB, [c] is composed from the ratio of A to B (that is C to D): & from the ratio A to C. But the ratio of the rectangle BC to the square D, is composed from the same ratios; for the ratio of the rectangle BC to the square D, is composed from the ratio B to D, that is A to C, & from the ratio C to D, that is A to B; hence the square A, the rect. BC, & the square D are continued quantities: which was to be shown.

*c 23 Sexti.*

## *PROPOSITIO LVI.*

Lineae GI, DF, AC ita sint divisae in H, E, B, ut ratio DE, ad EF duplicata sit rationis GH ad HI; & ratio AB, ad BC triplicata rationis GH ad HI; sintque praeterea GH, AB, in continua analogia.

Dico & rectangula GHI, DEF, ABC, in continua esse analogia.




**Prop.56. Fig. 1.**

***Demonstratio.***

Fiat ut DE ad GH, sic GH ad K, & ut EF ad HI, sic HI ad L. igitur rectangulum sub DE & K quadrato GH, & rectangulum sub EF & L, quadrato HI [c], aequale est; ergo rectangulum DEK, est ad rectangulum EFL, ut quadratum GH ad HI. Ex hypothesi, autem DE est ad EF, in duplicata ratione GH ad HI, hoc est, DE est ad EF ut quadratum GH, ad quadratum HI: ergo rectangulum sub DE & K, est ad rectangulum sub EF & L, ut DE ad EF. Ergo [d] K ad L lineae sunt aequales: quia autem ex hypothesi GH, DE, AB, & ex constructione K, GI, DE, sunt continuae; patet omnes quatuor K, GH, DE, AB esse continuas. Iam alias quoque lineas L, HI, EF, BC, dico esse continuas; si enim non sint, fiat tribus lineis L, HI, EF, quae ex constructione sunt continuae, quarta continue proportionalis, quaevis BM, maior vel minor quam BC, habemus ergo duas continuarum series K, GH, DE, AB; L, HI, EF, BM quae incipiant ab equalibus terminis K & L. quare ratio AB ad [e] BM, erit triplicata rationis GH ad HI; Atqui ex hypothesi etiam ratio AB ad BC, erat triplicata rationis GH ad HI; est ergo ut AB ad BC, ita AB ad BM; quod est absurdum. ergo L, HI, EF, BC etiam sunt continuae. Atque ratio composita ex rationibus GH ad DE & HI ad EF, hoc est, ratio rectanguli GHI [f] ad rectangulum DEF, eadem est cum ratione composita ex rationibus DE ad AB, & EF ad BC, hoc est cum ratione rectanguli DEF, ad rectangulum ABC. rectangula igitur GHI, DEF, ABC sunt in continua analogia. Quod erat demonstrandum.

*c 17 Sexti; d Ibid; e 27 huius; f 23 Sexti.*

## *L2.§1.* *PROPOSITION 56.*

The lines GI, DF, and AC are thus divided in H, E, and B, in order that the ratio DE, to EF is the square of the ratio GH ad HI; and the ratio AB to BC is the cube of the ratio GH to HI; and in addition GH, DE, and AB are in continued proportion.

I say that the rectangles GHI, DEF, and ABC are in continued proportion.

***Demonstration.***

The ratio DE to GH is thus made as GH to K, and EF to HI thus as HI to L. Therefore the rectangle under DE & K is equal to the square GH, & the rectangle under EF & L is equal to the square HI [c]; therefore rect. DE.K is to rect. EF.L, as the square GH is to the square HI. But from hypothesis, DE is to EF in the ratio of the square GH to HI; or DE to EF is as the square GH to the square HI. Therefore the rectangle under DE & K, is to the rectangle under EF & L, as DE to EF. Hence [d] the lines K to L are equal. Now, since from hypothesis GH, DE, and AB, and from construction K, GI, and DE are in continued proportion; it is apparent that all four lines K, GH, DE, AB are in continued proportion. Thus, I say that the other lines also: L, HI, EF, and BC are in continued proportion. For if they are not, then for the three lines L, HI, EF, which are in continued proportion, some line BM is made by construction as a fourth continued proportional, greater or less than BC. We have therefore series of continued proportionals : K, GH, DE, AB; and L, HI, EF, BM which begin from the equal terms K and L. Whereby the ratio AB to [e] BM, is the cube of the ratio GH to HI; but by the hypothesis the ratio AB to BC is also the cube of the ratio GH ad HI; hence as AB is to BC, so AB is to BM; which is absurd. Hence L, HI, EF, BC are indeed in continued proportion. The ratio composed from the ratios GH to DE & HI to EF, or, the ratio of rect.GHI [f] to rect. DEF, is the same as with the ratio composed from the ratios DE to AB, & EF to BC, that is with the ratio of rect.DEF to rect. ABC. Therefore the rectangles GHI, DEF, ABC are in a continued ratio. Q.e.d.

*c 17 Sexti; d Ibid; e 27 huius; f 23 Sexti.*

[DE/GH = GH/K and EF/HI = HI/L from construction; hence rect. DE.K = $GH^2$ and rect. EF.L = $HI^2$ ; hence rect.DE.K/rect.EF.L = $GH^2/HI^2$. But DE/EF = $GH^2/HI^2$; hence K = L. Now, GH/DE = DE/AB by hypothesis and K/GI = GI/DE by construction; hence K/GH = GH/DE = DE/AB; also, the other lines satisfy L/HI = HI/EF = EF/BC: for if the final length is some other magnitude BM, then BM/AB = K/DE/GH . HI/L.EF = HI/GH . EF/DE = $(HI/GH)^3$ or AB/BM = $(GH/HI)^3$ ; but by hypothesis, AB/BC = $(GH/HI)^3$ and so BM = BC. Again, GH/DE . HI/EF = GH.HI / DE.EF = rect. GHI/rect. DEF = DE/AB . EF/BC = rect. DEF/rect. ABC; hence rect. GHI/rect. DEF = rect. DEF/rect. ABC as required.]

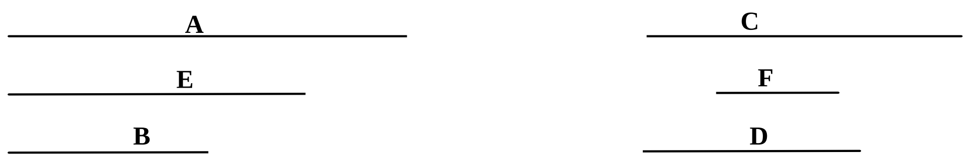
## *PROPOSITIO LVII.*

Sint ratio A ad B, eadem cum ratione C ad D, & inter utramque tam A, B quam C, D interponantur quaevis lineae: E quidem inter A, B; F vero inter C, D.

**[87]**

Dico rectangulum AE, ad EB rectangulum, eandem habere rationem, quam habet rectangulum CF ad FD rectangulum.

***Demonstratio.***




**Prop.57. Fig. 1.**

Ratio rectanguli AE ad EB rectangulum, est ea [a] quam habet A ad B; sed ut A ad B, ita ponitur C ad D; ergo rectangulum AE ad EB, est ut C ad D : sed rectangulum CF, ad FD rectangulum, etiam est ut linea [b] C ad D. igitur rectangulum AE, ad EB rectangulum, eandem habet rationes, quam habet CF ad FD rectangulum, Quod demonstrare oportebat.

*a 1 Sexti; b Ibid.*

## *L2.§1.* *PROPOSITION 57.*

Let the ratio A to B be the same as the ratio C to D, and between each, as for A and B, so for C and D: some line E is placed between A and B and some line F between C and D.

I say that the rectangle AE to the rectangle EB has the same ratio as the rectangle CF to the rectangle FD.

***Demonstration.***

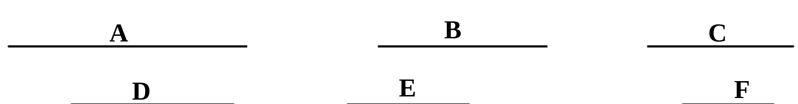
The ratio of rect.AE to rect.EB, is that which [a] A has to B; but as A to B, thus C to D is put; hence the rect. AE to the rect. EB is as C is to D : but rect.CF to rect.FD, is also as the line [b] C to D. Hence rect.AE to rect.EB, has the same ratios as rect.CF has to rect. FD. Which it was necessary to show.

*a 1 Sexti; b Ibid.*

## *PROPOSITIO LVIII.*

Ponantur tres lineae A, B, C; & aliae tres D, E, F; ut tam primae, quam secundae, suam analogiam licet diversam continuent.

Dico etiam rectangula AD, BE, CE esse in continua analogia.




**Prop.58. Fig. 1.**

***Demonstratio.***

Rectangulum sub A & D, ad rectangulum sub B & E habet [c] rationem compositam ex rationibus A ad B, & D ad E : proportio quoque rectanguli sub B & E ad rectangulum sub C & F composita ex rationibus B & C, hoc est A ad B, & ex ratione E ad F, hoc est D ad E; unde ratio rectanguli sub B & E, ad rectangulum CF, componitur ex iisdem, ex quibus ratio rectanguli sub A & D, ad rectangulum sub B & E est composita :

Quare cum proportiones ex iisdem rationibus compositae eaedem sint, erunt rectangula AD, BE, CF in contina analogia. Quod erat demonstrandum.
*23 Sexti.*

## *L2.§1.* *PROPOSITION 58.*

Three lines A, B, and C; and three other lines D, E, and F, are put in position so that each can continue in its own ratio.

I say that the rectangles AD, BE, and CE are in a continued ratio.

***Demonstration.***

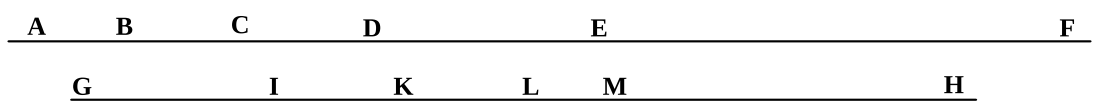
The rectangle under A & D to the rectangle under B & E has the ratio [c] composed from the ratios A to B, & D to E : the proportion also of the rectangle under B & E to the rectangle under C & F is composed from the ratio B & C, or A to B, & from the ratio E to F, or D to E; hence the ratio of the rectangle under B & E, to the rectangle CF, is componsed from the same, from which the ratio has been composed of the rectangle under A & D to the rectangle under B & E. Whereby as the proportions are composed from the same ratios, the rectangules AD, BE, and CF are in contined proportion. Q.e.d.
*c 23 Sexti.*

## *PROPOSITIO LIX.*

Sint duae diversarum rationum series continuè proportionalium AB, AC, AD, AE, AF; GH, IH, KH, LH, MH.

Dico rectangula ABGH, ACIH, ADKH, AELH, AFMH esse in continua analogia.

***Demonstratio.***




**Prop.59. Fig. 1.**

Rectangulum enim sub ABGH, ad rectangulum sub ACIH rationem habet compositam ex ratione AB ad AC, & ex ratione GH ad IH : & rectangulum ACIH ad rectangulum ADKH rationem compositam ex rationibus ex AC ad AD & IH ad KH. Quare cum ex datis sit ut AB ad AC, sic AC ad AD, & ut GH ad IH, sic IH ad KH: igitur ratio rectangularum ACIH & ADKH, ex iisdem rationibus componitur, ex quibus rectangulorum ABGH, ACIH, sunt igitur rectangula ABGH, ACIH, ADKH in contina analogia : idemque ita reliquis eodem discursu ostendetur.

**[88]**

***Corollarium.***

Porro cum per primium huius, etiam continuè proportionalium, differentiae sint in continua analogia suorum integrorum : manifestum est rectangula quoque ABGI, BCIK, CDKL, DELM, &c, esse continuè proportionalia.

## *L2.§1.* *PROPOSITION 59.*

There are two series in continued proportion with different ratios AB, AC, AD, AE, AF; and GH, IH, KH, LH, MH.

I say that the rectangles ABGH, ACIH, ADKH, AEFH, ALMH are in a continued ratio.

***Demonstration.***

Indeed the rectangle under AB.GH, to the rectangle under AC.IH, has a ratio composed from the ratios AB to AC, & GH to IH : & the rectangle AC.IH to the rectangle AD.KH has a ratio composed from the ratios AC to AD & IH to KH. Whereby as from what is given, AB is to AC, thus as AC is to AD, & as GH is to IH, so IH is to KH: therefore the ratio of the rectangles ACIH & ADKH, is put together fro, the same ratios, from which the rectangles ABGH & ACIH are composed. Hence the rectangles ABGH, ACIH, and ADKH are in acontinued ratio: and thus the remainder can be shown by the same discussion.

***Corollary.***

Again from the first proposition of this book, for continued proportions, the differences also of the wholes are in a continued ratio : the rectangles AB.GI, BC.IK, CD.KL, DE.LM, &c, too, can be shown to be in continued proportion.

## *PROPOSITIO LX.*

Sint in continua analogia AB, AC, AD.

Dico rectangulum ABC, ad ACD rectangulum, duplicatam habere rationem eius, quam habet AB ad AC lineam.

***Demonstratio.***


**Prop.60. Fig. 1.**

Quoniam DA, CA, BA ponuntur continuè proportionales, erit CD ad BC, ut CA ad BA, & invertendo BC ad CD, ut [a] AB ad AC : itaque cum rectangulorum ABC, [b] ACD ratio componatur ex laterum rationibus AB ad AC, & BC ad CD; quoque iam ostensae sunt aequales, constat eam esse duplicatam, rationis AB ad AC. Quod erat demonstratum. *a 1 Huius; b 23 Sexti.*

## *L2.§1.* *PROPOSITION 60.*

AB, AC, AD are in a continued ratio.

I say that the rectangle ABC, to the rectangle ACD, is in the square ratio of line AB to the line AC.

***Demonstration.***

Since DA, CA, and BA are put in continued proportion, CD is to BC, as CA is to BA, & on inverting, BC is to CD, as AB is to [a] AC : and thus the ratio of the rectangles ABC [b] and ACD is composed from the ratio of the sides AB to AC, & BC to CD; thus shown to be equal also, and it is agreed that the ratio is the square of the ratio AB to AC. Q.e.d. *a 1 Huius; b 23 Sexti.*

[DA/CA = CA/BA by hypothesis; hence CD/BC = CA/BA and BC/CD = AB/AC.
Rect. AB.BC/rect AC.CD = AB/AC . BC/CD = $(AB/AC)^2$ as required.]

## *PROPOSITIO LXI.*

Iisdem positis :

Dico ABC rectangulum ad ADC rectangulum, triplicatam habere rationem eius, quam habet AB ad AC lineam.

***Demonstratio.***

Ex datis ratio AB ad AD, duplicata est rationis AB ad AC : & ut patet ex praecedente & per primam huius ratio BC ad CD, aequalis est rationi AB ad AC; ergo ratio composita ex rationibus AB ad AD, & BC ad CD, triplicata est rationis AB ad AC. Quare cum rectangulorum ABC, ADC ratio ex proportionibus laterum AB ad AD, & BC ad CD c componatur, patet eam esse triplicatam rationis AB ad AC. Quod erat demonstratum. *c 23 Sexti.*

## *L2.§1.* *PROPOSITION 61.*

With the same positions.

I say that the rectangle ABC, to the rectangle ADC, is the cube of the ratio of line AB to the line AC.

***Demonstration.***

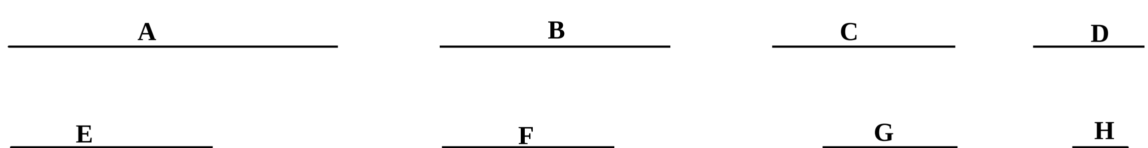
From what is given, the ratio AB to AD is the square of the ratio AB to AC. It is apparent from the preceeding proposition and from the first proposition of this book, that the ratio BC to CD is equal to the ratio AB to AC. Hence the ratio composed from the ratios AB to AD and BC to CD, is the cube of the ratio AB to AC. Whereby since the ratio of the rectangles ABC and ADC is composed from the proportions of the sides AB to AD and BC to CD [c]. It is apparent that it is the cube of the ratio AB to AC. Q.e.d. *a 23 Sexti.*
[$AB/AD = AB/AC \cdot AC/AD = (AB/AC)^2$ . Also, $BC/CD = AB/AC$;
hence $AB/AD \cdot BC/CD = (AB/AC)^2 \cdot AB/AC = (AB/AC)^3$ = rect. ABC /rect. ADC].

## *PROPOSITIO LXII.*

Ponatur duae series quatuor continuarum A, B, C, D ; & E, F, G, H.

Dico rectangulum AH, ad ED rectangulum, triplicatam rationem habere eius, quam habet BG rectangulum, ad rectangulum FC.

***Demonstratio.***




**Prop.62. Fig. 1.**

Rectanguli sub A & H, ad rectangulum sub E & D, ratio est composita, [d] ex ratione A & D, H & E : ratio vero rectanguli BG, ad rectangulum FC, composita est ex ratione B ad C, & G ad F : ratio autem A ad D, triplicata est ratione B ad C, & ratio H ad E etiam triplicata est rationis G ad F; igitur ratio rectanguli AH ad DE, triplicata est rationis eius, quam habet BG rectangulum, ad FC. Quod erat demonstratum. *c 23 Sexti.*

## *L2.§1.* *PROPOSITION 62.*

Two series in continued proportion are put in place: A, B, C, D ; & E, F, G, H.
I say that the ratio of rectangle AH to rectangle ED is the cube of the ratio that rectangle BG has to rectangle FC.

***Demonstration.***

The ratio of the rectangle under A & H, to the rectangle under E & D, is made from [d] the ratio A & D, and H & E : also, the ratio of rectangle BG, to rectangle FC, is made from the ratio B to C, & G to F. But the ratio A to D, is the cube of the ratio B to C, & the ratio H to E also is the cube of the ratio G to F; hence the ratio of rectangle AH to DE, is the cube of that which rectangle BG has to FC. Q.e.d. *c 23 Sexti.*

[Rect.A.H/rect E.D = A/D . H/E; also, rect. B.G/rect F.C = B/C . G/F; but A/D = $(B/C)^3$, and H/E = $(G/F)^3$; hence rect.A.H/rect E.D = $(\text{rect. B.G/rect C.F})^3$ as required.]

**[89]**

## *PROPOSITIO LXIII.*

Datae sint tres continuae proportionales AC, CD, DE ; & DE bifarium sit in B.
Dico quadratum AB, aequale esse quadratis AD, CB.

***Demonstratio.***


**Prop.63. Fig. 1.**

Quia AC, CD, DE sunt in continua proportione, quadratum CD [a] aequatur rectangulo ACDE ; hoc est (quoniam DE bisecta ponitur in B) rectangulo ACDB bis sumpto. Quare si utrisque/verisque ? commune addatur rectangulum CDB bis , erit quadratum CD, cum rectangulo CDB bis, aequale rectangulo ACDB bis, cum rectangulo CDB bis; quae quatuor rectangula constituunt [b] rectangulum ADB bis. Rursum ergo communi addito quadrato DB, erit quadratum CD, cum rectangulo CDB bis, & quadrato DB, id est [c] quadratum CB, aequale rectangulo ADB bis, cum quadrato DB : itaque communi addito quadrato AD, erit rectangulum ADB bis, cum quadratis DB, AD, id est [d] quadratum AB, aequale quadratis CB, AD : Quod erat demonstratum. *a 27 Sexti; b 2 Sexti ?; c 4 Secundi; d Ibid;*

## *L2.§1.* *PROPOSITION 63.*

Three continued proportionals AC, CD and DE are given ; & DE is bisected in B.
I say that the square AB is equal to the sum of the squares AD and CB.

***Demonstration.***

Since AC, CD and DE are in a continued proportion, then the square CD [a] is equal to the rectangle AC.DE ; or (as DE is bisected in B) to twice the rectangle AC.DB. Whereby if indeed twice the rectangle CDB is added in common, then the square CD with twice the rectangle CDB, is equal to the sum of twice the rectangle AC.DB and twice the rectangle CDB; which four [b] constitute twice the rectangle ADB. Again, therefore with the common square DB added, erit quadratum CD, cum rectangulo CDB bis, & quadrato DB, id est [c] quadratum CB, aequale rectangulo ADB bis, cum quadrato DB : itaque communi addito quadrato AD, erit rectangulum ADB bis, cum quadratis DB, AD, id est [d] quadratum AB, aequale quadratis CB, AD : Quod erat demonstratum. *a 27 Sexti; b 2 Sexti ?; c 4 Secundi; d Ibid;*

[Rect.A.H/rect E.D = A/D . H/E; also, rect. B.G/rect F.C = B/C . G/F; but A/D = $(B/C)^3$, and H/E = $(G/F)^3$; hence rect.A.H/rect E.D = $(\text{rect. B.G/rect C.F})^3$ as required.]

## *PROPOSITIO LXIV.*

Sint tres lineae in continuae analogia AB, BC, CD ; & dividatur CD bifarium in E. Dico quadratum AE, aequare quadratis AC, EB.

***Demonstratio.***


**Prop.64. Fig. 1.**

Cum sint continuae proportionales AB, BC, CD, erit ut BC [e] ad CD, sit AC ad DB. unde rectangula CBD, ACD aequantur. sed rectangulum CBD, est rectangulum CDB, cum quadrato DB, hoc est (quoniam ex datis CD bisecta est in E) rectangulum EDB bis, cum quadrato DB : rectangulum vero ACD, est rectangulum ACE bis; ergo rectangulum EDB bis, cum quadrato DB, aequatur rectangulo ACE bis, & communibus additis quadratis AC, CE, sive ED, erunt rectangulum EDB bis & quadrata BD, ED, AC simul sumpta, aequalia rectangulo ACE bis, & quadratis AC, CE : Atqui rectangulum EDB bis, cum quadratis DE, ED, AC, aequale est quadrato [f] EB, & quadrato AC; rectangulum vero ACE bis, cum quadratis AC, CE, aequantur [g] quadrato AE : ergo quadrata EB, AC aequantur quadrato AE. Quod erat demonstratum. *e 1 Huius; f 4 Secundi; g Ibid.*

## *L2.§1. PROPOSITION 64.*

There are three lines in continued proportion AB, BC and CD ; & CD is bisected in E. I say that the square AE is equal to the sum of the squares AC and EB.

***Demonstration.***

Since AB, BC, and CD are continued proportionals , thus BC [e] to CD is as AC to DB. Hence the rectangles CBD and ACD are equal; but rectangle CBD is equal to the sum of rectangle CDB and the square DB, or (as CD is given bisected in E) to the sum of twice the rectangle EDB and the square DB : but rectangle ACD is twice the rectangle ACE; hence the sum of twice the rectangle EDB and the square DB is equal to twice the rectangle ACE, and with the common squares AC and CE or ED added, the sum of twice the rectangle EDB and the squares BD, ED and AC taken together, is equal to the sum of twice the rectangle ACE and the squares AC and CE. But twice the rectangle EDB with the squares DE, ED and AC is equal to the square [f] EB and the square AC; thus twice the rectangle ACE with the squares AC and CE is equal to the square [g] AE : hence the sum of the squares EB and AC is equal to the square AE. Q.e.d.
*e 1 Huius; f 4 Secundi; g Ibid.*

[AB/BC = BC/CD; AC/BC = BD/CD or BC/CD = AC/BD and rect. CB.BD or rect. CBD = rect. AC.CD or rect. ACD. But rect. CB.BD = (CD + DB).BD = rect. CDB + $BD^2$ = 2.rect EDB + $BD^2$ ; also, rect. ACD = 2.rect. ACE; hence 2.rect EDB + $BD^2$ = 2.rect. ACE; on adding $AC^2$ + $ED^2$ to both sides: 2.rect EDB + $BD^2$ + $AC^2$ + $ED^2$ = 2.rect. ACE + $AC^2$ + $EC^2$ ; but 2.rect EDB + $BD^2$ + $ED^2$ = $BE^2$, and ] 2.rect. ACE + $AC^2$ + $EC^2$ = $AE^2$; hence $BE^2$ + $AC^2$ = $AE^2$, as required.]

## *PROPOSITIO LXV.*

Continuae proportionales sint AB, AC, AE, &c.; & ex BC, sunti possit BD aequalis AC.

Dico rectangulum sub BA & sub CD, AE, tamquam unam lineam constructum, aequali quadrato AD.

***Demonstratio.***


**Prop.65. Fig. 1.**

Rectangulum BACD, [h] aequatur rectangulo BCD, (id est [i] quadrato CD, & rectangulo BCD) una cum rectangulo ACD, sed (quoniam aequales sunt positae AC.BD)

**[90]**

aequalia sunt rctangula BDC, ACD; ergo rectangulum ABCD, aequatur rectangulo ACD bis, cum quadrato, CD: & quia sunt continuae BA, CA, EA, rectangulum BAE, aequale est quadrato CA : Igitur si rectangulo ABCD addas rectangulum BAE: & rectangulo ACD bis, cum quadrato CD, addas quadratum CA, erunt rectangula BACD, BAE, (id est rectangulum [a] ex BA in CDAE tanquam unam lineam) aequalia quadratis CD, CA, & rectangulo ACD bis, id est [b] quadrato AD. Quod erat demonstratum. *a 1 Secundi; b 4 Sextii.*

## *L2.§1.* *PROPOSITION 65.*

AB, AC, AE, etc. are continued proportionals, & from BC, BD is to be taken equal to AC.

I say that the rectangle under BA and under CD and AE, constructed as it were from one line, is equal to the square AD.

***Demonstration.***

The rectangle BACD [h] is equal to rectangle BCD, (that is [i] to the sum of the square CD and the rectangle BDC) together with rectangle ACD, but (since AC and BD are made equal ) the rectangles BDC and ACD are equal; hence rectangle ABCD is equal to the sum of twice the rectangle ACD and the square CD: and since BA, CA and EA are in continued proportion, rectangle BAE is equal to the square CA. Hence if you add rectangle BAE to rectangle ABCD: and if you add the square CA to the sum of twice rectangle ACD and the square CD, then the sum of the rectangles BACD and BAE, (that is the rectangle [a] from BA by CDAE considered as one line ) is equal to the sum of the squares CD and CA with twice the rectangle ACD , that is [b] to the square AD. Q.e.d. *a 1 Secundi; b 4 Sextii.*

[Rect. BA.CD = rect.BC.CD + rect. AC.CD, where rect.BC.CD = $CD^2$ + rect.CD.DB; But as AC = BD, rect.CD.DB = rect.AC.CD, then rect. BA.CD = 2.rect. AC.CD + $CD^2$. Again, as BA/CA = CA/ EA, rect.BA.AE = $CA^2$ , then rect. BA.CD + rect.BA.AE = rect.BA.(AE + CD) = 2.rect. AC.CD + $CD^2$ + $CA^2$ = $AD^2$ as required].

## *PROPOSITIO LXVI.*

Ponatur linea AB, divisa in tres proportionales AB, CB, DB; duabus autem CB, DB, in directum constituantur aequales, BF, BE.

Dico rectangulum CDF, rectangulo ADB aequale esse.

***Demonstratio.***


**Prop.66. Fig. 1.**

Quoniam AB, CB, DB, ponatur continuae, erit rectangulum ABD, (id est [c] rectangulum ADB cum quadrato DB) aequale quadrato CB; atqui etiam (cum ex datis CF bisecta sit in B) rectangulum CDF cum quadrato DB, aequatur [d] quadrato CB; ergo rectangulum ADB, cum quadrato DB, aequatur rectangulo CDF, cum quadrato DB. Quocirca ablato communi quadrato DB rectangulum, CDF, rectangulo ADB aequale erit. Quod erat demonstratum. *c 3 Secundi; d 5 Secundi.*

## *L2.§1.* *PROPOSITION 66.*

The line AB is established and divided in three proportionals AB, CB, DB; but the lines BF and BE are placed in order equal to the two lines CB and DB.

I say that the rectangle CDF is equal to the rectangleADB.

***Demonstration.***

Since AB, CB, and DB are placed in continued proportion, the rectangle ABD, (that is [c]rectangle ADB plus the square DB) is equal to the square CB; but also (since from what is given, CF is bisected in B) the sum of rectangle CDF and the square DB, is equal to [d] the square CB; hence the sum of the rectangle ADB and the square DB, is equal to the sum of the rectangle CDF and the square DB. Whereupon, by taking the common square DB, the rectangle CDF is equal to the rectangle ADB. Q.e.d. *c 3 Secundi; d 5 Secundi.*

[Since AB/CB = CB/ DB, rect.AB.BD = $CB^2$ = rect. AD.DB + $DB^2$, given BC = BF and BD = BE; also rect.CD.DF + $DB^2$ = $CB^2$ , from the difference of the squares; hence rect. AD.DB = rect.CD.DF, as required].

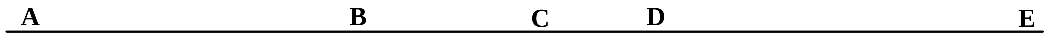
## *PROPOSITIO LXVII.*

Sit AB ad AC, ut AD ad AE.

Dico primo rectangulum sub primo AB, & DE differentia quartae & tertiae, aequari rectangulo sub BC, differentiae primae & secundae, & sub AD tertia:

Secundo rectangulum sub prima AB, & BE differentia primae & quartae, aequari duobus rectangulis sub AC, BD, & sub AB, BC.

***Demonstratio.***




**Prop.67. Fig. 1.**

Cum sit ut AB ad AC, sic AD ad AE, itaque invertendo & dividendo ut CB ad BA, sic ED ad DA. quare rectangulum EDAB aequatur rectangulo DACB. quod erat primum. Deinde cum sit ut EA ad DA, sic CA ad BA, ergo rectangulum EAB aequatur rectangulo DAC, hoc est [e] rectangulo DCA cum quadrato CA. Quod sit autem a rectangulo EAB abstuleris quadratum AB, remanet rectangulum EBA : & si idem abstuleris a rectangulo DCA cum quadrato CA, remanent [f] rectangulum DCA, rectangulum CBA bis cum quadrato BC. Itaque cum tota fuerint aequalia, erunt ablato communi, aequalia adhuc reliqua; rectangulum nempe EBA, & rectangula DCA semel, CBA bis, & quadratum BC simul sumpta. atqui rectangulum ACB, aequale est rectangulo ABC cum quadrato BC : cui si addas rectangulum ACD, orietur [g] rectangulum ACBD, quod cum rectangulo ABC aequale erit rectangulo ABC bis cum quadrato BC, & rectangulo ACD : quae cum simul sumpta aequalia esse ostendeum rectangulo EBA, erit quoque rectangulum AC, BD cum rectangulo ABC aequale rectangulo EBA, quod erat demonstratum. *e 3 Secundi; f 4 Secundi ; g 1 Secundi.*

## *L2.§1.* *PROPOSITION 67.*

The line AB is established and divided in three proportionals AB, CB, DB; but the lines BF and BE are placed in order equal to the two lines CB and DB.

I say that the rectangle CDF is equal to the rectangleADB.

***Demonstration.***

Since AB, CB, and DB are placed in continued proportion, the rectangle ABD, (that is [c]rectangle ADB plus the square DB) is equal to the square CB; but also (since from what is given, CF is bisected in B) the sum of rectangle CDF and the square DB, is equal to [d] the square CB; hence the sum of the rectangle ADB and the square DB, is equal to the sum of the rectangle CDF and the square DB. Whereupon, by taking the common square DB, the rectangle CDF is equal to the rectangle ADB. Q.e.d. *c 3 Secundi; d 5 Secundi.*

[Since AB/CB = CB/ DB, rect.AB.BD = $CB^2$ = rect. AD.DB + $DB^2$, given BC = BF and BD = BE; also rect.CD.DF + $DB^2$ = $CB^2$ , from the difference of the squares; hence rect. AD.DB = rect.CD.DF, as required].

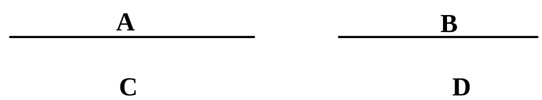
**[91]**

## *PROPOSITIO LXVIII.*

Si quatuor lineae proportionales fuerint, maximae & minimae, quadrata simul sumpta, maiora sunt reliquarum quadratis simul sumptis.

***Demonstratio.***

Cum enim quatuor lineae ponantur proportionales, etiam [a] earum quadrata erunt proportionalia, ita tamen ut quadratum maximae lineae maximum sit, quadratum vero minimae, sit minimum, ut patet ex elementis; igitur [b] constat propositum. Theorema eadem fere posita demonstratione quibusuis planis & solidis similibus applicari poterit. quod erat demonstratum. *a 22 Sexti; b 25 Quinti.*




**Prop.68. Fig. 1.**

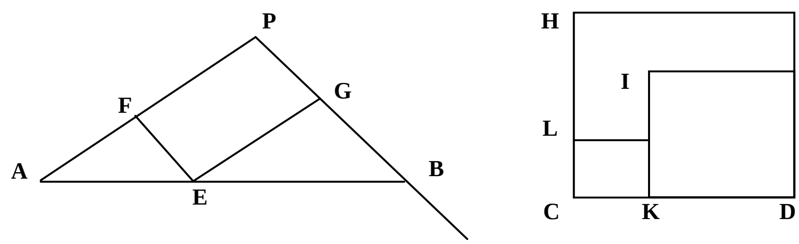
## *L2.§1.* *PROPOSITION 68.*

If there are four lines in proportion, then the squares of the maximum and minimum are larger and smaller than the squares of the other proportions.

### *Demonstration.*

Since indeed there are four lines placed in proportion, the squares of these are also in proportion, thus indeed as the square of the greatest line is the greatest, and the square of the least shall indeed be the least of the squares, as is apparent from elementary considerations. Therefore the truth of the propositon is agreed upon. It is possible for the same theorem to be applied generally for any similar demonstrations of planes and solids. *a 22 Sexti; b 25 Quinti..*

## *PROPOSITIO LXIX.*



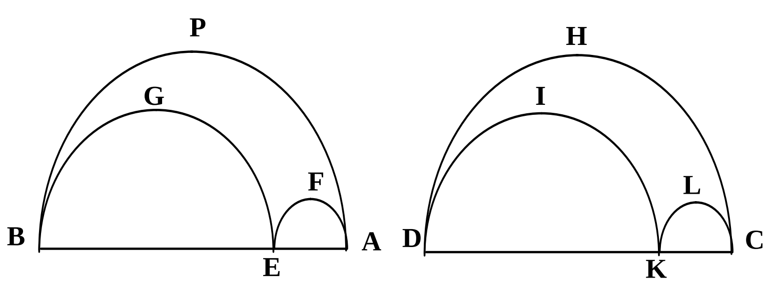
**Prop.69. Fig. 1.**

Duobus lineis AB, CD secundi eandem rationem divisis in E, & K, fiat super earum una AB quavis figura APB : & sub partibus duae similes ei quae fit a tota, nempe AFE, EGB. Deinde sub alia CD fiat quaecumque alia figura CHD, sive similis praecedentibus, sive dissimilis seu rectilinea seu curvilinea; sub partibus autem statuantur duae aliae CLK, KID, similes ei quae sit tota.

Dico ut APB ad CHD, sic duae AFE, EGB ad duas CLK, KID.

### *Demonstratio.*

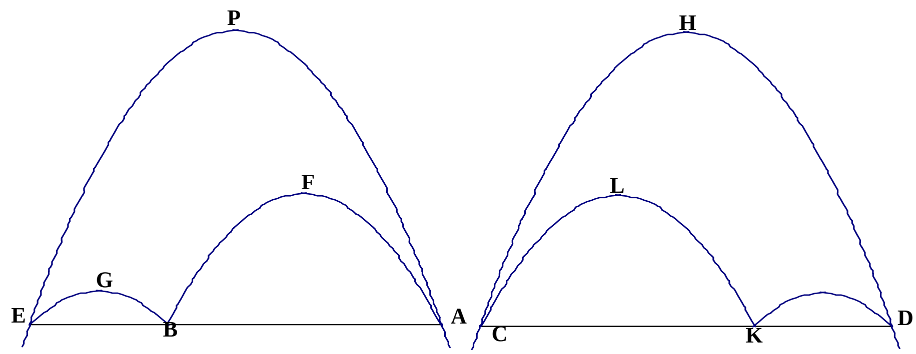
Comparimus primo rectilinea cum rectilineis : quia figurae similes sunt AFE, APB, erit AFE, ad APB, in duplicata ratione laterum [c] homologorum AE, AB : similiter quia CLK, CHD sunt figura similes, erunt in duplicata ratione CK ad CD, id est ex datis, in duplicata ratione AE ad AB : ergo AFE est ad APB, ut CLK ad CHD. igitur AFE d cum EGB, est as APB, ut CLK cum KID, ad CHD. Itaque permutando AFE est cum EGB, ad CLK cum KID, ut APB ad CHD.



**Prop.69. Fig.2.**

Comparimus deinde rectlinae cum curvilineis. Super CD constituatur segmentum circuli CHD: & super CK, KD, segmenta CLK, KID. similia segmento CHD: cum igitur similia circulorum segmenta, duplicatam habeant proportionalem subtensarum, si rectilinea lineae AB, cum segmentis lineae CD comparentor, eadem prorsus demonstratione concludetur propositum, qua usi fuimus in prima comparatione : Tertio si curvilinea est curvilineis eiusdem speciei conferantur, patet a fortiori propositum:

**[92]**




**Prop.69. Fig.3.**

Eodem modo si curvilinea, cum diversae speciei curvilineis, tres nempe parabolae similes, cum tribus hyperbolis similibus, conferantur, eadem his quoque demonstrandi ratio conveniet : cum tam parabolae similes, quam hyberbolae sint in duplicata ratione subtensarum. Constat igitur huius theorematis universalis veritas. *c 19 Sexti; d 24 Quinti.*

## *L2.§1.* *PROPOSITION 69.*

For two lines AB and CD, following the same ratio of division in E and K, there is constructed on the one line AB some figure APB, and within the parts of the division two figures AFE and EGB are made similar to the whole figure. Then within the other CD some other figure is made, either similar to the first mentioned, or dissimilar, either rectilinear or curvilinear; however two other figures CLK and KID are established within the sections of the line, which are similar to the whole figure.

I say that as APB is to CHD [i.e. as areas], thus the sum of ADE and EGB is to CLK and KID.

### *Demonstration.*

First we compare the rectilinear figure with the rectilinear figures : since AFE and APB are similar figures, the ratio of AFE to APB is in the square ratio of the homologous sides [c] AE and AB : similarly since CLK and CHD are similar figures, they are in the square ratio of CK to CD, that is from what is given, in square ratio AE to AB : hence AFE is to APB, thus as CLK is to CHD. Therefore the sum of AFE [d] and EGB, is to APB, as the sum of CLK and KID is to CHD. Thus by interchanging, as the sum of AFE and EGB is to sum of CLK and KID, thus APB is to CHD.

In the second case we can compare a rectlinear figure with a curvilinear one. Upon CD the segment of a circle CHD is set up: and upon the sections CK and KD, the segments CLK and KID are set up similar to the segment CHD. Therefore, since the ratio of the areas of similar segments of circles are as the squares of the subtended chords, if the rectilinear figures of the line AB are compared with the segments of the line CD, the same proposition can in short be conculuded, which we used in the first comparison.

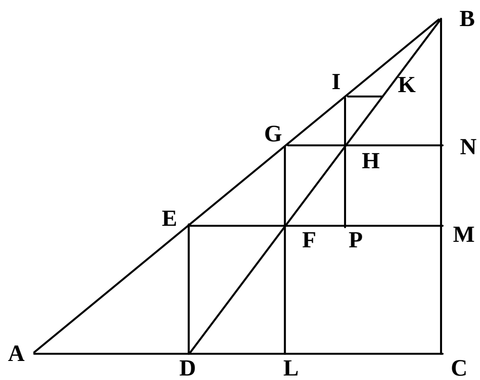
In the third case if a curvilinear figure is brought together with curvilinear figures of the same kind, the proposition is apparent from what has gone before : In the same manner, if a curvilinear figure is brought together with a different kind of curvilinear figure, indeed if three similar parabolas are brought together with three similar hyperbolas, for these too the same ratio can beshown to be agreed upon: for as with similar parabolas, so the areas under similar hyberbolas are in the ratio of the squares of the subtending chords. Thus the truth of this theorem can be agreed upon in all cases. *c 19 Sexti; d 24 Quinti.*

# *PROPOSITIO LXX.*

Sit ABC triangulum divisum rectam lineam DB, ducanturque lineae DE, EF, FG, GH, HI, IK, basi AC, & lateri BC parallelae quot libverit.

Dico omnes AD, EF, GH, IK, item DE, FG, HI, &c. esse in eadem continuata analogia.

***Demonstratio.***




**Prop.70. Fig. 1.**

Producantur enim lineae GF in L, EF in M, GH in N, & IH in P. Igitur cum parallelae sint AC, EM, erit AD ad DC , ut EF ad FM: & componendo AC ad CD, ut EM ad FM, id est ut DC ad LC; sunt igitur in continua ratione AC, DC, LC; quare & AD [a] ad DL, id est EF ut AC est ad DC. Similiter ostendam esse continuas EM, FM, PM : unde rursus est ut EM [b] ad FM (id est ut AC ad DC, id est AD ad EF) sic EF ad FP, id est GH. continue proportinales sunt igitur AD, EF, GH : eodem modo ostendam IK, & alias quotcumque in eadem serie esse continuas. Deinde cum AD ipsiEF, & DE ipsi FG sit parallela, patet similia esse triangula ADE, EDG: ergo ut AD ad EF, ita DE ad FG : similiter ostendam esse ut EF ad GH, ita FG ad HI. quare erunt etiam DE, FG, IH, &c. continuae, & quidem in ea ratione in qua sunt AD, EF, GH, Quod erat demonstrandum. *a 1 Huius; b Ibid.*

## *L2.§1.* *PROPOSITION 70.*

Triangle ABC is divided by the line DB, and the lines DE, EF, FG, GH, HI, and IK are drawn parallel to the base AC and to the side BC as often as it pleases.

I say that all the lines AD, EF, GH, IK, likewise DE, FG, HI, etc. are in the same continued ratio.

***Demonstration.***

For the lines are produced: GF in L, EF in M, GH in N, and IH in P. Therefore as the lines AC and EM are parallel, the ratio AD to DC is thus as EF to FM: and by adding AC is to CD as EM is to FM, or as DC is to LC; AC, DC, and LC are therefore in a continued ratio; and whereby AD [a] is to DL, or EF, as AC is to DC. Similarly it can be shown that EM, FM, PM are in continued proportion : thus again as EM [b] is to FM (or AC to DC, or AD to EF) thus EF is to FP, or GH. Hence AD, EF, anf GH are continued proportinals: in the same way it can be shown that IK and any othes in the same series are in proportion. Hence as AD to itself EF, & DE to FG itself are parallel, it is apparent that the triangles ADE and EDG are similar: hence as AD is to EF, thus DE is to FG : similarly it can be showm that as EF is to GH, thus FG to HI. Whereby DE, FG, IH, etc. are also in continued proportion, and indeed are in the same continued ratio as AD, EF, and GH. Q.e.d. *a 1 Huius; b Ibid.*

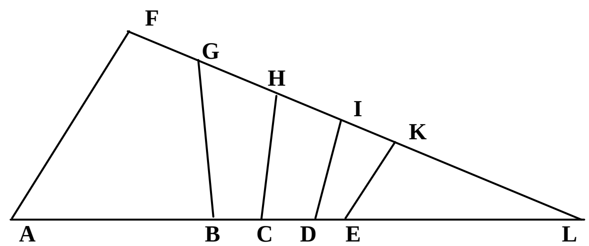
## *PROPOSITIO LXXI.*

Duae lineae AL, FL angulum facientes, sectae sint in continuae proportionales, quarumcumque rationes AL, BL, CL, DL, EL, &c. item FL, GL, HL, &c. dein opposita sectionum puncta lineis AF, BG, CH, DI, &c. coniungantur.

Dico triangula AFL, BGL, CHL, & caetera in infinitum esse in continua analogia.

**[93]**

***Demonstratio.***



**Prop.71. Fig. 1.**

Cum AL, BL, CL, &c. ponantur continuae proportinales, est ut AL ad BL, sic BL ad CL, & CL ad DL, &c. similiter cum ponantur continuae FL, GL, &c. erit ut FL ad GL, ita GL ad HL, atque ita semper : igitur ratio composita ex rationibus AL ad BL, & FL ad GL, eadem erit cum ratione composita ex rationibus BL ad CL, & GL ad HL; & composita ex rationibus BL ad CL, & GL ad HL, eadem erit, cum composita ex rationibus CL ad DL, & HL ad IL: Atqui trianguli AFL, ad triangulum BGL proportio composita est ex rationibus AL ad BL, & FL ad GL; & ratio trianguli BGL ad triangulum CHL, composita est ex rationibus BL ad CL, & GL ad HL, ut ex Commandino demonstrat Clavius, ad propositionem 23 sexti : eadem igitur est ratio trianguli AFL, ad triangulum BGL, quae huius, ad triangulum CHL. Similiter ostendentur reliqua triangula esse in analogia continua. Quod erat demonstrandum.

***Corollarium.***

Hinc consequitur etiam Trapezia AG, BH, CI, &c. esse in continua analogia, sunt enim trapazia, triangulorum continuae proportionalium differentiae, unde ex prima huius patet corollarii veritas.

## *L2.§1.* *PROPOSITION 71.*

The two lines AL and FL making the sides of an angle are cut in continued proportions, according to some ratio AL, BL, CL, DL, EL, etc., and likewise FL, GL, HL, etc., then the opposite points of the sections of the lines AF, BG, CH, DI, etc. are joined.

I say that the triangles AFL, BGL, CHL, and so on indefinitely are in continued proportion.

***Demonstration.***

Since the lines AL, BL, CL, etc. are placed in continued proportion, AL is to BL, thus as BL is to CL, and CL to DL, etc. Similarly, since FL, GL, etc, are put in proportion, FL is to GL thus as GL is to HL, and so on indefinitely: therefore the ratio composed from the ratios AL to BL, and FL to GL, is the same as that composed from the ratios BL to CL, and GL to HL; and that composed from the ratios BL to CL, and GL to HL, is the same as that composed from the ratios CL to DL, and HL to IL. But the proportion of triangle AFL to triangle BGL is composed from the ratios AL to BL, and FL to GL; and the ratio of triangle BGL to triangle CHL is composed from the ratios BL to CL, and GL to HL, as Clavius shows from Commandinus, by Proposition 23 of Book Six : hence, in the same manner is the ratio of triangle AFL to triangle BGL, and of that to triangle CHL. Similarly the rest of the triangles can be shown to be in continued proportion. Q.e.d.

[Note that in general the ratios on the two lines are different; however, $\Delta AFL/\Delta BGL = AL/BL \cdot FL/GL$, etc, insuring the truth of the proposition.

***Corollary.***

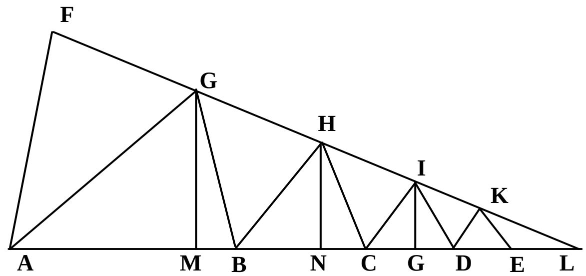
It hence follows that the trapezia AG, BH, CI, etc. also are in continued proportion, for the trapazia are the differences of the of the triangles in continued proportion, thus the truth of the corollary is apparent from the first part of this proposition .

## *PROPOSITIO LXXII.*

Iisdem positis ducantur in singulis trapeziis diametri, AG, BH, CI, &c

Dico triangula inde nata AGB, BHC, CID, &c, itemque triangula FAG, GBH, &c. esse continuae proportionalia.

***Demonstratio.***




**Prop.72. Fig. 1.**

Ex punctis enim G, H, I, &c. demittantur normales GM, HN, IG, &c. ratio trianguli AGB, ad triangulum BHC, componitur ex [a] rationibus AB ad BC, & altitudinis GM, ad altitudinem HN : sed quia AL, BL, CL, &c. ponuntur continuae proportinales etiam AB, BC, CD, [b] &c. erunt continuae in ratione suorum integrorum AL, BL, CL, &c., atque? GM, HN, IG, &c. ad AL normales sint? interse parallela sunt, erit GM ad HN, ut GL ad HL, hoc est ex datis, ut HL ad IL, igitur ratio trianguli AGB, ad triangulum BHC, eo aequitur ex rationibus BC ad CD, & HL ad IL, simili plane dIscursu ostendemus rationem trianguli BHC ad triangulum CID : ex iisdem rationibus esse compositam : igitur triangula AGB, GHC, CID, sunt in continua analogia, similiter de aliis idem demonstrabimus. Patet veritas propositionis.

*a Claudius ad 23 Sexti ; b 1 Huius.*

## *L2.§1.* *PROPOSITION 72.*

With the same lines and points in position, the diameters of the trapeziums AG, BH, CI, etc., are draw.

I say that the triangles thus produced AGB, BHC, CID, &c. and likewise the triangles FAB, GBH, &c. are in continued proportion.

***Demonstration.***

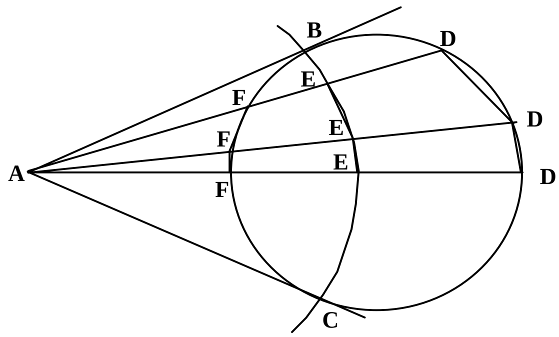
Normals GM, HN, IG, are sent from the points G, H, I, &c. The ratio of triangle AGB to triangle BHC is composed from the ratios [a] AB to BC and the altitude GM to the altitudine HN : but since AL, BL, CL, etc. are placed in continued proportion, AB, BC, CD, [b] etc. are also in proportionals in the ratio of the whole AL, BL, CL, etc. But GM, HN, IG, &c. are normal to AL and hence are parallel to each other, hence GM is to HN, as GL is to HL, or from what is given, as HL to IL. Therefore the ratio of triangle AGB to triangle BHC is equal to that from the ratios BC to CD and HL to IL. By a similar argument we can show that the ratio of triangle BHC to triangle CID is composed from the same ratios : therefore triangles AGB, GHC, and CID are in a continued ratio, similarly we can show the same for the other triangles, and the truth of the proposition is apparent.

## *PROPOSITIO LXXIII.*

Contingant circulum BCD duae lineae AB, AC, ex eodem puncto eductae ; & centro A intervallo B, C, describatur arcus BEC. deinde ex puncto A, ductis quotcumque lineis, secantibus AFED iungantur DD, EE, FF.

**[94]**

Dico triangula inde nata DAD, EAD, FAF, esse in totinua analogia.




**Prop.73. Fig. 1.**

***Demonstratio.***

Cum AB contingat circulum, erit [a] rectangulum DAF, aequale quadrato AB, hoc est quadrato AE. Unde DA, EA, FA, sunt continuae proportinales. Similiter reliquae omnes lineae DA, EA, FA, erunt in continua analogia. triangula igitur DAD, EAE, FAF [b] etiam in continua sunt analogia. Quod erat demonstrandum .

*a 36 Tertii ; b 72 Huius.*

***L2.§1.***

## *PROPOSITION 73.*

Two lines AB and AC are drawn from the same point A and touch the circle BCD; with centre A, and within the interval BC the arc BEC is described, then from the point A some lines are drawn with secants AFED and DD, EE, FF are joined.

I say that the triangles thus produced DAD, EAD, FAF are in complete proportion.

***Demonstration.***

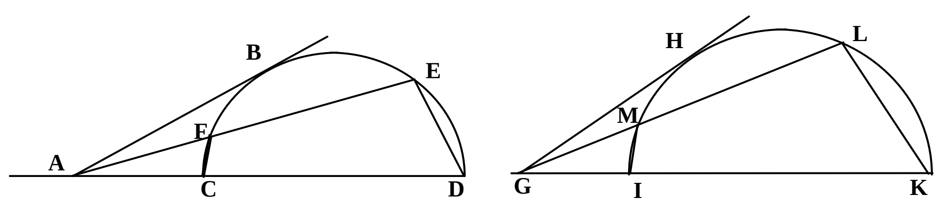
Since AB touches the circle, the rectangle DAF [a] is equal to the square AB, or to the square AE. Hence DA, EA, and FA are in continued proportion. Similarly the rest of all the lines DA, EA, FA are in continued proportion. Therefore the triangles DAD, EAE, and FAF also are in continued proportion. Q.e.d.

*a 36 Tertii ; b 72 Huius.*

## *PROPOSITIO LXXIV.*

Duos circulos inaequales CBD, IKL, contingant aequalea lineae AB, GH : & ex punctis A, G, educantur secantes ACD, AFE, GIK, GML continentes angulos aequales EAD, LGK : iunganturque FC, ED, MI, LK.

Dico reciprocam esse triangulorum EAD, KGL, MIG, FAC proportionem.




**Prop.74. Fig. 1.**

***Demonstratio.***

Quoniam tangentes AB, GH aequales sunt, patet [c] rectangula DAC, KGI aequalia esse: igitur & rationes AD ad KG, & IG [d] ad CA aequales sunt. Similiter cum rectangula EAF, LGM aequalium tangentium quadratis aequantur, inter se erunt aequalia: quare & rationes EA ad LG, MG ad FA eaedem sint, si igitur rationibus aequalibus AD ad KG, & IG ad CA, aequales addantur rationes, EA ad LG, & MG ad FA, erit ratio composita ex rationibus AD ad KG, & EA ad LG aequalis compositae ex rationibus IG ad CA, & MG ad FA; hoc est ratio trianguli EAD ad LGK triangulum aequalis rationi trianguli MGI, ad FAC triangulum; cum ob angulorum aequalitatum A G, rationem ex lateribus habeant compositam. Unde veritas patet propositionis. *c 36 Tertii ; b 24 Sexti.*

## *L2.§1.* *PROPOSITION 74.*

Two lines of equal length AB and GH are tangents to the unequal circles CBD and IKL ; from the points A and G the secants ACD, AFE, GIK, GML are drawn that contain equal angles EAD and LGK : the lines FC, ED, MI, and LK are joined.

I say that the triangles EAD, KGL; MIG, FAC are in reciprocal proportion.

***Demonstration.***

Since the tangents AB and GH are equal, it is apparent that the rectangles [c]DAC and KGI are equal: and therefore the ratios AD to KG, and IG [d] to CA are equal. Similarly, since the rectangles EAF and LGM are equal to the squares of the tangents, they are equal to each other: and whereby the ratios EA to LG and MG to FA equal. Therefore, if the equal ratios AD to KG, and IG to CA, are added the equal ratios EA to LG, and MG to FA, the common ratio of AD to KG and EA to LG is equal to the common ratio of IG to CA, and MG to FA; that is the ratio of triangle EAD to triangle LGK is equal to the ratio of triangle MGI to triangle FAC; also as the angles subtended A and G are equal, the ratio is composed from the sides alone. Thus the truth of the proposition is apparent. *c 36 Tertii ; b 24 Sexti.*

[ For $AB^2 = AC.AD = AF.AE = GH^2 = GI.GK = GM.GL$; and $AD/GK = IG/CA$ and $EA/LG = MG/FA$. Hence, $(AD/GK).(EA/LG) = EA.AD / LG.GK = \Delta EAD/\Delta LGK$ ;

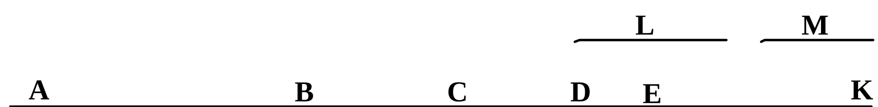
while $(IG/CA).(MG/FA) = MG.IG / CA.FA = \Delta MIG/\Delta FAC$. Hence $\Delta EAD/\Delta LGK = \Delta$**[95]**

# PROGRESSIONUM GEOMETRICARUM

# PARS SECUNDA

*Terminum cuiuscunque progressionis in infinitum continuatae designat:*

## *PROPOSITIO LXXV.*




**Prop.75. Fig. 1.**

Si fuerit magnitudo AB, ad magnitudinem BK, ut magnitudo BC ad magnitudinem CK.

Dico proportionem AB ad BC, sine termino continuati actu posse intra magnitudinem AK, ita ut numquam ad K perveniatur.

***Demonstratio.***

Fiat enim ut AB ad BC, sic BC ad L. quia igitur AB est ad BK, ut BC ad CK, erit alternando ut AB ad BC, id est ut BC ad L, sic BK ad CK : & rursum alternando, ut BC ad BK, sic L ad CK : quare cum BC ex datis, minor sit, quam BK, erit etiam L, minor quam CK : poterit ergo ipsi L, ex CK sumi aequalis CD : erant autem AB, BC, L, tres continuae proportionales; ergo & AB, BC, CD tres sunt continae. Fiat iam his tribus magnitudinibus continue proportionalibus AB, BC, CD, quarta proportionalis continua M : quoniam igitur paulo ante ostendi esse BK ad BC ut CK est ad L, sive CD, erit dividendo & invertendo, BC ad CK, ut CD ad DK : eodem plane discursu ostendam, M esse minoram ipsa DK, quo ante L ostendi esse minorem ipsa CK : poterit ergo ipsi M, ex DK, abscindi DE, aequalis. Sunt igitur quatuor magnitudines AB, BC, CD, DE continuae proportionales. Atque ita demonstrabimus proportionem AB ad BC, intra lineam AK sine termino posse actu continuari , ita ut nunquam ad K perveniatur. Quod erat demonstratum.

*A terminus can be designated to any progression continued to infinity.*

## *L2.§2. PROPOSITION 75.*

If the length AB is to the length BK, as the length BC is to the length CK, then I say that it is possible for the proportion of AB to BC to be continued by acting within the length AK without reaching K.

***Demonstration.***

For indeed the ratio BC to L can be made equal to the ratio AB to BC. Therefore since the ratio AB to BK is as BC is to CK, on rearranging, so AB is to BC or BC to L, thus BK is to CK : and again on rearranging, as BC to BK, thus L to CK. Whereby from what is given, BC is less than BK, and hence also L is less than CK : therefore L can be taken to equal a length CD in the interval CK : but as AB, BC, and L are continued proportionals; so therefore AB, BC, CD are three continued proportionals. Now a fourth continued proportional M can be constructed from these three lengths AB, BC, and CD: for we have just shown that BK to BC is as CK to L, or CD; hence on dividing and inverting, BC is to CK, as CD is to DK : from the same clear argument it can be shown that M is less than DK, from which before L was shown to be less than CK : hence M can be set equal to DE, less than DK. AB, BC, CD, and DE are hence four lengths in continued proportion. And thus we will show [in the next proposition] that the proportion AB to BC can be made to act within the line AK, thus without reaching K. Q.e.d.

[Set BC/L = AB/BC; again, as AB/BK = BC/CK, AB/BC = BK/CK = BC/L, or BC/BK = L/CK.
Now, BC < BK, hence L < CK; CD is made equal to L, and then AB, BC, CD are in continued proportion, etc.]

## *PROPOSITIO LXXVI.*


**Prop.76. Fig. 1.**

Si fuerit magnitudo AB, ad magnitudinem BK, ut magnitudo BC ad magnitudinem CK; & proportio AB ad BC, continuetur in magnitude AK, per plures terminos CD, DE, EF, &c.

Dico etiam CD, fore ad DK, & DE ad EK, &c. sic deinceps, ut AB est ad BK & BC ad CK, &c.

***Demonstratio.***

Quando quidem AB est ad BK, ut BC ad CK; erit alternando AB ad BC, hoc est ex datis BC ad CD, ut BK ad CK; & rursum alternando ac invertando: KB ad BC , uti KC ad CD; & dividendo ac invertendo, ut BC ad CK; sic CD ad DK: non aliter ostendimus ut CD ad DK; sic esse DE ad EK; atque ita deinceps in infinitum. Quod erat demonstratum.

## *L2.§2. PROPOSITION 76.*

If the length AB is to the length BK, as the length BC is to the length CK; and the proportion AB to BC is to be continued within the length AK through many terms CD, DE, EF, etc, then I say that CD also to DK, and DE to EK, etc, and thus henceforth, shall be as AB to BK, and BC to CK, etc.

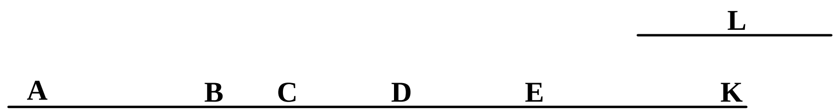
***Demonstration.***

Since AB is to BK as BC is to CK; AB to BC is on interchanging terms, or from what is given, BC to CD, is as BK to CK; and again on alternating and inverting: KB to BC, as KC to CD; and by dividing and inverting, as BC to CK; thus CD to DK: in the same way we can show that as CD is to DK, thus DE is to EK; and thus henceforth indefinitely. Q.e.d.

[Since AB/BK = BC/CK; AB/BC = BC/CD = BK/CK ; KB/BC = KC/CD giving BC/CK = CD/DK, etc.]

**[96]**

## *PROPOSITIO LXXVII.*




**Prop.77. Fig. 1.**

Data sit proportio quaevis minoris inaequalitatis AB ad AC.
Dico si haec continuetur, exhibendam magnitudinem quaevis data maiorem.

***Demonstratio.***

Detur enim magnitudo quaevis L : manifestum est si BC, excessus secundae magnitudinis AC, supra primam AB, aliquoties sumatur, maiorem fore magnitudine L : debeat ergo sumi BC quater, ut excedat L : continuetur ratio AB ad AC per quinque terminos AB, AC, AD, AE, AK; atque ita habebimus quatuor differentiae BC, CD, DE, EK. quoniam autem est ut a DA ad CA, sic DC ad CB, & cum DA maior sit CA, erit quoque DC maior quam BC : similiter, erit ED maior quam CD, & KE quam ED. ergo KB ex quatuor differentis composita maior erit quam BC quater sumpta. Quare cum BC quater sumpta maior ponatur quam L : erit KB multo maior quam L, ideaque AK adhuc multo quam L maior erit : constat igitur quod fuerat demonstratum.

## *L2.§2.* *PROPOSITION 77.*

The proportion is given with some inequality AB < AC.

I say that if this ratio is continued a number of times then a magnitude greater than any given can be produced.

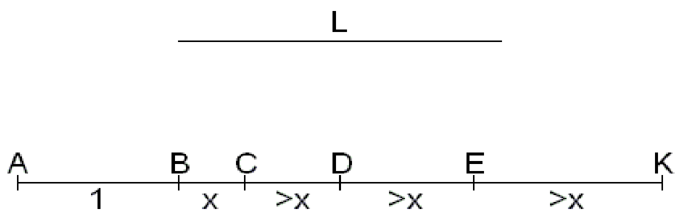
***Demonstration.***

Proof. Let *L* be any given magnitude: it is clear that if *BC*, the excess of the second magnitude above the first is repeated any number of times, it will exceed the magnitude *L*. [*Archimedean Order*] Therefore let us suppose that *BC*, taken 4 times will exceed *L*. Let the ratio *AB*:*AC* be continued through 5 terms *AB*, *AC*, *AD*, *AE*, *AK*. Thus we have four differences *BC*, *CD*, *DE*, *EK*. However, from Book II, Prop. I we have *DA* : *CA* :: *DC* : *CB*, and with *DA* > *CA*, *DC* > *BC*; similarly *ED* > *CD* and *KE* > *ED*, therefore *KB* made from four differences is greater than *BC* taken four times. Whence, since *BC* taken four times is greater than *L*, *AK* > *L*. [Translation supplied by Prof. Burn.]

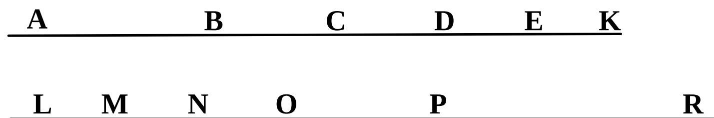
*a* Prop. I, this book.

[Helpful note due to Prof. Burn. : if a diagram with *ABCDK* is shown on a line with $4 \times BC$ > some given *L*, then calling $AB = 1$ and $BC = x$, we have $(1+x)^4 > 4x$, or in general $(1+x)^n > nx$, (this is a rudimentary form of Bernoulli's inequality) and by the Archimedean axiom $nx > L$ for some *n*.

Let there be given any proportion *AB* : *AC*, where *AB* < *AC*. I say that if this proportion is continued then it will exhibit a magnitude greater than any given magnitude.



## *PROPOSITIO LXXVIII.*




**Prop.78. Fig. 1.**

A magnitudine AK auferatur quaevis pars AB, & a residuo BK auferatur BC, ea lege ut sicut est AB ad BK, ita sit BC ad CK.
Dico si haec ablatio semper fiat, relinqui ex AK quantitatem data minorem *est universalis prima decimi*.

***Demonstratio.***

Detur enim quantitas LM aut alia quantumvis parva : dein ut KB ad KA, sic LM fiat ad LN: atque haec proportio per tot terminos continuetur, donec LR maior sit quam AK, hoc autem aliquando futurum est per praecedentem. Deinde quoties in M, N, O, P, divisa est LR, toties in ratione AB ad BK subdividatur AK in B, C, D, E. Quoniam ergo ex constructione: LM, LN, LO, &c. sunt continuae, erit ut LM ad LN, id est ut BK ad KA, sic LP ad LB. quare invertendo, ut AK ad BK, sic RL ad PL, & permutando ut AK ad RL, sic BK ad PL. Atqui ex constructione AK minor est quam RL, & ergo BK quam PL minor erit, deinde quoniam ex constructione est AB ad BK, ut BC ad CK; & CD ad DK. patet componendo omnes AK, BK, CK, EK esse continuas : itaque BK est ad CK, ut AK ad BK, id est ex constructione ut NL ad ML : sed PL est ad OL, ut NL ad ML; quia omnes RL, PL, OL, &c. sunt ex constructione continuae; ergo BK est ad CK , ut PL ad OL; & permutando ut BK est ad PL, sic CK est ad OL. atqui iam ostendimus BK minorem esse quam PL, ergo & CK quam OL minor erit. similiter demonstrabimus DR minorem esse quam NL, ac tandem EK esse minorem quam ML, quae erat data quantitas minor; ergo relinquitas quantitas: quod erat demonstratum.

## *L2.§2.* *PROPOSITION 78.*

From a magnitude AK some part AB is taken, and from the remainder BK is taken BC, according to the rule AB is to BK as BC is to CK.

I say that if this is always done by subtraction, then the amount to be left from the given AK *is always less than any magintude*. A generalisation of Euclid Book X.Prop I.

***Demonstration.*** [Translation supplied by Prof. Burn.]

**Let the quantity *LM* be given, or any other quantity however small**: then let $KB : KA :: LM : LN$ : and this proportion may be continued through so many terms until $LR > AK$; this follows from the previous proposition. Then as often as $LR$ is divided at $M$, $N$, $O$, $P$, so often $AK$ is subdivided in the ratio $AB : BK$ at $B$, $C$, $D$ and $E$. Since from the continued construction of $LM$, $LN$, $LO$, etc $LM : LN :: BK : KA :: LP : LR$. Wherefore by inversion $AK : BK :: RL : PL$, and on interchanging $AK:RL :: BK:PL$.
From the construction $AK < RL$ therefore $BK < PL$, then because of the construction $AB : BK :: BC:CK :: CD:DK :: DE :EK$, so *clearly* $AK$, $BK$, $CK$, $DK$, $EK$ are in continued proportion. Thus $BK:CK :: AK:BK$ and by construction $:: NL:ML$. But $PL : OL::NL:ML$ because $RL$, $PL$, $OL$ etc are all in continued proportion by construction; therefore $BK:CK :: PL:OL$ and by interchanging $BK:PL :: CK:OL$. We have already shown

that *BK* < *PL*, therefore *CK* < *OL.* Similarly we can show that *DK* < *NL* and at length *EK* < *ML*, which was less than the given quantity.

[Note. If AK, BK,… is $l$, $lr$, … decreasing, then for any ε; ε, ε(1/$r$), ε$(1/r)^2$,.. is increasing and so for some $n$, ε$(1/r)^n > l$, so ε > $lr^n$.]

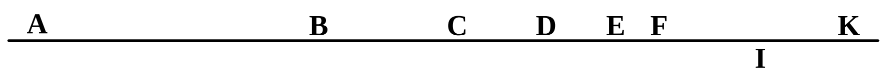
*Scholium.*

*Nota : dum in propositione dicitur*, si haec ablato semper fiat, dico relinqui ex AK quantitatum data minorem : *sensum propositionis nom esse, relinqui ex AK quantitatem data minorem, post ablationem terminorum in infinitum continuatum; sint post totam seriem absolutam, relinqui adhuc quantitatam data minorem; sed auferendo terminus ex AK, in ratione ante dicta, aliquando tot auferendos, ut residua pars totius AK, minor sit quantite data* quod in gratiam quorundam dictum sit.

*Scholium.*

While in the proposition it is said: "If this subtraction may always be done, the quantity to remain from *AK* is less than any given quantity." The meaning of the proposition is *not*: the quantity remaining from *AK* is less than any given quantity after subtracting the terms continuously to infinity; *nor* after the whole series is removed what is left of *AK* is less than any given quantity; but by taking away terms from *AK* in the aforesaid ratio, at length so much has been removed that the remaining part of *AK* is less than any given quantity.

## *PROPOSITIO LXXIX.*




**Prop.79. Fig. 1.**

Data sit magnitudo quaecumque AK : si fuerit

AB ad BK, ut BC ad CK.
AB ad AK, ut BC ad BK.
vel AB, BK, CK continuè proportionales.
AB ad BC, ut BK ad CK.
AB ad BC, ut AK ad BK.

Dico magnitudinem AK aequalem esse toti progressioni magnitudini, num continue proportionalium, rationis AB ad BC in infinitum continuatae terminum esse K.

***Demonstratio.***

Cum AB sit ad BK, ut BC ad CK, poterit ratio [a] AB ad BC, intra magnitudinem AK semper continuati, ita numquam perveniatur ad K, id est AK maior erit quaecunque serie finita terminorum; ergo AK non est minor serie tota rationis AB ad BC. Deinde quia AB est ad BK, ut BC ad CK, si ratio AB ad BC semper est continuetur, erit ut AB ad b BK, sive ut BC ad CK, sic CD ad DK, & DE ad EK, atque ita deinceps in infinitum : Itaque si continuetur semper ratio AB ad BC, relinquetur c tandem ex AK magnitudino quavis data minor. Quare AK nequit esse maior , serie rationis AB ad BC : nam si maior esset deberet aliquo excessu esse maior, ponatur is IK; igitur AI seriei rationis AB ad CD aequalis erit: ergo ratio AB ad BC quantumvis continuata non transiliet unquam I. ergo relinquetur ex AK magnitudo semper maior quam IK. ergo non minor quavis data, contra iam demonstrata. non erit igitur AK maior serie rationis AB ad BC: Quare cum neque minorem esse antea sit ostensum, aequalis sit necesse est. Quod erat demonstrandum.

Relique quorum hypothesum demonstrationes ad primam reducuntur. nam si fuerit AB ad AK, ut BC ad BK, erit dividendo AB ad BK, ut BC ad CK, ergo per primam demonstrationem, rationis AB ad BC semper continuatae terminus est in K.

Deinque si fuerit AB ad BC, ut BK ad CK, vel AK ad BK, erit permutando vel AB ad BK, ut BC ad CK, vel AB ad AK, ut BC ad CK. Unde iterum per primam demonstrationem conficitur proposition.

*a Prop. 75 huius; b Prop 76 huius; c Prop 78 huius*.

## *L2.§2.* *PROPOSITION 79.*

Some magnitude AK is given: if either of these ratios is true:

or

AB to BK, as BC to CK.
AB to AK, as BC to BK.
AB, BK, CK are continued proportionals.
AB to BC, as BK to CK.
AB to BC, as AK to BK.

I say that the magnitude AK is equal to the magnitude of the whole progression [i. e. equal to the limit of the series]: for if the proportions in the ratio AB to BC are continued indefinitely then K is the final amount.

***Demonstration.***

Since AB is to BK, as BC to CK, the ratio [a] AB to BC, within the length AK can always be continued, thus never reaching K, that is AK is larger than any finite series of terms ; hence AK is not less than the total series of the ratio AB ad BC. Hence as AB is to BK, as BC to CK, if the ratio AB to BC is always to be continued, then as AB to [b] BK, or as BC to CK, thus CD to DK, amd DE to EK, and thus henceforth to infinity. And thus if the ratio AB to BC is always continued, the length from AK that is left is at last smaller than any given length whatever [c]. Whereby AK is unable to be larger for the series of ratios AB to BC : for if AK is greater, then there is some length IK by which it is greater; therefore AI is equal to the series of ratios of AB ad CD: therefore the ratio AB to BC can be continued as far as you like without jumping beyond I at any time. Hence the remainder of the terms from the magnitude AK is always larger than IK, and therefore not less than any given, this is in contradiction to what has been shown. Therefore AK is not greater than the series of ratios AB to BC: whereas before it has been shown that it is not smaller either, then it shall be equal by necessity. Q.e.d.

The rest of these hypothesis can be demonstrated by being reduced to the first. For if AB to AK is as BC to BK, then on division AB is to BK as BC to CK, hence by the first demonstration, with the ratio AB to BC always to be continued, the end is in K.

And then if AB is to BC, as BK to CK, or AK to BK, by interchanging it will be either AB to BK, as BC to CK, or AB to AK, as BC to CK. Hence again the proposition is agreed upon by the first demonstration. *a Prop. 75 of this book; b Prop. 76 of this book; c Prop. 78 of this book*.

## *PROPOSITIO LXXX.*

**A M B C D E F K**

**Prop.80. Fig. 1.**

Data serie continue proportionalium AB, BC, CD, &c. quiuscumque propositionis, & quocumque in generis quantitatis; invenire magnitudinem quae omnibus terminis totius seriei datae in infinitum continuatur, sit aequalis:

**[98]**

***Constructio prima.***

Sit AM, differentia primorum duorum terminorum : fiatque ut AM ad BC secundum terminum, sit BC ad tertium quempiam CK. Dico CK, magnitudinem cum primo AB, & secundo termino BC, aequalem esse seriei universae AB, BC, CD, &c.

***Constructio secunda.***

Fiat ut AM duorum primorum terminorum differentia, ad AB primum terminum, ita secundus terminus BC, ad tertiam aliquam magnitudinem BK. Dico magnitudinem BK, cum primo termino, exhibere quantitam aequalem tot seriei.

***Constructio tertia.***

Differentiae duorum primorum terminorum AM, & primo termino AB, tertia proportionalis fiat AK. Dico AK totam seriam exhibere.

***Demonstratio.***

*Prima constructionis;* AM est ad BC, ut BC ad CK; igitur componendo AM cum BC, hoc est AB, erit ad BC, ut BK ad CK : & permutando, AB ad BK, ut BC ad CK. Quare [a] AK magnitudo, hoc est CK, cum AB & BC primis terminis, toti seriei aequalis est.

*Secunda constructionis;* AM est ad AB, ut BC ad BK ex constructione : igitur dividendo, ut AM est ad MB, id est ut AM ad BC, sic BC ad CK, itaque per demonstrationem primae constructionis, AK (hoc est BK una cum primo termino AB) aequatur toti seriei.

*Tertia constructionis;* Cum erit ex constructione AM ad AB, ut AB ad AK, erit invertendo, dividendo, rursusque invertendo AM ad MB, ut AB ad BK : & componendo AB ad MB, hoc est BC, ut AK ad BK. Quare AK b toti seriei aequalis est. Factum igitur est quod petebatur.

*Prima igitur constructione exhibere tota series praeter primum & secundum terminum. 2. constructione habetur seriei tota praeter primum terminum. 3. constructione simul tota producitur series. Huius autem problematis, aequalis esse aliquis eisdem universalitatem, amplius deductam habet in propositione 123, huius & corollario quarto ibidem.*

*a 79 of this book; b 79 of this book.*

## *L2.§2.* *PROPOSITION 80.*

Given a series of continued proportionals AB, BC, CD, etc. of some proposition, and for some quantities in general; to find the magnitude that is equal to all the terms of the given series continued to infinity.

***First construction.***

Let AM be the difference of the first two terms : and as AM is to BC the second term, BC is made to some third term CK. I say that the magnitude CK, together with the first AB and second term BC is equal to the whole series AB, BC, CD, &c.

***Second construction.***

As AM the difference of the first two terms, is made to AB the first term, thus the second term BC is to some other third BK. I say that the length BK, together with the first term, furnishes an amount equal to the whole series.

***Third construction..***

The difference of the first two terms AM, and with the first term AB, then the third of the proportionals AK is made. I say that AK presents the whole of the series.

***Demonstration.***

*First construction:* AM is to BC, as BC is to CK; therefore by adding AM with BC, or AB is to BC as BK ad CK : and by interchanging, AB is to BK, as BC is to CK. Whereby [a] the magnitude AK, or CK with AB and BC of the first terms, is equal to the whole series.

*Second construction:* AM is to AB, as BC is to BK by construction : therefore on dividing, as AM is to MB, or as AM is to BC, thus BC is to CK, and thus by the demonstration of the first construction, AK (or BK together with the first term AB) is equal to the whole series.

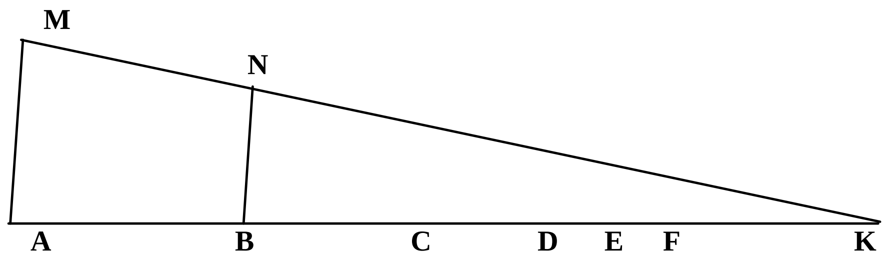
*Third construction:* as by construction AM is to AB, as AB is to AK, on inverting and dividing, and inverting again, AM is to MB, as AB is to BK : and on adding, AB is to MB, or BC, as AK is to BK. Whereby AK [b] is equal to the whole series. Therefore what was sought has been achieved.

*1. Therefore the whole series except the first and second terms are shown by the first construction.*

*2. The whole of the series except the first term is shown by the construction.*

*3. The whole series is produced at the same time from this construction. However this problem of producing the whole series has otherwise been more fully deduced in Proposition 123 of this book, and in the fourth corollary of the same.* *a 79 huius; b 78 huius*.

## *PROPOSITIO LXXXI.*




**Prop.81. Fig. 1.**

Quia vero ultimo ratio quantumcumque quantitatum ad lineas reduci potest, hinc etiam sequenti methodo, lineam toti seriei proportionalium linearum, reperiemus aequalae

***Constructio &demonstratio.***

Ex punctis A & B, erige ad quantum angulum, parallelas AM , BN, & quarum AB, BC ipses sint proportionales; & per punctis M & N ductur recta MN, concurrens cum ABC in K. Dico AK, toti seriei AB, BC, CD, &c. aequalem esse.

Quod autem MN, occurrere debeat ABC productae, patet ergo, quod BC ex hypothesi, minor sit quam AB;

**[99]**

and proinde etiam BN: cum AM, BN proportionales sint ipsis AB, BC; & igitur concursus in K; erit ut AB ad BC, sic MA ad NB; sed ut MA ad NB, sic AK ad BK, ergo AB est ad BC, ut AK ad BK. Unde [a] AK toti seriei aequalis est. Factum igitur est quod perebatur.

*a Ibid*.

## *L2.§2.* *PROPOSITION 81.*

Because the truly final ratio of any number of quantities can be reduced to line segments, by the following method, we may find a line segment equal to the whole series of proportional line segments.

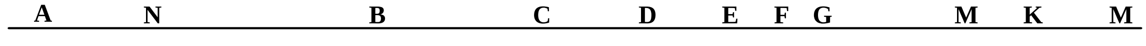
***Construction and demonstration.***

From the points A and B, you can raise parallel lines AM and BN to an angle of some size, and to which AB and BC are proportional; and through the points M and N the line MN is drawn crossing with ABC in K. I say that AK is equal to the sum of the series AB, BC, CD, etc.

For since MN ought to cross ABC produced, it is apparent that BC from hypothesis is less than AB; and hence BN also is less than AM, since AM, BN are in proportion with AB and BC themselves. Hence the line MN meets the line ABC in K: thus, AB is to BC as MA is to NB; but as MA is to NB, thus AK to BK, hence AB is to BC, as AK is to BK. Hence [a] AK is equal to the sum of the whole series. Hence what was required has been accomplished.

*a Ibid*.

## *L2.§2.* *PROPOSITIO LXXXII.*




**Prop.82. Fig. 1.**

Sit magnitudo AK : series tota rationis AB ad BC continuatae in infinitum.

AB ad BK, ut BC ad CK.

AB ad AK, ut BC ad BK.
Dico esse AK, BK, CK &c. continue proportionales.
AB ad BC, ut BK ad CK.
AB ad BC, ut AK ad BK.

***Demonstratio.***

Ex AB abscindatur NB, aequalis BC : si ergo non est AB ad BK, ut BC ad CK, neque permutando AB erit ad BC, id est BN, ut BK ad CK; ergo neque dividendo AN, erit ad NB, ut BC ad CK. Fiat igitur ut AN ad NB, sic BC ad aliam magnitudinem CM, maiorem vel minorem quam CK. Itaque componendo, AB erit ad NB, hoc est BC, ut BM ad CM: & permutando AB ad BM, ut BC ad CM. Quare totius seriei [b] rationis AB ad BC, terminus erit M; sive AM aequalis erit seriei rationes AB ad BC, quod est absurdum, cum AK maior vel minor quam AM, sit ex hypothesi aequalis seriei datae. non erit igitur alia ratios AB ad BK a ratione BC ad BK, ergo eadem, quod erat demonstrandum.

Reliquas autem assertionis partes ex prima deducemus. Cum enim iam demonstratum sit ex hypothesi theorematis, sequi AB esse ad BK ut BC ad CK, componendo erit AK ad BK, ut BD ad CK. quod erat secundum.

Et quoniam AK est ad BK, ut BK ad CK, igitur per constructionem rationis, AK est ad AB, ut BK ad BC; & invertendo AB ad AK, ut BC ad BK. quod erat tertium. rursum quoniam AB est ad BK, ut BC ad CK, erit permutando AB ad BC, ut BK ad CK, quod est quartium. denique quoniam ostensum est AB esse ad AK, ut BC ad BK, etiam permutando AB est ad BC, ut AK ad BK: quae omnia erant demonstranda.

***Corollarium.***

Ex quintae assertionis partem hoc theorema deducitur : data sit series magnitudinum continue proportionalium AB, BC, CD, &c. sine termino continuata. Dico esse ut una antecedentum, nempe AB, ad unam consequentium BC. sic omnes, hoc est infinitas antecedentes, sive AK, ad omnes sine infinitas consequentes, sive BK. *b 79 huius*.

## *PROPOSITION 82.*

The magnitude AK is given: the whole series of ratios of AB to BC is continued to infinitity.

I say that
- AB is to BK, as BC is to CK.
- AB is to AK, as BC is to BK.
- AK, BK, CK &c. are continued proportionals.
- AB is to BC, as BK is to CK.
- AB is to BC, as AK is to BK.

***Demonstration.***

From AB, NB is cut off equal to BC : if therefore AB to BK is not as BC to CK, neither by interchanging will AB be to BC or BN, as BK is to CK; and therefore nor on dividing will AN be to NB, as BC to CK. Hence AN to NB is made thus as BC to another length CM, greater or less than CK. And thus on adding, AB is to NB or BC, as BM to CM: and on interchanging AB is to BM, as BC to CM. Whereby of the whole series of ratios [b] AB to BC, the end is M; or AM is equal to the series of ratios AB to BC, which is absurd, for AK which is greater or less than AM, by hypothesis is equal to the given series. Therefore the ratio AB to BK is the same as the ratio BC to BK, q.e.d.

We can be deduce the remaining parts of the proposition from the first part. For indeed thus can be show from hypothesis of the theorem that is follows that AB is to BK as BC is to CK, on adding AK is to BK, as BD is to CK. Which is the second part.

And since AK is to BK, as BK is to CK, therefore by construction the ratio AK is to AB, as BK is to BC; and on inverting, AB is to AK, as BC is to BK. Which is the third part. Again, since AB is to BK, as BC is to CK, on interchanging AB is to BC, as BK is to CK, which is the fourth part. Hence as it was shown that AB is to AK, as BC is to BK, also on interchanging AB is to BC, as AK is to BK: whereby all the parts have been demonstrated. *b 79 huius*.

***Corollarium.***

From the fifth part of the proposition, this theorem can be deduced : a series of magnitudes is given in continued proportion AB, BC, CD, etc. to continue without end. I say that as one preceding term to another following term, such as AB to BC, thus the same ratio holds for all the terms. That is for all the preceding terms, or AK, to all the following terms or BK. *b 79 huius*.

## *L2.§2.* *PROPOSITIO LXXXIII.*

**A M B C D E F K**

**Prop.83. Fig. 1.**

Sit AK magnitudo producta ex ratione AB ad BC, in infinitum continuata; primi autem & secundi termini differentia sit AM;

Dico primo, differentiam AB, BC secundum terminum, CK totam seriem, (praeter duos primos terminos) in continua esse analogia.

**[100]**

Dico secundo, AM differentiam, ad primum terminum AB, esse ut BC secundus terminus, ad BK totam seriem, praiter primam terminum.

Dico tertio, differentiam AM, primum terminum AB, & totam seriem AK, in continua esse analogia.

***Demonstratio.***

**A M B C D E F K**

**Prop.83. Fig. 1.**

Quoniam AK magnitudino producta est ex ratione AB ad BC in infinitum continuata, erit per praecedentem AB ad BK, ut BC ad CK: igitur permutando ut BK ad CK, sic AB ad BC, hoc est MB, & dividendo AM ad MB, hoc est BC, ut BC ad CK. quod erat primum.

Rursum cum sit ut AM ad BC, hoc est MB, sic BC ad CK, erit componendo invertendo, ut AB ad MB, sic BK ad CK; and convertendo invertendo ut AM ad AB, ita BC ad BK. quod erat secundum.

Iterum cum sit ut AM ad AB, sic BC ad BK, erit permutando, ut AM ad BC, hoc est MB, sic AB ad BK, & invertendo componendo, & iterum invertendo ut AM ad AB, sic AB ad AK. quod erat tertio loco demonstrandum.

## *PROPOSITION 83.*

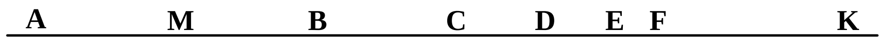
Let the length AK be producted from the ratio AB to BC continued indefinitely; also, the difference of the first and second terms is AM;

I say in the first case, that the difference AB, the second term BC, and the length CK of the total series (except the two first terms) are in continued proportion.

In the second place, I say that the difference AM, to the first term AB, is as BC the second term, to the whole series BK, except the first term.

In the third place, I say that the difference AM, the first term AB, and the whole series AK, are in continued proportion.

***Demonstration.***




**Prop.83. Fig. 1.**

Since the length AK has been produced from the ratio AB to BC continued indefinitely, then from the preceding proposition, AB is to BK, as BC is to CK: hence on rearranging, as BK is to CK, thus AB is to BC, and on dividing, AM to MB or BC is as BC is to CK. Which shows the first part.

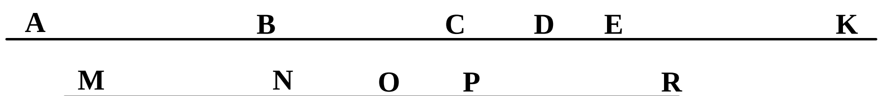
[AB/BC = BK/CK; AM/BC = BC/CK].

Again as AM is to BC, or MB, thus BC is to CK, then on adding and inverting, as AB is to MB, thus BK is to CK; and converting and inverting, as AM to AB, thus BC to BK. Which shows the second part.

[AM/MB = AM/BC = BC/CK; AB/MB = BK/CK ; MB/AB = CK/BK; AM/AB = BC/BK.]

Again since AM is to AB, thus BC is to BK, then on interchanging, as AM is to BC, or MB, thus AB is to BK, and on inverting and adding, and again inverting, as AM is to AB, thus AB is to AK. Which was to be shown in the third place.

## *L2.§2. PROPOSITIO LXXXIV.*




**Prop.84. Fig. 1.**

Datae sint binae series AK, MR (quas similes licebit appellare) magnitudinum continue proportionalium, in eadem proportione.

Dico seriem totam AK, esse ad seriem MR ut AB primus terminus seriei AK, ad MN, primum terminum seriei MR.

***Demonstratio.***

Per octavagesimam secundam huius, AK est ad BK, ut AB ad BC, hoc est , (cum ex hypothesi AB ad BC, MN ad NO, similes sint rationes) ut MN ad NO; sed per eandem etiam est MR ad NR, ut MN ad NO, igitur AK est ad BK ut MR ad NR. unde per conversionem rationis AK est ad AB, ut MR ad MN: & permutando series AK, est ad seriem MR, ut primus terminus AB primae seriei, ad primum terminum MN secundae seriei; quod erat &c.

## *PROPOSITION 84.*

Two series of magnitudes in the same continued proportion are given (which can be called similar).

I say that the whole series AK is to the whole series MR, as the first term AB of the series AK, is to the first term of the series MR.

***Demonstration.***

By the eighty-second proposition of this book, AK is to BK, as AB to BC, or , (since by hypothesis AB to BC and MN to NO, are similar ratios) as MN to NO; but from the same theorem also MR is to NR, as MN is to NO, therefore AK is to BK as MR is to NR. hence on converting the ratios, AK is to AB, as MR is to MN: and by interchanging, the series AK is to the series MR, as the first term AB of the first series, to the first term MN of the second series; quod erat &c.

[AK/BK = AB/BC = MN/NO; MR/NR = MN/NO; hence AK/BK = MR/NR, giving BK/AK = NR/MR and hence AB/AK = MN/MR or AK/AB = MR/MN. Hence AK/MR = AB/MN as required.]

## *L2.§2.* *PROPOSITIO LXXXV.*

*Quamquam ex iis, quae hactenus universaliter demonstravimus propositione octogesimam, & octagesimam primam, nota sit proportio totius seriei, ad primam magnitudinem; placuit tamen exercitii causa, notissimis quibusdam proportionibus applicare, maxime cum se illis saepe mentio futura sit.*

Primum igitur data sit, quocumque in genere quantitatis, proportio dupla, AB ad BC.
Dico totam seriem proportionis huius, sine termino continuatae, constituere magnitudinem, quae dupla sit primae magnitudinis.

**[101]**
***Demonstratio.***


**Prop.85. Fig. 1.**

Fiat enim ipsi BC, aequalis BI, & fiat ut AI ad AB, sic AB ad AK; erit K terminus [a] rationis AB ad BC, semper continuatae. & quoniam BA, dupla est BC, estque BI aequalia BC, erit quoque BA dupla IA; Quare cum sit ex constructione ut BA ad IA, sic AK ad AB, erit etiam AK, (id est tota series rationis AB ad BC) dupla BA, primae magnitudinis. Quod erat &c. *a 80 huius*.

*Nevertheless from these propositions, which we have demonstrated so far entirely from propositions eighty and eighty-one, the proportion of the whole series to the first term is to be noted. The reason for this being the case is still to be determined; however from certain of the more noteworthy proportions to be considered, this one ratio is referred to most often.*

## *PROPOSITION 85.*

The first term is given, and the proportion of AB to BC is two to one for all the terms in the series.
I say that the sum of this series of terms in proportion, continued without end, is double in size to the first term.

***Demonstration.***

Indeed BC is made equal to BI, and as AI is to AB, thus AB is to AK; K is the end of the ratios AB to BC [a], always continued. Since BA is double BC, and BI is equal to BC, BA is also double IA. Whereby as from construction, as BA is to IA, thus AK is to AB, and AK (which is the sum of the series AB to BC) is double BA, the first term. Q. e. d. *a 80 huius*.

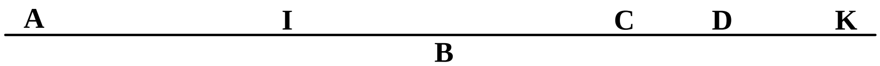
[This proposition is merely an illustration of Prop. 81. It is of interest to compare the geometrical result of Gregorius with the algebraic infinite sum formula for a G. P.: $S = a/(1 - r)$, where $a$ = AB, and the common ratio $r$ = BC/AB; for AB/BC = AK/BK, hence BC/AB = BK/AK, and AB/AK = (AB - BC)/AB; or $a/S = a(1 - r)/a$,
giving $S = a/(1 - r)$ ; In the present case, AB - BC = AI, and AI/AB = $^1/_2$, hence AK = 2.AB as required.]

## *L2.§2.* *PROPOSITIO LXXXVI.*

Detur deinde proportio tripla, AB ad BC.

Dico totam seriem , fore sesquialteram primae magnitudinis.

***Demonstratio.***




**Prop.86. Fig. 1.**

Fiat enim secundae magnitudini BC, aequalis BI; erit ergo BI, tertia pars AB; & AI excessu, seu differentia primae & secundae; unde si fiat ut AI ad AB, sic AB ad AK; erit AK [b] aequalis toti seriei; & quia IB, tertia pars est BA, erit BA sesquialtera ipsius AI : quare cum sit ut BA AI, sic AK ad AB, erit quoque AK, sesquialtera primae magnitudinis AB. Quod erat demonstrandum. *b 80 huius*.

## *PROPOSITION 86.*

In the next case, and the proportion of AB to BC is three to one.

I say that the sum of the series is one and a half of the first term.

***Demonstration.***

Indeed the second term BC is made equal to BI, and hence BI is equal to a third part of AB, and AI the difference of the first and second terms. Thus if AI to AB is made in the ratio AB to AK, then AK [b] is the sum of the whole series. Since IB is the third part of BA, BA is $^3/_2$ of AI : whereby as BA is to AI, thus AK is to AB, also AK is $^3/_2$ of AB. Q. e. d.

## *L2.§2.* *PROPOSITIO LXXXVII.*

Denique proportio data sit quadrupla, AB ad BC.

Dico totam seriem esse sesquitertiam primae magnitudinis : sive eam habere rationem ad primam magnitudinem, quam quatuor ad tria.

***Demonstratio.***


**Prop.87. Fig. 1.**

Fiat enim BC aequalis BI; erit ergo BI quarta pars AB, & consequentur BA sesquitertia ipsius IA. Fiat igitur ut AI ad AB, sic AB ad AK; erit AK [c] tota seriei. Quare cum KA sit ad BA, ut BA ad IA, erit KA sesquitertia primae magnitudinis BA. Quod erat demonstrandum. *c 80 huius*.

## *PROPOSITION 87.*

In the next case, the proportion of AB to BC is four to one.

I say that the sum of the series is one and a third of the first term : or the sum of the series to the first term has the ratio four to three.

***Demonstration.***

Indeed BC is made equal to BI, and hence BI is equal to a quarter part of AB, and AB is one and a third of IA. Thus if AI to AB is made in the ratio AB to AK, then AK [c] is the sum of the whole series. Whereby since KA is to BA, thus BA is to IA; AK is the four thirds part of BA. Q. e. d. *c 80 huius*.

[Again, AB/BC = AK/BK or (AB - BC)/AB = AB/AK; but BC = IB = $^1/_4$AB, hence AI = $^3/_4$AB and AK/AB = $^4/_3$ as required. ]

*Sed de his modo satis: poterit enim quilibet ex propositione ex octuagesima huius bene intellecta, data proportionis rationalis, seu numeri ad numerum, totam seriem, & consequnter rationem seriei, ad primum terminum exhibere.*

*Scholium.*

*Quod si constructionem secundam octuagesimae propositionis adhibere placuerit, habebitur unica operatione proportio primae magnitudinis ad reliquam seriam: si vero prima constructione utamur, exhibebitur proportio primae & secundae magnitudinis simul sumptarum, ad seriem reliquam.*

*Praesens materia memorem me facit eius, quod in argumento huius libri praefatus sum, cum mentio incideret Zenonici discursus, quo se credebat omnem motus rationem e medio tollere posse: nucleus autem argumenti tanta apud auctorem authoritatis exstitit, ut eundem Achillis invictissimi Ducis nomenclatura dignaretur : persuasum habens, cum nucleum usque ad eo duro tortice futurum, qui par foret omnibus Philosophicorum Elenchorum malleis sustinendis.*

*Repetam discursum Zenonis, iisdem verbis, quibus in praefatione huius libri sum usus. consistebat ille in duobus, quae moverentur : primo Achillis, velocissime currentis, altero testudinis tardissime reptantis,*

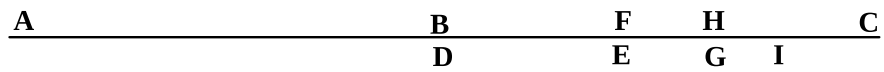
**[102]**


**Prop.87. Fig. 2.**

*Ponatur inquiebat ille, Achilles cursor pernicissimus, ex A puncto testudinem reptantem per semitam BC, lentissimo motu, velle assequi; quo tempore Achilles tendit ex A ad B, mota est testudo ad aliquod spatium, per veniens in D : igitur necdum Achilles assecuties est testudinem; iterum quo tempore Achilles ex B currit, ut assequatur testudinem existentem in D, mota est testudo ad F punctum; igitur Achilles existens in D nondum assecutus est testudinem; atque hoc in infinitum eveniet ; quoniam continuum divisibile est in infinitum, unde numquam Achilles assequetur testudinem. Incumbit igitur nobis hunc nucleum effringere, ex doctrina huius libri; quod assecutos nos esse cognosces, cum ipsissimum punctum assignaverimus, quo Achilles testudinem apprehendre.*

*Ut nodum hunc Gordium, ex principiis huius libri dissoluamus, supponemus, non minus Achillem, quam testudinem in suo cursu uniformiter procedure; ita ut celeritas, prima parte motu assumpta, perseveret in eodem statu, usque ad ultimum temporis momentum quo suae spatia decurrunt; supponemus insuper, (quoniam omnis motu species est quantitatis) duos hosce motus, cum uniformes ponantur, insuis partibus, sortiri inter se aliquam proportionem, quod necesse est eveniat inter omnes quantitates, quae in eadem specie versantur; ut sunt duo motus recti & uniformes.*

*Ponatur igitur proportio duorum horum mobilium, secundum celeritatem, consistere in ratione dupla; ita ut Achilles duplo celerius, spatium decurrat, quam testudo : igitur quo tempore testudo ad quartem partem stadii promota fuerit, medium stadium confecerit Achilles. Eductae itaque lineae AC, DC, in ratione dupla ex C puncto, dividentur in B, F, H, &c, & E, G, I secumdum rationem duplam ;ut AC dupla, sit BC, & DC dupla EC. Item BC dupla FC, &c. EC dupla GC, &c.*




**Prop.87. Fig. 3.**

*Consistat itaque Achilles in A. sitque AC semitam repraesentans stadio longitudine aequalem; testudo vero constituta in dimidio stadii, in puncto B, vel D, posita DC aequali ipsi BC. Quoniam Achilles ex A moveri incipit, quo tempore ex D inchoat cursum testudo; igitur pervenerit Achilles ex A in B, quo tempore ex D, testudo pertinget in E : & quo Achilles ex B pertinget in F, eadem pervenerit testudo ex E in G. & sic consequenter: quia vero terminum progressionis rationis AB ad BF terminatur in C, prout propositione octuagesima quinta demonstratum est; similiter cum progressio, sucundum rationem BF, ad FH, vel DE ad EG, finem sortiatur in puncto C, secundum eandum proportionem: igitur concursus duorum horum mobilium, Achillis scilicet & testudinis, continget in puncto C: Quod si loco proportionis duplae, assumatur proportio tripla, tunc assignabitur concursus per proportionem octaugesimam sextam hiuis. Si vero quadrupla, inserviet propositio octuagesima septima, & sic de reliquis.*

*Captiosus Zenonis discursus molestias creat non consideranti discrimen, quod in eo exsurgit, inter duplicem progressionem, qua argumentationis filum dubium facit; alia enim est progressio per partes aequales; alia per partes proportionales; hic utriusque cursus supponitur fieri per partes uniformes, sive per passus aequales, cum passus primus a secundo, vel tertio non discrepet, licet duos passus Achilles, verbi gratia, eadem contingant tempore quo unus passus testudinis; secundum vero hos passus sit utriusque*


**Prop.87. Fig. 4.**

*cursus : Zeno autem in decursu argumenti sui, distinguit motus cursorum per partes proportionales, secundum quas mobilia nullo modo moventur; ac proinde in idem eius discursus recidit, ac si dicat quis, eo tempore quo dividam lineam AE, in partes quatuor aequales, alius eam subdividet secundum aliquam seriem per partes proportionales, profecto citius assignabantur termini quatuor partium aequalium, qua infiniti termini partium proportionalium: Achilles enim & testudo decurrentes AE spatium, per partes aequales, suorum passuum aequalium terminum tandem acquirunt; Zeno vero dum hac contingunt, a cursoribus dividi vult spatium AE, in partes proportionales, secundum quas mobilia non succedunt.*

**[103]**

*Ad argumentum porro respondendum est, dum dicitur :* Priusquam Achilles ex A perveniat ad punctum B, mota est testudo ex B in F:


**Prop.87. Fig. 5.**

*Sensum huius propositionis coincidere cum hoc quo dicatur, prius debet Achilles assignare punctum B, quam notet punctum F. quid repugnat cursui secundum rationem motus; nam omnis assignatis in hac materia continet rationem subsistentiae, ut Mathematici sentiant, saltem secundem intellectam, ac proinde alicuis quietis, quae motui repugnat. Veram hac in gratium Philosophorum dicta sufficiant.*

*But enough has been said about this method : for anyone with a good understanding of Proposition 80 of this work can find the sum of the whole series, and consequently the ratio of the sum to the first term, either geometrically for a given ratio of proportion, or numerically in terms of one number to another.*

*Scholium.*

*For if one is resolved to put the second construction of the 80th proposition to use, one has in a single operation the proportion of the first term to the rest of the series : while indeed if we use the first construction, the proportion of the first two terms taken together to the rest of the series is found.*

*The present material brings to mind the argument that I have set out in the preface of the present book, concerning the discourse of Zeno, where he set out that all motion could be understood in terms of the average motion: but the heart of the argument really arose from the authority of the author, in order that a runner by the name of Achilles, the same as that most invincible leader, is deemed worthy to be persuaded to do battle with the hard- shelled one, for Zeno was an equal of the Elenctic philosophers* [*Those given to refuting or cross-examining matters*], *and sustained by their might*.
[*Note: I have paraphrased somewhat here, as the original narrative appears to be a pun on the word 'nucleus', which is the nut or kernel, while it is also the centre of the argument, and which cannot be reproduced sensibly.*]

*Repeating Zeno's argument, in the same words that I used in the preface of this book. The argument consists of two opponents who are in motion : first there is Achilles, who is running swiftly, and with the other a tortoise who is crawling most slowly. It may be said that Achilles, that most nimble of runners, is placed at the point A initially, and wishes to overtake the tortoise who crawls along the path from BC most slowly. By the time Achilles has gone from A to B, the tortoise has moved some distance to come to the point D, and Achilles has not yet overtaken the tortoise. Again, by the time Achilles has run from B to D, in order to overtake the tortoise present at D, the tortoise has moved to some point F; hence Achilles, now present at D, has not yet overtaken the tortoise; and this division of the interval between the two can be the outcome indefinitely: thus Achilles can never overtake the tortoise. It is incumbent therefore for us to break open this nut from the teaching of this book; since we know that Achilles really does overtakes the tortoise, we can assign the very point itself at which this happens.*

*In order that we can resolve this Gordian knot from the principles of this book, we can assume that not only Achilles but also the tortoise are thus to continue in their paths at uniform rates, so that their speeds for the first part of the motion are assumed to continue in the same state until the final moment of time when their distances* [*from A*] *are equal. We assume above, since any kind of motion can be specified by a number or quantity, that for these two motions each considered uniform in their own intervals, then some proportion can be chosen between the intervals, which necessarily comes about between all the intervals, as the two motions are along a straight line and uniform.*

*Therefore the proportion between these two intervals, according to the speed, is taken for argument's sake to be in the ratio two to one. Thus Achilles runs twice as fast as the tortoise: hence in the time taken for the tortoise to move forwards by a quarter length of the course, Achilles has made it as far as the middle. Hence the lines AC and DC drawn from the point C, are divided in the ratio two to one by the points B, F, H, etc. and E, G, I, etc., in order that AC is double BC, DC is double EC; likewise, BC is double FC, EC is double GC, etc.*

*So it is agreed that Achilles is at the point A, and AC is equal to the length of the course, representing the path. The tortoise is set in place in the middle of the course at B or D* [*Fig. 3*], *where DC is equal to BC. Achilles begins to move from A at the same time as the tortoise starts the course from D. Therefore in the time taken for Achilles to go from A to B, the tortoise reaches E from D; and in the time Achilles reaches F from B, the tortoise arrives at G from E, and thus consequently. Since the end of the progression of the ratio AB to BF ends in C, as has been shown in Prop. 85; similarly with the progression following the ratio BF to FH, or DE to EG, the end is chosen at the point C for the intervals following the same proportion. Therefore the two motions of Achilles and the tortoise run together to meet at the point C. But if in the place of the duplicate proportion, the triplicate proportion is assumed instead, then the point of concurrence is assigned by Prop. 85 of this book, and if the proportion is the quadruple, then Prop. 87 takes care of the point, and thus for the remainder.*

*Zeno's deceptive discourse gives rise to troubles by not discriminating between two series that arise in it, which leads to a doubtful thread in the argument. For one series is a progression by equal parts* [*or intervals*], *while the other is by proportional parts. Here in our discussion, each course is considered to be made in uniform intervals, or in equal steps from the first to the second, or the third without being out of step. For example, Achilles is allowed to have two steps that occur in the same time the tortoise has one step* [*of equal size*], *and the other course is indeed following these steps. However Zeno in the discourse of his argument, has distinguished the motion of the courses by proportional parts, following which the moving objects are not moved in any way [in that the time intervals tend towards zero]; and according to his discourse the same interval is diminished, and if one should say by what [amount], it is initially by that time in which the line AE has been divided into four equal parts, while the other is subdivided following some series of proportional parts; indeed the four boundaries of equal parts can be assigned more quickly than an infinite number of proportional parts. For Achilles and the tortoise are hurrying across the* *distance AE in steps of equal interval, and at last reach the end point, each in terms of their own equal steps: but Zeno, while they [actually] touch here, wants their courses in the distance AE to be divided into proportional parts, following which the objects in motion do not succeed in meeting.*

*When it is said: Before Achilles arrives at the point B from A, the tortoise has moved from B to F [Fig.5]; the response to the argument should be: The understanding of this proposition is to agree with that which is stated. Before Achilles can be assigned to the point B, the point F is noted; which disagrees with a course that follows [only] the ratio of the motion. For all intervals assigned in this matter contain a sustaining ratio, [i.e. the difference of the distances gone], as mathematicians think they have an understanding of the second method, and hence are quiet about other things which are in disagreement with the motion. Truly, with the grace of the philosophers, enough has been said here.*

*[It is not the business of the translator to interject his own ideas into the discourse of Gregorius. However, we should note that the paradox of Zeno is still with us, and lies at the heart of the limiting process. These arguments were produced far in advance of a time when any proper understanding could be forthcoming; for Zeno was upset by the idea that an infinite number of subdivisions had to be gone through before the moving objects could meet. Gregorius some 2000 years later dismissed these objections, and said essentially: Look, you are going about this problem in the wrong way: this is how you should think about it, in terms of a geometric progression; I can even sum the series of infinite intervals for you and find the point where the bold Achilles overtakes his adversary. We realise now that both men had a valid point to make in an understanding of the limiting process. ]*

## *L2.§2.* *PROPOSITIO LXXXVIII.*

Data AB, cuius tertia pars sit CB; fiat ut tota AB, ad CB tertiam sui partem, ita CB ad CD, & CD ad DE, & DE ad EF, atque ita semper.

Dico K terminum huius progressionis, bifariam dividere propositam magnitudinem AB.

***Demonstratio.***


**Prop.88. Fig. 1.**

Erit enim ex hypothesi AB cum BK, aequalis toti seriei proportionis triplae. atqui tota series rationis triplae, [a] sesquialtera est magnitudinis primae, ergo AB cum BK, sesquialtera est primae magnitudinis AB; ergo BK, est eiusdem dimidia. Quod erat demonstrandum. *c 86 huius*.

## *PROPOSITION 88.*

Given the line AB, the third of which is CB; the progression is constructed such that the whole length AB to CB is three parts to one, and thus CB to CD, CD to DE, DE to EF, and so on in the same ratio.

I say that the end point K of this progression divides the magnitude AB in two equal parts.

***Demonstration.***

For indeed AB and BK by hypothesis is equal to three times the sum of the whole series. But the whole series of ratios tripled [a] is one and a half times the first, hence AB plus BK is one and a half of the first magnitude AB; hence BK is the half of AB. Q. e. d. *a 86 huius*.

[From Prop. 86, the sum of the whole series is $1^{1}/_{2}$ times the first term; BK or S = $^{3}/_{2}$BC; for: AB + BK = 3.BC + BK = 3.BC + $^{3}/_{2}$ . BC = $^{9}/_{2}$. BC = 3.S. Hence AB + S = 3.S, or S (or KB) = AB/2.]

## *L2.§2.* *PROPOSITIO LXXXIX.*

Detur quantitatis AB, cuius quarta pars sit BC; fiat ut tota AB, ad quartam sui partem BC, ita BC ad CD, & CD ad DE, & DE ad EF, atque ita semper continuando, punctum K hiuis progressionis terminus emergati:

Dico BK esse tertiam partem propositae quantitatis AB.

***Demonstratio.***


**Prop.89. Fig. 1.**

Nam ex hypothesi AB cum BK, est tota series proportionis quadruplae. Atqui tota series rationis quadruplae, [b] est ad primam magnitudinem duem, ut quatuor ad trias, ergo AB cum BK ; est ad AB , ut quatuor ad tria. & dividendo KB ad AB, ut unum ad tria, hoc est KB tertia pars est ipsius AB. Quod erat demonstrandum. *b 87 huius*.

## *PROPOSITION 89.*

Given the amount AB, the quarter of which is BC; the total is constructed such that the whole length AB to BC is four parts to one, and thus BC to CD, CD to DE, DE to EF, and always to be continued thus, until the end of the progression appears.

I say that BK is the third part of the proposed quantity AB.

***Demonstration.***

For by hypothesis AB and BK is four times the sum of the whole series of proportions. But the whole series of ratios quadrupled [b] to the first term is in the ratio two to one, hence AB plus BK is to AB, as four is to three, and by division, KB to AB, as one is to three. Hence KB is the third part of AB. Q. e. d.
*a 87 huius*.

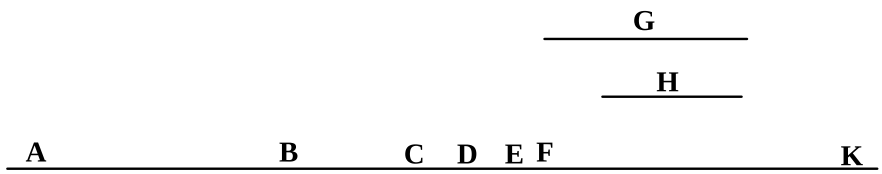
[From Prop. 87, the sum of the whole series is $1^{1}/_{3}$ times the first term, or BK or S = $^{4}/_{3}$BC; for: AB + BK = 4.BC + BK = 4.BC + $^{4}/_{3}$ . BC = $^{16}/_{3}$. BC = 4.S. Hence AB + S = 4.S, or S (or KB) = AB/3.]

## *L2.§2.* *PROPOSITIO XC.*

Datam magnitudinem AK, in duobus punctis B, C, dividere, ut AB ad BC, habeat rationem datam, G ad H. Ex proportionis AB ad BC continuatae progressio terminetur in K.

**[104]**

***Constructio &Demonstratio.***




**Prop.90. Fig. 1.**

Divide AK in B, ita ut sit AK ad BK, ut G ad H; & ut AK ad BK, sic AB fac ad BC. Dico factum esse quod petebatur. cum enim sit ut AK ad BK, sic AB ad BC, erit & reliquum BK [a] ad reliquum CK, ut AK ad BK: unde rationis AB ad BC [b] terminus est K : est autem AB ad BC ratio eadem cum ratione AK ad BK, it est G ad H. constat igitur propositum. *a Euclid V.19; b Prop. 79 huius*.

*Corollarium.*

Ex hac propositione manifestum est, omnem magnitudenem, omnes rationum seriei continere.

## *PROPOSITION 90.*

The given magnitude AK is divided by the two points B and C, in order that AB to BC has the given ratio G to H. The progression of the continued proportion AB to BC is terminated in K.

***Construction &Demonstration.***

Divide AK in B, so that the ratio AK to BK is as G to H; and as AK is to BK, thus make AB to BC. I say that what is sought has been done. For indeed as AK is to BK, thus AB is to BC, and the remainder BK [a] to CK, as AK to BK: hence the termination of the ratio AB ad BC [b] is the point K : but the ratio AB to BC is the same as the ratio AK to BK, that is G to H. Therefore the proposition is agrees upon. *a Euclid V.19; b Prop. 79 of this book*.

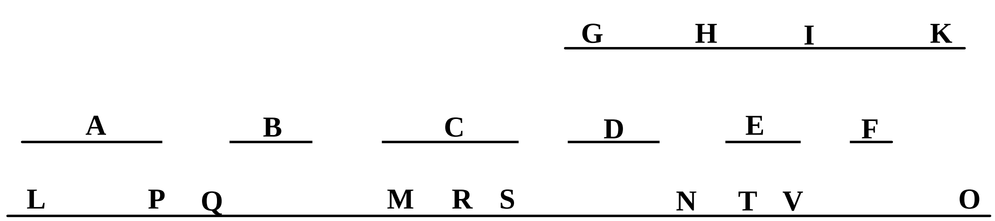
*Corollary.*

It is observed that every magnitude of all ratios of a series are contained in this proposition .

## *L2.§2. PROPOSITIO XCI.*

Datis quotcumque rationibus A ad B, C ad D, E ad F, datam magnitudinem LO, oporteat dividere in tot partes, quot datae sunt rationes, nempe in LM, MN, NO, ita ut partes illae eandem habeant rationem, quam primi datarum rationum termini, A, C, E, & praeter ea singulae partes LM, MN, NO singulis rationibus, sine termine continuatis, sint aequales.

***Constructio &Demonstratio.***




**Prop.91. Fig. 1.**

Fiat GHIK, omnibus A, C, E, aequalis, & ut divisa est GK, sic divide LO in M & N; denique per 90 huius LM ita divide in P & Q, ut sit LP ad PQ sic ut A ad B: & rationis LP ad PQ, terminus sit M, similiter MN, NO divide in punctis R, S, T, V secundum rationes C ad D, E ad F, ita ut rationum MR ad RS, NT ad TV, termini sint N & O. Dico factum quod petebatur. Demonstratio ex ipsa constructione est manifesta.

## *PROPOSITION 91.*

For some given ratios A to B, C to D, E to F, it is required to divide a given magnitude LO into as many parts as the ratios give, surely in LM, MN, NO, thus as the parts in each section have the same ratio as the first of the given terms of the ratio A, C, E, and beyond that the individual parts are equal to the individual ratios within LM, MN, and NO which are to be continued without end.

***Construction &Demonstration.***

Let GHIK be made equal to all the first parts of the ratios A, C and E. Thus as GK is divided, so divide LO in M and N; hence by Prop. 90 of this book, LM is thus to be divides in P and Q, in order that LP to PQ thus is as A is to B: & the termination of the ratios LP to PQ is M; similarly MN and NO are to be divided in the points R, S, T, V following the ratios C to D, and E to F, thus in order that the terminations of the ratios MR to RS, and NT to TV, are the points N and O, I say that what is sought has been done. The demonstration is clear from the construction itself.

## *L2.§2.* *PROPOSITIO XCII.*

Datis quotcumque rationibus AB ad BC, DE ad EF, GH ad HI, &c. magnitudinem invenire, quae omnes harum rationum, series progressionum adaequet.


**Prop.92. Fig. 1.**

***Constructio &Demonstratio.***

Per octogesimam huius inveniantur magnitudines, quae singularum rationum series adæquent; atque omnibus illis magnitudinibus, una fiat æqualis, hæc, ut patet, adæquabit omnes series datarum rationum.

## *PROPOSITION 92.*

For some given ratios AB to BC, DE to EF, GH to HI, etc. , to find the magnitude of the sum of the series of the progressions formed from these ratios.

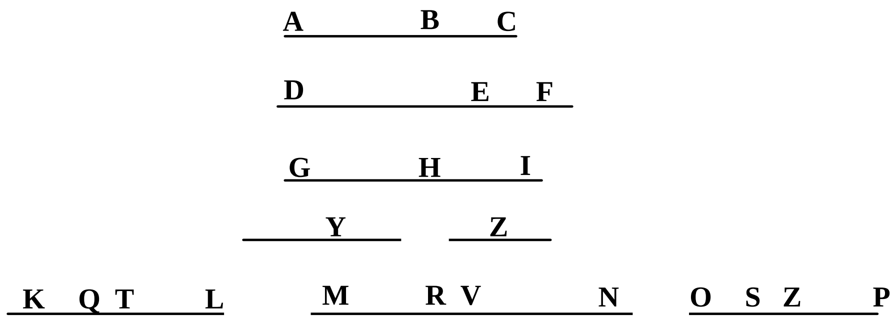
***Construction &Demonstration.***

By Prop. 80 of this work, the magnitudes can be found to which the individual series of ratios are equal, and the sum of all of these magnitudes, as is apparent, is made equal to the sum of the series of the given ratios.

**[105]**

## *L2.§2.* *PROPOSITIO XCIII.*

Datas quotcumque series diversarum rationum ita constituere, ut sint in continua analogia datae proportionis.




**Prop.93. Fig. 1.**

***Constructio &Demonstratio.***

Rationes diversae sint AB ad BC, DE ad EF, GH ad HI. & alia dato ratio Y ad Z. Fiant tres magnitudines KL, MN, OP continue proportionales, in ratione Y ad Z, & KL ita [a] dividatur in QT, &c. ut seriem constitunt rationis AB ad BC; & MN ita dividatur in RV, &c. ut adaequet seriem rationis DE ad EF : at demum dividatur similiter OP in S & Z, ut rationis GH ad HI constitunt seriem, factumque erit quod petebatur. *a 90 huius*.

## *PROPOSITION 93.*

You are given some series of different ratios thus to be put in place, so that they are continued ratios of a given proportion.

***Construction &Demonstration.***

The different ratios are AB to BC, DE to EF, GH to HI; and with the other given ratio Y to Z. Three magnitudes KL, MN, and OP are made in continued proportion, in the ratio Y to Z; KL is thus [a] divided in Q, T, etc, in order that AB to BC constitute a series of the ratio ; again MN is thus divided in R,V, etc, in order that DE to EF is equal to a series of the ratio : and finally OP is similarly divided in S and Z, in order that GH to HI constitute a series of ratios. What was required has been accomplished. *a 90 huius*.
[In modern terms, let the common ratios for the three lines be $r$, $s$, and $t$; while the fourth ratio is $R$. The line KL can be chosen at will, but MN = $R$.KL, and OP = $R^2$.KL; KL is the sum of the series with ratio $r$, and likewise MN and OP are the sums for the ratios $s$ and $t$. Thus, the first terms KQ, MR, and OS are chosen to accommodate the ratios $r$, $s$, and $t$, and the sums KL, MN, and OP ]

## *L2.§2.* *PROPOSITIO XCIV.*

Datam magnitudinem AE, semel sectam in B, ita rursum secare in C & D, ut tam progressio rationis AB ad BC, quam progressio rationis AD ad DB, terminetur in E.


**Prop.94. Fig. 1.**

***Constructio &Demonstratio.***

Reperiatur primo ipsius AE, BE, tertia proportionalis CE. Dico progressionem AB, BC terminari in E. patet ex septuagesima nona huius. Deinde inter AE, BE inveniatur media DE. patet rursum progressionem AD, DB terminari in E, per eandem propositionem : fecimus ergo quod petebatur.

## *PROPOSITION 94.*

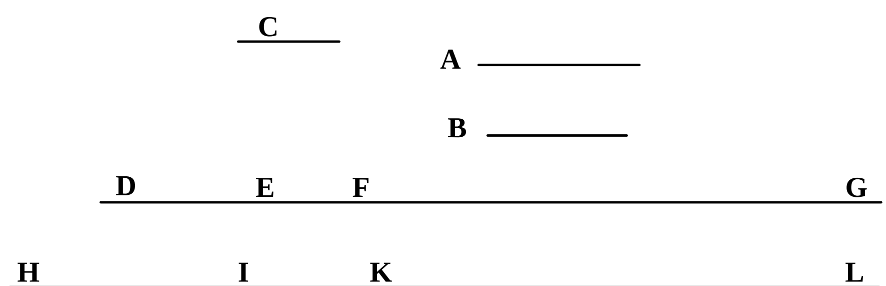
The given magnitude AE is cut once in B, and thus cut again in C and D, so that the progression of the ratio AB to BC, as for the progression of the ratio AD to DB, is terminated in E.

***Construction &Demonstration.***

First the third proportional CE of AE and BE is itself found. I say that the progression AB, BC is to terminate in E, as is apparent from Prop. 79 of this work. Then the mean DE is found between AE and BE; and again it is apparent that the progression AD, DB is to end in E, by the same proposition : we have done what was asked. [In this case, there are two progressions: the first is produced from the ratio BE/AE = CE/BE from AB/BC = AE/BE; while the second arises from the ratio DE/AE = BE/DE, etc.]

## *L2.§2.* *PROPOSITIO XCV.*

Data magnitudine DG, & proportione maioris inaequalitatis A ad B, itemque alia magnitudine C: examinare quomodo series progressionis datae rationis A ad B, cuius primus terminus sit C, se habeat ad datam DG magnitudinem.




**Prop.95. Fig. 1.**

**[106]**
***Constructio &Demonstratio.***

Fiat ut A ad B, sic DG ad EG. Primo igitur data magnitudo C, quae primus terminus progressionis esse debet, aequalis sit DE, erit series rationis A ad B habens primum terminum C aequalis datae magnitudini DG nam fiat DE (id est C ad EF, ut A ad B). quoniam igitur DG est ad EG ut A ad B, erit quoque DE ad EF ut DG ad EG. ergo progressio [a] rationis DE ad EF, (id est progressio rationis A ad B, habens primum terminum C) constituet magnitudinem DG.

Secundo si magnitudino C, quae debet esse primas terminus, maior sit quam DE, erit progressio AB, habens primum terminum C, maior quam DG. Fiat enim ipsi C aequalis HI, utque A est ad B, sic HI sit ad IK, & progressionis HI ad IK [b] reperiatur terminus L. Igitur [c] ut HI ad DE sic HL tota series progressionis HI, IK ? ad DG totam seriem progressionis DE, EF. atqui HI maior est quam DE, ergo HL maior est quam DG.

Si denique C minor sit quam DE, erit quoque progressio rationis A ad B, habens primum terminum C, minor quam DC; quod eodem modo ostendimus, quo primum. Fecimus igitur quod poscebatur. *a 79 huius; b 80 huius; c 84 huius.*

## *PROPOSITION 95.*

For the given magnitude DG, and with the proportion of the inequality A greater than B, and likewise with another magnitude C given: to examine in what manner the series of a given progression in the ratio A to B, of which the first term is C, can itself have the given magnitude DG.

***Construction &Demonstration.***

Thus DG to EG is made in the ratio A to B. In the first place therefore the magnitude C, which must be the first term of the progression, is equal to DE . There is thus a series of the ratio A to B having the first term C, to which DE is equal, which is equal to the given magnitude DG (i. e. C to EF, is as A to B). Therefore since DG is to EG as A to B, also DE is to EF is as DG to EG. Therefore the progression of [a] the ratio DE to EF, (i. e. the progression of the ratio A ad B, having the first term C) constitutes the magnitude DG. [DG is given, as is C and the ratio A/B; hence A/B = DG/EG = DE/EF = C/EF ].

In the second place the magnitude C, which must be equal to the first term of the series, is greater than DE; in this case there is a progression in the ratio A to B, having the first term C, which is greater than DG. For C is itself made equal to HI, and thus A is to B, as HI is to IK, and the last point or terminus L is found of the progression HI to IK [b] . Therefore [c] as HI to DE thus HL to the whole series of the progression DG: HL to DG, the sum of the series of the progression DE to EF. But HI is greater than DE, hence HL is greater than DG.

Hence if C is less than DE, there is also a progression of the ratio A ad B, having the first term C, which is less than DG; which we can show by the same method as the first. We have done what was requested. *a 79 huius; b 80 huius; c 84 huius.*

## *L2.§2.* *PROPOSITIO XCVI.*


**Prop.96. Fig. 1.**

Data sit progressio rationis AF ad FG, terminata in B, & alia magnitudino CE: Oportet in magnitudine CE, ita utrimque ad C & E continuere progressionem rationis AF ad FG (progressiones, nempe CH, HI, &c. & EK, KL, &c.) ut eundem habeant terminum D, qui ita dividat CE, ut AB, CD, DE, sint in continua analogia.

***Constructio &Demonstratio.***

Rectum CE ita divide per trigesimam sextam huius ut AB, CD, DE sint continue proportionales. Deinde per nonagesimam huius ita divide CD, in H & I, ut ratio CH ad HI, eadem cum ratione AF ad FG, ac simul progressio CH, HI, terminatur in D. Idem facito in ED. Factumque erit quod petebatur.

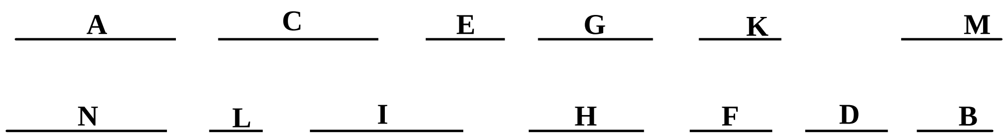
## *PROPOSITION 96.*

The progression of ratios AF to FG is given, to terminate in B, and also another magnitude CE: it is required in the length CE from C and E, to continue a progression of the ratio AF to FG thus from both ends of the line (the progressions are namely CH, HI, etc. & EK, KL, etc.) in order that they have the same termination D, which thus divides CE so that AB, CD, DE are in the continued ratio.

***Construction &Demonstration.***

The line CE is thus to be divided by Prop. 36 of this work in order that AB, CD, and DE are continued proportionals. Then by Prop. 90 of this work, CD is to be divided in H and I, in order that the ratio CH to HI is the same as AF to FG, and at the same time the progression CH, HI is terminated in D. In the same way make a series in ED. What was sought has been done.

## *L2.§2.* *PROPOSITIO XCVII.*




**Prop.97. Fig. 1.**

Si datum seriem AM, BN termini A, B, C, D, E, F, G, &c. & termini in continua sint analogia.

Dico illas inter se eam proportionem habere, quam primi termini.

***Demonstratio.***

Quoniam A, B, C sunt tres continuae proportionales; ergo A est ad C, in duplicata ratione A ad B; iterum cum C, D, E fior tres continuae, C erit ad E, in duplicata ratione C ad D; hoc est, ut patet ex datis A ad B: ergo cum rationes A ad C;

**[107]**

& C ad E eiusdem duplicatae sint, erunt A, C, E tres continue proportionales in ratione duplicata A ad B. Quare series AM est series rationis duplicatae rationis A ad B : deinde quia B, C, D sunt continuae proportionales, erit ratio B ad D, duplicata rationis B ad C. similiter ostendemus, rationem D ad F, duplicatam esse rationis B ad C. continuae proportionales sunt igitur B, D, F. unde series BN, est series duplicatae rationis B ad C, id est ex datis, A ad B: similes igitur series sunt AM & BN. quare sunt [a] inter se, ut primi termini A & B. quod erat demonstrandum. *a 84 huius*.

## *PROPOSITION 97.*

If the given series AM, BN of terms A, B, C, D, E, F, G, &c. are terms in a continued ratio.

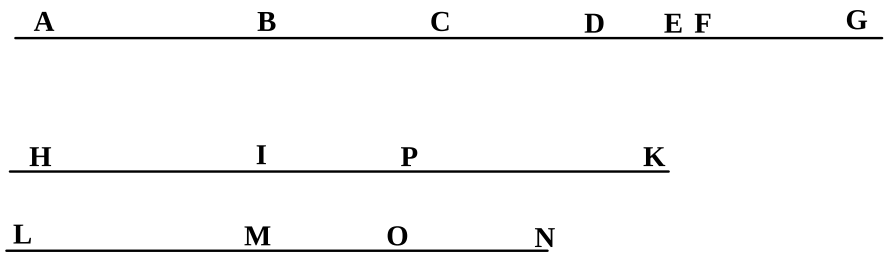
I say that [corresponding terms of] these series have the same proportion between each other as the first terms [of the two series].

***Construction &Demonstration.***

Since A, B, C are three continued proportionals, then A to C is the square ratio of A to B; again with the terms C, D, E, I can make three continued proportions, and C to E is the square ratio of C to D. That is, the ratios A to C and C to E are the same squared ratios of A to B. Hence A, C, and E are three continued proportionals in the square ratio of A to B. Whereby the series AM is a series of the squares of the ratio A to B: then since B, C, D are continued proportionals, the ratio B to D is the square of the ratio B to C. Similarly we can show that the ratio D to F is the square of the ratio B to C. Therefore B, D, F are continued proportionals. Hence the series BN is a series of the squares of the ratio B to C, or of the given ratio A to B: the series AM and BN are therefore similar, [a] whereby the corresponding terms of the two series are in the ratio of the first terms of the two series A and B. *a 84 huius*.

[A/B = B/C= C/D=D/E=....= 1/*r*, the common ratio in modern terms is *r*; then A/B .B/C = A/C = $A^2/B^2$ = $1/r^2$; similarly, C/E = $C^2/D^2$ = $1/r^2$; etc. Thus the original series $a, ar, ar^2, ar^3, ar^4, ar^5, ar^6$, ... is regrouped as AM or $a, ar^2, ar^4, ar^6$, ... and BN or $ar, ar^3, ar^5, ar^7$, ... , in which the common ratio *r* is maintained between corresponding terms. ]

## *L2.§2.* *PROPOSITIO XCVIII.*




**Prop.98. Fig. 1.**

Continuetur ratio AB ad BC, habeatque terminum G. deinde fiat seriei rationis AB ad CD, aequali magnitudo HK: seriei autem rationis AB ad DF, aequalis LN.

Dico HK magnitedinem maiorem esse quam sit LN.

***Demonstratio.***

Quia HK est series AB, CD, &c. & LN est series AB, DE, &c.. sumantur ex HK, partes HI, IP aequales ipsis AB, CD: ex LN vero partes LM, MO aequales ipsis AB, DE. Quoniam igitur seriei HI, IP terminus est K, erit HK [b] ad IK, ut HI ad IP : & quia seriei LM, MO, &c. terminus est N, erit LN [c] ad MN, ut LM ad MO. sed HI ad IP minorem habet rationem, quam LM, id est HI ad MO. Ergo HK ad IKM minorem habet, quam LN ad LM, & per conversionem rationis KH ad HI, maiorem habet quam LN ad LM. ergo permutando HK ad LN, maiorem habet, quam HI ad LM. Quare cum ex const. HI, LM sint aequales, necesse est HK maiorem esse quam LN. Quod erat demonstrandum.
*a 84 huius*.

## *PROPOSITION 98.*

The ratio AB to BC is continued [indefinitely] and has the termination [or endpoint] G. Then a series of ratios AB to CD is made to be equal to the magnitude [or length] HK; and also a series of the ratio AB to DE equal to the length LN.

I say that the magnitude HK is greater than LN.

***Construction &Demonstration.***

Since HK is the series AB, CD, &c. & LN is the series AB, DE, &c., the parts [or terms] HI and IP taken from HK are themselves equal to AB and CD: from LN likewise the terms LM and MO themselves equal to AB and DE. Therefore since the end point of the series HI, IP is K, HK [b] is to IK, as HI to IP : & since the end point of the series LM, MO, &c. is N, LN [c] is to MN, as LM is to MO. But HI to IP is a smaller ratio than LM (, or HI as both equal AB) to MO. Therefore HK to IK is less than LN to MN , & by the converse of the ratio KH to HI, is greater than LN to LM. [see note immediately below.] Therefore by interchanging, HK to LN is greater than HI to LM. Since from the construction, HI and LM are equal, it is necessary that HK is greater than LN. Q.e.d. *a 84 huius*.

[HI/IP = AB/CD = HK/IK ; and LM/MO = AB/DE = LN/MN ; but since CD > DE and LM = HI = AB, hence: the first set of equalities is less than the second set, and HK/IK - 1 < LN/MN - 1,
or IH/IK < LM/MN, inverting, we have IK/IH > MN/LM, and HK/IH > LN/LM, giving HK > LN as required.]

## *L2.§2.* *PROPOSITIO XCIX.*


**Prop.99. Fig. 1.**

Series rationis AB ad BC, terminetur in M; assumatur autem ex serie AM, terminus quicumque DE.

Dico rationis AB ad DE seriem, una cum serie rationis BC ad EF, ac serie rationis CD ad FG, aequari seriei rationis AB ad BC.

***Demonstratio.***

Quandoquidem omnes AB, BC, CD, DE, &c. in continua sint analogia, patet ex elementis AB, DE, GH, & sic in infinitum esse continue proportionales. Similiter BC, EF, HI, &c. itemque CD, FG, IK esse continue proportionales; ergo series trium rationum AB, DE, BC, EF, CD, FG, continuantur perpetuo; intra seriem AB, BC, CD, &c. ita ut in his tribus seriebus simul sumptis, nec plures, nec pauciores termini reperiantur, quam sint in serie AB, BC, CD, &c. manifestum igitur est has tres series, seriei AB, BC, &c. aequales esse. Quod erat demonstrandum.

**[108]**

## *PROPOSITION 99.*

The series in the ratio AB to BC is to be terminated in M; but some term DE is taken from the series AM.

I say that the series of the ratio AB to DE, together with the series of the ratio BC to EF, and the series of the ratio CD to FG, is equal to the given series of the ratio AB to BC.

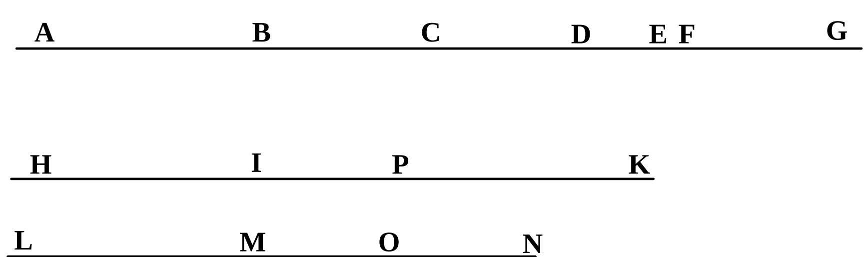
***Demonstration.***

Since all AB, BC, CD, DE, &c. are in continued proportion, it is apparent from elementary considerations that AB, DE, GH, & thus indefinitely are continued proportionals. Similarly, BC, EF, HI, &c. , and likewise CD, FG, IK are continued proportionals; therefore the three series of ratios AB, DE, BC, EF, CD, FG, are to continue indefinitely; within the series AB, BC, CD, &c. thus as with these three series taken together, neither more nor less terms can be found, than there are in the series AB, BC, CD, &c. It is therefore shown that the sum of the three series is equal to the series AB, BC, &c. Q.e.d.

## *L2.§2.* *PROPOSITIO C.*

Series rationis AB ad BC, terminetur in G; seriei autem rationis AB ad CD, aequalis sit HK; item seriei rationis BC ad EF, aequalis LN.

Dico seriem AG, duabus HK, LN, maiorem esse.




**Prop.100. Fig. 1.**

***Demonstratio.***

Series BC, EF, &c. [a] minor est serie BC, DE, &c. ergo series BC, EF, &c. cum serie AB, CD, &c. minor erit, quam series BC,DE, &c. una cum serie AB, CD, &c. atqui per praecedentem series BC, DE, &c. cum series AB, CD, &c. constituit seriem AB, BC, CD, DE, &c. hoc est magnitudinem AG. ergo series BC, EF, &c. cum series AB, CD, &c. minorem constituit, quam AG: ergo HK, LN series aequales seriebus AB, CD, &c. BC, EF, &c. minores sunt, quam series AG. Quod erat demonstrandum. *a 98 huius.*

## *PROPOSITION 100.*

The series of the ratio AB to BC is terminated in G; but HK is equal to a series of the ratio AB ad CD ; likewise LN is equal to a series of the ratio BC to EF.

I say that the series AG is greater than the two series HK and LN.

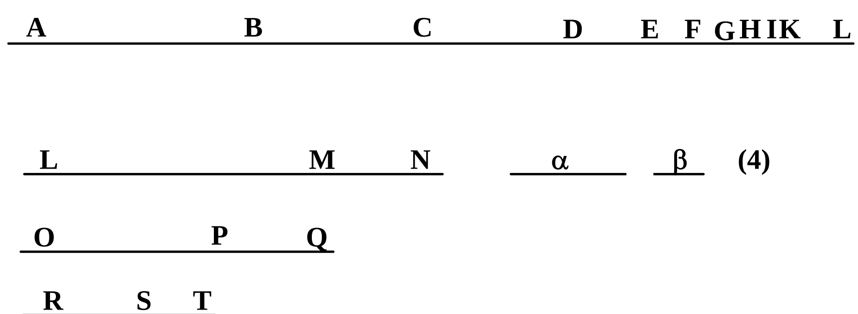
***Demonstration.***

The series BC, EF, &c. [a] is less than the series BC, DE, &c. Hence the series BC, EF, &c. summed with the series AB, CD, &c. is less than the series BC,DE, &c. together with the series AB, CD, &c. But by the preceding proposition, the series BC, DE, &c. taken with the series AB, CD, &c. makes the series AB, BC, CD, DE, &c., or the magnitude AG. Hence the series BC, EF, &c. taken with the series AB, CD, &c. makes a smaller sum than AG: hence the series HK and LN summed together is equal to the series with the ratios AB, CD, and BC, EF, which is less than the sum of the series AG. Q.e.d. *a 98 huius.*

## *L2.§2.* *PROPOSITIO CI.*

Series continue proportionalium AB, BC, CD, &c. terminetur in L. Detur autem proportio α ad β, multiplicata rationis AB ad BC, iuxta datum aliquem numerum (quatuor exempli causa) & quot sunt unitates in dato numero, tot una minus fac magnitudines LM, OP, RS, aequales datae seriei terminis AB, BC, CD.

Dico series, rationis α ad β, quarum primi termini sint L, O, R, simul sumptas, constituere eandem magnitudenem, quam series rationis AB ad BC.




***Demonstratio.***

Fiat ut α ad β, sic LM ad MN, & OP ad PQ, & RS ad ST; erunt ergo omnes hae rationes quadruplicatae rationis AB ad BC : & quoniam LM, AB, aequales sunt, eandem habebunt rationem ad MN; ergo & ratio AB ad MN, quadruplicata est rationis AB ad BC; est autem & ratio AB ad DE, quadruplicata rationis AB ad BC. ergo ut AB ad DE, ita AB ad MN, aequantur igitur DE & MN: series igitur rationis LM ad MN, est series rationis AB ad DE. Similiter ostendam seriem rationis OP ad PQ, esse seriem rationis BC ad EF, &

seriem rationis RS ad ST, seriem esse rationis CD ad FG, atqui tres[b] series simul sumptae, adaequant seriem rationis AB ad BC; ergo etiam & illae eandem adaequabunt. Quod erat demonstrandum. *b 99 huius.*

**[109]**

## *PROPOSITION 101.*

The series of continued proportions AB, BC, CD, &c. is terminated in L. While a proportion α to β which is a multiple [or power] of the ratio AB to BC is given, such as some given number, e. g, four, and for whatever the given number chosen is, one less than this number gives e. g. the three magnitudes LM, OP, RS equal to the terms AB, BC, CD of the whole given series.

I say that the series in the ratio α to β, the first terms of which are L, O, R taken together, give rise to the same magnitude as the series with the ratio AB to BC.

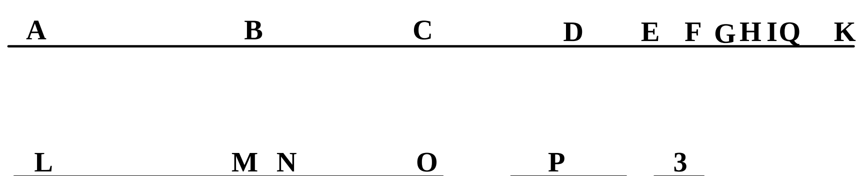
### *Demonstration.*

Thus LM to MN, & OP to PQ, & RS to ST are made in the ratio α to β ; therefore all these ratios are the quadruple [i. e. multiplied by themselves four times, or raised to the fourth power in modern terminology] of the ratio AB to BC : & since LM and AB are equal, they have the same ratio to MN; & hence the ratio to AB to MN is the fourth power of the ratio AB to BC; & whereas the ratio AB to DE is the fourth power of AB to BC, then it follows that AB is to DE thus as AB to MN, and hence DE & MN are equal. Hence the series of the ratio LM to MN is the series of the ratio AB to DE. Similarly I can show that the series of the ratio OP to PQ is the series of the ratio BC to EF, & the series of the ratio RS to ST is the series of the ratio CD to FG, but the three[b] series added together are equal to the series of the ratio AB to BC; and hence also these are equal to the same. Q.e.d. *b 99 huius.*

## *L2.§2.* *PROPOSITIO CII.*

Series rationis AB ad BC, continuatae, terminetur in K. Data autem sit LM aequalis AB, & quivis numerus (puta 3) deinde fiat ratio LM, ad MN multiplicata rationis AB ad BC, iuxta datum numerum ; sitque; rationis LM ad MN terminus O.

Dico seriem AK, ad seriem LO, eandem habere proportionem, quam totidem termini seriei AB, BC, CD quot sunt unitates in dato numero, habent ad primum AB.




**Prop.102. Fig. 1.**

### *Demonstratio.*

Cum, exempli causa, numerus datus ponatur ternarius; erit ratio LM ad MN; triplicata rationis AB ad BC; ostendendum nobis est, AK esse ad LO, ut tres primi termini DA ad primum AB. ex serie AK sume sex terminos AG: Igitur proportio AD[a] ad DG, triplicata est proportionis AB ad BC; aequalis igitur est rationi LM ad MN, hoc est[b] rationi LO ad MO. cum enim O sit terminus seriei LM, MN, erit LO ad MO, ut LM ad MN; Deinde quia K terminus est seriei AB, BC, erunt[c] tres AK, BK, CK continuae proportionales: adeoque omnes etiam sequentes erunt continuae: Quare & AK, DK, GK, inter quas par continue proportionalium numerus interijcitur, ex elementis patet esse continue proportionales. Unde AK,[d] est ad DK, ut AD ad DG, id est (quemadmodum iam ostendi) ut LO ad MO. Itaque per constructionem rationis

AK est ad AD, ut LO ad LM, & permutando AK ad LO, ut AD ad LM, id est ex datis AB . Quod erat demonstrandum. *a 32 huius; b 82 huius; a 32 huius; c Ibid; d 1 huius.*

## *PROPOSITION 102.*

The series of the ratio AB to BC continued is terminated in K. While LM is given equal to AB, and some number, taken as 3, then makes the ratio LM to MN equal to this power of the ratio AB to BC, for this given number; and O is the terminus of the given ratio LM to LN.

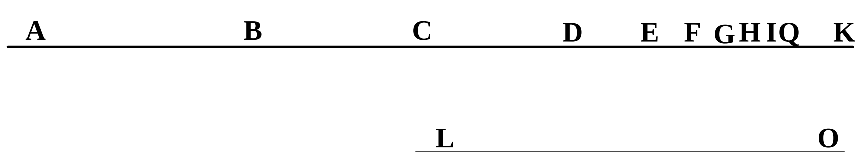
I say that the series AK has the same proportion to the series LO as the same number of terms AD (in the series AB, BC, CD, which is the number of units in the given number), has to the first term AB.

### *Demonstration.*

For the sake of an example, the given number is put as three ; the ratio LM to MN is the triplicate of the ratio AB to BC [i. e. raised to the third power]. It must be shown by us that AK is to LO, as the first three terms DA is to the first term AB. From the series AK six terms AG are taken: therefore the proportion AD [a] to DG is the triplicate [or third power] of the proportion AB to BC, which is therefore equal to the ratio LM to MN, or [b] to the ratio LO to MO; also, as O is the terminus of the series LM, MN: the ratio LO to MO is as LM to MN. Then, since K is the terminus of the series AB, BC, the three terms [c] AK, BK, and CK are in continued proportion: and thus also all the following terms are in continued proportion. Whereby AK, DK, GK, between which the number of continued proportions is inserted, from elementary considerations are apparent to be continued proportionals. Thus AK [d] is to DK, as AD to DG, that is (as has now been shown) as LO to MO. Hence from the construction of the ratio, AK is to AD, as LO to LM, & on interchanging, AK is to LO, as AD to LM, or to the given AB. Q.e.d. *a 32 huius; b 82 huius; a 32 huius; c Ibid; d 1 huius.*

[We are given $LM/LN = (AB/BC)^3$; it is required to prove that AK/LO = DA/AB or DA/LM, or (sum of whole series with ratio $r$) / (sum of series with ratio $r^3$) = (AB + BC +CD)/AB = $(a + ar + ar^2)/a = 1 + r + r^2$ in modern terms, from which the sum of the ratio $r^3$ follows. Geometrically, $AG/DG = (AB/BC)^3$ = LM/MN = LO/MO; since K is the end-point of the whole series, all the terms are in continued proportion with a ratio $r$: AK/BK = BK/CK = CK/DK = ..... = $1/r$; also, AK/DK = DK/GK = .... = $1/r^3$,
from which AK/AD = DK/DG or AK/DK = AD/DG. By construction, AK/AD = LO/LM, or
AK/LO = AD/LM.
Algebraically, this equality corresponds to $(a/1 - r)$ /(sum of series with ratio $r^3$) = $a(1 + r + r^2)/ar^3$, or the sum of series with ratio $r^3 = ar^3/(1 - r^3)$.]

## ***L2.§2.*** *PROPOSITIO CIII.*




**Prop.103. Fig. 1.**

Ex serie continue proportionalium AK, sumatur quivis terminus ut HI.

Dico seriem rationis AB ad BC, habere proportionem ad seriem rationis AB ad HI, quam habet HA (omnes nempe termini ipsum HI praecedentes) ad AB primum terminum.

Haec propositio, ut consideranti facile patebit, eadem est cum praecedenti, sed aliter & commodius fortasse proposita. Quare eadem erit utriusque demonstratio.

### *Corollarium.*

Ex hoc theoremate licebit praxim desumere, assignato quovis termino HI, in serie, AK, rationis AB ad BC; reperiendi magnitudinem toti seriei rationis AB ad HI aequalem. Nam si fiat ut HA ad AB, sic KA ad aliam LO, erit LO aequalis toti seriei rationis AB ad HI.

Fatior tamen opus non esse ad hanc praxim recurrere, cum universalem methodum, eamque facillimam reperiendi magnitudinem, toti seriei cuiuscumque rationis aequalem, propositio 80 huius suppeditet.

**[110]**

## *PROPOSITION 103.*

Some term such as HI is taken from the series of continuously proportional terms

I say that the [sum of the] series in the ratio AB to BC has the same proportion to the [sum of the] series in the ratio AB to HI, as HA has to the first term AB (truly all the terms preceding HI itself ).

This proposition, as it should be considered easy to show, is the same as the preceding one, although perhaps more easily established. Whereby the demonstration is the same as the other.

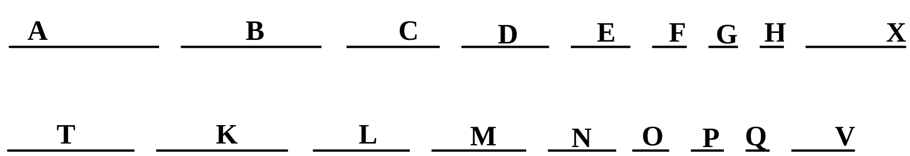
[One cannot of course generalise geometrically as is readily done algebraically, which is what this proposition tries to do for the previous proposition : one is always stuck with a particular instance according to the diagram.]

### *Corollory.*

From this theorem one might wish to choose an exercise: to designate some term HI in the series AK, of ratio AB to BC, to which the magnitude of the whole series in the ratio AB to HI can be found to be equal. For if the ratio is constructed whereby HA to AB is thus equal to the ratio of KA to a different magnitude LO, then LO is equal to the sum of the whole series of the ratio AB to HI.

Nevertheless there is no need to return to this exercise as I have composed a more general method by which the magnitude of any series of any ratio can be found , and which meets the needs of proposition 80 of this work.

## *L2.§2.* *PROPOSITIO CIV.*




**Prop.104. Fig. 1.**

Data sint continue proportionalium series binae, AX , KV rationum diversarum , ita tamen ut A, K, L, B etiam sint continuae.

Dico seriem A, K, L, M, N, &c. terminatam in V, eam habere proportionem ad seriem A, B, C, D, &c. terminatam in X , quam A, K, L simul sumpti ad A primum terminum.

### *Demonstratio.*

Addatur ipsi K terminus T in directum, aequalis ipsi A: ratio igitur (quod ex hypothesi colliges) T ad M, triplicata rationis T ad K. Quia autem A, K, L, B ponantur continuae proportionales, erit L ad B, ut K ad L: sed etiam L est ad M, ut K ad L; ergo L ad B, & M, eandem habet rationem: adeoque [a] B & M aequales sunt. Sunt vero etiam aequales AT, ergo ratio A ad B, eadem est cum ratione T ad M. quare ratio A ad B, triplicata est rationis T ad K. cum ergo utriusque seriei initium, idem sit terminus A, erit series T, K, L, &c.

id est series A, K, L, M, &c., ad seriem A, B, C, D, &c. ut tres primi termini L, K, T, hoc est L, K, A, simul sumpti ad T, hoc est ad A, primum terminum : quod erat demonstrandum. *a 9 Quinti; b 102 Huius.*

## *PROPOSITION 104.*

Two series of continuously proportional terms of different ratios are given, AX and KV, nevertheless in order that A, K, L, B are thus also in continued proportion.

I say that the series A, K, L, M, N, &c. terminating in V has the same proportion to the series A, B, C, D, &c. terminating in X, as the sum of A, K, L has to the first term A.

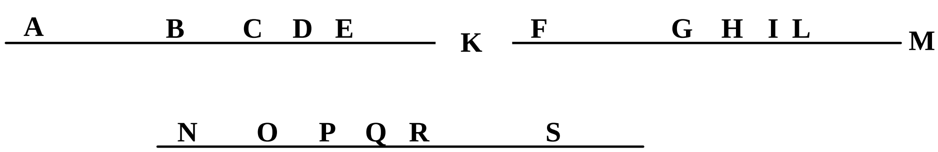
### *Demonstration.*

A term T equal to A is insertd next to K on the second line : therefore the ratio (which by hypothesis are collected together [according to A or T, K, L, B in proporton]) T to M, is the three-fold or cube of the ratio T to K. However since A, K, L, B are placed in continued proportion, L is to B, as K is to L: but also L is to M as K is to L; hence L to B and L to M are in the same ratio : hence [a] B & M are equal. Also indeed, A and T are equal; hence the ratio A to B is the same as the ratio T to M. Whereby the ratio A to B is the cube of the ratio T to K. Hence likewise for the term A for the start of the other series . Hence the series T, K, L, &c. (or the series A, K, L, M, &c.) is in the same ratio to the series A, B, C, D, &c. as the sum of the three first terms L, K, T, or L, K, A, is to T (or A), the initial term. Q. e. d. *a 9 Quinti; b 102 Huius.*

[T = A; A/K = K/L = L/B and K/L = L/M and hence M = B; hence $T/K \,.K/L \,.L/M = T/M = (T/K)^3$ ; again, $T/M = A/B = (T/K)^3$ . It is required to show that TV is to AX as TL is to A: A/B = AX/BX, while T/M = TV/MV, hence AX/BX = TV/MV, and AX/A = TV/TL or TV/AX = TL/A as required.

Algebraically, let the top series be $a, ar, ar^2, ar^3$, ..... so that AX/BX = BX/CX = .... = $r$; and the second series is $b, bt, bt^2, bt^3$, ..... so that KV/LV = LV/MV = .... = $t$. We are given, however, that $a/b = b/bt = bt/ar$, from A, K, L, B in proportion, hence $t = b/a$ and $ar = bt^2$ or $r = t^3 = b^3/a^3$. Hence, maintaining the second series, the first one can be written as $a, at^3, at^6, a^9$, ...
The required ratio TV/AX reduces to $1 + t + t^2$, and likewise for TL/A.]

## *L2.§2.* *PROPOSITIO CV.*




**Prop.105. Fig. 1.**

Data sint binae series continue proportionalium magnitudinum in diversis rationibus, terminatae in K & M ; & ab aequalibus terminis AB, FG incipiens. Fiat autem secundo termino BC, unius seriei aequalis NO; & GH secundo termino alterius seriei, aequalis OP; Deinde ratio NO ad OP (vel ratio OP ad NO, si OP maior sit quam NO) in infinitum continuetur.

Dico NO esse ad PQ, ut CD ad HI; & NO esse ad QR ut DE ad IL, atque ita in infinitum.

### *Demonstratio.*

Cum series AK, FM incipiant ab aequalibus terminis, erit ratio [c] CD ad HI, rationis BC ad GH, hoc est, ex constructione, rationis NO ad OP, duplicata; Atqui etiam ratio NO ad PQ, duplicata est, ex datis, rationis NO ad OP, eaedem igitur sunt rationes CD ad HI, & NO ad PQ; similiter ratio DE ad IL, triplicata est rationis BC ad GH, hoc est rationis NO ad OP: Quare cum & ratio NO ad QR, eiusdem rationis NO ad OP, sit triplicata, eaedem erunt rationes DE ad IL, & NO ad QR. Atque ita in infinitum, similis demonstratione procedemus. Patet igitur Theorematis veritas. *c 27 Huius.*

**[111]**

## *PROPOSITION 105.*

Two series of continuously proportional magnitudes in different ratios are given, terminating in K and M; and beginning from equal first terms AB and FG. But the second term of the first series BC is made equal to NO, and the second term GH of the other series is set equal to OP; from there on, the ratio NO to OP (or the ratio OP to NO, if OP is greater than NO) is continued indefinitely.

I say that NO is to PQ, as CD is to HI; & NO is to QR as DE to IL, and thus indefinitely.

### *Demonstration.*

As the series AK and FM begin with equal terms, the ratio [c] CD to HI, is the square of the ratio BC to GH, or from the construction, the square of the ratio NO ad OP. But also the ratio NO to PQ is the square of the ratio NO to OP, from what is given, and therefore the ratios CD to HI, & NO to PQ are the same; similarly the ratio DE to IL is the cube of the ratio BC to GH, or of the ratio NO to OP. Whereby the ratio NO to QR is the cube of the same ratio NO ad OP, and & NO to QR is the same ratio as DE ad IL. And thus indefinitely, we can proceed with a similar demonstration. Therefore the truth of the theorem is made apparent. *c 27 Huius.*

[$AB/BC = BC/CD = CD/DE = ....$; $FG/GH = GH/HI = HI/IJ = .....$; $AB = FG$;
$AB/BC.BC/CD = AB/CD = (AB/BC)^2$ and $FG/GH.GH/HI = FG/HI = (FG/GH)^2$ ;
hence $CD/HI = (BC/GH)^2 = (NO/OP)^2$ by construction; also, since $NO/PQ = (NO/OP)^2$,
$CD/HI = NO/PQ$. Similarly, $AB/DE = (AB/BC)^3$ and $FG/IL = (FG/GH)^3$ ,
hence $DE/IL = (BC/GH)^3 = (NO/OP)^3 = NO/QR$, etc.]

## *L2.§2.* *PROPOSITIO CVI.*

Data sint duae rationes similes, AB ad CD, & BC ad DE, quae terminis sic alternatim positis, continuentur.

Dico utriusque rationis in infinitum continuatae eundem terminum futurum.


**Prop.106. Fig. 1.**

### *Demonstratio.*

Quoniam ex hypothesi AB est ad CD, ut BC ad DE; erit permutando componendo, rursumque permutando AC ad CE, ut BC ad DE: similiter quia CD est ad EF, ut DE ad FG, erit permutando componendo, rursumque permutando, CE ad EG, ut DE ad FG; hoc est ex hypothesi ut BC ad DE, hoc est iam demonstratis, ut AC ad CE: sunt igitur AC, CE, EG continuae proportionales. Quod si rationes AB ad CD, & BC ad DE, in infinitum continuentur, ostendam pariter, rationem AC ad CE, in infinitum continuari, per terminos continue proportionales AC, CE, EG, &c. Inveniatur igitur [a] terminus seriei AC, CE, EG, &c. sitque K. Itaque non est punctum assignabile, inter puncta A & K, ultra quod non cadat aliquis terminus seriei AC, CE, EG. Quare cum rationum AB ad CD, & BC ad DE continuatarum, termini omnes ita contineantur in serie AC, CE, &c. ut singuli termini seriei AC, CE, &c. contineant unum terminum rationis AB ad CD, & unum terminum rationis BC ad DE; manifestum quoque est nullum punctum assignari posse inter A & K, ultra quos non cadat aliquis terminus, tam rationis AB ad CD, quam rationis BC ad DE; neutra igitur series terminabitur inter A & K: sed neque ulli dictarum rationum termini transilient K, cum perpetuo contineantur in serie AC, CE, EG, &c.(quae ex constructione non transilit unquam K) ergo binae series rationum AB ad CD, & BC ad DE, eundem habent terminum K. Quod erat demonstrandum. *a 8 Huius.*

## *PROPOSITION 106.*

Two like ratios are given, AB to CD, and BC to DE, are put in line, the terms of which thus alternate in position.

I say that each ratio continued indefinitely comes to the same terminus.

***Demonstration.***

Since AB is to CD, as BC is to DE by hypothesis ; on interchanging and adding, and interchanging again, AC is to CE, as BC is to DE: similarly since CD is to EF, as DE to FG, on interchanging and adding and again interchanging, CE is to EG as DE is to FG; that is from hypothesis as BC is to DE, now by demonstration it is as AC to CE: therefore AC, CE, and EG are continued proportionals. For if the ratios AB to CD, & BC to DE, are continued indefinitely, I can show equally that the ratio AC to CE, to be continued indefinitely, by means of the terms in continued proportion AC, CE, EG, &c. Therefore a terminus of the seris[a] AC, CE, EG, &c. can be found and it is K. Hence there is no point assignable between the points A & K, beyond which some term of the series does not fall AC, CE, EG. Whereby since all the terms of the continued ratios AB to CD, & BC to DE are thus contained in the series AC, CE, &c. in order that individual terms of the series AC, CE, &c. contain one of the terms of the ratio AB to CD, & one of the terms of the ratio BC to DE. It is also the case that no point can be assigned between A & K, beyond which some term does not fall, either of the ratio AB to CD, of of the ratio BC to DE; neither series therefore terminates between A & K: and yet none of the terms of the said ratios can jump across K, as they are always contained in the series AC, CE, EG, &c.(which by construction cannot jump over K at any time) . Hence the two series of ratios AB to CD, & BC to DE, have the same terminus K. Q. e. d. *a 8 Huius.*

[Since AB/CD = BC/DE = CD/EF = DE/FG = ..... , then AB/BC = CD/DE, AC/BC = CE/DE on adding, <u>AC/CE</u> = BC/DE = AB/CD on interchanging; similarly, BC/DE = CD/EF = DE/FG = <u>CE/EG</u> , hence AC/CE = CE/EG, and AC, CE, EG are in continued proportion. The rest then follows.
In terms of algebra, we are dealing with a geometric progression split into the odd and even terms $S_1$ and $S_2$, and thus, in an obvious notation: AB = $a$; BC = $ar$; CD= $ar^2$; DE = $ar^3$; EF = $ar^4$; FG = $r^5$; etc.
AB/CD = $1/r^2$ = BC/DE = etc.; while BC/DE = $1/r^2$ = CD/EF = etc.; thus both series have the same ratio.
The infinite sum of $S_1$ is $a/(1 - r^2)$ and the sum of $S_2$ is $ar/(1 - r^2)$; hence the sum of both series is
AK = $a/(1 - r^2) + ar/(1 - r^2) = a/(1 - r)$ as expected. On the other hand, the single equivalent progression considered AC, CE, EG, .... has the terms $a(1 + r)$, $ar^2(1 + r)$, $ar^4(1 + r)$,.....
for which the infinite sum is $a(1 + r)/(1 - r^2)$, or $a/(1 - r)$.]

## *L2.§2.* *PROPOSITIO CVII.*


**Prop.107. Fig. 1.**

Magnitudo AB bisecta sit in C; BC autem in D; & CD in E; & DE in F; & EF in G; & FG in I; atque hoc semper fiat:

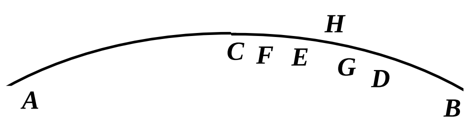
Dico alternae huius progressionis terminum fore in puncto, quo magnitudo AB, dividitur in partes, habentes rationem quam unum ad duo, sive terminum progressionis abscindere BG, tertium partem magnitudinis AB.

***Demonstratio.***

Quoniam DC dupla est CE, & BC dupla DC, erit BC, hoc est AC, quadrupla ipsius CE: similiter cum FE dupla sit GE, & DE dupla FE, erit & DE, hoc est CE, quadrupla EG; sunt igitur AC, CE, EG tres continuae in proportione quadrupla. Deinde cum DE ex hypothesi dupla sit DG, & CD dupla ED; erit iterum CD, hoc est BD, quadrupla DF: similiter quia GF dupla est FI, & EF dupla GF, erit EF, hoc est DF quadrupla FI: sunt igitur BD, DF, FI continuae in ratione quadrupla. Itaque si alterna illa bisectio sinc statu continuetur, constituerat utrimque progressio in infinitium proportionis quadruplae; & quoniam AC quadrupla est CE, erit & BC eiusdem CE quadrupla : est vero & CE, quadrupla EG, atque ita in infinitium progressio igitur AC, CE, EG, &c. eadem est cum progressione

**[112]**

BC , CE, EG, &c. & eundem terminum habet : Atque progressionis quadruplae BC, CE, &c. terminus H, secat [a] CB in ratione unius ad duo, ergo etiam terminus progressionis AC, CE secat CB, in ratione unius ad duo. Quare CH dimidia est ipsius HB, & CB sesquialtera HB : ideoque AB tripla ipsius HB; ac denique AH dupla HB; ergo terminus progressionis AC, CE, &c. secat AB, in ratione unius ad duo. ulterius cum progressiones AC, CE, &c. D, DF, FI, &c., eiusdem sint rationis, nempe quadruplae, erit tota series progressionis AC, CE, EG, &c. ad totam seriem progressionis BD, DF, &c.[b] ut AC ad BD : Quare cum AC dupla sit DB, erit quoque series progressionis AC, CE, &c. id est AH; dupla seriei BDM DF, &c. Atque AH iam ostendimus etiam duplam esse HB; ergo AH, ad seriem progressionis BD, DF, eandem habet rationem quam ad HB: unde progressionis BD, DF, series terminatur etiam in punct H. Quare per eadem procedens puncta, cum alterna illa bisectio constituat utramque progressionem, illius quoque terminus erit punctum H, quo dividitur AB in ratione unius ad duo: Quod erat demonstrandum. *a 89 huius; b 89 huius.*




**Prop.107. Fig. 2.**

### *Corollarium.*

Ex hocTheoremate reperietur arcus trisectio, si independenter a trisectione, alternae illius progressionis terminus inveniatur. Cum enim Theorema univerale sit, & in quavis magnitudine demonstratio allata valeat, si in arcu dato AB, similis alterna fiat bisectio, terminus quoque progressionis alternae H, abscindet tertiam arcus partem BH : proindeque reperto alia via ducto termino, arcus etiam dati trisectio reperietur.

## *PROPOSITION 107.*

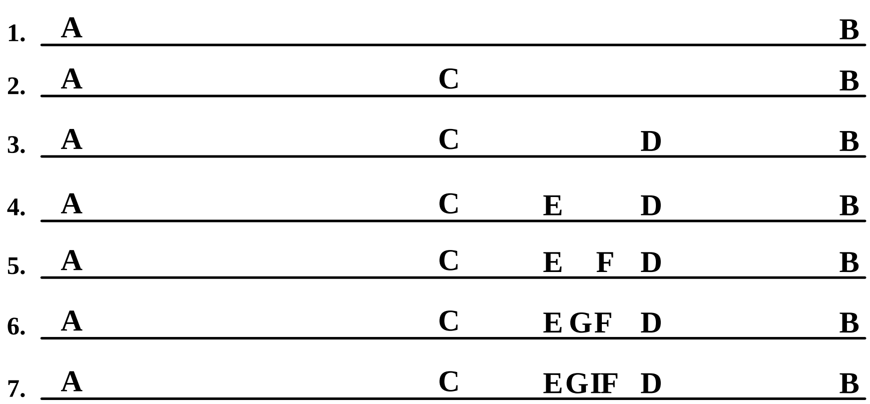
The magnitude AB is bisected in C; BC again in D; CD in E; and DE in F; and EF in G; and FG in I; and this is done indefinitely.

I say that the end [or terminus] of this alternating progression to be in a point, by which the magnitude AB is divided in parts having the ratio one to two, or the end of the progression cuts BG into a third part of the magnitude of AB.

#### *Demonstration.*

Since CD is twice CE, & BC double CD, BC or AC is four times CE itself: similarly as FE is twice GE, & DE twice FE, DE or CE is four times EG; therefore AC, CE, and EG are three lengths in four-fold continued proportion. Then since DE by hypothesis is double DG, & CD double ED; CD or BD is again four times DF: similarly since GF is twice FI, & EF twice GF, EF or DF is four times FI: therefore BD, DF, FI are continued proportions in a four fold ratio. Thus if this alternate bisection is continued without stopping, a progression is established indefinitely from both ends in the four fold proporton; and since AC is four times CE, and BC is the same four times CE : CE is truly four times EG, and thus AC, CE, EG, &c. are in the same infinite progression as BC, CE, EG, &c, and have the same terminus [or end-point]. But the terminus H of the four fold progression BC, CE, &c. cuts [a] CB in the ratio of one to two, hence the terminus of the progression AC, CE also cuts CB, in the ratio of one to two. Whereby CH is half of HB, & CB is one and a half of HB : thus AB is three times HB; and hence AH is twice HB; hence the terminus of the progression AC, CE, &c., cuts AB in the ratio of one to two. Since the progressions on either side AC, CE, &c. BD, DF, FI, &c., are of the same ratio, truly the quadruple, the sum of the series of the progression AC, CE, EG, &c. to the sum of the series of the progression BD, DF, &c.[b] is as AC to BD. Whereby as AC is twice DB, the series of the progression AC, CE, &c. or AH is also twice the series BD, DF, &c. But we have shown that AH is also thus twice HB; hence AH has the same ratio to the series of the progression BD, DF as it has to HB: thus the series of the progression BD, DF is also terminated in the point H. Whereby the proceding point, from the other bisection gives rise to the other progression, the terminus of that too is the point H, by which we conclude that AB is divided in the ratio of one to two. Q.e.d. *a 89 huius; b 89 huius.*

[This probem is analysed by Gregorius in terms of two series, whereas we would now consider a series of alternating terms with $r = -\,{}^{1}/_{2}$. Initially we follow the scheme of Gregorius, Fig. 3:



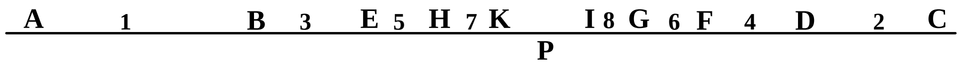

**Prop.107. Fig. 3.**

AC, CE, and EG are segments leading to the right in lines 2, 4, 6 (with which we could associate the lines AC, AE, and AG); and these segments are in the progression AC, AC/4, AC/16. Again, BD, DF, and FI are segments leading to the left in lines 3, 5, 7 (with which we could associate the line BD, BF, and BI), and these lines are in the progression BD, BD/4, BD/16. It is now established that each progression has an end-point, and from the similar nature of the progressions, these are shown to be equal to H. The 'forwards' progression involves the even line numbers in Fig. 3, and following the geometrical nature of summation, we have AH/CH = AC/CE , or AC/CH = BE/CE, or CH = AC.CE/BE = AC.(1/8)/(3/8) = AB/6; Hence AH = 2AB/3. The 'backwards' progression similarly follows the terms BD, DF, FI, and agrement is reached that the two progressions give the same result as required. In modern terms, if A is taken as the origin on a number line, and AB = $a$, then $r = -\,{}^{1}/_{2}$, and the sum is $a/(1-r) = 2a/3$]

***Corollary.***

The trisection of an angle can be found from this theorem, if independently from the trisection considered above, another example of a termination of this progression can be found. For indeed the theorem is universal, and the demonstration can be brought to prevail for any magnitude. Hence, if in the arc of a given circle, the bisection is made by a similar alternate series [i. e. choosing one half of each section repeatedly in a forwards and reverse manner as above], then the terminus H of the alternate progressions also cuts off a third part of the arc : and hence by finding both ways that leads to the same terminus, the trisection of the given arc is found. [One has to perform an infinite number of divisions of line segments by two, using compasses and ruler.]

## *L2.§2.* *PROPOSITIO CVIII.*




**Prop.108. Fig. 1.**

Data sit magnitudo AC utcunque secta sit in B; Deinde fiat AC ad BC, sic BC ad BD; & BD ad ED ; & ED ad EF; & EF ad HF; atque sic altera divisio semper fiat;

Dico utrimque constitutum in duas progressiones similes, magnitudinum AB, BE, EH, &c. & CD, DF, FG, &c. continue proportionalium in ratione duplicata proportionis AC ad BC.

***Demonstratio.***

Quoniam ex datis AC, BC, BD, ED, EF, &c. sunt continuae proportionales, erunt eurum differentiae [c] AB, CD, BE, DF, EH, FG, &c. etiam in continua analogia, & quidem eo ordine ut prima, tertia, quinta, septima, & sic deinceps (intermisso semper numero medio) constituant seriem A; secundo, vero quarta, sexta, octava, & sic deinceps (semper omisso numero medio) seriem C, conficiunt. igitur ut AB ad BE, prima ad tertiam, sic CD est ad DF, secunda ad quartam, & sic deinceps; adeoque rationes AB ad BE, & BE ad EH, &c. similes erunt rationibus

**[113]**

CD ad DF, & DF ad FG, &c. est autem AB ad BE, ut a AC ad BD, hoc est in ratione duplicata AC ad BC; & CD est ad DF, ut BC est ad ED secunda ad quartam, hoc est in ratione duplicata BC ad BD; hoc est AB ad BC; ergo duae illae progressiones, similes erunt, & in ratione duplicata AB ad BC. Quod erat demonstrandum. *c 1 huius; a ibid.*

## *PROPOSITION 108.*

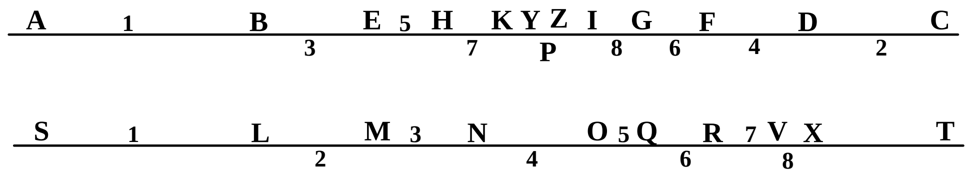
The magnitude AC is cut in some manner in B. Then as AC to BC, thus the ratio BC to BD; and BD to ED, and ED to EF, and EF to HF; and thus the division is always done for the following terms.

I say that the series on either side establish two like progressions of magnitudes AB, BE, EH, etc. & CD, DF, FG, etc., of continued proportionals in the square of the ratio AC to BC.

***Demonstration.***

Since AC, BC, BD, ED, EF, etc. are continued proportionals from the given ratios, the differences of these [c] AB, CD, BE, DF, EH, FG, etc. are also in continued proportion, and indeed the terms in the order of the first , third, fifth, seventh, and thus henceforth (with the middle number always missing) constitute serie A; while the second, the fourth, sixth, eighth, etc., terms, and thus henceforth (with the middle number always missing) make the series C. Thus as AB is to BE, the first to the third term, thus CD is to DF, the second to the fourth term, and thus henceforth ; hence the ratios AB to BE, & BE to EH, etc. are similar to the ratios CD to DF, & DF to FG, etc. But AB to BE is as [a] AC to BD, that is in the square of the ratio AC to BC; & CD is to DF, as BC is to ED the second to the fourth, or in the square ratio of BC to BD; or AB to BC; hence these two two progressions are alike, & in the square ratio of AB to BC. Q.e.d. *c 1 huius; a ibid.*

## *L2.§2.* *PROPOSITIO CIX.*




**Prop.109. Fig. 1.**

Iisdem positis progressio utraque AB, BE, EH, &c. CD, DF, FG, &c.; ex alterna illa divisione nata, terminum habebit eandem in magnitudine AC.

***Demonstratio.***

Sumatur ST aequalis AC; & capiantur ex ea magnitudines LT, MT, NT, OT, &c. quae aequales sint continue proportionalibus BC, BD, ED, EF; &c.; erunt igitur SL, LM, MN, NO, &c. ipsis quoque AB, CD, BE, FD, &c. aequales; cum enim tota AC, ST, & ablata BC, LT, aequalia sint; necesse est etiam reliqua AB, SL esse aqualia: & rursum quia tota BC, LT, & ablata BD, MT aequalia sunt, patet quoque reliqua CD, LM aequalia esse. Similiter ostendam & BE ipsi MN, DF ipsi NO, atque ita in infinitum, reliqua reliquis aequalia esse. ergo utraque progressio A & C. progressioni SL, LM, aequales sunt simul sumptae. & quoniam ST, LT, MT, &c. aequantur continue proportionalibus AC, BC, BD, &c. etiam ipsae erunt continuae [b] ideoque SL,LM, MN, &c. sunt continuae proportionales. atque ita sine termino continuatur ratio, SL ad LM; ergo progressio c SL ad LM, terminatur in T, sive constituit magnitudinem ST : quarecum utraque progressio A & C simul sumptae, aequentur progressioni SL, LM; etiam constituent magnitudinem ST, hoc est AC ex constructione: eundem igitur terminum habeant in magnitudine AC necesse est. nam si diversos habeant, sint illi Y & Z. vel inter utrumque terminum Y, Z superit media quaedam magnitudo, quae ad neutram seriem pertineat, vel aliqua erit magnitudo, utrique seriei communis, eritque Z terminus seriei A, & Y terminus seriei minorem quam AC; & posito altero maiorem quam AC. Quod utrumque repugnat modo demonstratis; non igitur diversos habebunt terminos dictae progressiones, sed eundem. Quod erat demonstrandum. *b 1 huius; c 79 huius.*

## *PROPOSITION 109.*

With the same points in place the progression AB, BE, EH, &c. and CD, DF, FG, &c. are given on each side; from that alternating form of division of the line AC another progression is produced ending in the same magnitude AC

***Demonstration.***

ST is taken equal to AC; & from that line the magnitudes LT, MT, NT, OT, etc are put in place which are equal to the continued proportionals BC, BD, ED, EF, etc; therefore SL, LM, MN, NO, etc and AB, CD, BE, FD, etc are respectively equal to each other; for indeed the whole lengths AC are equal ST, and if the equal lengths BC and LT are taken away, then it follows that the remainders AB and SL are also equal. Again, since the whole lengths BC and LT and the lengths taken BD and MT are equal, it is apparent that the lengths CD and LM are equal. Similarly I can show, for BE taken with MN, and DF with NO, and thus indefinitely, that the rest of the remainders are equal. Therefore both progressions A and C taken together are equal to the progression SL, LM. Since ST, LT, MT, etc. are equal to the continued proportions AC, BC, BD, etc. they are also in continued proportion. [b] and therefore SL,LM, MN, etc. are continuee proportionals. Thus the ratio SL ad LM can continue without end, and hence the progression [c] SL to LM can finish in T, or given the magnitude ST : whereby as both progressions A and C taken together are equal to the progression SL, LM; they give rise to the magnitude ST or AC from the construction: hence by necessity the two series have the same total length or termination given by the magnitude AC. For if they have different terminations, let these be the points Y and Z. Hence either between each of the terminations

Y and Z there is present some magnitude in the middle to which neither series belongs, or there is another magnitude which both series have in common, and Z is the end of series A and both series are greater than AC, or Y the end of a series and both series are less than AC. Which are both shown to be in disagreement in this manner; therefore the said progressions do not have different end-points, but the same. Q.e.d. *b 1 huius; c 79 huius.*

## *L2.§2.* *PROPOSITIO CX.*

Iisdem positis; geminae progressionis AB, BE; & CD, DF, ex alterna sectione natae, communis terminus P, magnitudinem AC, dividet in ratione AC ad BC.

***Demonstratio.***


**Prop.110. Fig. 1.**

Per praecedentem ponatur P esse communis utriusque terminus, quoniam igitur similes sunt progressiones A & C, erit tota series progressionis A, hoc est AP, ad totam seriem progressionis C hoc est

**[114]**

CP, [a] ut AB ad CD. quia autem ex hypothesi AC, BC, BD sunt continuae proportionales, erit AB ad CD, [b] ut AC ad BC; est igitur AP ad CP, ut AC ad BC. terminus ergo P utriusque progressionis. dividit AC, in ratione AC ad BC: quod erat demonstrandum. *d 108 huius; a 94 huius; b 1 huius.*

## *PROPOSITION 110.*

With the same points in place, for both of the progressions AB, BE, EH, &c. and CD, DF, FG, &c. coming from alternate sections; common terminus P, divides the length AC in the ratio AB to BC.

***Demonstration.***

By the preceding theorem, P is put in place as the common terminus of both series, and therefore as the progressions A and C are like, then the whole series A, or AP is to the whole series B or CP [a], as AB is to CD. But since from hypothesis, AC, BC, and BD are continued proportionals, AB is to CD, [b] as AC is to BC; therefore AP is to CP as AC is to BC: the end-point or the limit of both progressions divides AC in the ratio AC to BC. Q.e.d. *d 108 huius; a 94 huius; b 1 huius.*

[AB/AP = CD/CP ; given AC/BC = BC/BD, then AC/BC = AB/CD, and hence AC/BC = AB/CD = AP/CP as required. In terms of algebra, the left-hand series can be identified with $a, ar^2, ar^4$, ....., etc., while the right-hand series is given by $ar, ar^3, ar^5$, ...., etc. The sum of the first series $S_1 = a/(1 - r^2)$ = AP, while the second series is $S_2 = ar/(1 - r^2)$ = CP; hence $S_1 + S_2 = a/(1 + r)$ = AC, while AP/CP = $a/ar$ = AB/CD = AC/BC = $(a/1 - r)/(ar/1 - r)$. ]

## *L2.§2.* *PROPOSITIO CXI.*

Iisdem positis; terminus alternae sectionis, sive progressionis AC, BC, BD, ED, EF, &c., dividet magnitudinem AC, in ratione AC ad BC.

***Demonstratio.***


**Prop.111. Fig. 1.**

Utriusque progressionis AB, BE & CD, DF, termini simul sumpti sunt iisdem cum terminis progressionis AC, BC, BD, ED, &c. alternatim sumptis. ergo progressio alterna AC, BC, BD, &c., eundem habet terminum quem progressiones AB, BE & CD, DF. sed harum terminus per praecedentem, dividit AC, in ratione AC ad CB; ergo & alternae progressionis AC, BC, BD, &c., terminus in eadem ratione dividet magnitudinem AC. Quod erat demonstrandum.

## *PROPOSITION 111.*

With the same points in place, the terminus or end-point of the alternate progressions AC, BC, BD, ED, EF, &c., divides the magnitude AC in the ratio AC to BC.

***Demonstration.***

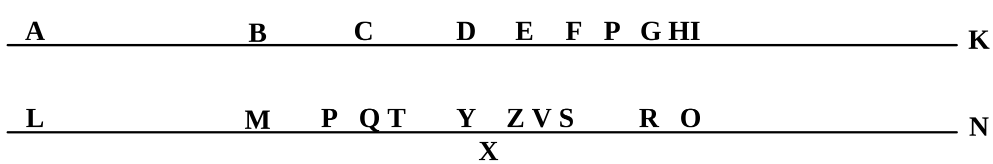
The terms of each progression AB, BE & CD, DF are the same as of the progression AC, BC, BD, ED, &c., taken alternately. Hence the alternate progression AC, BC, BD, &c., has the same terminus as the progressions AB, BE and CD, DF. But the terminus of these by the preceding theorem, divides AC in the ratio AC to CB; and hence the terminus of the alternate progressions AC, BC, BD, &c., divides AC in the same ratio. Q.e.d.

[AC/BC = BC/BD = BD/ED = ED/EF = ....etc., gives AB/BC = CD/BD = BE/ED = DF/EF = ....etc., or AB/BE = BC/ED ..... for the right-hand series, is equal to CD/DF = BD/EF = .... for the left-hand series. Hence both progressions have the same limit or terminus as required.
In terms of algebra:
AP is the series $a, ar^2, ar^4, ......$, and has the sum AP = $S_L = a/(1 - r^2)$; while
CP is the series $ar, ar^3, ar^5, ......$, and has the sum CP = $S_R = ar/(1 - r^2)$;
in which case AC = $S_L + S_R = a/(1 - r)$.
In terms of these lengths, BC = $ar/(1 - r^2)$; BD = $ar^2/(1 - r^2)$; ED = $ar^3/(1 - r^2)$; etc. Since this is an alternating series, the sum S = $(a/(1 - r)) /(1 + r) = a/(1 - r^2)$ = AP as required .
Hence, AP/PC = $1/r$ = AC/BC as required.]

## *L2.§2.* *PROPOSITIO CXII.*

Data sint continue proportionalium series, constituens magnitudinem AK: sit autem LN, aequalis AK, fiatque primae AB, aequalis LM; secundae vero BC, aequalis fiat NO; tertiae autem CD, sit aequalis MP, & quartae DE, aequalis OR : atque hoc alternatim semper fiat, ita ut omnes AB, CD, EF, GH, &c. sint ex parte L; omnes vero BC, DE, FG, HI, &c. sint ex parte N.

Dico terminum hiuis alternae progressionis, AB, NO, MP, OR, PQ, RS, &c.; dividere magnitudinem LN, in ratione AB ad BC.




**Prop.112. Fig. 1.**

***Demonstratio.***

Quoniam AB, BC, CD, DE, sunt continuae proportionales, etiam AB, CD, EF &c. sunt continuae, & quidem in ratione duplicata AB ad BC; ut patet ex elementis. Similiter in eadem ratione duplicata AB ad BC, erunt continuae proportionales omnes BC, DE, FG, &c atqui omnes AB, CD, &c. sunt ex parte L, & omnes, BC, DE, &c ex parte N. Igitur in magnitudine LN per alternam illam progressionem, constituuntur duae series appositae similes, eiusdem nempe rationis duplicatase AB ad BC. Quare series tota LM, [c] MP, &c. est ad seriem totam NO, OR, &c. ut LM ad NO, hoc est AB ad BC. Deinde quia per series AB, d CD, EF, &c. & series BC, DE, &c. sumul sumptae, aequabuntur seriei AB, BC, CD, DE, &c series quoque L & N , simul sumptae aequabunter seriei AB, BC, CD, DE, &c. Quare cum haec ex hypothesi constituat magnitudinem AK, id est ex hypothesi LN, etiam series L & N magnitudinem LN constituent. ergo eundem

**[115]**

in magnitudine LN, habeant terminum necesse est : si enim diversos habeant ut Y, Z; vel inter utrumque terminum superit media queadam magnitudo, quae ad neutram seriem pertineat; vel aliquid erit utrique commune; ita ut terminus seriei L, sit Z, terminus vero seriei N, sit Y. neutrum autem fieri potest: nam primo dato constitueret utraque series magnitudinem minorem quam LN, alterio autem posito maiorem ; quod utrumque iam demonstratis repugnat. eundem igitur terminum X, habebunt series L & N: cum igitur ostendum prius sit, seriem L esse ad seriem N, ut AB ad BC, utriusque seriei terminus communis X dividet magnitudinem LN, in ratione AB ad BC. Atqui alterna illa magnitudinum LM, NO, MP, OR, &c. progressio, constituit series L & N; ergo ipsius quoque terminus erit X; dividens LN in ratione AB ad BC: quod erat demonstrandum. *c 84 huius; d 99 huius;*

## *PROPOSITION 112.*

A series of continued proportionals is given, making the magnitude AK: moreover, let LN be equal to AK, and the first term AB is made equal to LM; the second truly BC is made equal to NO; the third CD is equal to MP, the fourth DE equal to OR: and this is always done alternately, thus in order that AB, CD, EF, GH, etc. are from the part L; and BC, DE, FG, HI, etc. are from the part N.

I say that the terminus of this alternating progression divides the magnitude LN in the ratio AB to BC.

### ***Demonstration.***

Since AB, BC, CD, and DE are continued proportionals, AB, CD, EF, etc., also are continued proportionals, and indeed are in ratio of the square AB to BC, as is apparent from elementary considerations. Similarly all the continued proportionale BC, DE, and FG etc. are continued proportions in the same square ratio AB to BC. But all AB, CD, etc. are from part L, and all of BC, DE, etc., are from part N. Hence in the length LN from the other progression, two similar series are set up opposed to each other which are indeed in the same square ratio of AB to BC. Whereby the whole series formed from LM, [c] MP, etc. is to the whole series from NO, OR, etc., as LM is to NO, or as AB is to BC. Hence, in accordance the series AB, [d] CD, EF, &c. and the series BC, DE, &c. summed together are equal to the series AB, BC, CD, DE, &c., and also the series L and N summed together are equal to the series AB, BC, CD, DE, &c. Whereby as this constitutes the magnitude AK from hypothesis, also by hypothesis the series L and N constitute the magnitude LN. Hence in the same magnitude LN, it is necessary to have the same terminus or limit : for indeed if they have different end-points, such as Y and Z; then either between both end-points there will be some middle length to which neither series belongs, or there will be some some length common to both series, so that the terminus of series L is Z, and the terminus of series N is Y. But neither is possible to be the case: for the first given gives rise to another series with magnitude less than LN, whereas the other put in place is greater; which both now disagree with the demonstration. Therefore the series L and N have the same terminal point X: hence as was to be shown before, the series L is to the series N, as AB to BC, and the common terminus X of the series divides the magnitude LN in the ratio AB to BC. But that other progression of magnitudes LM, NO, MP, OR, &c., constitute the series L and N; hence the terminus of that too is X; dividing LN in the ratio AB to BC: q.e.d. *c 84 huius; d 99 huius;*

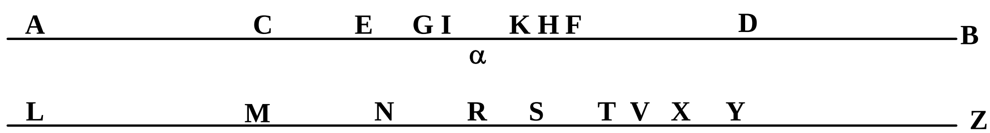
[This theorem is a re-run of the previous one, but with an extra line drawn, and a more detailed proof of a common limit point, which was assumed before.]

## *L2.§2.* *PROPOSITIO CXIII.*

*Centesima undecim aliter demonstratio.*

Data sit magnitudino AB utcunque divisa in C; fiat autem ut AB ad BC, sic BC ad CD, & CD ad DE, & DE ad EF, & EF ad FG: atque hoc semper fiat.

Dico alternae hiuis progressionis terminum α, dividere AB in ratione AB ad BC.




**Prop.113. Fig. 1.**

***Demonstratio.***

Sumatur enim LZ aequalis AB, & singulis in quas dividitur alternatim AB, continuis proportionalibus AB, BC, CD, DE, &c. aequalis fiant MZ, NZ, RZ, &c., erunt igitur etiam hae [a] ideoque & LM, MN, NR, &c. continuae proportionales; & progressionis huius LM, MN, &c. [b] terminus erit Z; & quoniam AB, LZ, & BC, M Z, aequantur, etiam AC, LM aequales erunt. rursus quia BC, MZ, & CD, NZ, aequales sunt, etiam BD, MN aequales erunt. Similiter ostendam CE, ipsi NR, & DF ipsi RS, & GE ipsi ST, & FH ipsi TV (& sic in infinitum) aequales esse : habemus igitur progressionem alternam magnitudinum AC, BD, CE, DF, &c. qualis in praecedente propositione proponebatur, cuius terminus X dividit AB in ratione LM ad MN. Igitur cum ex progressione alterna hic proposita AB, BC, CD, DE, EF, &c. illa altera oriatur, ita ut tam progressio AB, BC, CD, DE, &c. quam progressio AC, BD, CE, DE, &c. in punctis iisdem C, D, F, G, H, &c. dividant magnitudinem AB, huius quoque terminus erit α, dividens AB in ratione LM ad MN, [c] hoc est in ratione LZ ad MZ, hoc est ex constructionein ratione AB ad CB. Quod erat demonstrandum. *a 1 huius; b 79 huius; c 69 huius.*

### *Corollarium.*

Si vero fiat ut AC ad CB; sic BD ad DC, & CE ad ED, & DF ad FE, atque ita semper; hiuis quoque alternae progressionis terminus, dividet AB in ratione AB ad BC. cum enim sit ut AC ad CB, sic BD ad DC, & CE ad ED, &c. componendo erit AB ad BC, ut BC ad CD, & CD ad DE, &c. atqui terminus progressionis AB, BC, CD, &c. dividit AB in ratione AB ad BC: ergo & progressionis AC, CB, BD, DC, &c. terminus, dividet AB in ratione AB ad BC cum enim veraque haec progressio in punctis semper iisdem secet AB, eundem utraque terminum habere debet.

**[116]**

## *PROPOSITION 113.*

*Another Demonstration of Proposition One hundred and Eleven.*

The magnitude AB is given divided in some manner in C; moreover the division is made so that AB is to BC thus as BC is to CD, and CD to DE as DE to DF, and EF to FG: and this shall be done indefinitely.

I say that the terminus α of this alternating progression divides AB in the ratio AB to BC.

***Demonstration.***

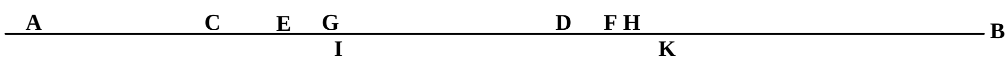
For LZ is taken equal to AB, and for every term in which AB is alternately divided by the continued proportions AB, BC, CD, DE, etc., the equal terms MZ, NZ, RZ, etc. are set out, and therefore these terms are also in [a]continued proportion: LM, MN, NR, &c.; and the terminus of this progression LM, MN, &c. [b] is Z; and since AB and LZ, and BC and M Z are equal, and also [ the differences] AC and LM are equal. Again since BC, MZ, and CD, NZ, are equal, BD and MN are also equal. Similarly I can show that CE is equal to NR, and likewise DF to RS, GE to ST, FH to TV (and so on indefinitely): we therefore have a alternating progression of magnitudes AC, BD, CE, DF, etc. such as are proposed in the preceding proposition, the terminus X of which divides AB in the ratio LM to MN [or AC to BD]. Therefore from the first progression proposed here: AB, BC, CD, DE, EF, etc., another is generated. Thus as the progression AB, BC, CD, DE, etc. , so also the progression AC, BD, CE, DE, etc., divide the magnitude AB in the same points C, D, F, G, H, etc., the terminus of which is $\alpha$ too. The point $\alpha$ divides AB in the ratio LM to MN, [c] or in the ratio LZ to MZ, or from the construction, in the ratio AB to CB. Q.e.d. *a 1 huius; b 79 huius; c 69 huius.*

## *Corollary.*

If indeed the ratio AC to CB is thus made as BD to DC, and CE to ED, and DF to FE, and thus indefinitely ; the terminus of this alternating progression also divides AB in the ratio AB ad BC. For indeed as AC is to CB, thus BD is to DC, & CE to ED, &c. By addition, AB is to BC as BC to CD, & CD to DE, &c., but the terminus of the progression AB, BC, CD, &c. divides AB in the ratio AB to BC: and hence the terminus of the progression AC, CB, BD, DC, &c. also divides AB in the ratio AB to BC; for indeed this progression always divides AB in the same points, and both must have the same terminus.

[This is easy to establish algebraically, but beware that the points are not assigned the same labels consistently between the diagrams for the different propositions!]

## *Lemma primum.*



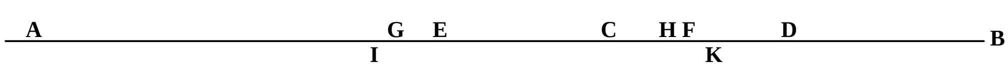

**Prop.113. Fig. 2.**

Data sit magnitudo AB sectam tres partes aequales, in I & K: & rursum aliter secta in C, inter A & I.
Dico bisectionem partis CB, cadere inter I & K in D; & bisectionem partis DA, cadere inter I & C in E; rursum bisectionem partis EB, contingere inter D & K, in F; ipsius autem FA bisectem, inter E & I in G : atque ita in infinitum.

### *Demonstratio.*

Quoniam CB maior est, quam IB dupla AI; erit ipsius CB demidia, maior quam AI. ergo bisectio ipsius CB, cadit ultra I, versus B. Iterum CB plus est, quam dupla KB, adeoque ipsius CB dimidia, maior quam BK; quare bisectio CB, cadit ultra K, versus A : adeoque cadit inter I & K in D. Deinde cum CB plus sit quam duae tertiae, ipsius AB, erit CD eius dimidia, plus quam una tertia ipsius AB; sit autem AC ex datis minor, quam una teria; ergo CD maior est AC. & bisectio ipsius DA, cadet ultra C versus B. similiter cum AI etiam maior sit quam DI, cadet bisectio ipsius DA ultra I versus A, adeoque inter C & I, in E; non aliter ostendemus reliqua, quae in assertione proposuimus. Constat igitur veritas lemmatis.

## *Lemma secundum.*




**Prop.113. Fig. 3.**

Data rursum sit AB sectam tres partes aequales, in I & K: & rursum aliter secta inter A & I, in C.
Dico bisectionem partis CB, cadere inter K & B in D; & bisectionem partis DA, cadere inter C & F in E; bisectionem autem partis EB, cadere inter K & D, in F; partis vero FA, bisectionem contingere inter E & I in G : atque ita in infinitum.
Demonstrato eadem prope quae lemmatis praecedentis.

***First Lemma.***

The magnitude AB is cut in three equal parts by the points I and K: and again cut by the point C in some other way.
I say that the bisection of the part CB lies at D between I and K; and the bisection of the part DA falls between I and C at E; again the bisection of the section EB lies at F, between D and K; moreover the bisection of FA lies at G between I and E, and thus indefinitely.

***Demonstration.***

Since CB is greater than IB, which is twice AI; then half of CB itself is greater than AI. Therefore the bisection of CB falls beyond I towards B at D. Again CB is more than twice KB, and hence half of CB is greater than BK; whereby the bisection of CB falls beyond K towards A: and thus D lies between I and K. Again, as CB is more than two thirds of AB, then half of this or CD is more than one third of AB; but it is given that AC is less than one third of AB; hence CD is greater than AC, and the bisection of DA lies beyond C towards B at E. Similarly, also as AI is greater than DI, the bisection of DA lies beyond I towards A, and thus E lies between C and I ; we will not otherwise demonstrate the rest of the terms, which we have presumed in the statement of the lemms, the truth of which is now established.

[Fig 2: AB is trisected by the points I and K into the equal sections AI, IK, and KB; AI is then cut in some manner at the point C. Now, $CB > IB = 2.AI$; hence $CB/2 > AI$, or the bisection point D viewed from the left end A lies beyond I towards B. Again, $CB > 2.KB$; hence $CB/2 > BK$, or the bisection point viewed from the right end B lies beyond K towards A. Hence D lies between I and K, or $AI < AD < AK$ in modern terms. This bisection at D starts a series in the middle section IK.

For the next term, which starts a progression in the first section AI: Since $CB > 2.AB/3$ then $CD = CB/2 > AB/3$ ; and it is given that $AC < AB/3$; hence $AC < CD$ gives $2.AC < AC + CD < 2.CD$ and hence the bisection of AD at E viewed from A lies beyond C towards B.
Again, as $AI > DI$, then $2.AI > AI + DI > 2.DI$, or the bisection of AD at E is less than I viewed towards A. Hence $AC < AE < AI$, establishing another term E in the series in the first section AI following C. Subsequently, $AD < AF < AK$ for next point F in the middle series, etc. ; and $AE < AG < AI$ for the next point G in the first series, and so indefinitely for the two series.]

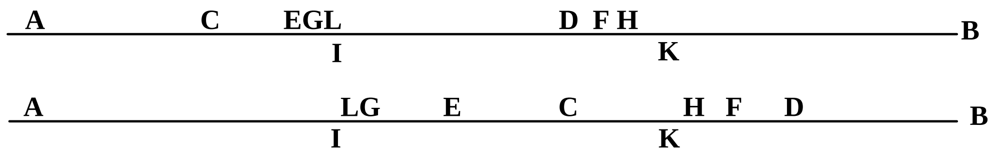
***Second Lemma.***

Again AB is given cut in three equal parts by the points I and K: and again cut by the point C in some way.
I say that the bisection of the part CB lies at D between K and B; and the bisection of the part DA falls between C and F at E; again the bisection of the section EB lies at F, between D and K; and truly the bisection of FA lies at G between I and E : and thus indefinitely.
This can be shown by almost the same method as the preceding lemma.

## *L2.§2.* *PROPOSITIO CXIV.*

Ex magnitudine AB secta in tres partes aequales in I & K, sumatur AC minor vel maior tertia parte totius AB, & bifariam dividatur CB in D, & DA bifariam in E, & EB in F, & FA in G. Rursum GB bifariam in H, & HA in L: atque hoc semper fiat.
Dico hiuis progressionis alternae terminos dividere magnitudinem AB in tres partes aequales.

***Demonstratio.***




**Prop.114. Fig. 1.**

Cum CB dupla sit DB, & EB dupla FB, erit CB ad DB, ut EB ad FB, ergo CE ad DF, [a] ut EB ad FB. Quare cum EB dupla sit FB, etiam CE, ipsius DF dupla erit. Deinde cum DA dupla sit EA, itemque FA dupla ipsius GA, erit

**[117]**

DA ad FA, ut EA ad GA; ac proinde DF [a] ad GE, ut DA ad EA. Quare cum DA ipsius EA dupla sit, etiam DF dupla erit EG. unde CE quadrupla est ipsius EG. Similiter ostendemus EG duplam esse FH, ipsam autem FH duplam esse GL, proindeque EG ipsius GL quadruplam esse : atque ita continuando sine statu, per alternam illam bisectionem constitui progressionem magnitudinum CE, EG, GL, &c. proportionis quadruplae. eadem autem discursu quo prius usi fuimus, demonstrabimus DF esse quadruplam FH, & FH quadruplam sequentis termini, ac proinde etiam hic progressione rationis quadruplae statui. Ulterius quoniam tam ratio CB ad DB, quam IB ad KB, dupla est, erit CB ad DB, ut IB ad KB, & [b] CI ad DK, ut IB ad KB. Itaque cum IB dupla sit KB, etiam CI ipsius DK dupla erit; similiter DK ipsius EI duplam esse demonstrabimus. Igitur CI quadrupla est EI: eadem methodo discurrendi, ostenditur [c] DK esse ad FK, ut DF est ad FH. Quare & progressionis DF, FH, &c. terminus erit K. dum igitur utraque progressio CE, EG, &c. DF, GH, &c. constituatur ab alterna illa bisectione, in propositione proposita, ipsius quoque termini erunt in I & K; ubi trifariam dividitur magnitudo AB. Quod erat demonstrandum. *a 19 Quinti; a ibid; b ibid; c 79 huius.*

*Assumpsimus AC minorem aut maiorem tertia parte magnitudinis data AB ; quia si aequalis uni tertia foret, bisectiones alterna in eadem semper puncta I & D inciderent; uti manifestum est, assertionem propositionis consideranti.*

## *PROPOSITION 114.*

From the magnitude AB cut into three equal parts by I and K, a section AC is taken either larger or smaller than the third part of the whole length AB, and CB is equally divided in D, and DA equally divided in E, and EB in F, and FA in G. Again GB is equally divided in H, and HA in L: and this bisection is made indefinitely.

I say that the alternate terms of this progression divide the magnitude AB into three equal parts.

***Demonstration.***

Since CB is twice DB, and EB is twice FB, then CB is to DB as EB is to FB, hence CE is to DF, [a] as EB is to FB. Whereby as EB is twice FB, also CE is twice DF. Hence as DA is twice EA, and likewise FA is the double of GA, DA is to EA as FA to GA; and hence DF [a] is to GE as DA to EA. Whereby as DA is the double of EA, also DF is the double of EG. Hence CE is four times EG. Similarly we can show that EG is twice FH, but FH is twice GL, and hence EG is four tims GL : and thus by continuation without stopping, the progression of the magnitudes CE, EG, GL, etc. of the quadruples of the proportions can be set up through bisecting the other section. Moreover by the same discourse which we used previously, we can show that DF is the quadruple of FH, and FH the quadruple of the following term, and thus also this progression of quadruple ratios is set up. Furthermore, since the ratios IB to KB and CB to DB are both 2, CB is to DB as IB to KB, and [b] CI to DK is as IB to KB. Thus as IB is twice KB, we can also show that CI is the double of DK; similarly DK is the double of EI. Therefore CI is four times EI: by the same kind of reasoning, it can be shown that [c] DK is to FK as DF is to FH. Whereby the terminus of the progression DF, FH, etc. is K, while each progression in the proposition proposed CE, EG, &c. DF, GH, &c. are thus established by the bisection of the sections of the one by the other, the end-points are also I and K, thus trisecting the magnitude AB. Q.e.d. *a 19 Quinti; a ibid; b ibid; c 79 huius.*

*Assumpsimus AC minorem aut maiorem tertia parte magnitudinis data AB ; quia si aequalis uni tertia foret, bisectiones alterna in eadem semper puncta I & D inciderent; uti manifestum est, assertionem propositionis consideranti.*

*We have assumed that AC is either less or greater than the third part of the magnitude AB; for if it should be equal to a third, alternate bisections always occur at the points I and D; a useful vindication of the theorm considered.*

[CB = 2.DB and EB = 2.FB; giving CB/DB = EB/FB from which CB/EB = DB/FB giving CE/EB = DF/FB and on re-arranging we have CE/DF = EB/FB, and so EB = 2.FB and CE = 2.DF;
Again: DA = 2. EA and FA = 2.GA and hence DA/EA = FA/GA or FA/DA = GA/EA
giving FD/DA = GE/EA or DF/GE = DA/EA; as DA = 2.EA and DF = 2.EG, then CE = 2.DF = 4.EG.
Similarly, EG = 2.FH and FG = 2.GL, hence EG = 4.GL; and thus the progression CE, EG, GL, .....is established where each term is $^1/_4$ of the previous term.
Again, DF = 4.FH; FH = 4....., etc; hence another similar progression can be set up.
Again, IB/KB = 2 and CB/DB = 2, then and CB/DB = IB/KB or CB/IB = DB/KB giving CI/DK = IB/KB = 2; thus IB = 2.KB and CI = 2.DK.
Similarly, DK = 2.EI and CI = 4. EI; since CI/EI = CE/EG = 4, and the limit of the progression is I.
Again, DK/FK = DF/FH , and thus the limit of the progression DF, FH, .... is K; hence the limit points I and K divide the line AB in the required ratio.
Analytically, we can set AB = 1 without loss of generality, and consider the point C to be at a distance $x_0$ from A, lying in the interval $0 < x_0 < {}^1/_3$. The point $y_0$, according to the construction, then lies at a distance $^1/_2\,(1 - x_0)$ from B: i. e. $y_0 = {}^1/_2\,(1 - x_0)$ ; subsequently, $x_1 = {}^1/_2\,(1 - y_0) = {}^1/_4\,(1 + x_0)$ ;
$y_1 = {}^1/_2\,(1 - x_1) = {}^1/_2\,({}^3/_4 - {}^1/_2\,x_0) = {}^1/_8\,(3 - x_0)$; $x_2 = {}^1/_2\,(1 - y_1) = {}^1/_{16}(5 + x_0)$;
$y_2 = {}^1/_2\,(1 - x_2) = {}^1/_{32}(11 - x_0)$ ; $x_3 = {}^1/_2\,(1 - y_2) = {}^1/_{64}(21 + x_0)$; $y_3 = {}^1/_2\,(1 - x_3) = {}^1/_{128}\,(43 - x_0)$, etc.

To generalise: the sequences $\{x_n\}$ and $\{y_n\}$ defined by $x_n = {}^1/_2\,(1 - y_{n-1})$ and $y_n = {}^1/_2\,(1 - x_n)$ for $n \geq 0$ and $x_0$ given in the interval $0 < x_0 < {}^1/_3$, obviously converge to the points I = $^1/_3$ and K = $^2/_3$.
For if we set $\lim x_n = X$ and $\lim y_n = Y$ for very large $n$, then $X = {}^1/_2\,(1 - Y)$ and $Y = {}^1/_2\,(1 - X)$, leading to X = Y = $^1/_3$.
In terms of the above ratios, CA = $x_0$; CB = $(1 - x_0)$; DB = $y_0 = {}^1/_2\,(1 - x_0)$ ; AD = $(1 - y_0) = {}^1/_2\,(1 + x_0)$;
AE = $x_1 = {}^1/_2\,(1 - y_0) = {}^1/_4\,(1 + x_0)$ ; EB = $1 - x_1 = {}^1/_4\,(3 - x_0)$ ; FB = $y_1 = {}^1/_8(3 - x_0)$; etc. ]

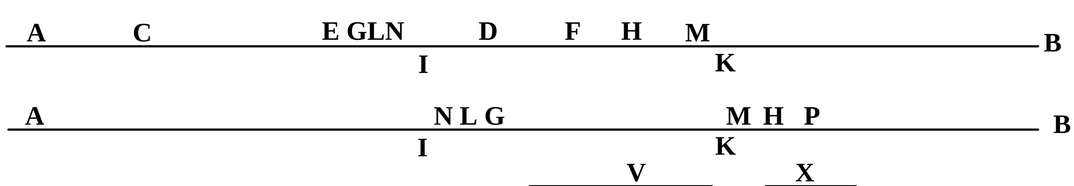
***Lemma.***

## *PARS PRIMA.*

Data sit magnitudo AB secta in I & K secundum rationem V ad X: ita ut AK sit ad KB, ut BI ad IA. divisa sit deinde AB adhuc aliter inter A & I in C.

Dico si CB divitatur in ratione V ad X, sectionem fiere ultra K in D, item si DA dividatur in ratione V ad X sectionem cadere ultra I in E: rursum si EB divitatur in eadem ratione, sectionem contingere inter K & D in F; & si FA, sectionem fore inter I & E in G. atque in infinitum.

***Demonstratio.***




**Prop.114. Fig. 2.**

Cum AK sit ad KB, ut BI ad IA, erit componendo AB ad KB, ut AB ad IA; ergo KB, IA , additoque communi IK etiam AK, IB aequantur: unde cum AK sit ad KA , ut V ad X; utique IB ad KBm in eadem ratione erit: quare CB (ex datis maior quam IB) maiorem habet rationem ad KB, quam V ad X; ergo sectio ipsius CB, in ratione V ad X, cadit ultra K in D: similiter cum AK sit ad KB, id est IA, sic ut V est ad X, erit DA ad eandem IA, in minori ratione, quam V ad X: unde sectio ipsius DA in ratione V ad X, cadet ultra I in E: Quod autem sectio ipsius EB cadet ultra K, eodem quo prius modo ostendatur: Item quod ultra D, versus B sic ostendo; facta CB ad DB, in ratione V ad X erit EB ad DB, in minori ratione quam V ad X.

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ergo sectio ipsius EB, in ratione V ad X, cadit ultra D, versus B: ergo cum etiam ultra K versus A cadat, inter D & K, contingat necesse est, nempe in F; similiter sectionem ipsius FA inter I & E, futuram in G demonstrabimus, atque ita in infinitum; discursus enim idem omnibus divisionibus sequintibus quadrat.

## *PARS SECUNDA.*

Iisdem positis; si C cadat inter I & K (na si inter B & K cadere, foret casus primae partis) simili plane discursu demonstrabimus eadem omnia contingere quae prius, hoc solum mutato, quod signa divisionum E, G, K, &c. DF, HM, &c. ad alterum latus ordine constituantur.

### *Lemma.*

## *First Part.*

The magnitude AB is given cut by the points I and K according to the ratio V to X : thus as AK is to KB, and so also BI to IA are as V to X. AB is then divided again in some manner at the point C lying between A and I.

I say that if [subsequently] CB is divided in the ratio V to X, then the section is made beyond K at D; likewise if DA is divided in the ratio V ad X then the section falls beyond I at E: again if EB is divided in the same ratio, then the section lies between K and D at F; and if FA is divided, then the section is between I and E at G; and so on indefinitely.

### *Demonstration.*

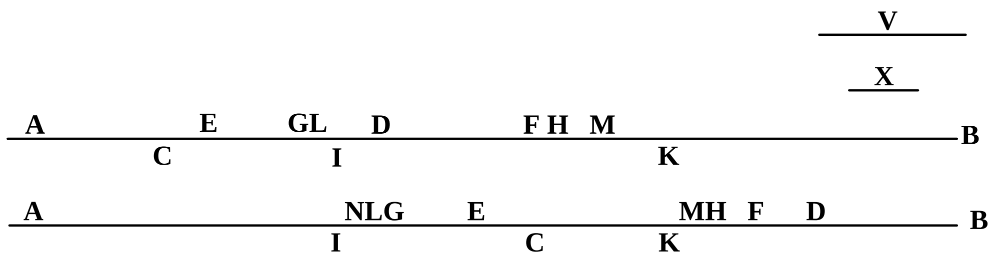
Since AK is to KB as BI is to IA, then the sum AB is to KB as the sum AB is to IA; therefore KB and IA are equal and on adding the common length IK to each, AK and IB are also equal: thus as AK is to KB as V is to X, then IB to KB is in the same ratio V to X. Whereby CB (which is given greater than IB) to KB, has a larger ratio than V to X; hence the section of CB, in the ratio V to X, lies beyond K at D. Similarly, AK is to KB (or IA) thus as Vis to X: hence DA is to the same IA in a smaller ratio than V to X, and thus the section of DA in the ratio V to X falls beyond I at E. But concerning the section of EB that lies beyond K at F, it can be shown in the same manner as established earlier, likewise as beyond D and towards B, I can thus show that CB made to DB in the ratio V to X results in a ratio EB to DB smaller than V to X that lies beyond D towards B . Hence the section of EB, in the ratio V to X, lies beyond D towards B : it also lies beyond K towards A, and is hence between D and K, and so it lies at F. Similarly the section of FA lies between I and E, we can show that it lies at G, and so on to infinity; indeed the same discourse for all quadruple divisions follows.

[We are given initially that $AK/KB = BI/IA = V/X$ from which $AB/KB = AB/IA$ and hence $AI = KB$ and $AK = IB$; and hence $BI/BK = AK/AI = V/X$. We are given $AC < AI$ initially, or equally, $BA > BC > BI$. Now, $BC > BI$ and hence $BC/BK > V/X = BI/BK$: hence the section of BC in the ratio V to X results in a point D that lies beyond K away from B; i. e. $BI > BD > BK$ or alternately $AI < AD < AK$.
In a like manner, $AK/AI = V/X$, and hence as $AD < AK$, then $AD/AI < AK/AI = V/X$, and the section of AD in the ratio V/X results in a point E such that $AC < AE < AI$, or alternately, $BC > BE > BI$.
Continuing the subdivision of the alternate interval BE : $BA > BC > BE > BI$, and hence subdivision of BE in the ratio V/X gives a point F such that $BI > BD > BF > BK$, etc. ]

## *Second Part.*

For the same positions; if C falls between I and K (for if it lies between B and K, it will be the case of the first part.) by the same clear discussion we can show that all the points lie as previously, but with this change only, that the marks of the divisions E, G, K, etc., and DF, HM, etc. are set up in order on the other side.

## *L2.§2.* *PROPOSITIO CXV.*




**Prop.115. Fig. 1.**

Data sit proportio V ad X, & magnitudo AB ita secta in I & K, ut AK sit ad KB, & BI ad IA, sicut V est ad X. Aliter deinde dividat ut AB in C; quocumque tandem loco cadat C, modo non incidat in I aut K:

Fiat autem CB ad DB, ut V ad X; & DA ad EA, ut V ad X: item EB ad FB, & FA ad GA, & GB ad KB, & HA ad LA, fuerint inter se ut V, est ad X: Atque hoc semper continuetur.

Dico alternae hiuis progressionis terminos, fore in I & K, ubi AB, dividitur in ratione V ad X.

***Demonstratio.***

Quoniam est ex constructione CB ad DB, ut EB ad FB (sunt enim utraque ad invicim in ratione V ad X) etiam CE reliquum [a], DF reliquum, erit ut CB ad DB, id est sicut V ad X: & quia DA est ad EA, ut FA ad GA (nempe in ratione V ad X) rursum erit DF ad EG, ut FA ad GA, hoc est ut V ad X : sunt igitur CE, DF, EG, tres continuae proportionales in ratione V ad X. ergo ratio CE ad EG duplicata est rationis V ad X. Similiter ostendimus EG, FH, GL esse continuae in ratione V ad X, ideoque rationem EG ad GL, duplicatam esse rationis V ad X. Cum ergo etiam ratio CE ad EG, sit rationis V ad X duplicata, erunt CE, EG, GL, in continua analogia; atque ita continuando sine statu alternam illam divisionem, demonstrabimus constitui progressionem magnitudinem CE, EG, GL, LN, &c. continue proportionalium in ratione duplicata V ad X, ab alterna vero parte, eodem plane discursu ostendemus DF, FH, HM, &c. esse continuas in ratione duplicata V ad X; ac proinde sic quoque constitui progresionem proportionis duplicatae V ad X. ulterius quia AK est ad KB, ut BI ad IA, componendo AB erit ad KB , ut AB ad AI; ideoque KB, AI aequantur: additaque IK communi, aequalis erunt IB, AK; ergo ut AK ad KB, id est ex constructione ut V ad X, sic IB ad KB: Quare cum & CB ad DB, sit ut V ad X, etiam CB erit ad DB, ut IB ad KB. allatis ergo IB, KB, CI erit ad DK, ut CB ad DB reliquum ad reliquum, hoc est ex constructione ut V ad X. similiter demonstrabimus DK esse ad EI, ut V ad X,

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erunt igitur CI, DK, EI continuae proportionales in ratione V ad X: ideoque ratio CI ad EI, duplicata erit rationis V ad X; quare cum & ratio CE ad EI, eiusdem ostensa sit esse duplicata, erit CI ad EI, ut CE ad EG, & permutando ut CE ad CI, sic EG ad EI, unde terminus [a] progressionis CE, EG, &c. est I. simili discursu ostentetur, etiam DK esse ad FK, ut DF est ad FH; quare huius quoque progressionis terminus erit K; Itaque cum utraque progressio CE, EG, &c. DF, FH, &c. ab alterna illa divisione constituatur; ipsius quoque termini erunt I & K; ubi magnitudo AB, dividitur in ratione V ad X. Quod erat demonstrandum.

*a 19 Quinti; b ibid; a 79 huius.*

***Scholium.***

*Hic quoque voluimus punctum C non incidere in I aut K; eo quod si in alterutrum incideret, divisiones quoque alterna, in eadem semper puncta I & K, deberent incidere; ut patet consideranti statum Theorematis.*

*Caeterum qui hanc propositione cum priori contulerit, facile intelliget hanc universalem esse, illam vero particularem casum complecti. placuit enim id subinde tum hic, tum alibi factitare, tum quia in particularibus casibus eiusdem Theorematis veritas, clarius non raro atque illustrius emicat, tum quia a particularem casum cognitione, facilius ad percipienda, universalium Theorematum demonstrationem proceditur.*

## *PROPOSITION 115.*

The proportion V to X is given, and the magnitude AB is thus cut in I and K, in order that AK is to KB, and BI is to IA, as V is to X. Following this, AB is divided by C in some other way; yet whatever point C falls on, it is not I or K:

Moreover CB is made to DB as Vto X; and DA to EA, as V to X: likewise EB to FB, and FA to GA, and GB to HB, and HA to LA, are each as V to X : and this construction is continued indefinitely.

I say that the alternating terminating points of this progression are I and K, as AB is divided in the ratio V to X.

***Demonstration.***

Since from the construction CB is to DB, as EB is to FB (for each in turn is equal to the ratio V to X) also the difference CE [a], and the difference DF are as CB to DB, or as V to X: and since DA is to EA, as FA to GA (surely in the ratio V to X) again DF is to EG as FA is to GA, or as V to X : therefore CE, DF, and EG are three continued proportionals in the ratio V to X. Therefore the ratio CE to EG is the square of the ratio V ad X. Similarly we can show that EG, FH, GL are continued proportionals in the ratio V to X, and thus the ratio EG to GL is the square of the ratio V to X. Also, as the ratio CE to EG is therefore the square of the ratio V to X then CE, EG, GL are in continued proportion; and thus by continuing without ceasing this alternate division , we can show that a progression of magnitude CE, EG, GL, LN, etc. of continued proportions can be established in the ratio of the square of V to X.

Indeed from the other part, by the same clear discourse, we can show that DF, FH, HM, &c. are continued in the square ratio V to X; and hence thus also constitute a progresion in the proportion of V to X squared. Beyond which AK is to KB, as BI to IA, on adding AB is to KB as AB is to AI; and hence KB and AI are equal: and on adding to the common length IK, IB and AK are equal. Hence as AK is to KB, or from the construction as V to X, thus IB is to KB. Whereby as CB to DB is as V is to X, also CB is to DB as IB is to KB, therefore on bringing together IB and KB, CI is to DK as CB is to DB, remainder to remainder, from the construction, or as V to X. Similarly we can show that DK is to EI as V to X, and therefore CI, DK and EI are continued proportionals in the ratio V to X: and hence the ratio CI to EI is the square of the ratio V to X; and the ratio CE to EI can be shown to be the square of the same ratio; hence CI to EI as CE to EG, and on re-arranging, as CE to CI, thus EG to EI, hence the terminus [a] of the progression CE, EG, etc. is I. By a like discourse it can be shown that DK also is to FK as DF is to FH; whereby the terminus of the progression is K. Thus as each progression CE, EG, erc., and DF, FH, erc. is established by this alternate division with termini I and K, just as the length AB is divided in the ratio V ad X. Q.e.d. *a 19 Quinti; b ibid; a 79 huius.*

***Scholium.***

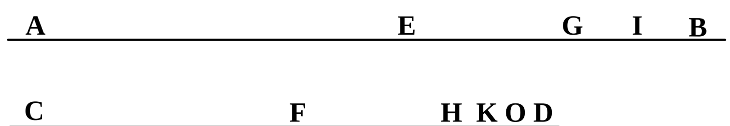
*In the present circumstances we have wished the initial point C not to fall on I or K; but if the point C happens to fall on in either of these, then the divisions alternate too, but always they have to fall on the same points I and K, as is apparent from the statement of the Theorem.*

*Otherwise, whoever brings this proposition and the previous one together, can easily understand that it is of a more universal nature than the particular case included. Indeed it may please one, immediately upon doing this example, to do one with a different ratio, for the truth of the Theorem arises most clearly from a study of particular cases; for from a study of the particular case, the general case should be made easer to understand, and the demonstration of the Theorem can proceed.*

[CB/DB = EB/FB = V/X; hence on subtracting, CB/EB = DB/FB, giving CE/EB = DF/FB and on re-arranging we have CE/DF = EB/FB = CB/DB = V/X.
Again, DA/EA = FA/GA = V/X is given; from which DA/FA = EA/GA and on subtracting, DF/FA = EG/GA or DF/EG = FA/GA = V/X and (CE, DF, EG) are lengths in continued proportion. Hence, CE/DF = DF/EG = V/X : and also (CE/DF).(DF/EG) = CE/EG = $(V/X)^2$.
Similarly, (EG, FH,GL) are in continued proportion in the ratio V/X, and EG/GL = $(V/X)^2$.
Also, from the underlined ratios: CE/EG = $(V/X)^2$ = EG/GL, (CE, EG, GL) are in continued proportion, and this sequence formed from the alternate divisions can be continued indefinitely in the ratio $(V/X)^2$.
We can thus show that a progression of magnitudes (CE, EG, GL, LN, .....) in the ratio $(V/X)^2$ can be established. Likewise, for the other series, (DF, FH, HM, ....) are continued in the ratio $(V/X)^2$.

We are also given AK/KB = BI/IA = V/X for the limit points I and K, from which on addition, AB/KB = AB/IA and hence KB = IA and also IB = AK on adding IK to each, as in the previous theorem. Hence AK/KB = IB/KB = V/X, and also CB/DB = V/X = IB/KB or KB/DB = IB/CB, and on subtraction, DK/DB = CI/CB or CI/DK = CB/DB = V/X . Again, DK/EI = V/X; for BI/IA = AK/ AI = V/X = DA/EA and so AK/DA = AI/EA giving DK/DA = EI/EA or DK/EI = DA/EA = V/X as required. Hence, (CI, DK, EI) are continued proportionals in the ratio V/X, and CI/EI = $(V/X)^2$ . From CI/EI = $(V/X)^2$ = CE/EG above, we have CE/CI = EG/EI = GL/GI = ...... , and hence the termination of the series of ratios is I. In a similar manner, DF/DK = FH/FK = HM/HK = ..., and the termination of the other series of ratios is K. Thus the series (CE, EG, GL, LN, .....) and (DF, FH, HM, ....) terminate in the points I and K, as does the length AB divided in the ratio V to X.

## *L2.§2.* *PROPOSITIO CXVI.*




**Prop.116. Fig. 1.**

Sint duae quantitates AB, CD; sitque AB divisa in E & G, ita ut AE, sit non minor dimidio AB, & EG non minor dimidio EB; eodem modo divisa sit CD in F & H, sintque AE, EG; CF, FH proportionales: & hoc semper fieri possit.

Dico totam AB esse ad totam CD, ut est AE ad CF.

***Demonstratio.***

Si enim non est proportio AB ad CD aequalis proportioni AE ad CF, erit vel maior vel minor: sit primo minor. cum ergo ponatur AB ad CD, minorem habere rationem, quam AE ad CF, habebit AB ad aliquam [b] minorem quam CD; nempe ad CK, eandem proportionem, quam AE ad CF: & quoniam ex quantitatibus AB, CD, earumque residuis semper non minus dimidio aufertur, si continuetur haec ablatio terminos, verbi gratia per tres CF, FH, HO, relinquetur tandem OD [c] minor quam KD ; ideoque CO erit maior quam CK: si iam ex AB totidem partes ad mentem proportionis AE, EG, GI, tollantur, erit ex hypothesi AE ad EG, ut CF ad FH, & EG ad GI, ut FH ad HO: ideoque permutando ut AE ad CF, sic EG ad FH, & ut EG ad FH, sic GI ad HO. ergo [e] ut AE una antecedentium, ad CF unam consequentium, sic omnes antecedentes, id est linea AI ad omnes consequentes, id est lineam CO: sed ut AE ad CF, sic est ex constructione AB ad CK; ergo AI, est ad CO, ut AB ad CK; quod est absurdum; ut patet ex elementis. non est igitur proportio AB ad CD minor proportione AE ad CF.

A E G K I B C F H O D

**Prop.116. Fig. 2.**

Sit iam, si fieri potest, proportio AB ad CD maior proportione AE ad CF : itaque aliqua E minor quam AK, habebit ad CD eandem rationem quam AE ad CF, & quoniam aufertur semper non minus dimidio, post aliquot partes, exempli gratia post tres AE, EG, GI, ablatas, relinquetur tandem IB minor quam AK; ideoque AI erit maior quam AK. Si iam totidem auferantur est quantitate CD, nempe partes CF, FH, HO, erit ex hypothesi, & permutando AE, ad CF, ut EG, ad FH,

**[120]**

item ut GI ad HO. ergo a ut AE una antecedentium, ad CF unam consequentium, ita omnes antecedentes, id est linea AI, ad omnes consequentes, nempe lineam CO. Atqui ex constructione ut AE ad CF, sic erit AK ad CD; ergo AI est ad CO ut AD ad CD, quod esse absurdum patet ex elementis. non est igitur ratio AB ad CD, maior ratione AE ad CF. patet ergo proportionis veritas. *b 8 Quinti; c 1 Decimi; d 8 Quinti ; e 8 Quinti; a 12 Quinti.*

***Corollarium.***

A E G I B

C F H K O D

**Prop.116. Fig. 3.**

A duabus quantitatibus AB, CD, auferri possint AE, CF, aequalia, & non minora dimidio ipsarum AB, CD; & a residuis EB, FD rursum auferri possint EG, FH, aequalia & non minora dimidio residuorum : sic hoc semper fieri possit, aequales erunt quantitates AB, CD. Patet ex demonstratione propositionis.

Quamquam fateat hoc Theorema aliud non continere, quam particularem casum propositionis prioris: tamen quia in libris sequentibus non semel usui veniet, visus mihi sum operae pretium facturus, si facilitatis causa explicite hic apponerem.

Similiter hoc quoque Theorema eiusdem proportionis universalis casus erit : si fuerint duae quantitates, a quibus auferri semper possint non minora dimidio, sic ut ablata singula unius, dupla perpetuo sint singulorum ex altera ablatorum, erit una quantitas alterius dupla.

Quod si ablata unius, semper tripla fuerint ablatorum alterius; erit una quantitas, alterius quadrupla. Atque ita in infinitum per proportiones quadruplam, quintupla, &c. licebit procedere.

## *PROPOSITION 116.*

There are two lengths AB and CD ; and AB is divided by E and G, so that AE is equal to half of AB, and EG is equal to half of EB; CD is divided by F and H in the same way, and AE, EG; CF, and FH are proportionals: and this is can be done indefinitely for both lines.

I say that the whole length AB to the whole length CD is as AE is to CF.

***Demonstration.***

If indeed the proportion AB to CD is not equal to the proportion AE to CF, it will be either greater or less than this ratio: first let us assume that it is smaller. Now, when AB to CD is made to have a smaller ratio than AE to CF, then AB will have the same ratio to some quantity less than [b] CD, surely CK, as AE to CF: and since from the quantities AB and CD, from these portions of the lines still remainding, a proportion always greater than a half is taken away, if these terms are continued in proportion by subtraction. For example, if the three terms CF, FH, and HO are constructed in the second series, and there remains finally a term KO less [c] than KD ; in this case CO is greater than CK. If now from the other series AB, the sum of the same terms according to the example of the proportion AE, EG, GI is taken away, then by hypothesis AE is to EG, as CF to FH, and EG is to GI, as FH to HO: and thus on interchanging, as AE is to CF thus EG is to FH, and as EG is to FH, thus GI is to HO. Hence, [e] the ratio of AE, the first term of the first series to CF, the first term of the second series, is thus to the sum of these terms of the first series or the linea AI, to the sum of the corresponding terms of the second series or the line CO: [i. e. AE/CF = AI/CO] but as AE is to CF, thus by construction AB is to CK [ i. e. we have assumed that AE/CF = AB/CK]; hence AI is to CO as AB is to CK; which is absurd, as is apparent from basic principles. [ i. e. AI/CO = AB/CK : the partial sum CO is greater than the whole assumed sum CK]. Therefore the proportion AB to CD is not less than the proportion AE to CF.

Now, if it is possible, the proportion AB to CD is greater than the proportion AE ad CF: and hence some [e] lesser length such as AK, has the same ratio to CD as AE to CF, and as the amounts taken away are always greater than half the amount left, after some number of terms, for example after the three terms AE, EG, and GI taken away, there remains finally a length IB less than AK; and thus AI is greater than AK. If now the same sum of terms is taken away from the quantity CD, surely the lengths CF, FH and HO, and by hypothesis and on interchanging AE is to CF as EG is to FH, and likewise as GI is to HO. Hence [a] as AE, the first term of the first series to CF the first term of the second series, thus the sum of the first series or the line AI, to the sum of the second series, surely the line CO. But from the construction as AE is to CF, thus AK is to CD; hence AI is to CO as AD to CD, which is absurd from elementary considerations.

hence the ratio AB to CD in not greater than the ratio AE to CF. The truth of the proportion is thus apparent. *b* Euclid V.8*; c* Euclid X.1*; d* Euclid V.8 *; e* Euclid V.8*; a* Euclid V.12

***Corollory.***

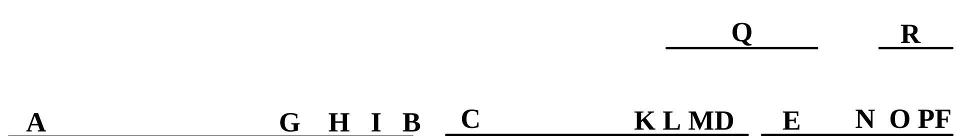
From two quantities AB and CD, it may be possible to take away equal amounts AE and CF, and which are not less than the half of AB and CD; and from the remainders EB and FD again to take the equal amounts EG and FH, and again the remainders are not less than half these sections : thus this is can be done indefinitely, and hence the quantites AB and CD are equal. This is apparent from the demonstration of the proposition.

Nevertheless it must be admitted that this Theorem does not include another [application] as a particular case of the prior proposition: nevertheless in the following books many uses spring to mind, and it seems to me that I am soon to have a reward for the work, if I can set out with ease the cause explicitely as here described .

Similarly too this Theorem is an instance of the same general proportion : if there are two quantities, from which it is always possible to take away not less than half the section, thus as a certain amount is taken from the first [term of the first series], twice this amount is always taken from [the corresponding term] of the other series, then one quantity is twice the other. [For the amounts taken are $a(1 - r)$ and $2.a(1 - r)$ for the second term from the first, for example].

For what is taken from the first, three times as much always can be taken from the other; or there is the first quantity, and four times the other; or four of the other, or five times as much, and as you please to procede.

## *L2.§2.* *PROPOSITIO CXVII.*




**Prop.117. Fig. 1.**

Data sint tres magnitudines, aut plures AB, CD, EF & a singulis auferri possit non minus dimidio, ita ut ablata AG, CK, EN sint in continua analogia Q ad R. Deinde a residuis auferri possit iterum non minus dimidio, ita ut ablata GH, KL, NO sint continue proportionalia in ratione eadem Q ad R: & hoc semper fieri possit.

Dico propositas magnitudenes AB, CD, EF esse in continua analogia.

***Demonstratio.***

Quoniam AG est ad CK, ex hypothesi ut Q ad R; & GH ad KL, & HI ad LM, ut Q ad R, erunt partes ablatae AG, CK, GH, KL; HI, LM, atque ita in finitum invicem proportionales : Quare cum hypothesi etiam singulae sint non [b] minores dimidiis suorum integrorum, erit AB ad CD, ut AG ad CK, hoc est ex datis ut Q ad R. Similiter ostendam CD esse ad EF, ut CK ad EN, hoc est ex datis ut Q ad R. erunt igitur AB, CD, EF, continuae proportionales magnitudines. Quod erat demonstrandum. *b 116 Huius.*

**[121]**

## *PROPOSITION 117.*

There are three or more lengths given AB, CD, EF, and not less than half of each can be taken from the individual sections, in order thus that the parts taken AG, CK, EN are in the continued ratio Q to R. Then from the remaining lengths not less than half of each can be taken away, in order that the remaining lengths GH, KL, NO are in continued proportion in the given ratio Q to R : and this process can be continued indefinitely.

I say that the magnitudes AB, CD, and EF are in continued proportion.

***Demonstration.***

Since AG is to CK, from hypothesis as Q to R; and GH to KL, and HI to LM, as Q to R; then the parts taken AG, CK, GH, KL; HI, LM, and thus indefinitely are in turn proportionals. Whereby by hypothesis the individual terms also are not[b] less than half of their wholes, hence AB is to CD as AG to CK, that is given as Q to R. Similarly I can show that CD is to EF as CK to EN, or as Q to R from what is given. Therefore AB, CD, and EF are magnitudes in continued proportion. Q.e.d. *b 116 Huius*

## *L2.§2.* *PROPOSITIO CXVIII.*

**A G H I B C KLMD E N O PF Q R**

**Prop.118. Fig. 1.**

Propositae sint tres, aut plures magnitudines, AB, CD, EF & ratio Q ad R quaecumque, minoris inaequalitatis: Auferantur a singulis AG, CK, EN, ita ut ablata AG, CK, EN, sint ad sua tota, in ratione Q ad R & in eadem ratione Q ad R inter se continuae proportionalia;

Dico hoc semper fieri possit, propositas magnitudenes AB, CD, EF esse in continua analogia.

***Demonstratio.***

Quia ex hypothesi AG est ad AB, ut GH ad GB, erit etiam [a] reliquum GB ad reliquum HB, ut tota AB ad totam GB; sunt igitur AB, GB, HB & eodem discursu etiam IB reliquaeque in infinitum continuae proportionales : unde etiam [b] ablata AG, GH, HI, &c. sunt in continua analogia, & [c] terminus huius progressionis AG, GH, &c. est B. similiter ostendam ablata CK, KL, LM, &c. esse in continuae analogia, cuius terminus sit D. & quoniam ex hypothesi AG est ad AB, ut Q ad R; & CK ad CD, ut Q ad R, erit AG ad AB ut CK ad CD; & invertendo ac per conversionem rationis AB ad GB, ut CD ad KD; Atqui AG, GH, HI, &c, sint [d] continuae proportionales in ratione AB, ad GB; hoc est ut iam ostendi CD ad KD; & per eandem CK, KL, &c. sunt etiam continuae in ratione CD ad KD: Igitur AG, GH, &c. CK, KL, &c.similium rationum series sunt. Quare AB est ad CD [e], ut AG ad CK. simili prorsus discursu ostendam CD esse ad EF ut CK ad EN, sunt autem ex datis AG, CK, EN, tres continuae proportionales; ergo AB, CD, EF in continua sunt analogia. Quod erat demonstrandum. *a 19 Quinti; b 1 Huius; c 79 huius; d 6 huius; e 84 huius.*

## *PROPOSITION 118.*

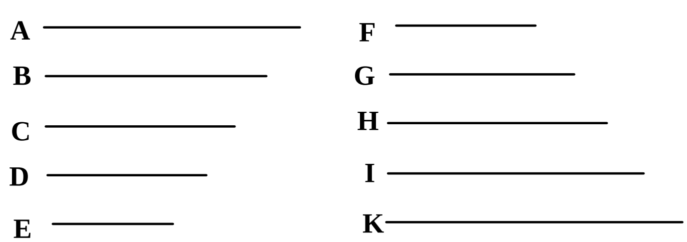
Three or more magnitudes [or lengths] AB, CD, EF, &c. are proposed,and some ratio Q to R of the smaller inequality [less than one]: AG, CK, and EN are to be taken from the individual lengths in order that the terms taken AG, CK, EN are in the ratio Q to R to their own total lengths, and each series is in continued proportion in the same ratio Q to R;

I say that this can always be accomplished, and the proposed magnitudes AB, CD, EF are in continued proportion.

***Demonstration.***

Since from hypothesis AG is to AB, as GH to GB, also the remainder [a] GB is to the remainder HB, as the total AB to the total GB; therefore AB, GB, HB, and by the same discourse IB and the rest also, are part of an infinite series of continued proportionals : hence al so [b] the remainders AG, GH, HI, &c. are in continued proportion, and [c] the terminus of this progression AG, GH, &c. is B. Similarly I can show that the remainders CK, KL, LM, &c. are in a continued progression, the terminus of which is D. Since from hypothesis AG is to AB as Q to R; and CK to CD as Q to R, then AG is to AB as CK to CD, on inverting and conversion of the ratio AB to GB as CD to KD. But AG, GH, and HI, &c, are [d] continued proportions in the ratio AB to GB; this is as now shown CD to KD; and in the same manner CK, KL, etc. are also continued in the ratio CD to KD. Therefore AG, GH, etc. and CK, KL, etc. are series of similar ratios. Whereby AB is to CD [e] as AG to CK. In short by a similar argument I can show that CD is to EF as CK to EN; but AG, CK, EN are given as three continued proportionals; hence AB, CD, and EF are in continued proportion. Q.e.d. *a 19 Quinti; b 1 Huius; c 79 huius; d 6 huius; e 84 huius.*

## *L2.§2.* *PROPOSITIO CXIX.*




**Prop.119. Fig. 1.**

Si data quaelibet proportio maioris inaequalitatis A ad B, continuetur perpetuo; devenietur tandem ad magnitudinem data minorem.

Dico hoc semper fieri possit, propositas magnitudenes AB, CD, EF esse in continua analogia.

***Demonstratio.***

Ponatur enim magnitudo quaevis F; & fiat ad altam G ut B ad A; continueturque ratio F ad G, donec per septuagesimam septimam huius habeatur K magnitudo, maior A magnitudine; & per totidem terminos continuetur ratio A ad B. Dico E minorem esse quam F. est enim ut A ad B, sic G ad F; & ut B ad C, sic H ad G, &c. ergo ex aequo in proportione perturbata, ut A ad E, sic K ad F; sed A minor est quam K, ergo & E[f] quam F. Quod erat demonstrandum. *f* Euclid V.14.

*Scholium.*

*Huc etiam pertineret propositio septuagesima septima, nisi illam, quod ad terminum progressionis inveniendum esset necessaria, coacti essemus citeriori loco collocare.*

**[122]**

## *PROPOSITION 119.*

If some proportion is given of the greater inequality of A to B, and it is continued repeatedly, then finally it will come to a given smaller magnitude.

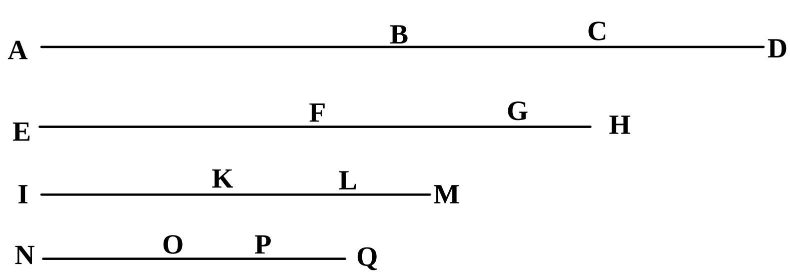
***Demonstration.***

For some magnitude F is put in place to a greater G and made in the same ratio as B to A; and the ratio F to G is continued, until according to Prop. 77 of thus book, it can have a magnitude K greater than the magnitude A, and the ratio A to B is continued throughout the terms. I say that E is less than F: for as A is to B, thus G is to F; and as B is to C, thus H is to G, etc., and hence from the equality of the re-arranged proportions, as A is to E, thus K is to F; but A is less than K, and hence[f] E is less than F. Q. e. d. *f 14 Quinti.*

*Scholium.*

*Up to this point too, the theorem is related to the 77th proposition, except that it was necessary there to continue to the terminus of the progression, and we are able to gather terms to a nearer place here.*

## *L2.§2.* *PROPOSITIO CXX.*




**Prop.120. Fig. 1.**

Sit magnitudo aliqua AD, secta in tres partes AB, BC, CD: ablata media BC vel alterutra extremarum, residuis AB, CD, fiat aequalis EH, quae dividatur in tres partes EF, FG, GH in eadem ratione qua secta est AD: se hoc continuetur;

Dico relinqui tandem magnitudinem data minorem.

***Demonstratio.***

Quoniam AB est ad BC, ut EF ad FG, & BC ad CD, ut FG ad GH; igitur permutando AB ad EF, ut BC ad FG, & CD ad GH: ergo [a] ut AB ad EF, sic AD ad EH: & permutando AB ad AD, ut EF ad EH, similiter demonstrabimus DC esse ad DA, ut HG ad HE: ergo AB [b] cum CD, ad AD, ut EF cum GH ad EH: & invertendo AD ad AB cum CD, id est ex hypothesi EH, ut EH ad EF cum GH, id est ex hypothesi IM. sunt igitur AD, EH, IM in continua ratione minoris inaequalitatis. non aliter demonstrabimus NQ, caeteraque residua in infinitum cum prioribus eandem proportionem minoris inaequalitatis continuare. Quare [c] relinquetur tandem magnitudo data minor. Quod erat demonstrandum. *a 12 Quinti; b 24 Quinti; c 119 Huius.*

## *PROPOSITION 120.*

Some magnitude AD is cut into three parts AB, BC, and CD; and the middle part BC or one of the end parts is taken away, and EH is made equal to the sum of the remaining parts AB and CD. This remaining part is divided into three parts EF, FG, and GH in the same ratio as which AD is cut: and this process is continued indefinitely among the parts.

I say that the magnitude can be diminished to any given [smaller] amount.

***Demonstration.***

Since AB is to BC as EF is to FG, and BC is to CD as FG is to GH; therefore on permuting, AB is to EF, as BC is to FG, as CD to GH: hence[a] , as AB is to EF, thus AD is to EH: and on permuting, AB is to AD, as EF is to EH. Similarly, we can show that DC is to DA as HG is to HE: hence[b] (the sum of AB and CD) to AD is as (the sum of EF and GH) to EH: and on inverting AD is to (the sum of AB and CD, or EH by hypothesis), as EH is to (the sum of EF and GH, or IM by hypothesis ). Therefore AD, EH, and IM are in a continued ratio of the smaller inequality [i. e. in the ratio < 1]. In the same way can we show that NQ, and the remaining remainders are to continue indefinitely with the same smaller proportion as before. Whereby[c] at last a given small magnitude is left. Q. e. d. *a 12 Quinti; b 24 Quinti; c 119 Huius.*

[AB/BC = EF/FG, and BC/CD = FG/GH; therefore on permuting, AB/EF = BC/FG = CD/GH = $k$: hence AB = $k$.EF; BC = $k$.FG; CD = $k$.GH; and hence AD = $k$.EH giving AB/EF = AD/EH = $k$: and on permuting, $\underline{\text{AB/AD}}$ = $\underline{\text{EF/EH}}$.
Similarly, $\underline{\text{CD/AD}}$ = $\underline{\text{GH/EH}}$: hence $\underline{\text{(AB + CD)/AD}}$ = $\underline{\text{(EF + GH)/EH}}$ on equating the underlined terms: and on inverting AD/(AB + CD, or EH) = EH/(EF + GH, or IM ). Hence, AD, EH, and IM are in the continued proportion of the inequality less than one .
In modern terms, let AB = $a$; BC = $ar$; CD = $ar^2$ ; then AD = $a.(1 + r + r^2)$ ; EH = $a.(1 + r^2)$;
$k$ = AD/EH = $(1 + r + r^2)/(1 + r^2)$; $\underline{\text{GH}}$ = CD/$k$ = $ar^2.(1 + r^2))/(1 + r + r^2)$ ; $\underline{\text{EF}}$ = $a.(1 + r^2)/(1 + r + r^2)$ ;
IM = $\underline{\text{EF + GH}}$ = $a.(1 + r^2)^2/(1 + r + r^2)$ = EH$^2$/AD as required.
The lesser ratio is EH/AD = $a.(1 + r^2)/(1 + r + r^2)$ .]

## *L2.§2.* *PROPOSITIO CXXI.*


**Prop.121. Fig. 1.**

Data sit magnitudo utcumque secta in C, ac inter AB, CB media proportionalis ponatur DB; rursum inter DB, CB media sit EB, & hoc continuetur.

Dico ex AC relinqui tandem magnitudinem data minorem.

***Demonstratio.***

Per primium huius ; AD est ad DC ut AB ad DB; atqui AB maior est quam DB, ergo etiam AD est maior quam dimidia AC: Similiter quoniam DB, EB, CB sunt continuae, erit DE ad EC, ut DB ad EB: quare DE maior est quam EC, ideoque & maior quam dimidis DC. Eodem modo probabitur EF esse plus dimidio ab

EC, atque ita in infinitum semper plus dimidio ab AC, eiusque; residuis auferetur. Quare [d] relinquetur tandem magnitudo data minor. Quod erat demonstrandum. *d 1 Decimi.*

**[123]**

## *PROPOSITION 121.*

The given magnitude AB is cut at C, and the mean of the proportions DB is placed between AB and CB; again the mean proportional EB is placed between DB and CB : and this process is continued among the parts.

I say that the magnitude to be left at last from AC is less than some given amount.

***Demonstration.***

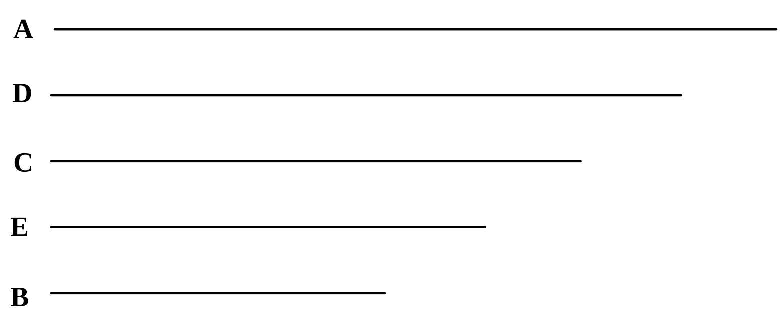
According to the first part of this above; AD is to DC as AB is to DB; but AB is greater than DB, hence also AD is greater than the half of AC. Similarly, since DB, EB and CB are continued proportions, DE is to EC as DB is to EB: whereby DE is greater than EC, and likewise greater than half DC. In the same way it is agreed that EF is more than half from EC, and thus indefinitely always more than half from AC, and the remainder is always taken from this amount. Whereby [d] at last the magnitude left is less than a given magnitude. Q. e. d. *d 1 Decimi.*

[ As AB, DB, and CB are continued proportions, hence AB/DB = DB/CB; from which on subtraction, AD/DB = DC/CB, giving DB/CB = AD/DC = AB/DB as required; as AB > DB then AD > DC or AD is greater than half AC. Again, DB/EB = EB/CB, as DB, EB, and CB are continued proportions also; from which as just demonstrated, DE/EB = EC/CB or EB/CB = DE/EC = DB/EB; since DB > EB then DE > EC or DE is greater than half DC; and so similarly, EF is greater than half EC.]

## *L2.§2.* *PROPOSITIO CXXII.*

Inter duas magnitudines inaequales A, B, inveniatur media proportionalis C, & inter has tres A, C, B, inveniantur duae mediae D & E: rursum inter illas quinque, quatuor statuantur mediae, & hoc semper fiat.

Dico hac praxi tandem exhibendas lineas quae simul sumptae maiorem sint data quavis magnitudine.




**Prop.122. Fig. 1.**

***Demonstratio.***

Quoniam C media est inter A & B , habebit A ad B maiorem rationem, quam ad C; ergo C maior est quam B; ergo tres magnitudines A, C, B maiores erunt quam tripla ipsius B. Similiter ostendam D & E maiores esse singulas, quam B: ac proinde A, D, C, E, B simul sumptas maiores esse, quam quintupla ipsius B; atque ita demonstrabimus si plures semper mediae reperiantur, summam magnitudinem, excessuram B magnitudinem determinatam, secundum quemvis numerum assignabilem. ex quo liquet magnitudines illas simul sumptas, futuras quavis data quantitate maiores.

## *PROPOSITION 122.*

Between two unequal magnitudes A and B, the mean C of the proportions is found; and between these three A, C, B are found the two means D and E: again, betweem these five proportionals, four means are put in place, and this construction can always be made.
I say that this practise finally produces lines which have a sum greater than any given magnitude.

***Demonstration.***

Since C is the mean of A and B, the ratio A to B is greater than the ratio A to C; hence C is greater than B. and hence the sum of the three magnitudes A, C, and B is greater than three times B. Similarly, I can show that the further means D and E are each greater than B, and hence the sum of A, D, C, E, B is greater than five times B; and thus we can show that if always more means are found, then their sum surpasses the magnitude determined by B, according to whatever number is assigned for that quantity: from which it is proven that the sum of these magnitudes is soon greater than any given quantity.

**[124]**

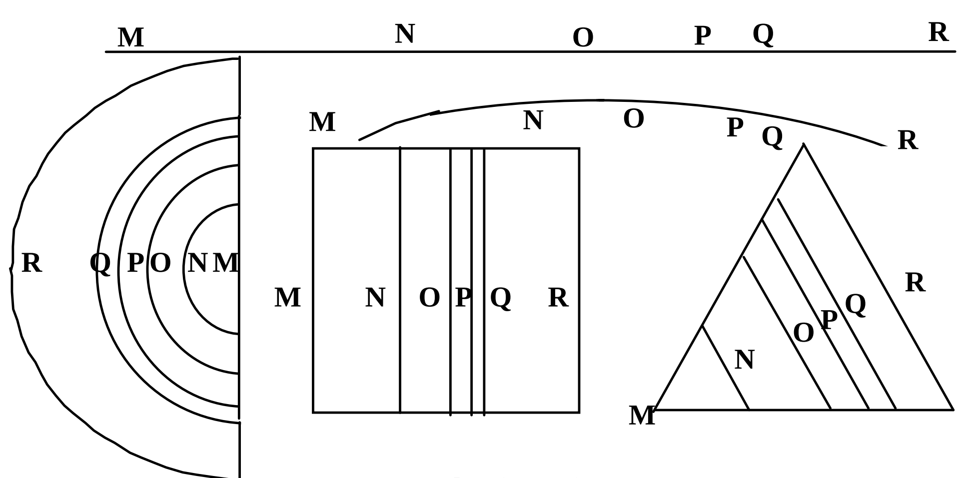
# PROGRESSIONUM GEOMETRICARUM

# PARS TERTIA

*Progressiones terminatas planis applicat, praesertim similibus.*

*The Third Part is applied to terminations [i. e. limits] of progressions for plane figures, especially those which are similar.*

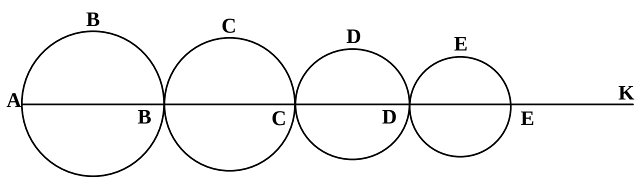
*Qua de progressionibus Geometricis secunda parte hactenus demonstravimus, absque ullo discrimine lineis, superficiebus, corporibusque conveniunt: hac enim de causa nomen magnitudinis, non linea perpetuo assumpsimus, ut propositionum universalitatis indicaretur. quia tamen superficierum corporumque similium similiterque positorum progressiones, si extra invicem in directum constituantur, singulares habent proprietates non paucas, visum est operae pretium futurum illas hac tertia ac quarta partes explicare.*




**Introduction: Figures 1.**

*As far as the geometric progressions are concerned that we have already demonstrated in the second part, we may note that the methods introduced there can be applied directly to any lines, planes or areas, and solid bodies or volumes. The following development is more concerned then, with particular instances where such lines are parts of plane or solid figures, rather than with properties of the progressions of lines themselves. The propositions developed in the first part are assumed to have the same universal application, as the proposition presented about general use may indicate. Hence, for progressions of series derived from similarly placed similar surfaces and bodies, if these are set up in turn for shapes involving several lines, then the individual terms have more properties than those already considered for ratios along a single line: this is the interest in performing the present work, to be set out in this and the following part.*

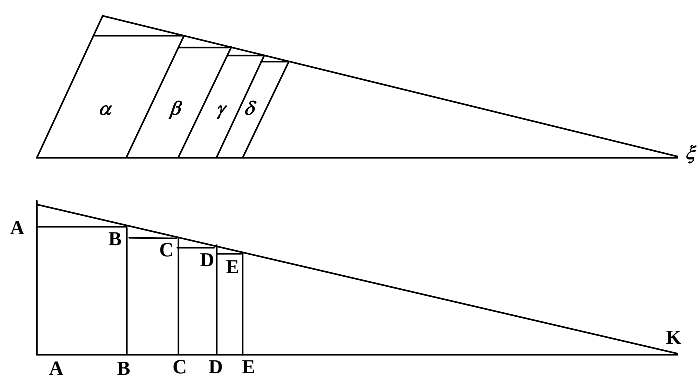
*Nota, duplici modo planorum ac corporum progressiones infstitui posse. primo quidem ut termini progressionis simul sumpti, unam magnitudinem continuam, ac homogeneam componant: ut in figuris appositis exhibitur. secundus enim terminus NO cum primo MN, unam magnitudinem MO componit; & tertius OP, cum secundo ac primo, constituit unam magnitudinem MP ; omnes denique termini simul sumpti unam componunt magnitudinem MR, continuam ac homogeneam.*




**Introduction Fig. 2.**

*Secundus modus est quando termini progressionis similes inter se sunt, similiterque positi, neque iuxta positionem qua dantur, constituunt simul sumpti unam magnitudinem: huiusmodi progressiones (quas quidem in sequentibus prosequemur) exhibent figurae oppositae A, B, C, D, E, K : in quibus termini omnes similes sunt similiterque positi, ac in directum constituti; ita ut neque secundus terminus BC, cum primo AB, neque tertius CD, cum primo & secundo, neque caeteri subsequentes cum praecedentibus componant unam magnitudinem.*

**[125]**




**Introduction Fig's. 3.**

*& figures sic positis terminum quidem ad quem bases illarum figurarum excurrent scilicet K, per praecedentia reperiemus; in heterogeneae vero illius seriei (ita enim lubet appellare) quam figurae similes similiterque; & extra se positae componunt, cognitionem non veniemus, nisi figuras has similes similiterque extra se positas, ad illas revocando, quarum prima cum secunda, & secunda cum tertia, & sic deinceps unam aliquam magnitudinem constituit. Ut si exempli gratia sint figurae AB, BC, CD, & similes similiterque & extra se invicem positae; harum termini in infinitum continuatarum sumpti*

*magnitudinem non unam aliquam, sed aggregatum quoddam figurarum constituent: si igitur magnitudo huic seriei figurarum aequalis quaeratur, oportebit figuras similes AB, BC, CD, DE, &c. ad figuras MN, NO, OP, &c. revocare. quae figurae MN, NO, OP magnitudinem unam constituunt; & si proportio MN ad NO, & NO ad OP continuata terminetur in R : erit MR toti seriei progressionis figurarum AB, BC, &c. aequalis; quae omnia sequenti propositione demonstrata fient clariora.*

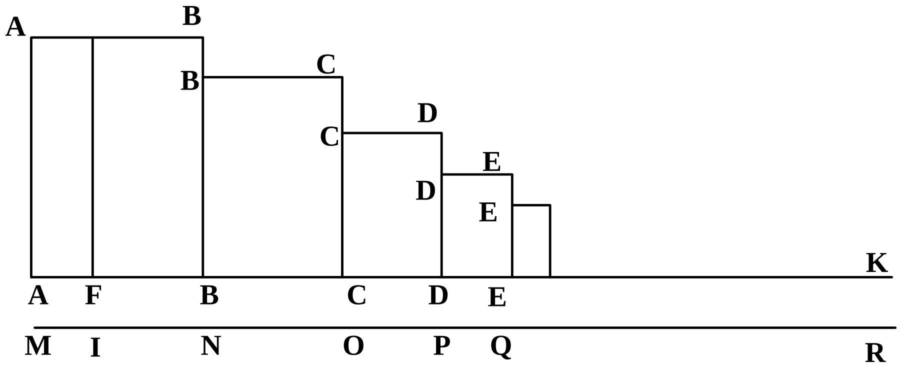
*Note: There are two ways in which progressions can be established for planes and solid bodies. According to the first way, in order that the terms of a progression can be summed together, successive terms of the same kind of a single magnitude are placed together, as is show in the above figures* (Introduction: Fig. 1): *for the second term NO taken with the first term MN gives a single magnitude MO; again, the third term OP summed with the second and the first, gives a single magnitude MP; all the terms summed in the same manner give a single magnitude MR of the same kind in the succession of terms.*

*The second manner in which a progression can be formed is one in which the figures are similar to each other, and similarly placed in given positions just touching each other, but the sum of the terms does not give a single magnitude of the same kind. Progressions of this kind (which indeed we shall describe later in detail) are set out for the figures A, B, C, D, E, K opposite* [Introduction:Fig.2]*: in which all the similar terms or corresponding figures are similarly put in place, and set together on a given line. However, in this case, neither the sum of the second BC with the first AB, nor the third with the second, nor those remaining in turn with the ones that have gone before, can be* [simply] *added together give a single similar figure of some magnitude.*

*Indeed, from the preceding arguments for series of the first kind, we find that the bases of such figures, if the terms are thus put in place in a geometric progression, do extend to meet in some point K. The different forms of these series (as thus they may be called) consist of similar figures similarly placed, and the series are formed by putting these terms in place. It is recognised that if more of these similar figures likewise are placed beyond the diagram, then we recall that the sum of the first and second, and following with the third, and thus henceforth do not in this case constitute some similar single figure of some magnitude. As for example, if the figures AB, BC, CD, and like ones are placed similarly in turn beyond, as in Fig.2, then the sum of these continued indefinitely do not constitute a single figure, but rather an infinite aggregate of such particular figures. Therefore if a magnitude equal to the sum of this series of figures is sought, then it is necessary to recall figures MN, NO, OP, &c. similar to the figures AB, BC, CD, DE, &c., where the sum of the figures MN, NO, OP, &c. has a given magnitude. If the continued proportions MN to NO,NO to OP, &c. terminate in R, then MR is equal to the sum of the progression of the figures AB, BC, &c., [from which the sum of the other series of figures follows]. It is this point which all the following demonstrations of the proposition try to make clear.*

## *PROPOSITIO CXXIII.*

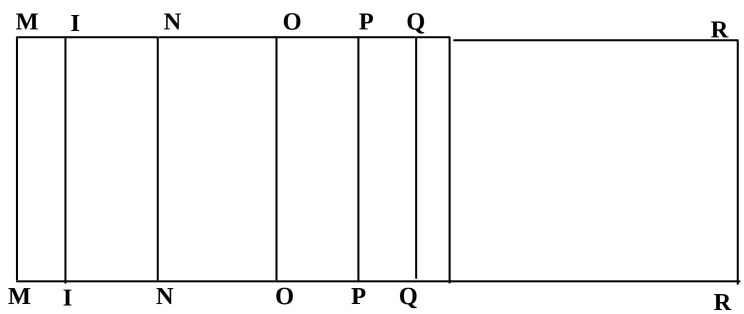
Data igitur sit secundi generis progressio AB, BC, CD, DE, &c. conflata ex similibus terminis similiterque; & in directum positis, sive planis, sive solidis, & quidem planis vel solidis cuiuscumque generis.




**Prop.123. Fig. 1.**

Dico omnes progressionum proprietates superiori parte demonstratas huiusmodi etiam progressionibus convenire, ac proinde propositiones, in quibus illae proprietates demonstrantur, prorsus universales esse: in hac igitur demonstratione, progressio posterior sive heterogenae, reducitur ad priorem sive homogeneam

**[126]**
***Demonstratio.***




**Prop.123. Fig. 2.**

Si enim primae magnitudini AB, aequalis quaecumque alia MN; & ut AB ad BC, ita sit MN ad aliam NO, quae cum MN, unam magnitudinem continuam & homogeneam componat: continueturque ratio MN ad NO, in infinitum per plures semper terminos OP, PQ, &c. qui perpetuo cum praecedibus terminis unam magnitudinem componant. Quoniam igitur aequales sunt AB, MN, erit AB ad NO, ut MN ad NO; sed MN est ad NO, ut AB ad BC; ergo AB est ad NO, ut AB ad BC: aequales ergo sunt BC, NO; ergo BC est ad OP, ut NO ad OP; sed NO est ad OP, ut MN ad NO, id est ut AB ad BC, id est ut BC ad CD; ergo BC est ad NO, ut BC est ad CD: aequales ergo sunt CD, OP. similiter ostendam singulos utriusque progressionis terminos inter se aequari in infiniti. Quare & series tota AB, &c. aequalis est toti seriei MN, &c. utpote constans aequalibus sive iisdem terminis: atqui quaecumque toto secundo libro demonstrata sunt de progressionum proprietatibus, conveniunt progressioni MN, NO, &c.: ergo etiam conveniunt progressioni AB, BC, &c. Quod erat demonstratum. Verum ut res clarius pateat, idipsum per aliquot consectaria seu corollaria explicabimus.

## *L2.§3.* *PROPOSITION 123.*

A progression AB, BC, CD, DE, &c. of the second kind is therefore given, put together from similar terms, and similarly placed together on a line either in a plane or in space, and the progression consists indeed of some kind of plane or solid shapes.

I say that all the properties of the progressions shown above are also appropriate for progressions of this kind, and that the propositions by which these properties are demonstrated are hence of general use. Therefore, in this demonstration, a heterogenous progression of the second kind can be reduced to a homogenous progression of the first kind.

### *Demonstration.*

For the first magnitudes, AB is set equal to some other MN; and subsequently as AB is to BC, so MN is to the other NO, which can be added to MN to give the single continuing and homogenous magnitude MO. Ratios of the form MN to NO can be continued indefinitely by adding more terms such as OP and PQ, etc., and these can in turn be added to the preceding terms to give single magnitudes.
Therefore, since AB and MN are equal, AB is to NO, as MN is to NO; but MN is to NO, as AB to BC; hence AB is to NO, as AB to BC: and hence BC and NO are equal. In like fashion, BC is to OP, as NO to OP; but NO is to OP as MN to NO, or as AB to BC, or as BC to CD; hence BC is to NO as BC is to CD, and hence CD and OP are equal.
Similarly, I can show that the individual successive terms of both progressions are equal to each other indefinitely. Whereby the sum of the series AB, etc. is equal to the sum of the series MN, etc., for they are in agreement by having all their consecutive terms equal to each other. But whatever sum is agreed upon, according to the properties of progressions set out in the second book, for the terms of the progression MN, NO, the sum of the terms of the progression AB, BC, &c., is also in agreement. Q.e.d.
However, in order that the idea becomes more apparent, we next set out some logical consequences or corollaries.

### *Corollarium primum.*

Ex his igitur (iisdem positis) infero primo : progressionem universam magnitudinum AB, BC, &c. producere eandem determinatam magnitudinem sive quantitatem quam series MN, NO.

### *Demonstratio.*

Series enim AB, BC, &c. aequalis est seriei MN, NO, &c. ergo eandem producit quantitatem: atqui series MN constituit [a] finitam & determinatam quantitatem (exempli causa MR) ergo & series AB producit quantitem finitam MR.

### *First Corollary.*

In the first place, I can infer from these points, lines, etc. (in the same positions) that a general progression of the magnitudes AB, BC, etc. produces a sum of the same size as the series MN, NO, etc.

### *Demonstration.*

For the series AB, BC, etc. is equal to the series MN, NO, etc., term by term; hence the same quantity or sum is produced by each; but the series MN sets up a finite bounded amount [a] (such as MR) , and hence the series AB also sets up the same finite quantity MR.

### *Corollarium secundum.*

Iisdem positis infero secundo, series AK (id est omnes antecedentes) est ad seriem BK (id est omnes consequentes) ut AB ad BC unam consequentem.

Et series BK est ad seriem CK ut AB ad BC.
Et tres series AK, BK, CK, sunt in continua analogia, & similiter alia inferemus quae prop. octuagesima secunda habentur.

### *Demonstratio.*

Sit MR aequale toti seriei MN, NO, &c. erit ergo & tota series, AB, BC, &c. etiam, ut ostensum antia, aequalis ipsi MR: cum igitur & AB aequalis sit MN, erit quoque series BC, CD, &c. aequalis ipsi NR. Igitur series AK est ad seriem BK ut MR ad NR: Atqui MR [b] est ad NR, ut MN ad NO. ergo series AK est ad seriem

**[127]**

BK ut MN ad NR, id est ut AB ad BC. quod erat primum. Similiter reliquas quoque demonstrabimus.
*a 79 huius; b 82 huius.*

### *Second Corollary.*

With everything as above I can infer in the second place that the series AK (or the sum of the terms in the initial part of the ratio) is to the series BK (or the sum of the terms in the following part) as AB is to BC or the first term to the next term. Again, the sum of the series BK is to the sum of the series CK as AB is to BC. The three series AK, BK and CK are in continued proportion, and similarly we can infer the rest that we have found from Prop. 82.

### *Demonstration.*

Let MR be equal to the whole series MN, NO, &c. Hence the whole series AB, BC, &c. also, as I have already shown, is equal to MR: and since AB is equal to MN, the series BC, CD, &c. too is equal to the series NR. Therefore the series AK is to the series BK as MR is to NR: But MR [b] is to NR as MN is to NO. Hence the series AK is to the series BK as MN is to NR, or as AB to BC, which shows the first part of the corollory. We can show the rest in a similar manner.
*a 79 huius; b 82 huius.*

### *Corollarium tertium.*

Iisdem positis infero tertio; AF differentia primi & secundi termini, AB primus terminus, tota series AK, sunt in continua analogia.
Et AF differentia, est ad AB primum terminum, ut BC secundus terminus ad totam seriem dempto primo termino nempe ad seriem BK : & AF differentia , BC secundus terminus, tota series demptis duobus primis terminis; (series nempe CK) sunt in continua analogia.

### *Demonstratio.*

Sit MN differentia primi & secundi termini, in progressione MR: quoniam igitur AB, MN, BC, NO aequantur, excessus quoque AF, MI, aequales erunt : & quia iam AF ipsi MI, & AB ipsi MN; aequalis est, ipsis AF, AB eadem magnitudo erit tertia proportionalis, quae ipsis MI, MN, ut patet ex elementis: atqui ipsis [a] MI, MN, tertia proportionalis MR: est tota series rationis MN ad NO; ergo ipsis etiam AF, AB eadem tota series MR, tertia proportionalis erit: atqui series AB, BC, per corollorium primum eandem producit magnitudinem MR, quam series MN, NO, ergo etiam productum seriei AB, BC, &c. sive tota series AK, erit tertia proportionis ipsis AF, AB: sunt itaque AF differentia, AB primus terminus, tota series AK in continua analogia. Quod erat primum. Similiter reliquas corollarii partes demonstrabimus.
*a 83 huius.*

### *Third Corollary.*

For the same situation, I can infer in the third place that the difference AF between the first and second terms, the first term AB, and the whole series AK, are in continued proportion.
And the difference AF is to the first term AB, as the second term BC is to the whole series with the first term removed, surely the series BK. The difference AF, the second term BC, and the whole series with the first two terms removed (that is, the series CK) are in continued proportion.

***Demonstration.***

Let MI be the difference of the first and second terms in the progression MR: hence, since AB is equal to MN, and BC is equal to NO, then the differences AF and MI are equal too : and now, since AF is equal to MI, and AB is equal to MN; it follows that AF, AB has the same magnitude for the third of the proportions as MI, MN, as is apparent from first principles: but MR is the third of the proportionals [a] MI and MN: and MR is the sum of the series of ratios MN to NO; hence also the sum of the series MR is the same third proportion of AF, AB : but the series AB, BC, by the first corollary gives the same magnitude MR, as the series MN, NO; hence the sum of the series AB, BC, &c. that is AK, is also the third of the proportions of AF, AB: hence the difference AF, the first term AB, and the sum of the series AK are in continued proportion. Which demonstrates the first part of the corollary. Similarly, the remaining parts of the corollary can be demonstrated. *a 83 huius.*

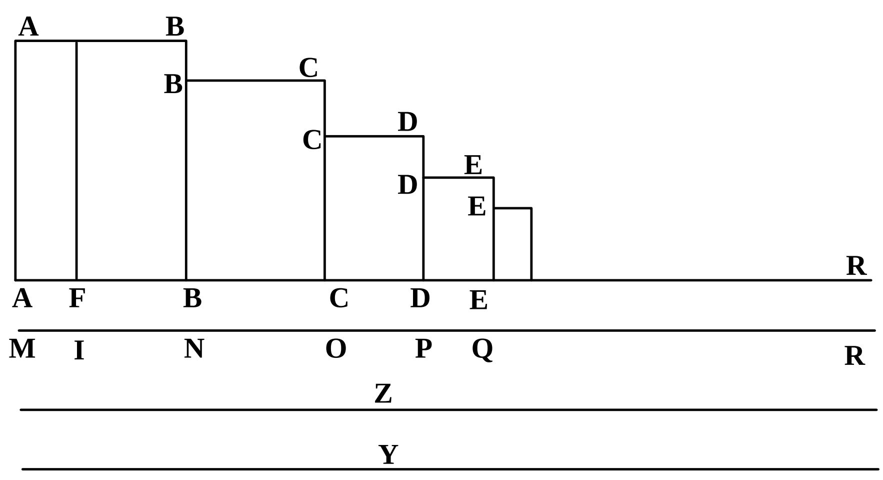
***Corollarium quartum.***

Iisdem positis infero quatro, hic etiam valere universalem illam ac triplicem constructionem qua propositione datae seriei magnitudinem aequalem invenimus.
Fiat enim ut AF differentia primoram terminoram ad AB, sic AB ad aliam magnitudinem Z. Dico, Z aequalem esse toti similium magnitudinum seriei AK.

***Demonstratio.***

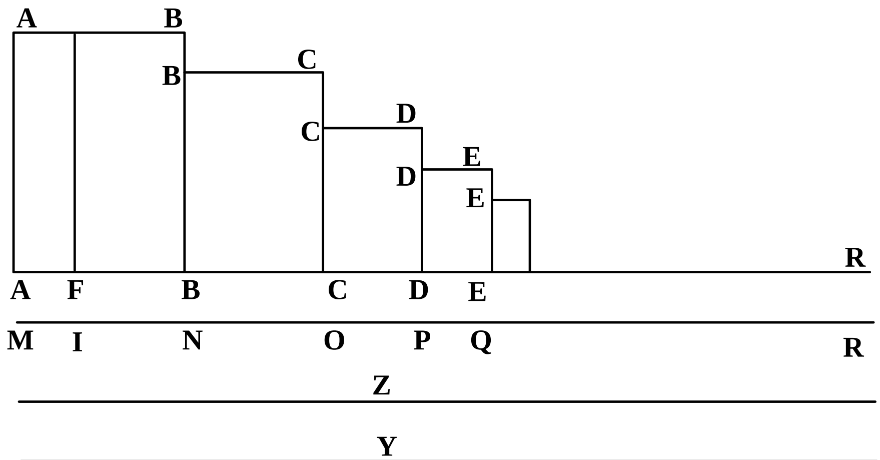
Si non est Z aequalis toti seriei, ergo alia magnitudo maior vel minor quam Z ipsi aequalis erit; (aliqua enim magnitudo per Corollarium primum toti seriei AK aequalis est) sit illa Y. ergo per corollarium praecedens AF, AB, Y, sunt continuae. atqui etiam ex constructione AF, AB, Z sunt continuae, ergo AB est ad Z ut AF est ad AB; & AB est ad Y, ut AF est ad AB. eandem igitur AB ad Z & Y rationem habet: aequales igitur sunt Z & Y contra hypothesim: ponebatur enim Y maior aut minor quam Z. non erit ergo alia minor maioruc quam Z, aequalis seriei AK. ergo Z aequalis erit. similiter duas alias propositionis octavagesimae huius constructioni demonstrabimus.

**[128]**




**Prop.123. Fig. 3.**

### *Fourth Corollary.*




**Prop.123. Fig. 3.**

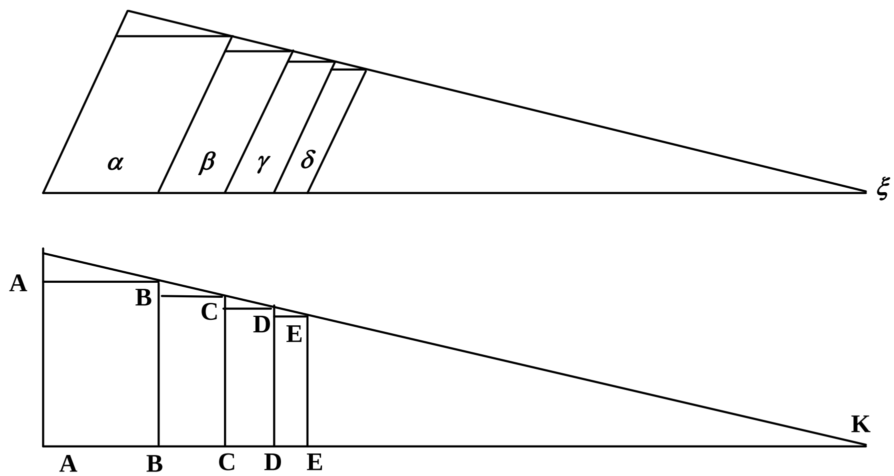
For the same situation, in the fourth place, I can infer that the general theorem prevails here also, and we find three constructions for which the magnitude of the series for the given proposition is equal.
For it happens that as AF the difference of the first terms AB and BC, is to AB, thus AB is to a different magnitude Z. I say that Z is equal to the sum AK of the series of similar magnitudes.

### *Demonstration.*

If Z is not equal to the sum of the whole series, then the sum is equal to a different magnitude which is larger or smaller ; (for by the first Corollary the sum of the series AK is equal to some magnitude) let that quantity be Y. Hence by the preceding corollary AF, AB, and Y are in continued proportion. But also from the construction AF, AB, and Z are in continued proportion, hence AB is to Z as AF is to AB; and AB is to Y as AF is to AB. Therefore AB to Y and Z have the same ratio: therefore Z and Y are equal contrary to the hypothesis whereby Y is made greater or less than Z. Hence there is not a different quantity less or greater than Z that is equal to the series AK, and thus the series is equal to Z. Similarly, we can demonstrate the two other arguments of proposition eighty for this construction.

### *Corollarium quintum.*

Iisdem positis infero quinto: si fuerit progressio AB, &c. similitum magnitudinum itemque alia progressio similium inter se magnitudinem α, β, γ, δ, &c. sive similes illae sint terminis alterius sive dissimiles; sit autem progressio utraque eiusdem proportionis: infero inquam totas series AK, α, ξ eam habere rationem inter se, quam primi termini A, B, &c.




**Prop. 123: Fig. 4.**

### *Demonstratio.*

Per corollarium secundum series AK, est ad seriem BK, ut AB ad BC : sed ex hypothesi AB est ad BC, ut α ad β; ergo series AK est ad seriem BK, ut α ad β, hoc est per idem corollarium ut series αξ ad seriem βξ. Igitur per constructionem rationis series AK est ad AB, ut series αξ, ad α : & permutando series AK est ad seriem αξ ut AB ad α. Quod erat demonstrandum.

### *Corollary five.*

With the same points and lines in position I can infer in the fifth place that if there is a progression AB, &c. of similar magnitudes and likewise another progression of magnitudes similar between themselves α, β, γ, δ, &c. , and these terms are either similar or dissimilar with the terms of the other progression; but each progression is of the same proportions: then I can indeed infer that the sum of the series AK, α, and ξ have the same ratio between themselves as the first series A, B, etc.

### *Demonstration.*

By the second corollary, the series AK is to the series BK as AB is to BC : but by hypothesis AB is to BC, as α is to β; hence the series AK is to the series BK, as α is to β, or by the same corollary as the series αξ is to the series βξ. Therefore from the construction of the ratios, the series AK is to AB as the series αξ is to α : and on interchanging, the series AK is to the series αξ as AB is to α. Q. e. d.

***Corollarium sextum.***

Et quamquam hactenus solum assumpserimus progressionem planorem, corporumve similium similiterque positorum, non est tamen quod existimet lector, quae hactenus demonstrata sunt non subsistere, si planorum aut corporum non similium statuatur progressio, eadem quippe utrobique, ut cuilibet rem expedenti manifestum est, & veritas est & veritatis demonstratio, idcirco autem figuras similes assumere placuit, quod & usus earum frequentior, & magis sint ad demonstrandum accommodatae.

Ex his hunc in modum demonstratis manifestum est progressionum proprietates, secunda parte explicatas progressionibus magnitudinum in directum positarum, quas deinceps prosequimur, non minus quam aliis convenire : ac proinde propositiones superioris partis in quibus illae tractantur, prorsus universales esse. Quare has deinceps uti revera tales in sequentium theorematum demonstrationibus citabimus

***Corollary six.***

And though up to this point, we have only assumed progressions of similar plane or solid figures placed in some regular arrangement, the reader should not think that this is the end of the matter. For until now we have not stopped to demonstrate the case of a progression of plane or solid figures set in place which is not directly similar to a progression along a line, though obviously there is agreement between the terms in both places, in order that the desired outcome can be shown and there is a demonstration of the true. Hence more general series are to be considered which will be used more often, and these are arranged here for further explanation.

From these in this manner the characteristics of progressions are to be made clear, as the following section explains the magnitudes of progressions placed in given directions, which we can then examine in detail to establish that they are in agreement no less than for the others: and hence the propositions set out in the above sections are shown in short to be quite general. Whereby we set in motion these progressions that are actually to be used as such in the demonstrations of the theorems to follow.

## *PROPOSITIO CXXIV.*

Data sint proportionales continuae AB, BC, CD, &c. & super iis constructa plana similia.

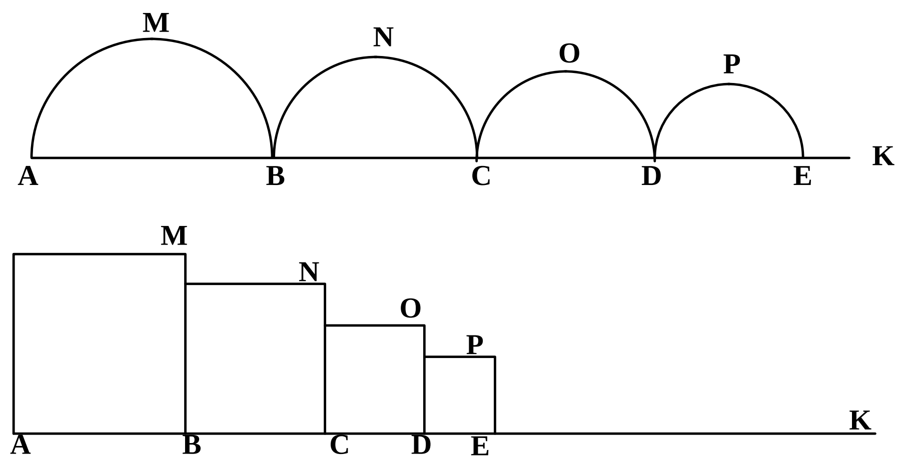
Dico plana esse in continua analogia : & si plana dentur continue proportionalia:

Dico etiam bases fore continue proportionales.

**[129]**

***Demonstratio.***

Planum AM est ad planum BN, in duplicata ratione AB ad BC; & planum BN est ad planum CO, in duplicata ratione BC ad CD, id est ex datis AB ad BC : similiter planum CO est ad planum DO, in duplicata ratione CD ad DE, id est rursus AB ad BC: similiter ostendam omnia reliqua inter se esse in duplicata ratione AB ad BC : manifestum est igitur omnia esse in continua analogia : Quod erat primum. secunda pars simili plane discursu ostendatur; patet igitur veritas propositionis.




**Prop.124. Fig. 1.**

## *L2.§3.* *PROPOSITION 124.*

The continued proportionals AB, BC,CD, etc. are given, and above these similar plane figures are constructed.

I say that the plane figures are in continued proportion; and if the planes figures are given on continued proportion, then I say that the bases are also in continued proportion.

***Demonstration.***

The plane figure AM is to the plane figure BN in the square ratio AB to BC; and the plane figure BN is to the plane figure CO in the square ratio BC to CD, that is from the given ratio AB to BC. Similarly the plane figure CO is to the plane figure DO, in the square ratio CD to DE, or again as AB to BC. Similarly I can show that all the remaining terms are in the square ratio AB to BC to each other: which demonstrates the first part, that the plane figures are in proportion; the second part can be shown by a similar argument, and so the truth of the proposition is apparent.
Q.e.d.

## *PROPOSITIO CXXV.*

Eadem posita figura data sint duo plana similis, basibus homologis in directum positis, AM maius, BN, minus. Petitur inveniri terminus longitudinis, ad quem proportio dictorum planorum sine statu continuata excurret.

***Constructio & Demonstratio.***

Per octavagesima huius inveniatur progressionis basium AB, BC terminus; sitque; K.

Dico etiam K terminum esse longitudinis ad quem series planorum excurret : plana enim similia quae fient super terminis progressionis basium, per praecedentium erunt continuae proportionalia, ac proinde linearum planorumque; in infinitum progressio, pari passu procedent: quare utriusque terminus erit K. Quod erat demonstrandum.

***Corollarium.***

Idem igitur punctum, terminus est progressionis basium, & terminus longitudinis, quem habet series figurarum similium : sive, quod idem est, linea quae aequalis est seriei basium, est longitudo seriei figurarum similium, super basibus descriptarum.

## *L2.§3.* *PROPOSITION 125.*

With the same given figure position, two sets of similar plane figures are given, with bases of the same kind placed along a line, with AM larger than BN, etc. It is required to find the terminus of the length to which the proportions of the said plane figures extends on being continued indefinitely.

***Construction & Demonstration.***

By Prop. 80 of this book, the terminus of the progression of the bases is found to be K.
I say that the terminus K is also the length to which the series of plane figures extends: for the plane figures constructed on the bases of the progression are in continued proportion, by the preceding theorem, and hence both the lines and the plane figures are in an infinite progression, proceeding with like steps; whereby the terminus of both series is K. Q.e.d.

***Corollary.***

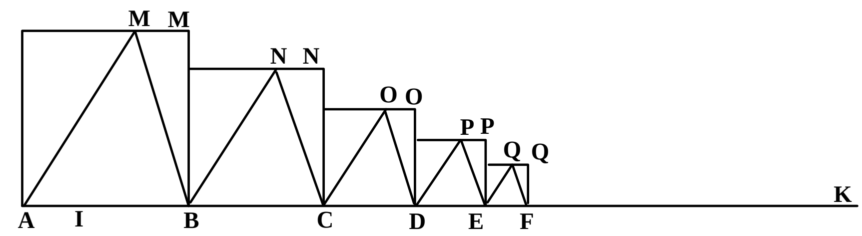
Therefore the terminus of the progression of basis is the same point as the terminus of the length of the series of similar figures; or, what amounts to the same thing, the line equal to the sum of the infinite series of bases is also the length of the line of the sum of the series of similar figures described on the bases.

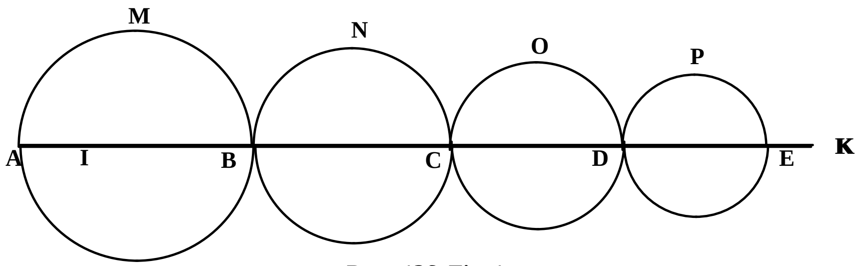
## *PROPOSITIO CXXVI.*

Data sit planorum similium continue proportionalium series, homologis basibus AB, BC, CD, &c. in directum positis, habens terminum longitudinis punctum K;

Dico, totam planorum seriem MK esse ad primum terminum AM, ut est tota series basium imparium AB, CD, EF, &c. ad primam AB.

**[130]**




**Prop.126. Fig. 1.**

***Demonstratio.***

Per octuagesimam secundam huius, tota planorum series MK est ad seriem NK, ut planum AM ad planum BN : atqui cum plana sint similia, duplicatam habent ratione basium AB, BC; ergo series MK ad seriem NK duplicatem habet rationem AB ad BC : quia autem per corollarium proposit. praecedentis, progressionis basium AB, BC, &c. terminus est K, erunt igitur AK, BK, [a] CK tres continuae proportionales. unde ratio AK ad CK, duplicata est rationis AK ad BK, id est rationis AB ad BC : ergo series MK est ad seriem NK, ut AK ad CK: quare per conversionem rationis series MK, est ad planum AM, ut AK ad CA : deinde quia series AB, BC, CD, DE, &c. id est linea AK, est ad seriem AB, CD, &c. ut CA ad BA, erit alternando AK ad CA, ut series AB, CD &c. ad AB : sed series planorum MK, est ad planum AM, ut AK ad CA, ergo series MK est ad planum AM, ut series AB, CD, &c. ad AB. Quod erat demonstrandum.

*a 82 huius; b 1 huius; c 103 huius.*

***Manifestum.***

Ex demonstrationis discursu patet totam planorum seriem esse ad primum planum, ut KA ad CA. Quod quia postea usui veniet, sigillatim notare placuit.

## *L2.§3.* *PROPOSITION 126.*

A series is given of similar continued proportions, with bases of the same kind placed on a line, having the terminus of the length [taken up by the series] at the point K.

I say that the sum of the series of plane figures MK is to the first term AM as the sum of the odd bases AB, CD, EF, &c. is to the first term AB.

***Demonstration.***

By Prop. 80 of this book, the sum of the plane figures MK is to the series NK, as the plane figure AM is to the plane figure BN : but as the plane figures are similar, they are in the ratio of the squares of the bases AB, BC; hence the ratio of the series MK to the series NK is as the square of AB to BC : however, by a corollary of the preceding proposition, the terminus of the progression of the bases AB, BC, &c. is K. Hence [a] AK, BK, and CK are three lines in continued proportion, from which it follows that the ratio AK to CK is the square of the ratio AK to BK, or AB to BC. Hence the series of similar plane figures MK is to the series of figures NK as AK is to CK; whereby on rearranging the ratio by subtraction, the series of plane figures MK is to the plane figure AM, as AK is to CA. Hence, since the sum of the series AB, BC, CD, DE, &c., or line AK, is to the series AB, CD, &c, as CA is to BA, then on alternating the ratio, AK is to CA as the series AB, CD, &c. is to AB. But the series of plane figures MK, is to the plane figure AM, as AK is to CA; hence the series of plane figures MK is to the plane figure AM as the series AB, CD, &c. is to AB. Q.e.d.

[Initially we have: (Sum of plane figures MK)/(Sum of plane figures NK) = area AM/area BN = $AB^2/BC^2$; however, AK/BK = BK/CK = .....; and on subtraction, AB/BK = BC/CK; and AK/BK = AB/BC; hence AK/CK = $AK^2/BK^2 = AB^2/BC^2$ ;
hence (Sum of plane figures MK)/(Sum of plane figures NK) = AK/CK, from which it follows that (area of figure AM)/(Sum of plane figures MK) = AC/AK; (changing from NK and CK to MK and AK).
Again: (sum of AB, BC, CD, ..... )/(sum even terms AB, CD, EF, .... ) = CA/BA, on application of Prop. 103,
then AK/CA = (sum of even terms AB, CD, EF, etc.)/BA = (Sum of plane figures MK) /(area of figure AM), as required.
In modern terms, $S(r)/S(r^2) = a/(1 - r) \times (1 - r^2)/a = 1 + r = a(1 + r)/a$ = CA/BA; whence $S(r)/a(1 + r) = aS(r^2)/a^2$ = S(MK)/area AM]

***Conclusion.***

From the discussion of the demonstration, it is apparent that the ratio of the sum of the series of plane figures to the first plane figure is in the same ratio as KA to CA**.** Since later use will come about for these propositions, it has pleased us to note them individually.

## *PROPOSITIO CXXVII.*

Iisdem positis sit AI differentia primae AB, & tertia CD.

Dico totam similium planorum seriem esse ad primum planum AM, ut AB prima basis, ad AI primae & tertiae differentiam.

***Demonstratio.***

Fiat lineis AI, AB tertia ST continue proportionalis. Igitur ST [d] aequalis est tota seriei basium imparium AB, CD, EF, &c. ergo per praecedentem tota series planorum MK est ad planum AM ut ST ad AB : Atqui ex constructione AB est ad AI, ut ST ad AB; ergo tota series est ad planum AM, ut AB ad AI. Quod erat demonstrandum. *d 79 huius.*

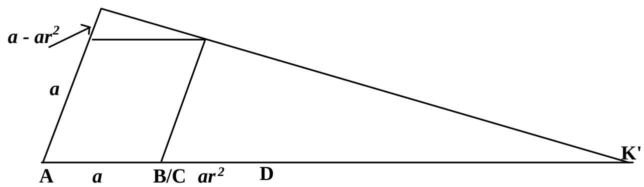
## *L2.§3.* *PROPOSITION 127.*

With the same points and figures in place, AI is the difference of the first and third terms AB and CD.

I say that the sum of the series of similar plane figures MK is to the first plane figure AM, as the first base AB is to AI, the difference of the first and third.

***Demonstration.***

ST is made the third proportion with the lines AI and AB in continued proportion. Therefore ST [d] is equal to the sum of the series of the odd [ *i. e.* in the sense the first, the third, etc. terms, though these actually correspond to even powers of the common ratio, which is not of course in use here] bases AB, CD, EF, &c. Hence by the preceding proposition, the sum of the series of plane figures MK is to the plane figure AM as ST is to AB : But from the construction AB is to AI, as ST is to AB; hence the sum of the series is to the plane figure AM as AB is to AI. Q.e.d.




**Prop.127. Fig. 2.**

[From the last proposition:
AK/CA = (sum of even terms AB, CD, EF, etc.)/BA
= (Sum of plane figures MK) /(area of figure AM) = ST/BA .
Now, AI, AB - CD, AB, and AK' = ST are in continued proportion, (from similar triangles, see Fig. 2, not in original text), hence AB /(AB - CD) = ST/AB; from which it follows that
AB/AI = ST/AB = (Sum of plane figures MK) /(area of figure AM), as required.]

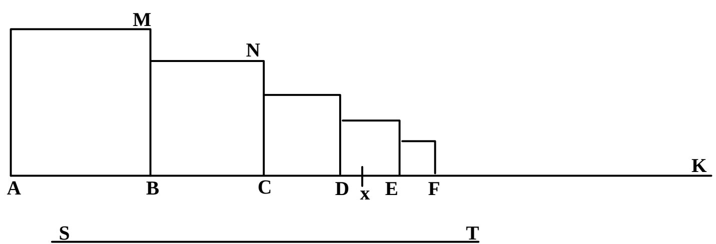
## *PROPOSITIO CXXVIII.*

Eadem manente figura, data sit quadratorum series basibus in directum positis, terminum habens longitudinis, punctum K : fiat autem per octuagesima huius ST, aequalis seriei basium imparium AB, CD, &c.

Dico rectangulum super ST in altitudine AB aequali toti seriei quadratorum MK.

**[131]**

***Demonstratio.***




**Prop.128. Fig. 1.**

Per propositionem centesimam vegesimam sextam hiuis, tota series quadratorum MK, est ad primum quadratum AM, ut ST ad AB; atqui rectangulum super ST in altitudine AB, est ad quadratum AM, ut [a] ST ad AB; ergo series quadratorum MK est ad quadratum AM, ut rectangulum ST AB ad quadratum AM : aequalia sunt igitur rectangulum, & tota series. Quod erat demonstrandum. *a 1 sexti ?.*

***Corollarium.***

Hinc sequitur quadratum ST, totam seriem MK, & quadratum AM in continua esse analogia; nam quadratum ST ad rectangulum super ST & AB, est ut ST linea ad AB lineam: sed rectangulum idem, hoc est tota series MK, est ad AM quadratum in eadem ratione; ergo, &c.

## *L2.§3. PROPOSITION 128.*

Continuing with the same figure, a series of squares is given with bases placed on the line having the terminus at the point K along the line : while by proposition eighty of this work, ST is set equal to the series of the odd numbered bases AB, CD, &c.

I say that the rectangle erected on ST with altitude AB is equal to the sum of the series of squares MK.

***Demonstration.***

By proposition 126 of this book, the sum of the series of squares MK is to the first square AM, as ST is to AB; but the rectangle upon ST with altitude AB is to the square AM, as [a] ST is to AB; hence the series of squares MK is to the square AM, as the rectangle ST AB is to the square AM: therefore the rectangle and the sum of the series are equal. Q.e.d. *a 1 sexti ?.*

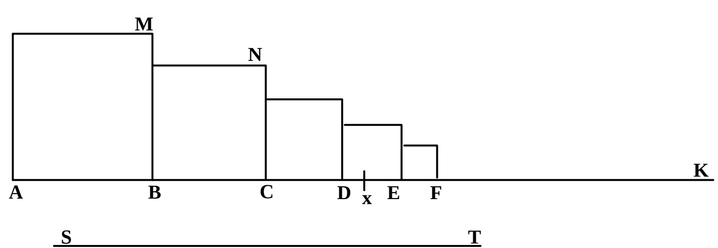
[From the last proposition, AK/CA = (sum of even terms AB, CD, EF, etc.)/BA
= (Sum of plane figures MK) /(area of figure AM) = ST/BA .
Now, Sum of plane figures MK = (area of figure AM) × ST/BA = $AB^2$ × ST/BA = AB × ST, as required.]

## *PROPOSITIO CXXIX.*

Iisdem positis ut AB ad BC, sic fiat AX ad XK.

Dico rectangulum XAB toti quadratorum seriei aequale esse.

***Demonstratio.***




**Prop.129. Fig. 1.**

Quia AX est ad XK, ut AB ad BC, erit invertendo ac componendo, KA ad XA, ut CA ad BA: Atqui etiam KA b est ad seriem linearum AB, CD, ut CA ad BA; ergo KA eandem habet rationem ad XA, & ad seriem AB, CD, aequales sunt igitur series AB, CD & linea XA. unde per praecedentem rectangulum XAB, toti quadratorum seriei est aequale. Quod erat demonstrandum. *b 103huius.*

***Corollarium.***

Ex duabus propositionibus colligere modum licet, quo quadratorum datae seriei repertiri possit quadratum unum aequale. nimirum si inter AB & ST, vel inter AB, &c. AX, media fiat proportionalibus; erit haec latus quadrati, toti seriei ut patet ex duobus iam demonstratis theorematibus : Verum luculentius & universalius hoc Theorema sequenti propositione construemus.

## *L2.§3.* *PROPOSITION 129.*

With the same points in position, as AB is to BC, thus AX is made to XK.
I say that the rectangle XAB is equal to the sum of the series of squares.

***Demonstration.***

Since AX is to XK as AB is to BC, on inverting and adding, KA is to XA as CA is to BA: But also KA [b] is to the series of the lines AB, CD, as CA is to BA; hence KA has the same ratio to XA and to the series AB, CD: therefore the series AB, CD and the line XA are equal. Hence by the preceding, the rectangle XAB is equal to the sum of the whole series of squares. Q.e.d. *b 103huius.*

[AX/XB = AB/BC = 1/*r*, hence KA/XA = CA/BA = 1 + *r* ;
but CA/BA = KA/XA = (Sum of plane figures MK) /(sum of square terms AB, CD, EF, etc.); hence XA is equal to sum of square terms AB, CD, EF, etc.]

***Corollary.***

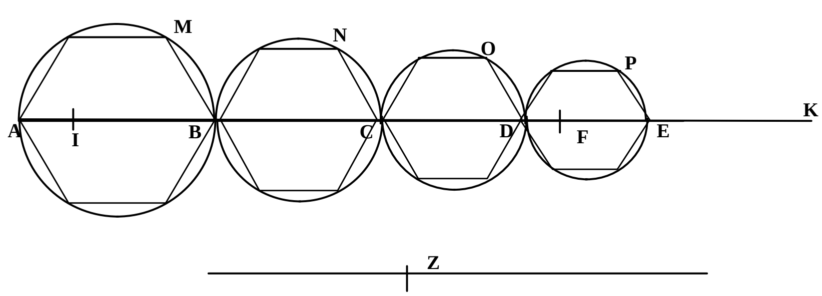
From the two propositions taken together the rule follows that a single square can be found equal to the sum of the given squares of the series: for without doubt between AB and ST, or between AB, &c. and AX, the mean of the proportionals can be found; this will be the side of the square of the sum of the whole series, as is apparent from the two theorems now demonstrated : The following proposition that we construct will show the truth and universal nature of this theorem most clearly.

## *PROPOSITIO CXXX.*

Data sit series planorum quorumcumque basibus in directum AB, BC, CD, &c. positis, ac terminum habens longitudinis punctum K. petitur planis seriei universae planum aequale ac simile exhiberi.

**[132]**

***Constructio ac demonstratio prima.***




**Prop.130. Fig. 1.**

Fiat ut AC ad AK, sic primum seriei planum AM, ad aliud simile cuius diameter vel basis sit Z. Dico hoc toti seriei aequale esse.

Per manifestum propositionis 126 huius, tota planorum series MK, est ad planum AM, ut AK ad AC, id est ex constructione ut Z, ad AM planum : ergo planum Z est ad planum AM, ut tota series MK, ad idem planum AM. aequantur igitur inter se planum Z, & tota series MK. Cum itaque etiam simile sit ex constructione planum Z, planis seriei datae MK, perfecimus quod in problemate petebatur.

***Constructio & demonstratio secunda.***

Sumatur AI primae AB, ac tertiae CD, basium differentia: fiatque ut AI ad AB, sic primum seriei planum AM, ad aliud sibi simile Z:

Dico Z planum satisfacere problemati. Nam tota series MK est ad planum AM ut [a] AB ad AI: Atqui ex constructione etiam planum Z est ad idem planum AM, ut AB ad AI; ergo planum Z, & tota series aequalia sunt. Invenimus igitur datae planorum similium seriei, planum aequale ac simile. Quod erat demonstratum.

*a 127 huius*.

***Constructio & demonstratio tertia.***

Fiat ut AB ad seriem basium imparium AB, CD, &c. sic primum planum ad aliud simile Z.
Dico hoc seriei planorum datae aequari. vel (quod idem est) Fiat ut AB ad BC, sic AF ad FK, utque AB est ad AF, sic planum primum fiat ad aliud simile.
Dico etiam hoc conficere problema: demonstratio eadem est quae primae ac secundae constructionis, ea tantum differentia, quod propositio 126. huius, sit adhibenda. Dixi autem secundum huius tertiae constructionis modum coincidere cum primo, eiusdem constructionis tertiae, quod ex praecedenti manifestum sit, FA aequalem esse seriei basium imparium.

## *L2.§3.* *PROPOSITION 130.*

A series of some kind of plane figures is given with bases AB, BC, CD, etc., placed along a line and having the terminus point at a distance K.

It is required that a similar plane figure is produced that is equal in area to the sum of the whole series.

***First construction and demonstration.***

As AC is to AK, thus the first plane figure of the series AM is made in the same ratio to a different like figure, the base or diameter of which is Z. I say that this figure is equal in area to the whole series.
As proposition 126 of this book has shown, the sum of the series of the plane figures MK is to the plane figure AM, as AK is to AC, or by construction as Z to the plane figure AM : hence the plane figure Z is to the plane figure AM, as the sum of the series MK is likewise to the plane figure AM. Therefore the figure Z is equal to the sum of the whole series of figures MK. Since the plane figure Z is also similar by construction to the plane figures in the given series MK, we have accomplished what was demanded in the problem.

[ (Sum of plane figures MK) /(first plane figure AM.) = AK/AC = Z /(first plane figure AM.), etc.
For, as we have already shown above, the sum $Z$ of the series of plane figures is proportional to the sum of the squares of the diameters of the figures, AB, CD, etc. ;
hence algebraically, $Z$ = (area AM). $(a/(1 - r))/a(1 + r)$ = (area AM)/$(1 - r^2)$ = (area AM)( $1 + r^2 + r^4 + ....$)]

***Construction & second demonstration.***

The difference AI is taken of the bases of the first and third terms AB and CD : and as AI is to AB, thus the first plane figure of the series AM is made to in the same ratio to some other similar plane figure Z:

I say that the plane figure Z provides a satisfactory answer to the problem. For the sum of the series of plane figures MK is to the plane figure AM as [a] AB is to AI: But from the construction the plane figure Z also is to the same plane figure AM, as AB is to AI; hence Z is equal to the sum of the series. Therefore we have found a plane figure equal to the sum of the similar given plane figures. Q. e. d.
[In this case, AI = AB - CD, and AI/AB = area AM/area Z; but from Prop. 127:
AB/AI = ST/AB = (Sum of plane figures MK) /(area of figure AM), from which the result follows at once.]

### *Third construction & demonstration.*

As the ratio of AB is made to the sum of the bases of the series of odd terms [i. e. even powers] AB, CD, etc., thus the first plane figure is made in the same ratio to some similar plane figure Z.
I say that this figure Z is equal to the given series of plane figures, in the sense that it has the same area. As AB is to BC, thus AF is to FK, and as AB is to AF, thus the first plane figure is made to some other similar figure.
I also say that in solving this problem: the demonstration of the first and second constructions is the same, however the difference, according to Prop. 126 of this book, is to be applied. In addition I say that the method of this subsequent third construction coincides with the first, for from the preceding, FA can be shown to be equal to the sum of the bases of the odd series.

[In this case, AB/sum of even powers = $1/(1 + r^2 + r^4 + ..) = 1 - r^2$ = area AM/Z, as previously; however, in this case, AB/BC = AF/FK, from which AC/BC = AK/FK on addition;
hence, FK = AK.BC/AC = $a/(1 - r) . r/(1 + r) = ar/(1 - r^2)$ and AF = $a/(1 - r) - ar/(1 - r^2) = a/(1 - r^2)$.]

### *Scholium.*

*Adverte constructionem illam triplicem propositionis octuagesimae huius, cum universalis sit, huic etiam series convenire : verum quia in progressionibus huius generis, faciliores subinde ac magis expedita constructiones suppetunt, visum est opera pretium, illas tum hoc loco, iam aliis etiam deinceps in medium proferre.*

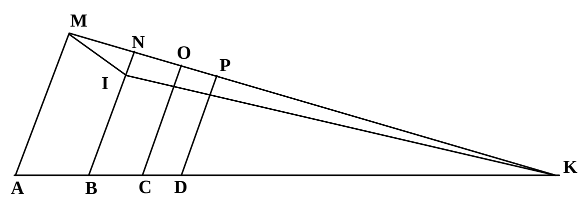
**[133]**

### *Scholium.*

*By directing one's attention to this triple construction instead of making use of the eightieth proposition directly, which is general, we note that the series is also in agreement with the proposition : however, for progressions of this kind, it is usually easier and more expedient to make use of some construction at hand, which is seen to be of value, not only for the present case, but also for other cases henceforth as they come to pass.*

### *Lemma.*

Esto linearum AB, BC, CD progressio terminata in K, & ex punctis A, B, C, &c. erigantur parallelae AM, BN, CO, &c. quae proportionales sint ipsis AB, BC, CD, &c. , ducaturque; ex puncto M ad terminum progressionis K, linea MK. Dico hanc per omnium parallelarum extrematates, N, O, P, &c. transire.



**Prop.130. Fig. 2.**

### *Demonstratio.*

Consideremus primo lineam BN; si ergo MK non transit per N, secabit lineam BN supra aut infra N, in I. Erit ergo MIK una recta. Et quoniam progressionis AB, BC, &c. terminus est K per 82. huius, erit ut AK ad BK sic AB ad BC, hoc est, ex datis AM ad BN: atqui etiam ut AK est ad BK, sic AM ad BI; ergo AM est ad BN, ut AM ad BI maiorem aut minorem quam BN, quod est impossibile. Non ergo secabit MK ipsam BN supra aut infra N, ergo in N; similiter ostendemus rectam NK (hoc est rectam MNK, ostendimus enim modo puncto MNK esse in una recta) transire per O. & sic de ceteris in infinitum: Patet igitur veritas lemmatis.

***Lemma.***

Let the progression of the lines AB, BC, CD be terminated in K, and from the points A, B, C, &c. the parallel lines AM, BN, CO, etc. are erected, which are themselves proportional to AB, BC, CD, etc.; and a line MK is drawn from the point M to the terminus of the progression K. I say that this line passes through the ends of all the parallel lines N, O, P, etc.

***Demonstration.***

We consider the first line BN; for if MK does not pass through N, then it cuts the line BN either above or below N at the point I. Therefore MIK is a single line. And since the terminus of the progression AB, BC, etc. is K by Prop. 82, AK is to BK thus as AB is to BC, or from what is given, as AM is to BN: but also as AK is to BK, thus AM is to BI; hence AM is to BN as AM is to BI, which is larger or smaller than BN, which is impossible. Hence MK does not cut BN itself above or below N, but instead passes through N; similarly we can show that the line NK (or the line MNK, as we have shown in this way that the point lies on a single line MNK) passes through O, and thus for the other points indefinitely: The truth of the lemma is thus established.

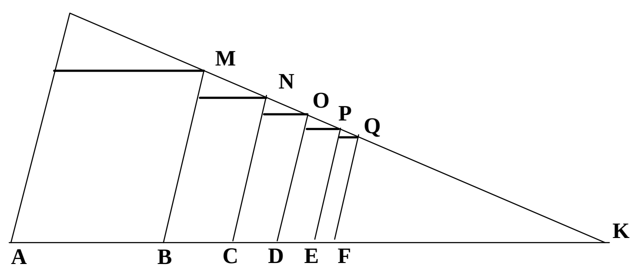
## *PROPOSITIO CXXXI.*

Esto planorum rectilineorum similium similiterque positorum series MK, basibus in directum collatis, terminum habens longitudinis punctum K. Ex vertice autem M, ad K ducatur recta MK.

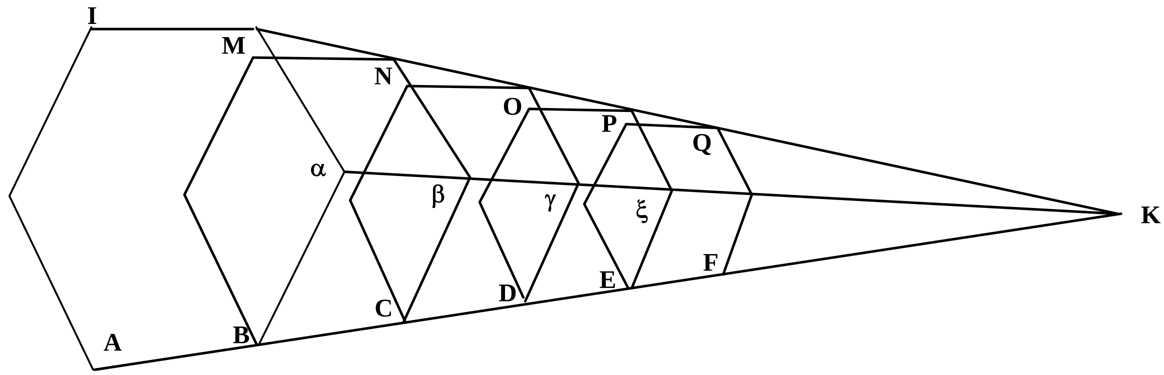
Dico hanc per omnes omnium angulorum totius seriei vertices N, O, P, Q; &c. transire : sive totam planorum seriem angulo AKM inscriptam esse.

***Demonstratio.***

Data sit primum series figurarum trium quatuorque laterum. Cum igitur ex hypothesisi planorum series terminetur in K, basium quoque progressonis terminus erit K per 124. huius: & quoniam plana AM, BN, &c. similia sunt, similiterque; posita, ex elementis patet latera BM, CN, DO, &c. inter se parallela esse, ipsisque AB, BC, &c. proportionalia. ergo per lemma praecedens MK transit per omnes vertices N, O, P, Q, &c. Quod erat demonstrandum.

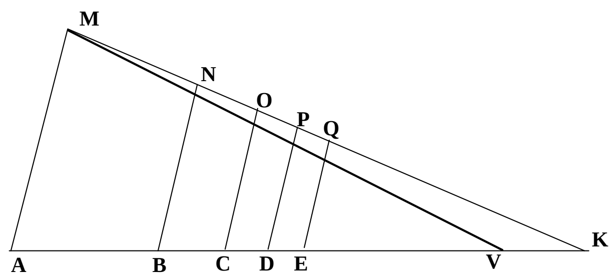



**Prop.131. Fig. 1.**




**Prop.131. Fig. 2.**

Sint iam plurium laterum figurae quarum series terminetur in K. Quoniam igitur plana AM, BN, &c. similia sunt similiterque posita, erunt Bα, Cβ, Dγ, Eξ, parallelae inter se, ipsisque AB, BC proportionales. quare per lemma precedens puncta α, β, γ, ξ &c. cum puncto K sunt in una eademque linea αK, quae cum similiter divisa sit, ac linea BK, manifestum est progressionem linearum αβ, βγ, &c. terminari quoque in K. Sunt autem ex punctis α, β, γ, &c. erecta parallela latera αM, βN, γO, &c. quae lateribus BC, CD, DE, hoc est ipsis αβ, βγ, γξ, sunt proportionalia (quae omnia patent ex eo quod AM, BN, CO, &c. similia plana sint similiterque posita) ergo per lemma, linea MK transit per omnia puncta M, N, O, P, Q, &c. Quod erat demonstrandum.




**Prop.131. Fig. 3.**

***Lemma.***

Esto linearum AB, BC, CD, &c. progressio terminata in K. & ex punctis A, B, C, D, &c. erigantur AM, BN, CO, DP, &c. inter se parallelae; & proportionales ipsis AB, BC, CD, &c. assumptisque duabus lineis AM, CO, per M & O ducatur recta MO.

Dico hanc productam incidere in K ac per omnium reliquarum extremitates transire.

***Demonstratio.***

Si enim ita non sit; igitur MO producta, cis vel ultra K in V concurret cum linea AK; & quoniam progressionis AB, BC, CD terminus est K, erunt AK, BK, [a] CK, DK, EK proportionales continuae. quare etiam AK, CK, EK ex aequo sunt continuae. unde [b] ut AK ad CK, sic AC ad CE. Atque [c] AC est ad CE, in duplicata rationis AB ad BC, id est ex hypothesi rationis AM ad BN; ergo AK est ad CK, in duplicata rationis AM ad BN : sed & ratio AM ad CO (ut ex datis colligitur) duplicata est rationis AM ad BN, ergo AM est ad CO, hoc est MOV ad OV, ut AK ad CK; quod est impossibile: non igitur occurret MO ipsi AK cis vel ultra K. Quod erat primum, hoc autem sic demonstrato, patet secunda pars ex lemmate propositionis preacedentis. Quae erant demonstranda. *a 82 huius; b 1. huius; c 106. huius.*

## *L2.§3.* *PROPOSITION 131.*

Let MK be a series of similar rectilinear plane figures similar placed in position in a like manner with their bases arranged along a line, having the terminal point at a distance K. The line MK is drawn from the vertex M to K.

I say that this line passes through the vertices N, O, P, Q, etc. of the whole series of angles of the complete series : or the whole plane figure is inscribed within the angle AKM.

### *Demonstration.*

A first series of figures of three and four sides is given. Since by hypothesis, the series of plane figures is terminated in K, the terminus of the bases of the progression is K too, by Prop. 124 of this work : and since the plane figures AM, BN, &c. are similar to each other, and placed in a similar manner, then it is apparent from elementary considerations that the sides BM, CN, DO, &c. are parallel to each other, and proportional to AB, BC, etc. Hence by the preceding lemma, MK passes through all the vertices N, O, P, Q, &c. Q. e. d.

Now there are figures of many sided for which the series terminates in K. Therefore, since the plane figures AM, BN, etc. are similar and similarly placed, then the lines B$\alpha$, C$\beta$, D$\gamma$, E$\xi$ are parallel to each other and proportional to AB and BC. Whereby by the preceding lemma the points $\alpha$, $\beta$, $\gamma$, $\xi$ etc. with the point K lie on the same line $\alpha$K, and the line BK which as it is likewise divided, shows a progression of lines $\alpha\beta$, $\beta\gamma$, etc. to be terminated too in K. However, from the points $\alpha$, $\beta$, $\gamma$, etc., there are erected parallel lines $\alpha$M, $\beta$N, $\gamma$O, etc. which are in proportion with the sides BC, CD, DE, or $\alpha\beta$, $\beta\gamma$, $\gamma\xi$ (which are all apparent from this, since AM, BN, CO, etc. are similar plane figures similarly positioned ) hence by the lemma, the line MK is cut by all the points M, N, O, P, Q, etc. Q. e. d.

### *Lemma.*

The progression of lines AB, BC, CD, etc. terminates in K. From the points A, B, C, D, etc. the lines AM, BN, CO, DP, etc. are erected parallel to each other; and from the proportions AB, BC, CD, etc. and from the two lines taken AM, CO, the line MO is drawn through M and O.

I say that this line produced is to be incident in K, and to pass through all of the other extremities.

### *Demonstration.*

For if this is not the case, then MO produced meets the line AK either on the near or far side of K in V; and since K is the terminus of the progression AB, BC, CD, then AK, BK, [a] CK, DK, EK are in continued proportion. Whereby AK, CK, EK are also in continued proportion from equality. Hence [b] as AK is to CK, thus AC is to CE. But [c] AC is to CE in the square of the ratio AB to BC, that is from the hypothesis for the ratios AM to BN; hence AK is to CK in the square of the ratio AM to BN : but the ratio AM to CO (as combined from what is given) is the square of the ratio AM to BN, hence AM is to CO, or MOV to OV, as AK is to CK; which is not possible : hence MO does not cut AK on the near or far side of K. Which establishes the first part of the proof, with this explained thus, the second part is apparent from the lemma of the preceding proposition. Q. e. d.

## *PROPOSITIO CXXXII.*

Esto planorum rectilineorum similium similiterque positorum series MK, uti prius; & terminus sit K, per quorumlibet autem duorum planorum AM, CO vertices ducatur recta MO.

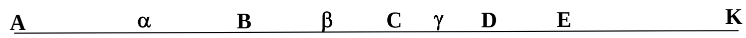
**[135]**

Dico hanc productam cadere in terminum K ac per omnium reliquorum vertices N, P, Q, &c. transire; sive totam planorum seriem angulo AKM inscriptam esse.

### *Demonstratio.*

Discursus demonstrationis plane idem erit qui propositionis praecedentis. Nam quemadmodum illic per lemma illi propositioni appositum demonstravimus propositum, ita hic per lemmatis proximi applicationem propositionis veritatem concludemus.

### *Lemma.*




**Prop.132. Fig. 1.**

Sint AK, BK, CK, DK, &c. in continua analogia, ac differentiae illarum nempe AB, BC, CD, DE, &c. bisectae sint in $\alpha$, $\beta$, $\gamma$, &c. Dico etiam $\alpha$K, $\beta$B, $\gamma$C, &c. esse continuas.

### *Demonstratio.*

Quia AK, BK, CK, &c. sunt continuae per primam huius, AB est ad BC, hoc est A$\alpha$ est ad B$\beta$, ut AK ad BK. cum ergo ablatum A$\alpha$ sit ad ablatum B$\beta$ ut totam AK ad totam BK, erit & reliquam $\alpha$K ad reliquam $\beta$K, ut totam AK ad totum BK. similiter ostendemus $\beta$K esse ad $\gamma$K ut BK est ad CK; hoc est ex hypothesi ut AK ad BK, hoc est ex demonstrationis ut $\alpha$K ad $\beta$K. Sunt igitur $\alpha$K, $\beta$B, $\gamma$C continuae. Quod erat demonstrandum.

## *L2.§3. PROPOSITION 132.*

Let MK be a series of similar rectilinear plane figures similar placed as before; and the terminus is K. The line MO is drawn through any two of the place figures AM and CO as you wish.

I say that this line produced ends in the terminus K and also passes through the vertices N, O, P, Q, etc. : or the whole plane figure is inscribed within the angle AKM.

### *Demonstration.*

The discussion of the demonstration clearly is the same as that for the preceding proposition. For since there we demonstrated the adjacent proposition from the lemma of that proposition, thus here we can conclude the truth of the proposition by an application of the next lemma.

### *Lemma.*

AK, BK, CK, DK, &c. are in continued proportion, and the differences of these are surely AB, BC, CD, DE, etc., which are bisected in $\alpha$, $\beta$, $\gamma$, etc. I say that $\alpha$K, $\beta$B, $\gamma$C, etc. are also in continued proportion.

### *Demonstration.*

Since AK, BK, CK, etc. are in continued proportion from the first part of this prop., then AB is to BC, or A$\alpha$ is to B$\beta$, as AK is to BK. Therefore when A$\alpha$ is taken and B$\beta$ is taken in the ratio of the whole length AK to the whole length BK, thus the remainder $\alpha$K is to the remainder $\beta$K, as the whole length AK to the whole length BK. In the same way we can show that $\beta$K is to $\gamma$K as BK is to CK; or from hypothesis as AK is to BK, or from the demonstration as $\alpha$K is to $\beta$K. Therefore $\alpha$K, $\beta$B, $\gamma$C are in continued proportion. Q. e. d.

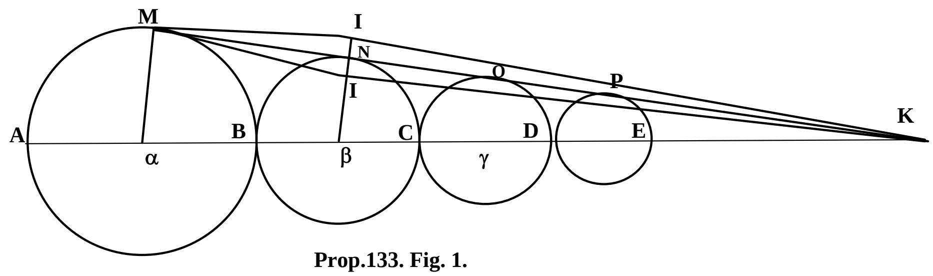
[For $A\alpha/B\beta = AK/BK$ gives $A\alpha/AK = B\beta/BK$ from which $\alpha K/AK = \beta K/BK$ or $\alpha K/\beta K = AK/BK$, etc.]

## *PROPOSITIO CXXXIII.*

Esto circulorum progressio diametris in directum positis, terminum habens longitudinis punctum K: & ex K ducta linea KM tangat quemvis datae seriei circulum, verbi gratia circulum AMB.

Dico lineam KM totam circulorum seriem contingere.

### *Demonstratio.*




**Prop.133. Fig. 1.**

Ex centro α ad contactum ducatur αM linea, cui ex centro β parallela sit βX secans circulum BNC in N. Recta igitur MK occurrit ipsi βX vel in puncto N, vel supra aut infra N in I. non autem supra aut infra posse occurrere enim, sic demonstro, occurrat emim, si fieri potest, supra vel infra N in I. quoniam igitur dantur circuli in continua analogia, etiam per 124 huius AB, BC, CD, &c. sunt continuae : ergo AB est ad BC, ut BC ad CD, hoc est ut Bβ ad Cγ : sed & AB est ad BC, ut αβ ad βC; ergo ut AB est ad BC, sic αβ est ad βγ. Praeterea quoniam K terminus est progressionis basium AB, BC, &c. per corollarium 125, huius erunt per 82, huius continuae proportionales AK, BK, CK, DK, quare per lemma etiam αK, βK, γK erunt continuae. ergo per primam huius ut αK ad βK, sic est αβ ad βγ. hoc est sicut ante ostendimus AB ad BC, hoc est αM ad βN. Atqui eiam est ut αK ad βK, sic αM ad βI : sunt enim ex

**[136]**

constructione lineae αM, βI parallelae; ergo αM est ad βI, ut αM ad BN maiorem aut minorem quam BI, quod est absurdum. Non igitur recta MK occurret rectae βI supra vel infra N: ergo in N. Quoniam autem ex centro ad contactum angulis KMα [a] rectus est; quare cum ex constructione βN parallela sit αM, angulus quoque KNβ rectus erit: ergo [b] linea KM tangit circulum BNC. simili ratiocinatione demonstrabimus KM reliquos etiam omnes circulos contingere. Quod erat demonstratum. *a 17 quinti; b 16 tertii.*

### *Corollarium.*

Quod in hoc Theor. de circulis demonstravimus, etiam de similibus Ellipsibus, Parabolis, Hyperbolis demonstrari potest.

## *L2.§3.* *PROPOSITION 133.*

A progression of diameters of circles is set out along a line, having the terminus at a distance K. The line KM is drawn to touch one or other of the given series of circles, such as the circle AMB, for example.

I say that the line KM touches all the series of circles.

### *Demonstration.*

From the centre α to the point of contact the line αM is drawn, to which the line βX is drawn parallel from the centre β, cutting the circle BNC in N. Therefore the line MK crosses βX either at the point N, or above or below N in I. I demonstrate as follows that it is not possible for the point to be above or below N. Indeed if we assume that it is cut above or below N in I, then as the circles are given in continued proportion, it follows by Prop. 124 that AB, BC, CD, etc. are in continued proportion : hence AB is to BC as BC is to CD, or as Bβ is to Cγ : but AB is to BC as αβ is to βC; hence as AB is to BC, thus αβ is to βγ. In addition, since K is the terminus of the progression of the bases AB, BC, etc., by the corollary of Prop.

125 and Prop. 82, AK, BK, CK, DK are continued proportionals. Whereby by the lemma αK, βK, γK are also continued proportionals. Hence as αK is to βK, thus αβ is to βγ. or as we showed before, as AB is to BC, or αM to βN. But also as αK is to βK, so αM is to βI : for the lines αM, βI are parallel from the construction; hence αM is to βI as αM is to BN, larger or smaller than BI, which is absurd. Therefore the line MK does not cut βI above or below N: and thus it cuts in N. But as the angle KMα [a] from the centre of the circle to the point of contact is right; since from the construction βN is parallel to αM, the angle KNβ is also right: hence [b] the line KM is a tangent to the circle BNC. By similar reasoning we can show that KM is also a tangent to all the other circles. Q. e. d. *a 17 quinti; b 16 tertii.*

***Corollary.***

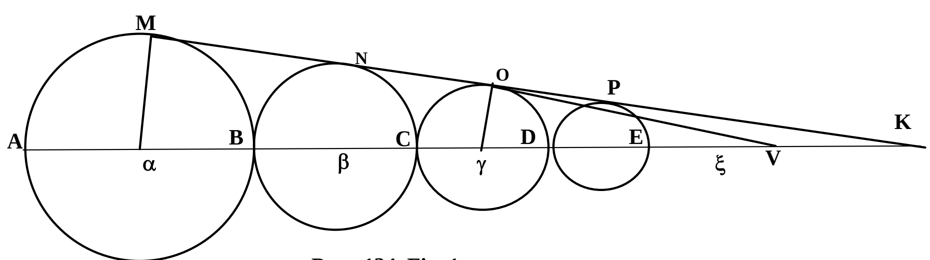
What we have shown for circles in this theorem can also be demonstrated for similar ellipses, parabolas, and hyperbolas.

## *PROPOSITIO CXXXIV.*

Iisdem positis duos quoscumque circulos seriei datae circulos, verbi gratia AMB, COD tangat recta MO.

Dico hance productam cadere in terminum longitudinis K, & reliquos circulos omnes contingere.

***Demonstratio.***




**Prop.134. Fig. 1.**

Centra circulorum sint α, β, γ, ξ, etc., tum ex α ac γ ad contactus ducantur αM, γO. si igitur negetur assertio, tangens MO producta occurret lineae AK cis vel ultra K in V. & quoniam per corollarium propositionis 125. huius terminus progressionis basium est in K, erunt [c] AK, BK, CK, DK, EK, &c. ac proinde per lemma propositionis praecedentis etiam αK, βK, γK, in continua analogia. Quare patet ex aequo etiam αK, βK, γK esse continuas, ergo αK est [d] ad γK, ut αγ est ad γξ. Atqui αγ est ad γξ, ut AC ad CE (quod simili modo ostendemus quo in praecedenti ostendimus AB esse ad BC ut αB ad Bγ) & AC est ad CE in [e] duplicata rationis AB ad BC, id est ut AB ad CD; & AB est ad CD ut αM ad γO, ergo αK est ad γK ut αM ad γO. Deinde cum αM, γO ex centris ad contactus ductae sunt, erunt perpendiculares ad MV, ideoque parallelae inter se. Quare MV erit ad OV, ut αM ad γO, hoc est per iam demonstrata ut αK ad γK, quod est absurdum. Non igitur MO occurret ipsi AK cis vel ultra sed in K. Quod erat primam. Quo demonstrato, patet per praecedentem secunda pars, quae erant demonstranda.

Quod si loco circuli COD assumator circulus BNC, ita ut linea duos vicinos circulos contingat, demonstratio longe erit facilior, quam proinde omisimus. *c 82 huius; d 1 huius; e 106 huius.*

**[137]**

***Corollarium.***

Hoc quoque Theorema non tantum circulis, sed aliis similibus sectionibus applicari potest.

## *L2.§3.* *PROPOSITION 134.*

With the same points in position, any two circles of the given series, for example, AMB and COD are tangents to the line MO.

I say that this line produced passes through the terminal point at a distance K, and that the rest of the circles in the series have the line MO as a tangent.

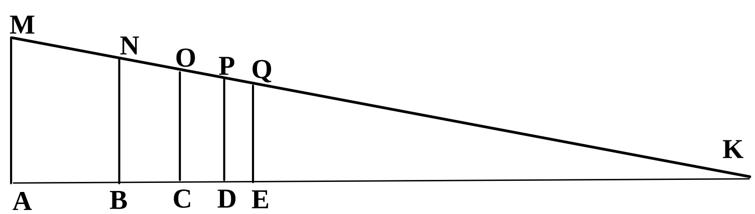
***Demonstration.***

The centres of the circles are α, β, γ, ξ, etc., then from α and γ to the point of contact the lines αM and γO are drawn. Therefore, if the proposition is not true, then the tangent MO produced meets one side or the other of the line AK in V. Since, by the corollary of proposition 125, the terminus of this progression of bases is K, the lines [c]AK, BK, CK, DK, EK, &c. are in a continued ratio, as also are the lines αK, βK, γK, by the lemma of the previous proposition. Whereby it is apparent from the equality that αK, βK, γK are in continued proportion, hence αK is [d] to γK, as αγ is to γξ. But αγ is to γξ, as AC is to CE (as we can show by a similar way as we have shown for the preceding that AB is to BC as αB to Bγ) and AC is to CE in [e] the square of the ratio AB to BC, or as AB ad CD; and AB is to CD as αM is to γO, hence αK is to γK as αM is to γO. Hence as αM and γO have been drawn from the centres of the circles to the points of contact, they are perpendicular to MV, and therefore parallel to each other. Whereby MV is to OV, as αM to γO, or as just demonstrated, as αK to γK, which is absurd. Therefore the line MO does not meet the line AK in one side or the other, but in K itself, which proves the first part. The second part is apparent from the first, and these had to be shown.
For if the place of the circle COD is taken by the circle BNC, since the line is a tangent to two neighbouring circles, the demonstration is much easier, and hence has been omitted. *c 82 huius; d 1 huius; e 106 huius*

***Corollary.***

This theorem can also be applied to other figures, as well as circles, with similar sections.

## *PROPOSITIO CXXXV.*




**Prop.135. Fig. 1.**

Linearum AB, BC, CD, &c. progressio terminatur in K: erectaque ad quemvis angulum recta AM, ducatur MK: Deinde ex singulis punctis in progressione AK repertis, ad AM, parallelae erigantur BN, CO, DP, atque ita semper.

Dico ex triangulo AMK relinquendum tandem triangulum aliquod quavis data superficie minus.

***Demonstratio.***

Quoniam similia sunt triangulia AMK, BNK, COK, &c. erit duplicata eorum proportio, proportionis laterum AK, BK, CK, &c. quia autem progressionis AB, BC terminus est K, erunt [a] AK, BK, CK, &c. continuae proportionales. unde similiter triangula AMK, BNK, &c. etiam sunt in ratione continua & dividendo trapezium AM NB ad triangulum BNK, ut trapezium BN OC ad triangulum COK, & trapezium CP ad triangulum DPK: atque ita semper. Igitur [b] relinquetur tandem triangulum dato minus. Quod erat demonstrandum. *a 82 huius; b 78 huius.*

## *L2.§3. PROPOSITION 135.*

The progression of the lines AB, BC, CD, etc. terminates in K: and the line AM is erected at right angles to AK at some angle to the line AK, and the line MK is drawn. Hence, from the individual points of the progression AK, the lines BN, CO, DP are drawn parallel to AM, indefinitely.

I say that by taking away parts of triangle AMK, a triangle is given at last with a surface less than some given area.

***Demonstration.***

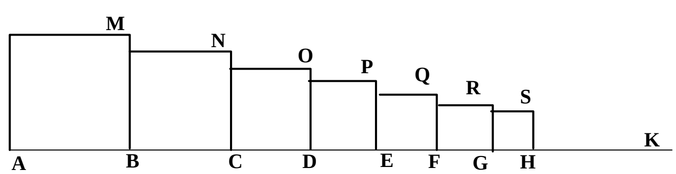
Since the triangles AMK, BNK, COK, etc. are similar, the areas are in the square ratio to the sides of the triangles AK, BK, CK, etc. But since the terminus of the progression AB, BC is K, [a] AK, BK, CK, etc. are continued proportionals. Hence, the triangles AMK, BNK, etc. also are similarly in a continued ratio, and on dividing, the trapezium AM NB is to the triangle BNK as the trapezium BN OC is to the triangle COK, & the trapezium CP is to the triangle DPK: and thus always indefinitely. Therefore [b] at last there remains a triangle less than some given surface. Q. e. d. *a 82 huius; b 78 huius.*
[ΔAMB/ΔBNK = ΔBNK/ΔCOK, etc., from which trap. AMNB/ΔBNK = trap. BNOC/ΔCOK, etc.]

## *PROPOSITIO CXXXVI.*

Esto planorum series lateribus homologis in directum constitutis, terminus autem longitudinis sit K.

Dico ex ablatione continuata planorum AM, BN, CO, &c. relinqui residium seriei, quovis dato plano minus.

***Demonstratio.***



**Prop.136. Fig. 1.**

Per octuagesimam secundam huius, AM est ad reliquam seriem planoram NK, ut planum BN est ad reliquam seriem OK : & per eandem planum BN est ad reliquam seriem OK, ut planum CO, est ad seriem reliquam PK: Atque ita in infinitum, ergo ex perpetua planorum AM, BN, &c. ablatione residuum seriei propositae [c] erit tandem qouvis dato plano minus. Quod erat demonstrandum. *c 78 huius.*

**[138]**

## *L2.§3. PROPOSITION 136.*

A series of plane figures with sides in correspondence is set up along a straight line, while the terminus of the length of the line is K.

I say that by the successive removal of the plane figures AM, BN, CO, etc, a part of the series is left smaller than some given plane figure.

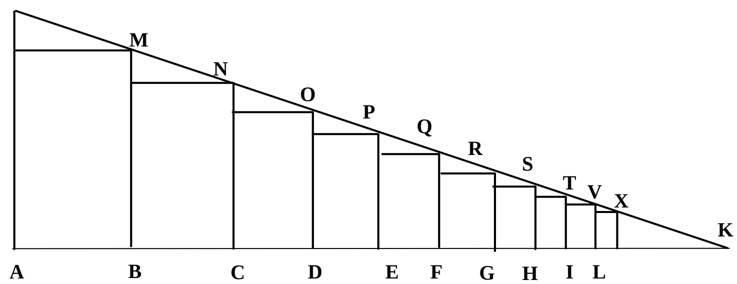
***Demonstration.***

By Prop. 82, the ratio of AM to the rest of the plane series NK, as the same manner BN is to the rest of the series OK, as the plane figure CO is to the rest of the series PK. Thus continued indefinitely, hence from the endless nature of the supply of plane figures, the remaining part of the proposed series is finally by subtraction less than some given plane figure. Q. e. d. *c 78 huius*

## *PROPOSITIO CXXXVI.*

Detur series planorum similitium homologis basibus in directum collocatis, terminum habens longitudinis punctis K: sumpto autem quovis plano CO, numerentur alia plana in infinitum EQ, GS, IV, &c. totidem semper intermissis quot inter planum CO & primum seriei planum AM intercedunt. Petuntur omnia plana AM, CO, EQ, GS, &c. ex proposita serie auferri.

### *Constructio & Demonstratio.*




**Prop.137. Fig. 1.**

Quoniam ex hypothesi plana AM, BN, CO, DP, EQ, FR, GS, sunt in continua ratione, etiam ex aequo plana AM, CO, EQ, IV, in continua sunt analogia. Igitur per propositionem 126 huius, seriei planorum continue proportionalium AM, CO, EQ, &c. inveniatur planum aequale. hoc si aufers ex plano, quod per eandem propositionem 126, factum fuerit aequale seriei datae MK, habebitur propositum.

## *L2.§3.* *PROPOSITION 137.*

A series of similar plane figures with bases in a given continued ratio are set out together along a line, the terminal point being at a distance K: by taking some plane CO, the other planes EQ, GS, IV, &c. are counted indefinitely by always putting the same total between the planes as there are placed between the plane CO and the first plane AM. It is required to show that all the [odd] planes AM, CO, EQ, GS, &c. are taken from the proposed series.

### *Demonstration.*

Since by hypothesis the planes AM, BN, CO, DP, EQ, FR, GS, etc. are in a continued ratio, then from the equality of the ratio, the planes AM, CO, EQ, IV, etc., are also in a continued ratio. It follows from proposition 126 that a plane figure can be found equal to the sum of the series of plane figures in continued proportion AM, CO, EQ, etc. If this plane figure is taken from the sum of all the plane figures, which by prop. 126 is equal to the sum of the given series MK, then the proposition is shown to be true.
[Prop. 126 asserts that (first figure AM)/(whole series of figures MK) = (first base AB)/(sum of odd bases);

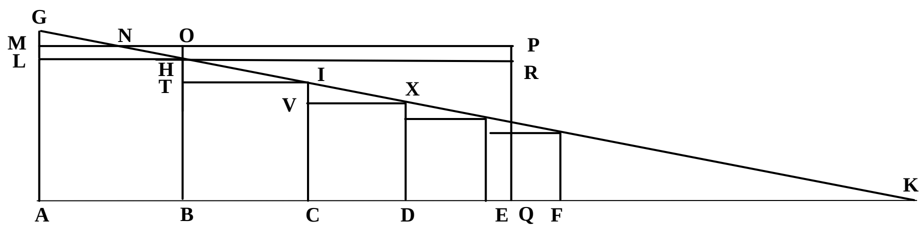
## *PROPOSITIO CXXXVIII.*

Detur progressio quadratorum habens bases in directum, & terminum longitudinis punctis K: & ex K per H ducatur recta KH, quae per propositionem 131 huius contingat omnia serie quadrata & concurrat cum AL in G.

Dico triangulum AGK, toti trapeziorum GB, HC, ID, &c. sive utrique quadratorum AH, BI, &c. ac triangulorum LGH, THI, &c. progressioni, aequale esse.

**[139]**

***Demonstratio.***




**Prop.138. Fig. 1.**

Per corollarium propositionis 125. huius series basium AB, BC, CD terminatur in K; unde AK, [a] BK, CK, &c. sunt continuae proportionales, & triangula [b] AGK, BHK, CIK, &c. in continua sunt analogia. Quare toti seriei proportionis quam habet trapezium GB ad trapezium HC, sint statu continuatae, triangulum AGK aequale est. Atqui trapezia ID, XE, &c. continuant rationem trapezii GB, ad trapezium HC; (cum enim triangula AGK; BHK; CIK; &c. sint continue proportionalia, eorum quoque differentiae nempe dicta trapezia erunt continua) ergo triangulum AGK, trapezium seriei universae est aequale: Quod erat demonstrandum. *a 82 huius; b 124 huius; c 89 huius.*

## *L2.§3. PROPOSITION 138.*

A progression of squares having bases along a line is given, and a terminus of length K: a line KH is drawn from K through H, which by Prop. 131 touches the whole series of squares and meets AL in G.

I say that the triangle AGK of the sum of all the trapeziums GB, HC, ID, etc. or of the sum of the progression of each of the squares AH, BI, etc. and the triangles LGH, THI, etc. are equal.

***Demonstration.***

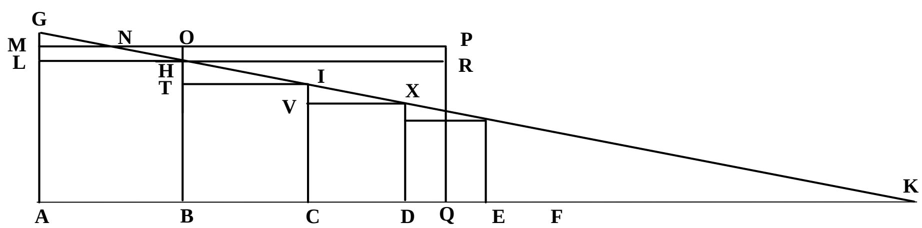
According to the corollary of Prop. 125, the series of the bases AB, BC, CD terminates in K; hence AK, [a] BK, CK, &c. are ratios in continued proportionals, and the triangles [b] AGK, BHK, CIK, etc. are in a continued ratio. Whereby the sum of the series of proportions that continues without end in the ratio of the trapezium GB to the trapezium HC, is equal to the triangle AGK. But the trapeziums ID, XE, etc. are continued in the ratio of the trapezium GB to the trapezium HC; (as the triangles AGK; BHK; CIK; etc. are continued proportionals, the differences of these too, surely the said trapeziums, are in continued proportion). Hence the triangle AGK is equal to the sum of the series of the trapeziums: Q. e. d. *a 82 huius; b 124 huius; c 89 huius.*

# *PROPOSITIO CXXXIX.*

Esto quadratorum series HK; bases habens in directum, & terminum longitudinis punctis K: & ex K per H ducatur recta KH, tangat per propositionem 131 huius omnia seriei quadrata, ac concurrat cum AL in G; inventa deinde per propos. 80.huius linea AQ, quae aequalis sit seriei basium imparium AB, CD, EF, divisaque; bifarium LG in M, fiat rectangula QAMP.

Dico hoc triangulo AGK, aequale esse.

### *Demonstratio.*




**Prop.139. Fig. 1.**

Producatur LH in R, & BH in O; linea MNO parallela est ex datis ad AQ, quae aequidistat LH; ergo MNO parallela est ad LH. & quia GM aequalis est ML, etiam GN aequalis erit NH; sunt autem & OH, ML, id est MG aequales, & anguli OHN, MGN, aequales : ergo aequantur triangula MGN, NOH: additoque communi LMNH, triangulum LGH, rectangulo LO aequale est : quare quadratum AH est ad triangulum LGH, ut idem quadratum, ad rectangulum LO, hoc est ut linea AL ad lineam LM. Praeterea triangula LGH, THI, VIX, similia sunt, & homologa latera LH, TI, VX; adeoque in duplicata ratione laterum LH, TI, VX, &c. Igitur cum quadrata AH, BI, &c. sint in eorundem laterum duplicata ratione; patet triangula dicta, quadratis proportionalia esse. unde permutando ut quadratum AH ad triangulum LGH, sic quadratum BI ad triangulum THI; & quadratum CX, ad triangulum VIX: atque ita in infinitum. Ergo ut quadratum AH ad triangulum LGH, hoc est ex ante demonstratis ut AL ad LM,

**[140]**

[a] sic omnia quadrata, ad omnia triangula. Atqui ut AL ad LM sic etiam est rectangulum AR ad rectangulum LR. Igitur ut omnia quadrata, ad omnia triangula, sic rectangulum AR ad rectangulum LP. Atqui rectangulum AR, aequale est toti [b] quadratorum seriei, (est enim AQ aequalis seriei basium imparium AB, CD, &c. & AL, AB) ergo cum series quadratorum ostensa sit aequalis esse rectangulo AR; etiam tota triangulorum series rectangulo LP [c] aequalis erit. quare totum rectangulum AP, utrique seriei quadratorum ac triangulorum, hoc est seriei trapeziorum GB, HC, ID, &c. aequabitur : sed trapeziorum series, [d] constituit triangulum AGK. ergo rectangulum AP triangulo AGK aequale est. Quod erat demonstrandum. *d 1 sexti; e 19 sexti; a 12 quinti; b 79 huius; c 14 quinti; d 136 huius .*

### *Corollorium.*

From the discourse of the demonstration it can be gathered, that the rectangle MLRP is equal to the series of triangles GLH, HTI, &c. : truly the rectangle GLR, is twice the sum of the progression of the triangles.

## *L2.§3.* *PROPOSITION 139.*

Let HK be a series of squares, having bases along a line, and the terminus of the points at a distance K. The line KH is drawn from K through H , touching all the terms in the series of squares, according to Prop. 131, and is concurrent with the line AL in G. Hence, by Prop. 80, a line AQ is found, which is equal to the sum of the series of the odd bases AB, CD, EF, etc, and bisecting LG in M, from which the rectangle QAMP is made.

I say that this rectangle QAMP is equal to the triangle AGK.

### *Demonstration.*

LH is extended to R, and BH to O; from the given, the line MNO is parallel to the given line AQ, to which LH is equidistant ; hence MNO is parallel to LH; and since GM is equal to ML, also GN is equal to NH. Moreover OH and ML (or MG) are equal, and the angles OHN and MGN are equal : hence the triangles MGN and NOH are equal: and on adding to the common LMNH, the triangle LGH is equal to the rectangle LO. Whereby the square AH is to the triangle LGH as the same square is to the rectangle LO, or as the line AL is to the linea LM. In addition the triangles LGH, THI, and VIX are similar, and the homologous sides are LH, TI, and VX; and [the areas are] in the square ratio of the sides LH, TI, VX, &c. Therefore since the squares AH, BI, &c. are in the square ratio of the same sides, it is apparent that the said triangles are in proportion with the squares. From which on interchanging, as the square AH is to the triangle LGH, thus the square BI is to triangle THI; and the square CX is to triangle VIX: and thus indefinitely. Hence as the square AH is to triangle LGH, (or from the previous demonstration, as AL is to LM,)[a], so the sum of all the squares is to the sum of all the triangles. But as AL is to LM thus also rectangle AR is to rectangle LP. Therefore, as the sum of all the squares is to the sum of all the triangles, thus the rectangle AR is to the rectangle LP. But the rectangle AR is equal to the sum of all the squares in the series [b] of squares, (as AQ is equal to the sum of the series of the odd bases AB, CD, &c. & AL is equal to AB) hence as the series of squares is shown to be equal to the rectangle AR; so also the whole series of triangles is equal to the rectangle LP [c]. Whereby the whole rectangle AP is equal to the sum of the series of squares and triangles, or of the series of trapeziums GB, HC, ID, etc. : but the series of trapeziums [d] constitutes the triangulum AGK. Hence the rectangle AP is equal in area to the triangle AGK. Q. e. d. *d 1 sexti; e 19 sexti; a 12 quinti; b 79 huius; c 14 quinti; d 136 huius .*

### *Corollary.*

From the discourse of the demonstration it can be gathered that the area of the rectangle MLRP is equal to the sum of the progression of triangles GLH, HTI, etc.: and the rectangle GLR is indeed equal to double the area of the same progression of triangles.

[The lines MNO, LHR, and AQ are parallel; GM = ML, GN = NH, and OH = ML = MG; the angles OHN and GNM are equal; hence the triangles MGN and ONH are equal, and ΔMGN + trap.LMNH = ΔLGH = ΔONH + trap.LMNH = rect.LO; sq.AH/ΔLGH = sq.AH/rect.LO = AL/LM (*).

In the same manner, since Δ's LGH, THI, and VIX are similar, with homologous sides LH, TI, and VX, then the areas are in the ratio $LH^2$ to $TI^2$, and $TI^2$ to $VX^2$, etc, (as the altitude of each is in the same proportion to the base); also, the squares AH, BI, etc are also in the same square ratio of the bases, and hence the triangles are in proportion to the squares. Hence: sq.AH/ΔLGH = sq.BI/ΔTHI = sq.CX/ΔVIX, etc, and sq.AH/ΔLGH = (sum of squares)/(sum of triangles), from proportionality, which can be extended indefinitely.

Also from (*), AL/LM = AL.AQ/LM.AQ = <u>rect.AR/rect.LP</u> = sq.AH/ΔLGH = (sum of squares)/(sum of triangles). But as rect. AR = sum of squares, (since AQ = AB + CD + EF + ..., and AL = AB); so also rect.LP = sum of triangles, from which the whole rectangle AP = sum of squares + sum of triangles = sum of trapeziums GB, HC, ID, etc, = ΔAGK.]

## *PROPOSITIO CXL.*

Esto, ut prius, quadratorum series HK; inscripta triangulo AGK; sitque AQ aequalis seriei basium imparium AB, CD, &c.

Dico rectangulum sub AQ, & lineis GA, AB, tamquam una contentum, rectangulo GAK aequale esse.

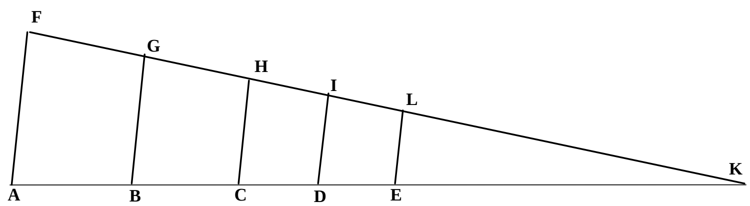
***Demonstratio.***

Dividatur enim LG bifariam in M: igitur linea composita ex GA, AB, bis continet lineas LM, LA. quare composita ex GA, AB dupla est linea AM; ergo rectangulum sub AQ & composita ex GA, AB, duplum est [c] rectanguli sub AQ & AM. Atqui rectangulum sub AQ & AM, aequatur [f] triangulo AGK: ergo rectangulum sub AQ & GA, AB tanquam una linea, duplum est trianguli AGK. Quare cum & rectangulum GAK, eiusdem trianguli GAK sit duplum, aequabuntur inter se, rectangulum sub AQ & composita ex GA,AB, & rectangulum GAK. Quod erat demonstrandum. *e 1 sexti; f 139 huius .*

***Lemma.***

Latus AK trianguli AFK, sit divisum per lineas lateri AF parallelas , in continue proportionales AK, BK, C, etc.

Dico trapezia FB, GC, HD, &c. esse similia.




**Prop.140. Fig. 2.**

**[141]**

***Demonstratio.***

Ob linearum AF, BG, CH aequidistantiam, angulum FAB, angulo GBC; & GFA, angulo HGB, & BGF, angulo CHG, & ABG, angulo BCH, aequalis est. Deinde quia AK, BK, CK, &c. sunt continuae proportionales, erit AB ad BC, [a] & BC ad CD, ut AK ad BK. Cum igitur etiam FA sit ad GB, ut AK ad BK, erit FA ad GB, ut AB ad BC, & permutando FA ad AB, ut GB ad BC: similiter cum AB sit ad BC [b] ut AK ad BK, hoc est ex hypothesi, ut BK ad CK, hoc est ut BG ad HC; erit permutando AB ad BG, ut BC ad CH : non aliter etiam ostendamus BG esse ad GF, ut CH ad HG, & GF ad FA, ut HG ad GB. quare cum trapeziorum FB, GC, & anguli omnes sint aequales, & latera circa aequales angulos proportionalia. Trapezia [c] FB, GC & omnia reliqua, erunt similia. Quod erat demonstrandum. *a 1 huius; b ibid; c Def. sexti.*

## *L2.§3. PROPOSITION 140.*

As before, let HK be a series of squares inscribed in the triangle AGK; and let AQ be equal to the series of odd bases AB, CD, &c.

I say that the rectangle formed from AQ and the lines GA and AB taken as one line, is equal to the rectangle GAK.

***Demonstration.***

Since LG is bisected in M, it follows that the line formed from the sum of GA and AB is twice the sum of the lines LM, LA. Whereby the sum of the lines GA and AB is twice the line AM; hence the rectangle under AQ and the sum of GA and AB is twice the [c] rectangle under AQ and AM. But the rectangulum under AQ and AM is equal to [f] the triangle AGK : hence the rectangle under AQ and GA, AB as one line, is twice the triangle AGK. Since the rectangle GAK is twice the same triangle GAK, then the rectangle under AQ and the sum of GA and AB, is equal to the rectangle GAK. Q. e. d. *e 1 sexti; f 139 huius .*

[GA + AB = 2.(LM + LA) = 2.AM, since LM = MG;
hence rect.AQ.(GA + AB) = 2.rect.AQ.AM; but
AQ.AM = ΔAGK, by the previous proposition;
hence rect.AQ.(GA + AB) = 2.ΔAGK = rect.AGK, as required.]

***Lemma.***

The side AK of the triangle AFK is divided by lines parallel to the side AF in the continued proportionals AK, BK, C, etc.
I say that the trapeziums FB, GC, HD, &c. are similar.

***Demonstration.***

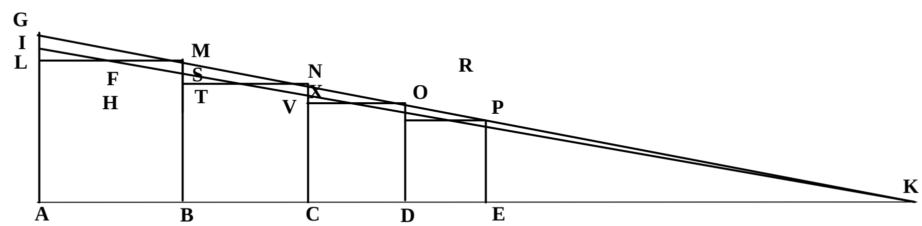
On account of the equidistant lines AF, BG, and CH, the angles FAB and GBC; GFA and HGB, BGF and CHG, ABG and BCH are equal. Hence since AK, BK, CK, etc. are continued proportionals, AB is to BC, [a] and BC to CD, as AK to BK. Therefore also as FA is to GB as AK is to BK, then FA is to GB as AB is to BC, and on interchanging, FA is to AB as GB is to BC. Similarly, since AB is to BC [b] as AK is to BK, that is by hypothesis, as BK is to CK, or as BG is to HC, then on interchanging, AB is to BG as BC is to CH : we can also show in the same way that BG is to GF as CH is to HG, and GF is to FA as HG is to GB. Whereby since the sides FB, GC of the trapeziums and the angles are all equal, and the sides around equal angles are in proportion. The trapeziums [c] FB, GC and all the others are similar. Q. e. d. *a 1 huius; b ibid; c Def. sexti.*

## *PROPOSITIO CXLI.*

Data sit quadratorum series HK; habens bases in directum, & terminum longitudinis K; inscripta triangulo AGK; ac primi quadrati latere LM bisecto in F, per F ducatur recta FK, occurrens ipsi AG in I.

Dico triangulum AIK seriei quadratorum; triangulum vero IGK, seriei triangulorum LGM, TMN, &c. aequale esse.

***Demonstratio.***




**Prop.141. Fig. 1.**

Cum LM bisecta sit in F, & SM, IL, ex hypothesi sint parallelae, patet triangula IFL, SFM aequalia esse, ac proinde addito communi ALFSB, trapezium AISB quadrato AM aequale. Deinde cum per corollarium propositionis 125. huius, progressionis basium AB, BC terminus sit K, erunt [d] AK, BK, &c. continuae: quare cum etiam AI, BS, CX, &c. sint parallelae, erunt per lemma, trapezia IB, SC, XD, &c. similia inter se. unde & [e] trapezia IB, SC, XD, &c., sunt in duplicata ratione laterum homologorum AB, BC, CD, &c. atqui & quadrata AM, BN, CO, &c. sunt in dictorum laterum duplicata ratione, ergo trapezia sunt quadratis proportionalia, & ut primum [f] trapezium IB, ad primum quadratum, ita tota trapeziorum series, ad quadratorum seriem. Atqui primum trapezium, ostendimus primo quadrato aequale esse, ergo trapeziorum etiam & quadratorum series aequales sunt. Deinde series trapeziorum [g] GB, MC, ND, constituit triangulum AGK, ergo & series trapeziorum IB, SC, triangulo AIK aequalis erit; Quare triangulum AIK, quadratorum seriei aequabitur: Quod erat primum; ex quo etiam patet secundum.
Quod erat demonstrandum. *d 82 huius*; *e 20 sexti; f 12 sexti; g 138 huius .*

## *L2.§3.* *PROPOSITION 141.*

A series of squares HK is given; having bases arranged on a line in a progression as usual, and the terminus of the series at a length K. The squares are inscribed in the triangle AGK; and with the side of the first square bisected in F, through F is drawn the line FK, crossing AG in I.

I say that the triangle AIK of the series of squares, and triangle IGK formed from the series of triangles LGM, TMN, &c. are equal.

***Demonstration.***

Since LM is bisected in F, and SM, IL, from hypothesis are parallel, it is apparent that the triangles IFL and SFM are equal, and hence by adding the common area ALFSB, the trapezium AISB is equal to the square AM. Following the corollary of proposition125, since the terminus of the progression of bases AB, BC, etc. is K, then [d] AK, BK, &c. are in continued proportion. Whereby since AI, BS, CX, &c. are also parallel, the trapeziums IB, SC, XD, &c are similar to each other according to the lemma. Hence the [e] trapezia IB, SC, XD, &c., are in the square ratio of the homologous sides AB, BC, CD, &c. But the squares AM, BN, CO, &c. are in the square ratio of the said sides, and hence the trapezia are proportional to the squares. As the first [f] trapezium IB is to the first square, thus the whole sum of the series of the trapezium series is to the sum of the series of squares. But the first trapezium has been shown to be equal to the first square, hence the sum of the series of trapeziums is also equal to the sum of the squares. Then the series of trapeziums [g] GB, MC, ND, constitute the triangle AGK, and hence the series trapeziums IB, SC, is equal to the triangle AIK. Whereby the triangle AIK is equal to the series of squares. Which demonstrates the first part, from which the second part follows. Q.e.d. *d 82 huius*; *e 20 sexti; f 12 sexti; g 138 huius .*

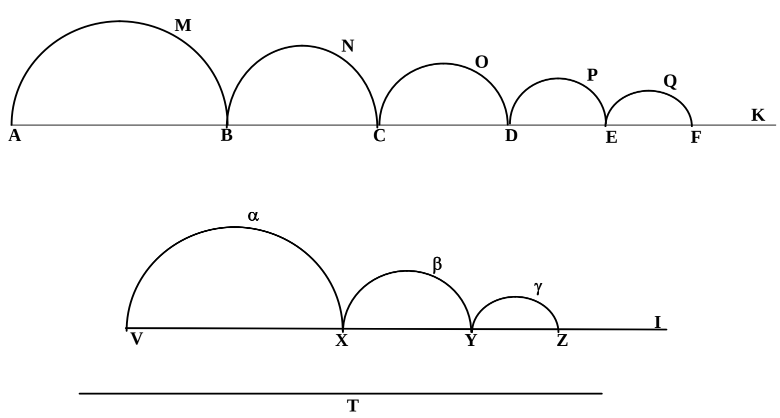
## *PROPOSITIO CXLII.*

Data sit planorum similium series habens bases homologas in directum, & terminum longitudinis punctum K; sitque seriei rationis AB primae, ad tertiam CD, aequalis linea VI: linea vero T seriei rationis AB primae, ad EF quintam sit aequalis;

**[142]**

Dico seriem totam planorum AM, BN, CO, DP, &c. esse ad seriem planorum imparium AM, CO, EQ, &c. ut linea VI est ad lineam T.

***Demonstratio.***




**Prop.142. Fig. 1.**

Series basium imparium AB, CD, EF, etc. constituatur separatim, ut VX ipsi AB, & XY ipsi CD, & YZ ipsi EF sit aequalis : itemque plana super his facta planis AM, CO, EQ, aequalia sint, & similia. patet igitur series rationis AB ad EF, & rationis VX ad YZ, eadem esse. quare cum T aequalis sit seriei AB, EF, etiam seriei VX, YZ, aequalis erit. Deinde series [a] planorum AM, BN, CO, est ad planum AM, ut linea VI ad AB : & series planorum Vα, Xβ, &c. est ad planum VX, id est ad planum AM, ut T ad VX, id est AB: Atqui ut VI ad AB, sic rectangulum sub VI & AB, ad quadratum AB: & ut T ad AB, sic rectangulum sub T & AB, ad quadratum AB: ergo series planorum AM, BN, &c. est ad planum AM ut rectangulum sub VI & AB, ad quadratum AB: & series planorum Vα, Xβ, &c. est ad planum Vα, hoc est AM, ut rectangulum TAB ad quadratum AM. Igitur permutando series AM, BN, est ad rectangulum sub VI, AB; ut planum AM, ad quadratum AB: itemque permutando series Vα, Xβ est ad rectangulum TAB, ut idem planum AM ad idem quadratum AB. ergo series AM, BN est ad rectangulum sub VIAB ut series Vα Xβ ad rectangulum TAB: & permutando series AM, BN, est ad seriem Vα, Xβ, id est ex constructione ad seriem AM, CO, EQ, ut rectangulum sub VI AB ad rectangulum TAB, hoc est ut linea VI ad lineam T: Quod erat demonstrandum. *a 126 huius .*

## *L2.§3.* *PROPOSITION 142.*

A series of similar plane figures is given, having homologous bases arranged on a line [in a progression as usual], and the terminus point of the series at a distance K. The line VI is equal to sum of the series of the first term AB to the third term CD, etc., and the line T is equal to the sum of the series of the first term AB to the fifth term EF, etc.

I say that the sum of the whole series of plane figures AM, BN, CO, DP, &c. is to the sum of the series of odd plane figures AM, CO, EQ, &c., as the line VI is to the line T.

### *Demonstration.*

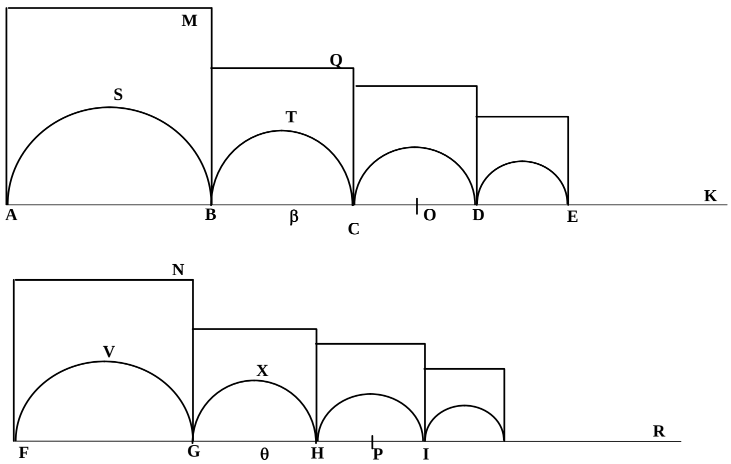
The series of odd bases can be set up separately, so that VX is equal to AB, XY is equal to CD, and YZ is equal to EF. In the same manner, the plane figures constructed on these lines are equal to AM, CO, EQ, and similar to each other. It is therefore apparent that the series formed from the ratios AB to EF and VX to YZ are the same. Whereby, since T is equal to the sum of the series AB to EF, it is also equal to the sum of the series VX toYZ. So the series [a] of plane figures AM, BN, CO, is to the plane figure AM, as the line VI is to AB : and the series of plane figures Vα, Xβ, &c. is to the plane figure Vα, (or to the plane figure AM), as T to VX, (or AB) [original text has Vα and VX interchanged]. But as VI is to AB, thus the rectangle VI.AB is to the square AB: and as T is to AB, thus the rectangle T.AB is to the square AB: hence the series of plane figures AM, BN, &c. is to the plane figure AM as the rectangle VI.AB is to the square AB: and the series of plane figures Vα, Xβ, &c. is to the plane figure Vα, or AM, as the rectangle T.AB is to the square AB. Therefore on interchanging, the series AM, BN, is to the rectangleVI.AB, as the plane figure AM is to the square AB. Likewise, on interchanging, the series Vα, Xβ is to the rectangle T.AB as the same plane figure AM to the same square AB. Hence, the series AM, BN is to the rectangle VI.AB as the series Vα, Xβ is to the rectangle T.AB: and on interchanging, the series AM, BN is to the series Vα, Xβ, (or from the construction to the series AM, CO, EQ), as the rectangle VI.AB is to the rectangle T.AB, or as the line VI is to the line T: Q.e.d. *a 126 huius .*

[From previously, S(all areas on AK)/AM = $1/(1 - r^2)$ = S(odd terms of AK)/AB = VI/AB; in the same manner, S(areas on line VI)/(plane figure Vα) = $1/(1 - r^4)$ = S(odd terms of VI )/VX = T/AB.
Following Gregorius: VI/AB = VI.AB/AB$^2$ ; and T/AB = T.AB/AB$^2$ .
Hence S(all areas on AK)/AM = VI/AB = rect.VI.AB/sq. AB
and S(areas on line VI)/(plane figure Vα) = T/AB = rect.T.AB/sq.AB.
Consequently, S(all areas on AK)/rect.VI.AB = AM/sq.AB ;
and likewise, S(areas on line VI)/rect.T.AB = (plane figure Vα or AM)/sq.AB.
Hence, S(all areas on AK)/rect.VI.AB = S(areas on line VI)/rect.T.AB, or
S(all areas on AK)/S(areas on line VI) = rect.VI.AB /rect.T.AB = VI/T as required.]

## *PROPOSITIO CXLIII.*

Datae sint quadratorum series binae, quae habeant bases in directum, & longitudinum terminos puncta K & R. sit autem AK divisa in O, in ratione AB ad BC; & FR divisa sit in P; secundum rationem FG ad GH.

Dico seriem quadratorum NK, ad seriem quadratorum NR, rationem habere compositam ex rationibus AB ad FG, & AO ad FP.



Prop.143. Fig. 1.

**[143]**

***Demonstratio.***

Series quadratorum MK, [a] aequatur rectangulo OAB, & series quadratorum NR, rectangulo PFG aequalis est. Atqui ratio rectangulorum OAB, PFG ex lateribus AB, FG, & AO, FP, rationibus componitur, ergo etiam serierum MK, NR, ex iisdem rationibus proportio componitur. Quod erat demonstratum. Sed hoc Theorema universale reddamus. *a 139 huius .*

## *L2.§3. PROPOSITION 143.*

Two series of squares are given, which have bases on a line, and end-points of the series are of lengths K & R from the start. Moreover, AK is divided by O in the ratio AB to BC; and FR is divided by P, in the second ratio FG to GH.

I say that the sums of the series of squares NK to the squares NR has the ratio of the sum of the ratios AB to FG and AO to FP.

***Demonstration.***

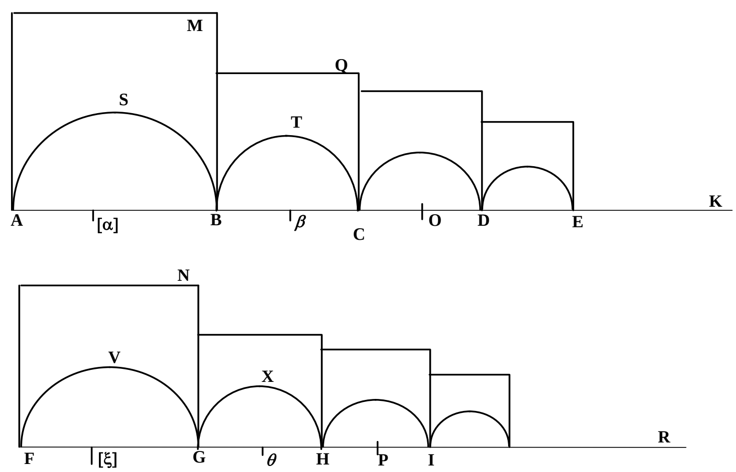
The series of squares MK [a] is equal to the rectangle OAB, and the series of squares NR is equal to the rectangle PFG . But the ratio of the rectangles OAB to PFG is composed of the ratios from the sides AB, FG, and AO, FP; hence also the ratio of the series MK, NR is composed from the same ratios. Q. e.d. But we can make this theorem general. *a 139 huius .*

## *PROPOSITIO CXLIV.*

Eadem manente figura, datae sint series binae planorum similium, quarum longitudines sint AK, FR : sit autem AK divisa in O, in ratione AB ad BC, & FR in P, in ratione FG ad GH.

Dico seriem planorum MK, ad seriem planorum similium NR, habere rationem compositam ex rationibus AB ad FG, & AO ad FP.

***Demonstratio.***




**Prop.144. Fig. 1.**

Super iidem basibus fiant binae quadratorum series, & lineis CD, HI, aequales fiant $\alpha$B, $\xi$G: deinde ut A$\alpha$ ad AB, & F$\xi$ ad FG fiat quadratum AB, ad A$\beta$ quadratum, & quadratum FG ad quadratum Fθ. itaque quadratum super A$\beta$ aequalitur [b] seriei quadratorum MK: planum vero super A$\beta$ simile planis AS, BT, FV, GX, &c. aequabitur seriei planorum SK : similiter ab altera parte quadratum super F$\theta$ seriei quadratorum NR, & planum simile super eadem F$\theta$, planorum seriei VR aequalia erunt. Itaque series planorum SK est ad seriem planorum VR, ut planum super A$\beta$, ad planum F$\theta$: Item series quadratorum MK est ad seriem quadratorum NR, ut quadratum super A$\beta$, ad quadratum F$\theta$; (cum enim tam quadrata, quam plana ex constructione sint similia, utraque sunt in duplicata rationis A$\beta$ ad F$\theta$: ) ergo series planorum SK est ad seriem planorum VR, ut series quadratorum MK ad seriem quadratorum NR. Atqui per praecedentem, series quadratorum MK ad seriem NR, rationem habet compositam, ex rationibus AB ad FG, & AO ad FP, ergo & planorum series SK ad seriem planorum VR, rationem habet ex iisdem rationibus compositam: quod erat demonstrandum. *b 130 huius .*

## *L2.§3.* *PROPOSITION 144.*

From the same figure as above, two series of similar plane figures are given, of which the lengths [or total sums] are AK, FR : while AK is divided by O in the ratio AB to BC, and FR in P in the ratio FG to GH.

I say that the series of plane figures MK to the series of similar plane figures NR has a ratio composed from the ratios AB to FG and AO to FP.

***Demonstration.***

Two series of squares are made upon the same bases, and with the lines CD, HI made equal to αB, $\xi$G: Then, as A$\alpha$ is to AB, and F$\xi$ is to FG, the square AB is made to the square A$\beta$, and the square FG to the square F$\theta$. Hence the square upon Aβ is equal [b] to the series of squares MK: truly the square upon A$\beta$ with the similar plane figures AS, BT, FV, GX, etc. is equal to the series of plane figures SK : similarly from the other part, the square upon F$\theta$ of the series of squares NR, and the similar plane figure upon the same F$\theta$, are equal to the series of plane figures VR. Thus the series of plane figures SK is to the series of plane figuresVR, as the plane figure upon A$\beta$ is to the plane figure F$\theta$: Likewise, the series of squares MK is to the series of squares NR, as the square upon A$\beta$ to the square upon F$\theta$; ( indeed as for for the squares, so the plane figures are similar from construction , and both are in the square ratio of A$\beta$ to F$\theta$: ); hence the series of plane figures SK is to the series of plane figures VR, as the series of squares MK is to the series of squares NR. But according to the preceding theorem, the series of squares MK to the series NR has the ratio composed from the ratios AB to FG and AO to FP. Hence the series of plane figures SK to the series of plane figuresVR has a ratio composed from the same ratios: q.e.d. *b 130 huius* .

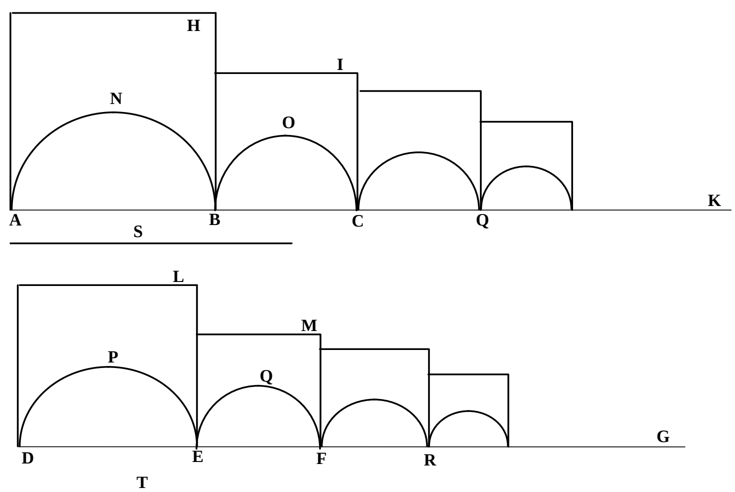
## PROPOSITIO CXLV.

Datae sint quadratorum series HK, LG, in quibusvis rationibus, ab aequalibus quadratis incipientes: sitque linea S seriei rationis primae AB ad tertiam CQ, itemque linea T, seriei rationis DE primae ad tertiam FR aequalis.

Dico series quadratorum eam proportionem habere quam lineae, S, T.

**[144]**

***Demonstratio.***




**Prop.145. Fig. 1.**

Series HK [a] aequatur rectangulo SAB, item series LG aequatur rectangulo TDE, hoc est, (quoniam quadrata AH, DL, ideoque & lineae AB, DE, ex hypothesi aequantur) rectangulo TAB. eadem igitur est serierum, & SAB, TAB rectangulorum ratio. Atqui rectangulorum SAB, TAB, eadem est ratio [b] quae rectarum S, T, ergo & serierum eadem quae S, T, linearum est proportio. Quod erat demonstrandum. *a 128 huius; b 1 sexti.*

## *L2.§3.* PROPOSITION 145.

The series of squares HK and LG are each given in any ratio, starting from equal squares : the line S is set equal to the ratio AB to CQ of the first series, and likewise the line T is equal to the ratio of the first DE to the third FR of the second series.

I say that the series of of squares have the same proportion as the lines S and T.

***Demonstration.***

The series HK [a] is equal to the rectangule SAB, and likewise the series LG is equal to the rectangule TDE, or (since the squares AH and DL, and thus the lines AB and DE are equal by hypothesis) to the rectangule TAB. Therefore the ratio of the series is the same as the ratio of the rectangles SAB and TAB. But the ratio of the rectangles SAB and TAB is the same as the ratio of the lines [b] S, T. Hence the proportion of the series is the same as of the lines S and T. Q.e.d. *a 128 huius; b 1 sexti.*

[For S(squares AH)/S(squares LG) = AH/sum(odd terms DE, FR, ...) × sum(odd terms AB, CQ, ...) /DL = AB.S/DL.T = S/T.]

## *PROPOSITIO CXLVI.*

Eadem manente figura; dentur planorum similium binae series quarumvis proportionum NK, PG, quae ab aequalibus incipiant planis, sintque lineae S, T, aequales seriebus AB, CQ, &c. AB, FR, &c.

Dico eandem esse serierum & rectarum S, T, proportionem.

### *Demonstratio.*

Ad huius Theorematis demonstrationem, eandem lector constructionem ac ratiocinationem si adhibeat, qua propositione praecedenti fuimus vsi, non aliter proportionis praesentis veritatem ex praecedenti deducet, quam propositionis 144, ex 143 huius deduximus.

## *L2.§3.* *PROPOSITION 146.*

With the same figure kept; two series of any similar plane figures whatever are given in the proportion NK to PG, and for which with equal initial figures, the lines S and T are equal to the sums of the series AB, CQ, etc., and AB, FR, etc.

I say that the proportion of the series and the lines S and T is the same.

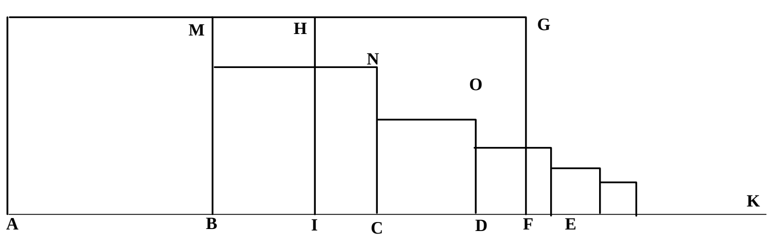
### *Demonstration.*

The reader can use the same construction and reasoning for the demonstration of this theorem as for the previous proposition, and in the same manner the truth of the present proposition can be deduced from the previous theorem, as we deduced propositon 144 from proposition 143.

## *PROPOSITIO CXLVII.*

Esto rectangulum altera parte longius AG, a quo AM quadratum ablatum sit. Petitur exhiberi series quadratorum, quae aequalis sit rectangulo AG, & incipiat a quadrato AM.

### *Constructio & demonstratio.*




**Prop.147. Fig. 1.**

Ut AF ad BF, sic AB fiat ad BI : & ex I ducta IH, parallela ad BM rectangulo IBMH, aequale fac quadratum BN, basi BC in directum posita cum AB : cum progressionis quadratorum AM, BN, [c] continuatae inveniatur K terminus longitudinins. Dico seriem quadratorum MK, problemati satisfacere.

**[145]**

Nam cum ex constructione AB sit ad BI, ut AF ad BF; erit AF aequalis [a] seriei rationis AB ad BI. Deinde ex constructione rectangulum BH aequatur quadrato BN, ergo quadratum AM ad rectangulum BH, & quadratum BN, eadem habet rationem; unde cum quadratum AM, sit ad quadratum BN ut AB ad CD, erit quoque quadratum AM ad rectangulum BH, ut AB ad CD. atqui ut quadratum AM ad rectangulum BH, sic AB ad BI : ergo AB est ad CD, ut AB ad BI : aequantur igitur CD, BI, ergo AF etiam aequalis est seriei rationis AB ad CD : quare rectangulum FAB, id est rectangulum AG [b] aequale est seriei quadratorum AM, BN, CO, &c. hoc est seriei MK. factum igitur est quod postulabatur.

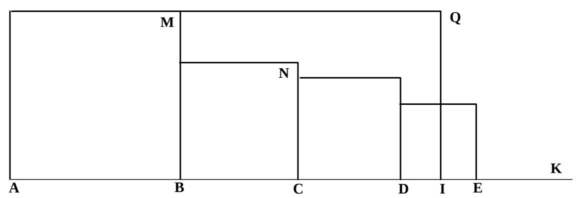
*c 125 huius; a 79 huius; b 128 huius.*

***Demonstratio alia.***

Tota series quadratorum MK est per 82 huius ad reliquam seriam seriem NK, ut quadratum AM ad quadratum BN, hoc est ex const. ad rectangulum BH. Atqui ex constructionis, rectangulum AG est ad rectangulum BG, ut rectangulum AM ad rectangulum BH, ergo series MK est ad seriem NK, ut AG ad BG. Igitur dividendo quadratum AM, est ad reliquam seriem NK, ut quadratum idem AM ad reliquum rectangulum BG: ergo series NK, & rectangulum BG aequantur. Quare communi addito quadrato AM, tota series MK, & rectangulum AG sunt aequalia.

***Corollarium.***

Esto linea AI secta in B. Petitur addi IK, ut AI sit ad IK, ut AK ad BK.



**Prop.147. Fig. 2.**

***Constructio & demonstratio.***

Super AI fiat in altitudine AB rectangulum AG, cui aequalis [c] inveniatur series quadratorum basibus in directum positis MNK; incipiens a quadrato AB, sive AM, & terminum habens longitudinis K: Dico factum quod petebatur. Cum enim rectangulum AG ex constructione sit aequale seriei MK, AI erit ad IK, ut AB ad BC, uti ex 129 huius facile demonstrari potest: & quia terminus longitudinis seriei quadratorum est K, progressionis etiam basium terminus [d] erit K; ergo AK est ad [e] BK, ut AB ad BC, hoc est per demonstrata ut AI ad IK. Factum igitur est quod petebatur.

*c 147 huius; d 125 huius; e 82 huius.*

## *L2.§3.* *PROPOSITION 147.*

Let there be a rectangle with the longer side AG, from which the square AM is taken away. It is required to demonstrate a series of squares which is equal to the rectangle AG, and which starts from from the square AM.

***Construction and Demonstration.***

As AF is to BF, thus AB is made to BI: and from I the line IH is drawn parallel to BM; make the square BN equal to the rectangle IBMH, with base BC placed on the line with AB : since the terminus of the continued progression of squares AM, BN, [c] can be found at a distance K. I say that the series of squares MK solves the problem.

For since from the construction AB is to BI, as AF is to BF; AF is equal to the sum of the series of ratios [a] AB to BI. Hence from the construction, the rectangle BH is equal to the square BN; hence the rectangle BH and the square BN have the same ratio to the square AM. Hence since the square AM is to the square BN as AB to CD, also the square AM is to the rectangle BH, as AB to CD. But as the square AM to the rectangle BH, thus AB to BI : hence AB is to CD, as AB to BI : therefore CD is equal to BI. Hence AF is equal to the series of ratios AB to CD also: whereby the rectangle FAB, or the rectangulum AG [b] is equal to the series of squares AM, BN, CO, &c. or to the series MK. Thus what was postulated has been done.

*c 125 huius; a 79 huius; b 128 huius.*

[AF/BF is the ratio of the lengths of sides of consecutive squares, of which F is the limit point. The ratio of consecutive squares is also constructed to be sq.AM/rect.BH = AB/BI = AB/CD ; from which BI = CD. Thus, BI is the same as the term CD in the series of odd lengths AB, CD, FE, etc. that sum to AF. Hence, starting from the square AM, consecutive rectangles of the form BH can be taken from the rect. AG, each of which is equal to a corresponding square such as BN, and in the limit, the sum of the squares is equal to the area of the rectangle AG.]

***Another demonstration.***

The sum of the series of square MK is to the sum of the rest of the series of squares NK by Prop. 82 as the square AM is to the square BN, (or by construction) as the rectangle BH. But from the construction, rect.AG is to rect.BG as rect.AM is to rect.BH, and hence the series MK is to the series NK as rect. AG is to rect.BG. Hence on subtraction, the square AM is to the remainder of the series NK, as likewise the square AM is to the remaining rect. BG: hence the series NK is equal to the rectangle BG. Whereby by adding the common square AM, the sum of the series MK and the rect. AG are equal.

***Corollarium.***

Let the line I be cut in B. The line IK is sought to be added in order that AI is to IK, as AK is to BK.

***Construction & demonstration.***

Upon AI the rectangle AG is conctructed with altitude AB, on which an equal series of squares with bases on the line MNK are placed, beginning with the square AB, or AM, and having the terminus point K at a distance K. I say that what was required has been effected. For since the rectangle AG by construction is equal to the series MK, AI is to IK, as AB to BC, as can be easily found from from Prop. 129: and because the terminus of the series of squares is at a length K, also the terminus of the progression of bases is [d] K; hence AK is to [e] BK, as AB is to BC, or as has been shown, as AI to IK. Thus what was sought has been accomplished. *c 147 huius; d 125 huius; e 82 huius.*

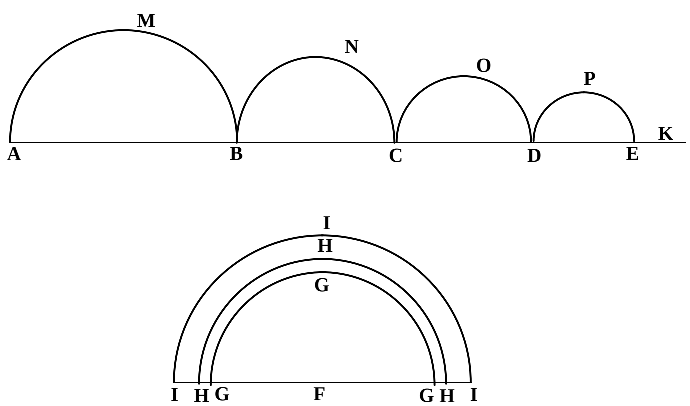
## *PROPOSITIO CXLVIII.*

Esto figura plana quaecunque FI, a qua similis auferatur FG, petitur exhiberi series planorum similium, quae incipiat a plano ablato FG, & dato plano FI sit aequalis.

**[146]**

***Constructio & demonstratio.***

Ut planum FI ad segmentum GI, sic planum FG, fiat ad segmentum aliquod GH, deinde plano FG fac simile & aequale AM, & segmento GH aequale sit planum BN, simile autem planis AM, FG, FI. Deindeque progressionis planorum AM, BN, continuarae [a] inveniatur terminus longitudinis K. Dico seriem planorum similium MK postulato satisfacere. est enim tota series planorum MK, ad reliquam seriem NK, ut [b] planum AM ad planum BN, hoc est ex constructione, ut planum FG ad segmentum GH; sed rursum ex constructione planum FI est ad segmentum GI, ut planum FG ad segmentum GH, ergo series MK, est ad reliquam NK, ut planum FI ad segmentum GI. Quare per conversionem rationis series MK, est ad planum AM, ut planum FI ad planum FG: sed ex constructione plana AM, FG aequalia sunt. Quod erat faciendum . *a 125 huius; b 82 huius.*



**Prop.148. Fig. 1.**

## *L2.§3. PROPOSITION 148.*

Let FI be some plane figure, from which a similar figure FG is taken, it is required to exhibit a series of similar plane figures, which begin with the plane figure FG, and the sum of which is equal to the given plane figure FI.

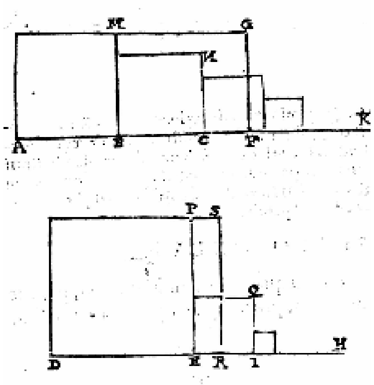
***Construction & demonstration.***

As the plane figure FI is to the annulus GI, thus the plane figure FG is made to some annulus GH [the text refers to GH as a segment, but this word now given another meaning]; then make the plane FG similar and equal to AM, in which case the plane figure BN in turn becomes equal to the annulus GH, and moreover similar to the figures AM, FG, and FI. Hence the terminus K of the continued progression of figures AM, BN [a] can be found. I say that the series of similar figures MK satisfy the postulate. For indeed the sum of the series of the figures MK is in the same ratio to the rest of the series NK, as [b] the figure AM is to the figure BN, or from the construction, as the figure FG is to the annulus GH; but again from the construction the figure FI is to the segment GI, as figure FG is to the segment GH, hence the series MK, is to the remainder of the series NK, as the figure FI is to the annulus GI. Whereby from the conversion of the ratio, the series MK is to the figure AM, as the figure FI is to the figure FG: but from the construction the figures AM and FG are equal. Which was to be established. *a 125 huius; b 82 huius.*

## *PROPOSITIO CXLIX.*

Data sit progressio quadratorum MK, & quadratum aliud DP, quod minus esse debet serie MK. Petitur exhiberi alia series quadratorum, incipiens a quadrato DP, aequalis seriei MK.

**Prop. 149, Fig. 1**




***Constructio & demonstratio.***

Fiat rectangulum AG in altitudine AB, aequale seriei datae MK[c]. Deinde, (quoniam quadratum DP ponitur minus serie MK, id est ex constructione rectangulo AG,) auge quadratum DP, rectangulo ES, ut rectangulum totum DS, aequale sit rectangulo AG : tum rectangulo DS [d], inveniatur aequalis series quadratorum PQH, incipiens a quadrato DP. Dico seriem PQH solvere Problem. Nam ex constr. Series quadratorum PQH incipit a quadrato dato DP, & aequalis est rectangulo DS, hoc est ex constructione rectangulo AG, hoc est rursum ex constructione seriei datae MK. Factum igitur est quod petebatur.
*c 129 huius*; *d 147 huius*.

## *L2.§3. PROPOSITION 149.*

For a given progression of squares MK, & some other square DP, which should be less than the series MK. It is desired to show another series of squares, starting from the square DP, equal to the series MK.

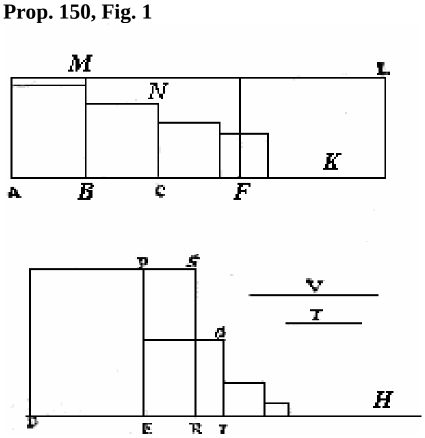
***Construction & demonstratin.***

Let the rectangle AG be made equal in height to AB, equal [in area] to the given series MK[c]. Then, (since the square DP is put less than the series MK, that is from the construction to the rectangle AG,) increase the size of the square DP by the rectangle ES, in order that the total DS is equal to the rectangle AG : for then the rectangle DS [d] is found equal to the series of squares PQH, beginning from DP. I say that the series PQH solves the problem. For from the construction, the series of squares PQH that begins from the given square DP, and is equal to the rectangle DS, that is from the construction equal to the rectangle AG, again this is from construction equal to the given series MK. Therefore what was sought has been accomplished. *c 129 huius*; *d 147 huius*.

**[p. 147]**

## *PROPOSITIO CL.*

**Prop. 150, Fig. 1**




Data sit iterum quadratorum series MK, & aliud quadratum DP, item ratio quaevis sive maioris sive minoris inaequalitatis V ad T, serie MK. Petitur exhiberi series quadratorum incipiens a quadrato DP, & habens ad seriem MK, rationem datam V ad T.

***Constructio & demonstratio.***

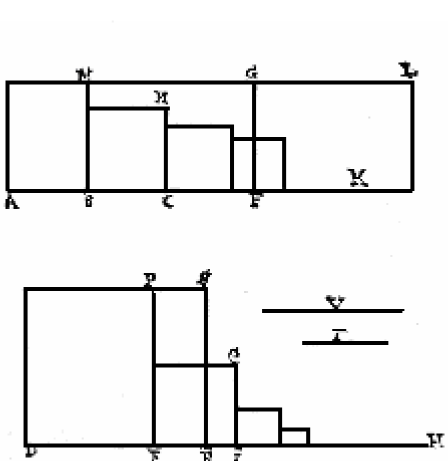
Fiat [a] rectangulum AG aequale seriei MK, & in eadem altitudine rectangulum AL, quod ad rectangulum AG, datam habeat rationem; si iam quadratum DP maius sit aut aequale rectangulo AL, impossibile est problem. Minus ergo sit oportet quadratum DP rectangulo AL. Itaque augeatur rectangulo ES, ita ut totum DS [b] rectangulo AL aequale sit. Tum rectangulo DS inveniatur aequalis series, quadratorum PQH incipiens a quadrato DP. Dico hanc solvere problema.

Ex constructione enim series PQH incipit a quadrato DP, & aequalis est rectangulo DS, id est ex constructione rectangulo AL; quare cum rectangulum AL ex constructione sit ad rectangulum AG, ut V ad T, etiam series PQH erit ad rectangulum AG, id est rursum ex constructione ad seriem datam MK, ut V ad T. Fecimus ergo quod petebatur. Nunc vero utramque propositionem praecedentem universalem faciamus. *a 129 huius*; *b 147 huius*.

## *L2.§3.* *PROPOSITION 150.*

The series of squares MK is given again, and another square DP likewise with some other unequal ratio V to T, which is either greater or less than that for the original series. It is desired to find the series of squares beginning from the square DP, & and having the given ratio to the series V to T.

**Prop. 150, Fig. 1**



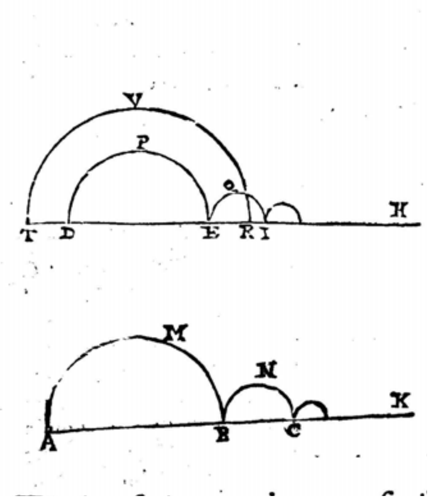
***Construction & demonstration.***

The rectangle AG is made equal to [the sum of] the series MK[a], & the rectangle AL is made with the same height, which has the given ratio to the rectangle AG ; now if the square DP is greater or equal to the rectangle AL, then the problem cannot be solved. Hence, it is necessary that the square DP is less than the rectangle AL. Hence it is augmented by the rectangle ES, thus in order that the rect. AL is equal to the rect. DS [b]. Then the rect. DS is found to be equal to the series of the squares PQH, beginning from the square PQ. I say that the problem is solved.

Indeed from the construction, the series PQH begins with the square DP, & is equal to the rectangle DS, that is by construction equal to the rectangle AL; whereby since the rectangle AL, from the construction, is to the rectangle AG, as V to T, and then also the series PQH is to the rectangle AG, (that is again by construction to the given series MK), as V to T. We therefore have constructed that which was desired. Now truly we can make general each of the preceding propositons [by considering other plane shapes].
*a 129 huius*; *b 147 huius*.

## *PROPOSITIO CLI.*

Data sit planorum similium progressio ut prius disposita MK, & aliud planum DP simile planis seriei MK : Petitur exhiberi similium planorum series, incipiens a dato plano DP, aequalis vero seriei datae MK.




**Prop. 151, Figure 1.**

***Constructio & demonstratio.***

Planis [a] seriei MK aequale ac simile fiat planum TVR, si iam planum datum DP, aequale aut maius sit plano TVR, fieri problema non poterit : minus ergo sit necesse est. Itaque inveniatur series [b] planorum similium PQH, quae aequalis sit plano TVR, & incipiat a plano dato DP. Dico hanc solvere problema.

Nam ex constructione, series PQH incipit a dato plano DP, & aequalis est plano TVR, id est ex constructione seriei datae MK; Factum igitur est quod postulabatur. *a 130 huius*; *b 148 huius*.

## *L2.§3.* *PROPOSITION 151.*

A progression of similar plane shapes MK is given in order as above, and another plane figure DP similar to the series of plane figures MK is given: It is desired to exhibit a series of plane figures, starting from the given plane figure DP, that is truly equal to the given series MK. [i.e. which has the same sum.]

***Construction & demonstration.***

The plane figure [a] TVR is made equal [to the sum of]and is similar to the series of plane figures MK, now if the given plane figure DP is equal to or greater than the plane figure TVR, then the problem cannot be solved : therefore it must be less. Thus the series of similar plane figures [b] PQH can be found, which is equal to the plane figure TVR, and which starts from the given plane figure DP. I say that this has solved the problem.

Now by the construction, the series PQH starts from the given plane figure DP, and [the sum of the series] is equal to the plane figure TVR, that follows from the construction of the given series MK ; therefore what was demanded has been done. *a 130 huius*; *b 148 huius*.

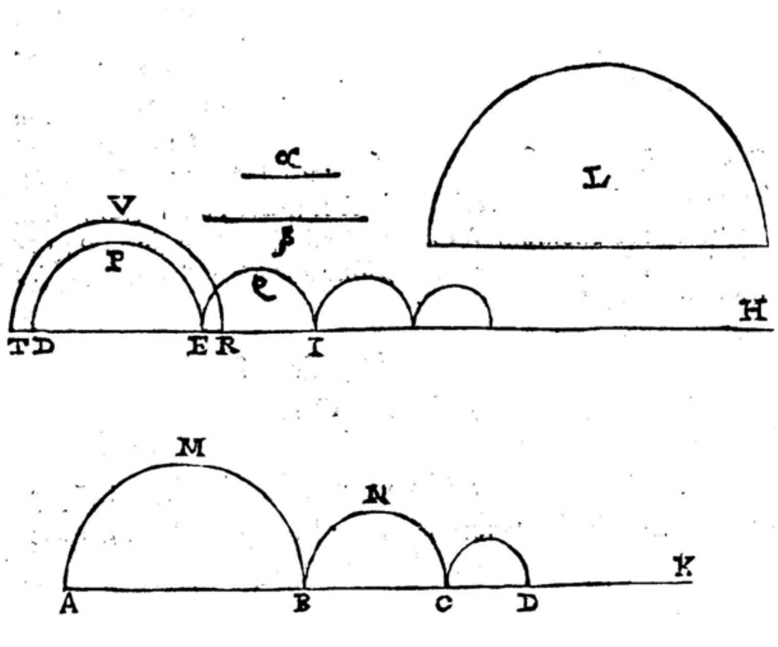
## *PROPOSITIO CLII.*

Data sit iterum planorum similium series MK, & aliud planum DP, simile datae seriei planis itemque ratio quaevis, sive maioris, sive minoris inaequalitatis $\alpha$ ad $\beta$. Petitur exhiberi series planorum similium, quae incipiat a plano DP, & ad seriem MK, datam habeat ratione.

***Constructio & demonstratio.***

Fiat planis seriei MK [c] aequale planum L, deinde ut $\alpha$ ad $\beta$, sic fiat planum simile TVR ad planum L, si iam planum datum DP, aequale vel maius est problema non poterit. Minus ergo planun DP sit opererteт, plano TVR : Itaque inveniatur [d] series planorum similium PQH, aequalis vero plano TVR. Dico hanc problema solvere, nam ex constructione series PQH, incipiat a plano dato DP, & aequalis est plano TVR, quare cum planum TVR ex constructione datam habeat rationem ad planum L, etiam series PQH ad planum L, hoc est rursum ex constructione ad seriem datam MK, habebit rationem datam : Factum igitur est quod petebatur.

*c 130 huius*; *d 148 huius*.




**Prop. 151, Figure 1.**

## *L2.§3.* *PROPOSITION 152.*

The series of similar plane figures is given again, and another plane figure DP, similar to the given series of plane figures, and likewise for some either greater or less unequal ratio $\alpha$ to $\beta$. It is desired to show the series of similar plane figures, which starts from the plane figure DP, and which has the given ratio to the series MK.

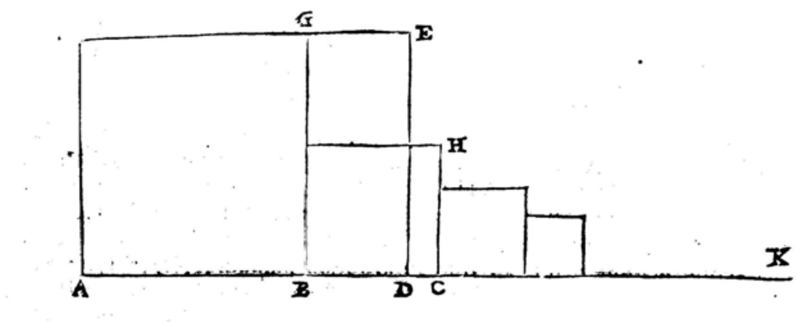
***Construction & demonstration.***

The plane figure L is constructed equal to the [sum of the] series of plane figures MK [c], thus the figure TVR is made in the ratio $\alpha$ to $\beta$ to the plane figure L; if the plane figure DP is now given equal or greater than this, then the problem cannot be solved. Hence it is necessary that the plane figure DP is smaller than the figure TVR : Thus a series [d] of similar plane figures PQH can be found, truly equal [in area] to the plane figure TVR. I say that this problem has been solved; for from the construction , the series PQH, beginning from the given plane figure DP, is equal to the plane figure TVR, and whereby, since the plane figure TVR, from the given construction, is in the given ratio to the plane figure L, then also the series PQH has the same ratio to the plane figure L, that is again from the construction, to the given series MK. Therefore, what was sought has been accomplished. *c 130 huius*; *d 148 huius*.

## *PROPOSITIO CLIII.*

Esto progressio quadratorum GHK, basibus in directum positis, & terminum longitudinis habens punctum K.

Dico quadratum super tota AK factum, ad seriem datam, proportionem habere compositam, ex ratione KA ad BA, & CA ad BA.




**Prop. 153, Figure 1.**

***Demonstratio.***

Fiat enim AD ad DK, ut AB ad BC. Igitur rectangulum DAB, sive AE, [a] aequatur seriei GK. Ergo quadratum AF, eandem ad seriem GK, & ad rectangulum AE habet rationem quoniam autem AD est ad DK, ut AB ad BC, erit invertendo ac componendo KA ad DA, ut CA ad BA. Ergo cum quadratum AF ad rectangulum AE, rationem habeat [b] compositam ex rationibus KA ad DE, & KA ad DA, habebit quoque quadratum AF, ad idem rectangulum, compositam ex rationibus KA ad DE, & CA ad BA : quare cum series GK ad rectangulum AE, aequalia sint, habebit quoque quadratum AF ad seriem GK rationem compositam ex rationibus KA ad DE, hoc est BA, & CA ad BA. Quod erat demonstrandum. Placet quoque theorema universaliter demonstrare.
*a 129 huius; b 23 sextius;*

## *L2.§3.* *PROPOSITION 153.*

Let GHK be a progression of squares with bases placed along a line, and having the end point of length K .

I say that the square constructed upon AK, for the given series, has the proportion composed from the ratios KA to BA, & CA to BA.

***Demonstration.***

For let AD to DK be made as AB to BC. Therefore the rectangle DAB, or AE, [a] is equal to the sum of the series of squares GK. Hence the square AB has the same ratio to the series GK and to the rectangle AE; moreover since AD is to DK as AB is to BC, by inverting and putting the ratios together KA is to DA, as CA is to BA. Hence since the square AB to the rectangle AE, has the ratio [b] composed from the ratios KA to DE, & KA to DA, also the square AB to the same rectangle composed from the ratios KA to DE, & CA to BA : whereby since the series GK is equal to the rectangulum AE, also the square AB to the series GK has a ratio composed from the ratios KA to DE, that is BA, & CA to BA. Q.e.d. It is also pleasing to demonstrate the theorem generally. *a 129 huius; b 23 sextius;*
[The reader can refer back to the explanations around Prop. 129, or note that in algebraic terms, if *r* is the common ratio of the 1-D geometric progression of which the first term is *a*,

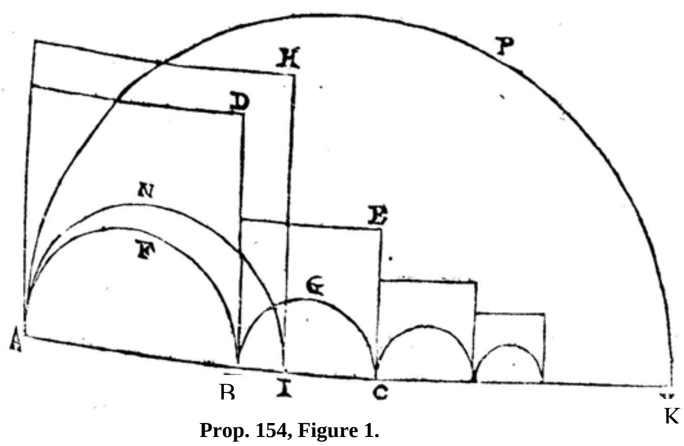
then the rectangle DAB $= \frac{a^2}{(1-r^2)}$, in which case the ratio AB$^2$ : sum of squares = $a^2$ : rect. DAB, from which the result follows. Thus, the sum for 2-D shapes in a geometric progression follows from that for 1-D line segments in a similar geometric progression.]

## *PROPOSITIO CLIV.*

Data sit planorum similium quorumcunque progression FGK, habens bases homologas in directum, & terminum longitudinis K, datae progressionis.

Dico planum APK ad totam seriem FGK habere rationem compositam, ex rationibus KA ad BA, & CA ad BA. [p. 150]

***Demonstratio.***




**Prop. 154, Figure 1.**

Super iisdem basibus AB, BC, &c. construatur quadratorum series DEK : fiatque quadratum AD ad aliud AHI, ut AC ad AK, quod toti [a] seriei quadratorum DEK aequabitur. Super AI vero fac planum ANI siile plano AF. Itaque planum AF est ad planum ANI [b] ut quadratum AD ad quadratum AHI, hoc est ex constructione ut AC ad AK. Quare etiam planum ANI, seriei planorum similium FGK [c] aequale erit; ergo series FGK est ad planum ANI, ut series DEK ad quadratum AHI : atqui planum ANI est ad planum APK, [d] ut quadratum AHI ad quadratum totius AK : igitur ex aequalitate series FGK est ad planum APK, ut quadratum AK ad seriem DEK, sed quadratum AK [e] ad seriem DEK, proportionem habet compositam ex rationibus KA ad BA, & CA ad BA. Ergo & planum APK ad seriem FGK, proportionem habet ex iisdem rationibus compositam; quod erat demonstrandum.

*a* 130 *huius*; *b* 22 *sexti; c* 130 *huius; d* 22 *sexti; e* 153 *huius*.

## *L2.§3.* *PROPOSITION 154.*

FGK is any progression of similar plane figures set out in order along a line, having homologous bases, and K is the terminal point of the given progression on the line.

I say that the ratio of the plane figure APK to the whole series FGK is composed from the ratios KA to BA & CA to BA.

***Demonstration.***

A series of squares DEK is constructed on the same bases AB, BC, &c. : and the square AD is made to another square in the ratio AD to AHI, which is as AC to AK, which is equal to the whole series of squares DEK [a]. Upon AI truly make the plane figure ANI similar to the plane figure AF. And thus the plane figure AF is to the plane figure ANI [b] as the square AD is to the square AHI, that is from the construction as AC

to AK. Whereby also the plane figure ANI, is equal to the series of plane figures FGK [c] ; hence the series FGK is to the plane figure ANI, as the series DEK is to the square AHI : but the plane figure ANI is to the plane figure APK, [d] as the square AHI is to the square of the whole length AK : therefore from the equality, the series FGK is to the plane figure APK, as the series DEK is to the square AK [this sentence has been inverted from the original], but the square AK [e] to the series DEK has the proportion composed from the ratios KA to BA, & CA to BA. Hence the plane figure APK to the series FGK, has a proportion composed from the same ratios; q.e.d.
*a* 130 *huius*; *b* 22 *sexti; c* 130 *huius; d* 22 *sexti; e* 153 *huius*.
**[**AD/AHI = AC/AK**; [**or $\frac{a^2}{rect.AHI} = \frac{a(1+r)}{a/(1-r)} = 1 - r^2$ ], hence rect. AHI [ $= \frac{a^2}{(1-r^2)}$ ], which is the sum of the squares DEK to infinity.
Again, fig.ANI/fig.AF = sq.AD/rect.AHI = AC/AK; whence fig.ANI is the sum of the figures FGK to infinity.
Hence (sum of plane figures FGK)/ (plane figure ANI) = (sum of squares DEK)/(rect.AHI), as these are identically equal; Again, (fig.ANI/fig. APK) = (rect.AHI)/($AK^2$), which amounts to (series FGK)/(fig. APK) = (series DEK)/( $AK^2$); but (series DEK)/( $AK^2$) [ $= \frac{a^2}{(1-r^2)} / \frac{a^2}{(1-r)^2} = \frac{1-r}{1+r}$ ] = BA/KA× BA/CA; we have not inverted the ratios as Gregorius has done.**]**

# *PROPOSITIO CLV.*

Esto quadratorum series habens bases in directum, & terminum logitudinis K, inscripta triangulo AGK, iuxta propositionem 131 huius, & completo rectangulo AI, latera quadratorum producantur in LS, &c. item in Q, O, R, &c.

Dico ex hac laterum productione perpetua, oriri progressionem rectangulorum MI, NL, &c. similium, & continue proportionalium, quae progressioni quadratorum quoque sit aequalis.

### *Demonstratio.*

Quoniam ex hypothesi K terminus est longitudinis quadratorum seriei, etiam K termins [f] erit progressionis basium AB, BC, &c. igitur BK [g] est ad CK, ut [p. 151] AB ad BC, hoc est ut PM ad HN, hoc est ut GM ad MN, hoc est (quia QGM, HMN similia sunt triangula) ut QM ad MH, hoc est denique ut QM ad ON : a primo igitur ad ultimum, BK est ad CK, hoc est ML ad NS, ut QM ad ON : rectangula igitur MI, NL, proportionalia habent latera. Quare cum sint & aequiangula, erunt similia; quod erat primum.

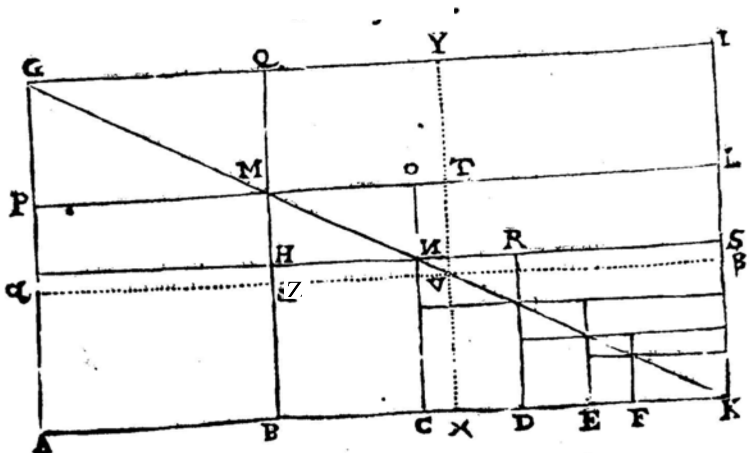
Deinde dicta rectangula complementa sunt eorum, que circa diametrum sunt. Ergo singula quadratis singulis datae seriei [a] aequantur. ergo & progressio tota toti progressioni aequalis erit; ex quo etiam secundum patet. Cum enim quadrata ex hypothesi sint in ratione continua, etiam complementa illis aequalia, in continua erunt analogia; quae erant demonstranda.

*f 125 huius; g 82 huius*; *a 43 primi.*

## *L2.§3.* *PROPOSITION 155.*

Let a series of squares having their bases ordered along a line and a terminal point of length K, be inscribed in the triangle AGK, as in proposition 131 of this section, and on completing the rectangle AI, the sides of the squares are produced in LS, &c. likewise in Q, O, R, &c.

I say that from this continued extension of the sides, a progression of similar rectangules MI, NL, &c. arises in proportion, which is also equal to the progression of squares.




**Prop. 155, Figure 1.**

Since by hypothesis, the terminal point K is the length of the series of squares, K is also the terminal point of the progression of base lengths AB, BC, &c. ; therefore BK [g] is to CK, as AB to BC, that is as PM to HN, that is as GM to MN, that is (since QGM and HMN are similar triangles) as QM to MH, and that is hence as QM to ON : therefore from the first to the last, BK is to CK, that is ML to NS, as QM to ON : therefore the rectangles MI and NL have sides in proportion. Whereby since the have equal angles, then they are similar figures; this is true for the first two rectangles.

Then the said rectangles are the complements of these which lie around the diagonal of the main rectangle. Hence the individual rectangles are equal to the given squares of the series [a] . Hence the whole progression of rectangles is equal to the whole progression of squares ; from which the following is also apparent. Since indeed the squares are in a continued ratio by hypothesis, also the complements of these are equally in an analogous continued ratio; which it was required to show. *f 125 huius; g 82 huius*; *a 43 primi.*
[Thus, from sim. triangles, PM/ML = PG/KL; or $PM^2 = \ldots$,etc.]

## *PROPOSITIO CLVI.*

Iisdem positis fiat AX b aequalis seriei rationis AB ad CD, & ex puncto X, ducta normalis XY secet lineas PL, GK, in T & V.

Dico rectangulum sub BKVY, toti complementorum seriei MI, NL, &c. aequari.

***Demonstratio.***

Ducatur enim AK per punctum V parallela αβ. Igitur rectangula [c] VI, AV, itemque rectangula MY, αM, aequalia sunt inter se. Igitur figura MQIβV figurae MPAXV aequalis est. quare communi addito rectangulo ZT, erunt rectangular ZI, AT aequalia. Atqui rectangulum ZI est rectangulum sub Zβ, id est BK, & sub ZQ, id est VY; rectangulum vero AT seriei quadratorum, [d] est aequale, ergo rectangulum sub BKVY, seriei quadratorum, hoc est per praecedentem seriei complementorum est aequale. Quod erat demonstrandum. *b 80 huius*; *c 43 primi; d 128 huius.*

## *L2.§3.* *PROPOSITION 156.*

With the same figure in place, AX [b] is made equal to the sum of the series of ratios AB to CD, and from the point X, the normal line XY cuts the lines PL and GK in T and V.

I say that the rectangle under BKVY is equal to the whole of the complementary series of rectangles MI, NL, &c.

***Demonstration.***

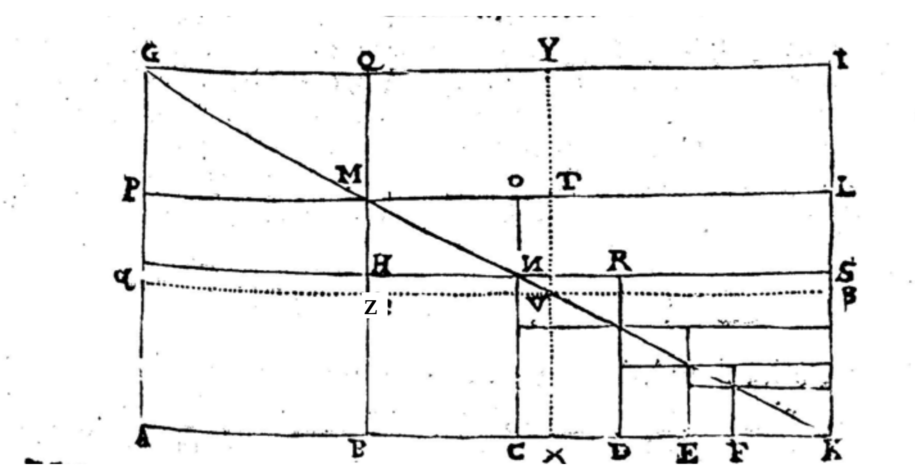
For $\alpha\beta$ is drawn through the point V parallel to AK. Therefore the rectangles [c] VI and AV, and likewise the rectangles MY and $\alpha$M are equal to each other amongst themselves [from similar triangles]. Therefore the figure MQI$\beta$V is equal to the figure MPAXV. Whereby on adding the common rectangle ZT, the rectangles ZI and AT are equal. But the rectangle ZI is the rectangle under Zβ, that is BK, and under ZQ, that is VY; truly the rectangle AT is equal to the series of squares [d]. Hence the rectangle under BKVY is equal to the series of squares, that is by the preceding, to the series of complements. Q.e.d. *b 80 huius*; *c 43 primi; d 128 huius.*

[It is a tedious but elementary process to show that both these expressions XV.BK and the sum of the odd squares are equal to $\frac{abr}{(1-r^2)(1-r)}$, where AB = $a$; PG = $b$, and $r$ is the common ratio for BC/AB, etc. Geometrically, it has been shown above in *d* that the rect. AT is the sum of the even squares, while here it is shown that the rect. AT = XV.BK. ]

## *PROPOSITIO CLVII.*

Iisdem positis, qua supra,

Dico ea quae circa diametrum sunt rectangula PQ, HO, & esse similia inter se & continue proportionalia; rectangulum vero sub AB & VY, toti eorum progressioni esse aequale.




Prop. 157, Fig. 1.

Ut PM est ad MO, sive HN, sic GP est ad ON, ob triangulorum GMP, MON, similitudinem : quare cum rectangula PQ, HO, & latera habeant proportionalia, & angulos aequales, erunt similia; ac proinde & reliqua omni eodem discursu similia erunt : quod fuit primum. Deinde cum similia sint dicta rectangula, erunt in duplicate homologorum laterum PM, HN, &c. ratione. Quare cum & quadrata sint in eorumdem laterum ratione duplicata, eadem erit rectangularum ac quadratorum proportio. Atqui haec ex hypothesi sunt in continua analogia, ergo & illa; quod erat alterum. Denique; rectangulum αM, aequatur [e] rectangulo MY. Addito [p. 152] igitur communi PQ, aequabitur rectangulum α Q, hoc est rectangulum sub AB & VY, rectangulo PY : Atque rectangulum PY, [a] duplum est progressionis triangulorum GPM, MHN, &c. ergo & rectangulum sub AB & VY, progressionis triangulorum est duplum : quare cum progressio rectangulorum PQ, HO, &c. etiam sit progressionis triangulorum dupla, rectangulum sub AB & VY, & progressio rectangulorum aequalia erunt : Quod postremum fuit eorum, quae erant demonstranda.

*e 43 primi ; a 139 huius;*

## *L2.§3.* *PROPOSITION 157.*

With the same figure in place as above:
I say that regarding those rectangles PQ, HO, etc., which lie around the diagonal GK, which are similar and in continued proportion to each other; truly the whole sum of this progression is equal to the rectangle under AB by VY.

### *Demonstration.*

As PM is to MO, or HN, thus GP is to ON [note that the 'N' has been reversed on the original diagram included here], on account of the similar triangles GMP and MON : whereby since the rectangles PQ and HO have sides in proportion, and the angles are equal, then they too are similar; and hence the remainder of the rectangles are similar by the same reasoning: which shows the first part of the proposition. Then when the rectangles PQ and HO are given as similar, then they are in the ratio of the squares of the sides PM and HN, &c. Whereby since the given squares are in the same ratio of the sides, then the rectangles and the squares are in the same proportion. But these by hypothesis are in continued proportion, and hence what is true for one is also true for the other. Finally : the rectangle αM is equal to the rectangle [e] MY. Therefore by adding the common rectangle [p. 152] PQ, then the rectangle αQ, or the rectangle formed by AB and VY, which is equal to the rectangle PY : And the rectangle PY, [a] is twice the progression of triangles GPM, MHN, &c. ; hence the rectangle formed by AB and VY is twice the progression of triangles : whereby since the progression of rectangles PQ, HO, &c. is also twice the progression of triangles, then the rectangle formed by AB and VY, and the progression of rectangles are equal : Which was the last of these, which were to be demonstrated.

*e 43 primi ; a 139 huius;*

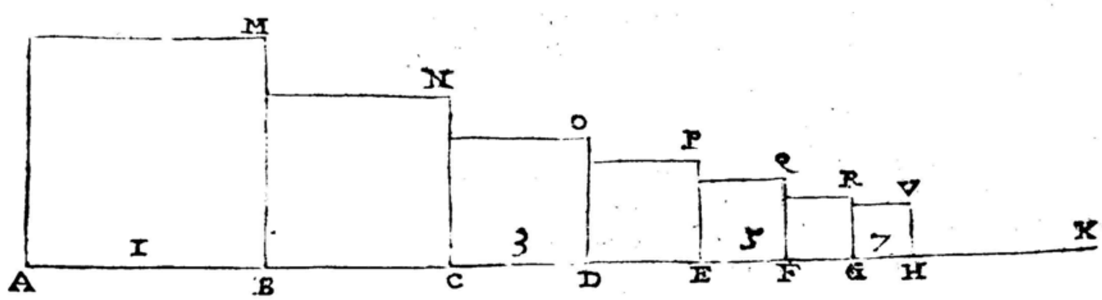
[Again, it is easy to show that both expressions are equal to $\frac{ab}{1-r^2}$ . Geometrically, the rectangles PQ, HO, etc, are equal to the areas $ab$, $abr^2$, $abr^4$, etc, which are thus in proportion to the odd squares.]

## *PROPOSITIO CLVIII.*

` Data sit quadratorum series, basibus in directum positis, & terminum habens longitudinis K : impares autem quadratorum bases AB, CE, EF, &c. notentur numeris imparibus 1.3.5.7. &c.

Dico, primum quadratum AM esse ad secundum BN, ut AB ad CD : & rursum primum quadratum AM esse ad tertium CO, ut AB ad EF; & ad quartum DP, ut AB ad GH. Atque ita in infinitum, bases notatae imparibus numeris, sunt primo quadrato cum subsequentibus comparato, proportionales.

***Demonstratio.***




**Prop. 158, Fig. 1**

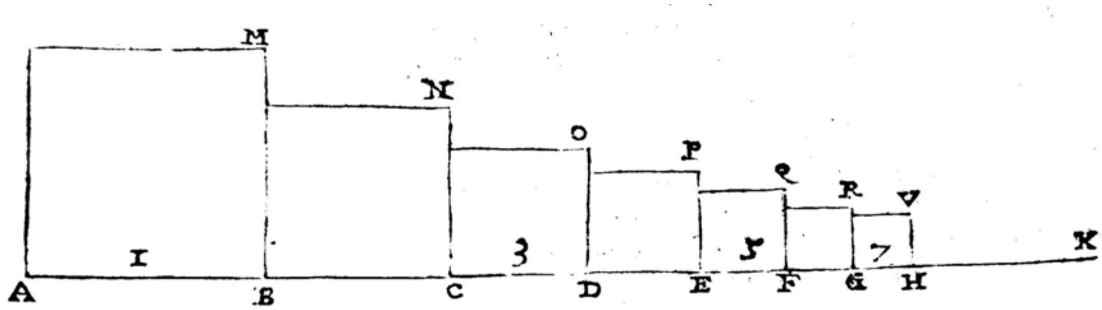
Comparimus exempli gratia quadratum AM, cum tertio CO; Quoniam quadrata omnia AM, BN, CO, &c. in continua sunt analogia, erunt[b] & bases continue proportionales ;ex aequalitate igitur etiam AB, CD, EF, erunt continuae; (cui inter ipsas aequalis continue proportionalium numerus intercedit) ergo quadratum AM est ad quadratum CO, ut AB ad EF; (cum ratione tam quadrati ad quadratum, quam lineae AB ad lineam EF, sint rationis AB ad CD duplicatae) eadem valebit demonstratio, si quadratum AM cum quovis alio comparetur, Constat ergo propositionis conclusio. *b 124 huius.*

## *L2.§3.* *PROPOSITION 158.*

` A series of squares is given, with bases ordered along a line and with its terminal point at a distance K from A : moreover the odd squares bases AB, CD, EF, &c. are to be noted by the odd numbers 1.3.5.7. &c.

I say that the first square AM is to the second square BN, as AB to CD : and again the first square AM is to the third square CO, as AB to EF; and to the fourth DP, as AB to GH. And thus without end, the bases are to be noted by the odd numbers, are proportional to the first square by comparison with the following squares.

***Demonstration.***




**Prop. 158, Fig. 1**

For the sake of an example we compare the square AM with the third CO; Since all the squares AM, BN, CO, &c are in a continued ratio, and the bases are in an analogous continued proportion [b]; from the equality selected therefore also AB, CD, EF, etc, are in a continued ratio; (for the [odd] numbers interceding are themselves in continued proportion) hence the square AM is to the square CO, as AB to EF; (so the ratios of square to square, as of the line AB to the line EF, are of the square of the ratio AB to CD) the same demonstration will prevail, if the square AM is compared with any other square. Therefore the conclusion of the proposition is agreed upon. *b 124 huius.*

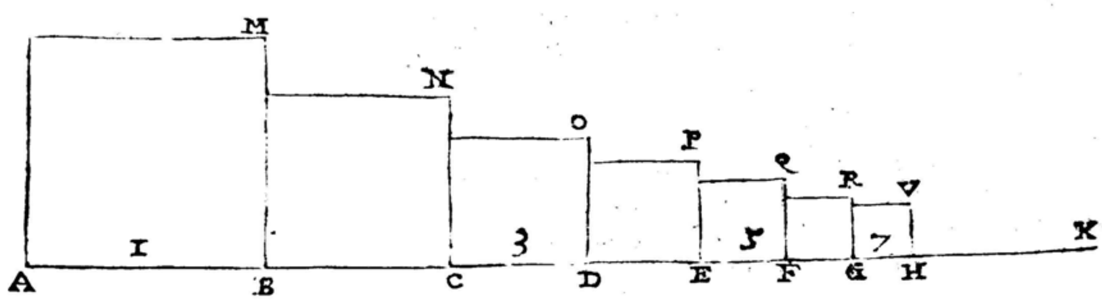
[If we let the geometric series on the line be $a, ar, ar^2, ar^3, ar^4, ar^5, ar^6$, ................,then the two dimensional series of squares are in the ratio $a^2, a^2r^2, a^2r^4, a^2r^6, a^2r^8, a^2r^{10}, a^2r^{12}$, ................; it is evident that sq.AM/sq.BN = $1/r^2$ = AB/CD; sq.AM/sq.CO = $1/r^4$ = AB/EF; sq.AM/sq.DP = $1/r^6$ = AB/GH; etc. ]

## *PROPOSITIO CLIX.*

Eadem posita figura; data sit planorum similium series MK, habens bases homologas in directum positis, & terminum longitudinis K.

Dico seriem MK ad nullam sui partem, verbi gratia ad seriem NK aut seriem OK, vel seriem PK, &c. eam habere rationem, quam inter se habent duae quacumque in hac basium serie, rectae lineae, inter quas par linearum numerus intercedit.

***Demonstratio.***




**Prop. 158, Fig. 1**

Series enim data MK, cum ea sui parte comparatur, ut inter utriusque primum terminum, vel par intercedat planorum numerus, vel impar; comparentur primo series MK & OK, inter quarum initia, impar terminorum numerus intercedit; quia igitur [p. 153] plana sunt in continua analogia, etiam bases AB, BC, &c. erunt a continuae proportionales. Quare ex aequo etiam AB, CD, EF, erunt continuae : ergo planum AM est ad planum CO, ut [b] AB ad EF. Atqui series MK [c] est ad seriem OK, ut planum AM ad planum CO, (sunt enim similium rationum series) ergo series MK est ad seriem OK, ut AB ad EF; inter quas impar numerus basium intercedit, nempe 3. Atqui in tota serie basium, non possunt reperiri duae aliae lineae, quae eandem rationem habeant, quam AB, EF, nisi illae inter quas idem ternarius linearum intercedit, uti ex elementis demonstratur; igitur nullae lineae ex serie basium, inter quas impar linearum numerus inter iicitur, eandem habent rationem, quam series MK ad sui partem OK.

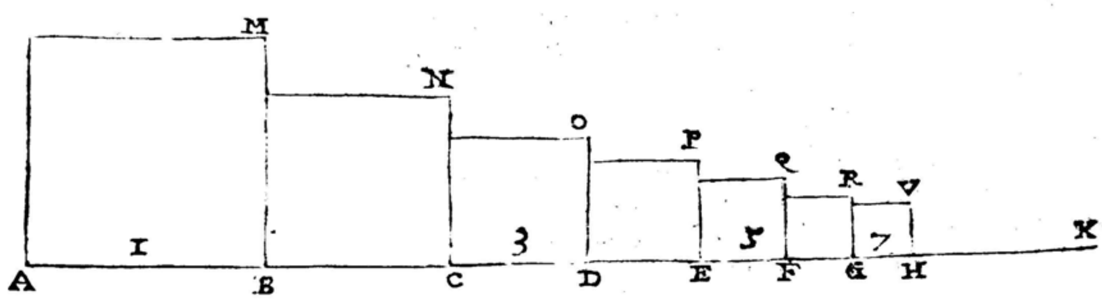
Comparentur modo duae series MK, PK, inter quarum initia par planorum sit numerus : Rursus igitur ostendemus uti prius seriem MK esse ad seriem PK, ut AB ad GH. quare cum inter AB & GH, impar linearum sit numerus, nempe 5; tota demonstratio primae partis huic etiam quadrat : unde patet proportionis veritas. *a 124 huius; b 20 sexti; c 84 huius*

## *L2.§3.* *PROPOSITION 159.*

With the same figure in place; a series of plane figures MK is given, having bases ordered along a line and with its terminal point at a distance K from A

I say that the series MK, taken in comparison with another series formed from any part of itself, for instance either to the series NK, or the series OK or PK, &c., always has that ratio between any two corresponding terms, corresponging to the intercession of an odd number of lines.

***Demonstration.***




**Prop. 159, Fig. 1**

Indeed the given series MK is compared with that part of itself, in order that betwen the first term of the other [sub-series], either an even or odd number of plane figures intercede. In the first place, the series MK is compared with the series OK, between the start of which, an odd number of terms of the original series intercede; therefore since the plane figures are in analogous proportion, [p. 153] also the bases AB, BC, &c. are continued proportionals [a]. Whereby from the equality, also AB, CD, EF, are in continued proportion : hence the plane figure AM is to the plane figure CO, as [b] AB to EF. But the series MK [c] is to the series OK, as the plane figure AM is to the plane figure CO, (for they are both series of similar ratios) hence the series MK is to the series OK, as AB to EF; between which an odd number of of base lengths intercede, truly 3. But in the whole series of bases, no two other lines can be found, which have the same ratio that AB has to EF, except these between which likewise have three interceding lines, as is demonstrated from [Euclid's] Elements ; therefore no other lines from the series of bases, between which an odd number of lines can be placed between, have the same ratio as the series MK has to its own part OK.

In the same manner the two series MK and PK can be compared, between which there is an even number of plane figures : Therefore again we can show, that the prior series MK is to the series PK, as AB is to GH, whereby since between AB and GH, there is an odd number of lines, surely 5; the whole demonstration rests upon the squaring of the base element : thus the truth of the proposition is apparent. *a 124 huius; b 20 sexti; c 84 huius*

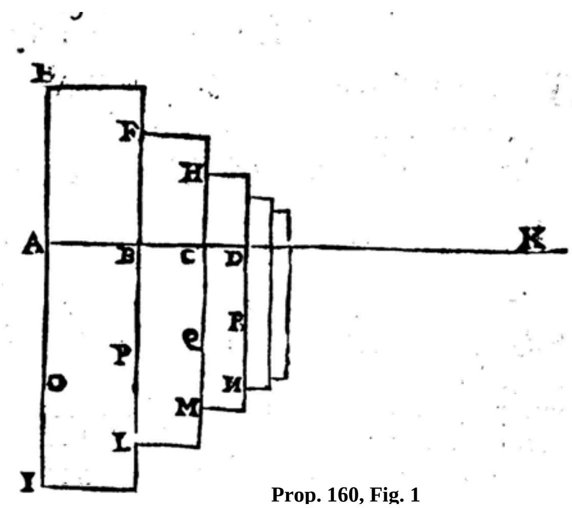
[If we let the geometric series on the line be $a, ar, ar^2, ar^3, ar^4, ar^5, ar^6$, ................,then as above, the two dimensional series of squares are in the ratio $a^2, a^2r^2, a^2r^4, a^2r^6, a^2r^8, a^2r^{10}, a^2r^{12}$, ................ If we now consider the series OK that starts from the term OC, corresponding to the length $ar^2$ , and with the area $a^2r^4$ corresponds to the length EF, for which there are 3 interceding terms BC,DC, and DE. Again, PF has 5 interceding terms, etc. ; thus, the squared elements always lie on even values of the index *r*, leading to the odd number of spaces,]

## *PROPOSITIO CLX.*

Data sit quadratorum series habens bases in directum & terminum longitudinis K. Deinde ex singulis punctis B, C, D, &c. erectae sint perpendiculares AI, BL, CM, &c. proportionales continuae in ratione dimidiata proportionis AB ad BC; & super illis perpendicularibus in altitudine linearum AB, BC, &c. fiant rectangula IB, LC, MD, &c.

Dico seriem rectangulorum IK, ad seriem rectangulorum LK, triplicatam habere proportionem rationis AI ad BL; cuius quadruplicatam habet series quadratorum EK, ad seriem quadratorum FK.

***Demonstratio.***

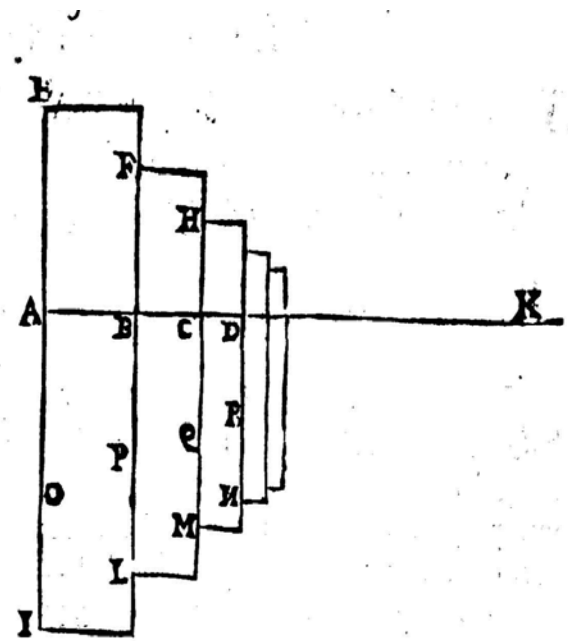



**Prop. 160, Fig. 1**

Primum enim rectangula IB, LC, &c. esse in continua analogia sic ostendo. Ratio rectanguli IB ad LC, coponitur ex rationibus AI ad BL, hoc est hypothesi BL ad CM, & AB ad BC, hoc est BC ad CD; Atqui etiam rectangulorum LC, MD, ratio componitur ex rationibus BL ad CM, & BC ad CD; ergo eadem est rectanguli IB ad LC, & LC ad MD ratio : ergo illa rectangula sunt in continua analogia, habeturque progressio continue proportionalium rectangularum I, L, M, N, K: quare series IK est [d] ad seriem LK, ut rectangulum IB, ad rectangulum LC : Atqui ratio rectanguli IB ad LC, componitur ex ratione AI ad BL, & ex ratione AB ad BC, quae ponitur esse duplicata rationis AI ad BL; ergo ratio rectanguli IB ad LC, hoc est sicut modo ostendimus, ratio seriei IK ad seriem LK, est triplicata rationis AI ad BL. Series autem quadratorum EK est ad seriem FK, ut quadratum BE ad quadratum CF, hoc est in duplicata rationis AB ad BC, [e] ut quadratum BE ad quadratum CF, hoc est (ut ex hypothesi colligitur) in duplicata rationis AI ad BL, cuius triplicata ratio seriei rectangulorum IK, ad seriem LK. Quod erat demonstrandum. *d 82 huius; e ibid.*

## *L2.§3.* ***PROPOSITION 160.***

A series of squares is given, having ordered bases set out along a line and the terminus of length K. Then from the individual points B, C, D, &c, perpendicular continued proportionals AI, BL, CM are erected in the ratio of the square of AB to BC; and upon these perpendiculars at the heights of the lines AB, BC, &c, the rectangles IB, LC, MD, &c are made.




**Prop. 160, Fig. 1**

I say that the series of rectangles IK, to the series of rectangles LK, to be in the cubic ratio to the ratio AI to BL; and the series of squares EK to the series of squares FK has the quadruple of this ratio.

***Demonstration.***

Indeed I show thus that the first rectangles IB, LC, &c. are in continued proporiton by analogy. The ratio of the rectangle IB to LC, is composed from the ratios AI to BL, that is by hypothesis as BL to CM, and from AB to BC, that is, from BC to CD; [Thus, rect.IB/rect.LC = AI/BL × AB/BC = AI/BL × BC/CD]

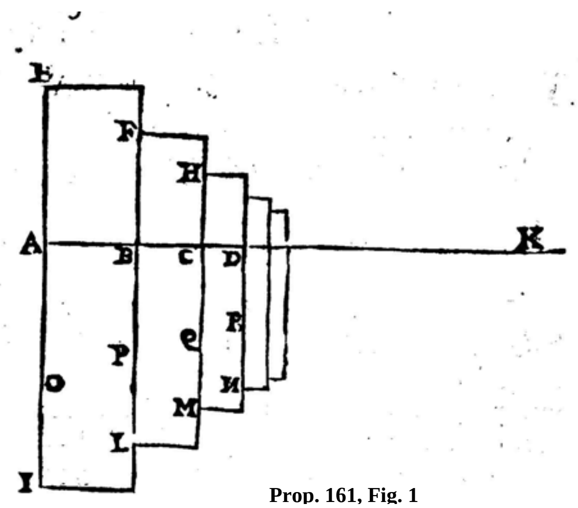
But also the ratio of the rectangles LC and MD, is composed from the ratios BL to CM, and BC to CD; hence the ratio is same for the rectangle IB to LC, and for the rectangles LC to MD : hence these rectangles are in analogous continued proportion, and a progression of rectangles in continued proportions is obtained : I, L, M, N, K; whereby the series IK is to [d] the series LK, as the rectangle IB is to the rectangle LC. But the ratio of the rectangle IB to LC, is composed from the ratio AI to BL, and from the ratio AB ad BC, which is put equal to the square of the ratio of AI ad BL; hence the ratio of the rectangle IB to LC, thus as we have shown in this way, or the ratio of the series IK to the series LK, is as the cube of the ratio of AI ad BL.
[Thus, series IK/series LK = rect.IB/rect.LC = AI/BL × BC/CD = $(AI/BL)^3$]
Moreover the series of squares EK is to the series FK, as the square BE is to the square CF, that is in the square ratio of AB to BC, [e] (as can be gathered from the hypothesis) or as the square BE to the square CF, or in the quadruple ratio of AI to BL, of which the cubic ratio is the ratio of the series of rectangles IK to the series of rectangles LK. Q.e.d. *d 82 huius; e ibid.*
[Thus, series EK/series FK = sq.BE/sq.CF = $(AB)^2/(BC)^2 = (AI/BL)^4$].

[p. 154]

## *PROPOSITIO CLXI.*

Iisdem positis loco rectangulorum intelligatur super normalibus AI, BL, &c. construi series quadratorum.

Dico seriem quadratorum EK, ad seriem FK, duplicatam habere rationem eius, quam habet series quadratorum AI, BL, CM, &c. ad series quadratorum BL, CM, DN, &c.

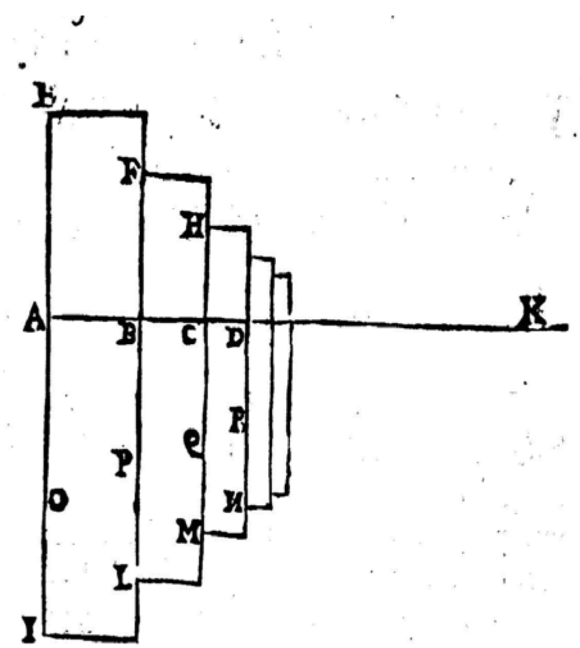


**Prop. 161, Fig. 1**

***Demonstratio.***

Series EK est ad seriem [a] FK ut quadratum BE ad quadratum CF, hoc est in duplicata rationis AB ad BC. Similiter ratio fieri quadratorum AI, BL, &c. ad seriem quadratorum BL, CM, &c. eadem est quae quadrati AI ad quadratum BL, hoc est ex hypothesi ratio seriei quadratorum AI, &c., ad seriem quadratorum BL, &c. Quare cum ostensum sit rationem seriei EK, ad seriem FK, esse duplicatam rationis AB ad BC, erit quoque duplicata rationis, quam habet series rectangulorum IK, ad seriem rectangulorum LK : Quod erat demonstrandum. *a ibid.*

## *L2.§3.* *PROPOSITION 161.*



With the same figure above, and in place of the rectangles, a series of squares is understood to be constructed with normals AI, BL, etc.

I say that the series of squares EK to the series FK, has the square ratio of that which the series of squares AI, BL, CM, &c. has to the series of squares BL, CM, DN, etc.

***Demonstration.***

The series EK is to the series [a] FK as the square BE to the square CF, that is, in the ratio of the squares of AB to BC. Similarly the ratio of the squares AI, BL, etc., to the series of squares BL, CM, &c. is the same as that of the square of AI to the square of BL, that is, from the hypothesis, the ratio of the series of squares AI, etc. to the series of squares BL, etc. Whereby since it can be shown that the ratio of the series EK to the series FK, is the square of the ratio of AB to BC, it is also the square of the ratio, that the series of rectangles IK has to the series of rectangles LK : Q.e.d. *a ibid.*

## *PROPOSITIO CLXII.*

Iisdem positis quae supra; inter BA, AI, CB, BL, DC, CM, &c. inveniantur mediae proportionales AO, BP, CQ, &c.

Dico quadrata AO, BP, CQ, &c. cubis AI, BL, CM, &c. ad series quadratorum BL, CM, DN, &c. esse proportionalia.

***Demonstratio.***

Quoniam BA, AO, AI, sunt continuae proportionales, erit quadratum AO, rectangulo BAI, [b] aequale : similiter reliqua quadrata BP, CQ, &c. reliquis rectangulis CBL, DCM, &c. erunt aequalia . Quare cum rectangula [c] dicta sunt continue proportionalia in ratione triplicata AI ad BL, quadrata quoque AO, BP, &c. erunt in dictorum laterum AI, BL, &c. triplicata ratione continue proportionalia. Atqui etiam cubi AI, BL, &c. sunt in laterum AI, BL, &c. [d] triplicata ratione; ergo quadrata AO, BP, &c. cubis AI, BL, &c. sunt proportionalia. Quod erat demonstrandum.

*a ibid; b 17 sexti;c 160 huius ; d 33 undecimi.*

## *L2.§3. PROPOSITION 162.*

With the same in place as above; the mean proportionals AO, BP, CQ, &c. are found between BA, AI, CB, BL, DC, CM, &c.

I say that the squares AO, BP, CQ, &c. with the cubes AI, BL, CM, &c. to the series of squares BL, CM, DN, &c are in proportion.

***Demonstration.***

Since BA, AO, AI, are continued proportionals, the square AO is equal to the rectangle BAI [b] : similarly the remaining squares BP, CQ, &c. are equal to the remaining rectangles CBL, DCM, &c. Whereby since the said rectangles [c] are in continued proportion in the ratio of the cube of AI to BL, also the squares AO, BP, &c.are in continued proportion in the ratio of the cubes of the sides AI, BL, &c. But also the cubes AI, BL, &c. are in the cubic ratio of the sides AI, BL, &c. [d] ; hence the squares AO, BP, &c. are proportional to the cubes AI, BL, &c. Q.e.d.

*a ibid; b 17 sexti;c 160 huius ; d 33 undecimi.*

[Thus, from above, rect.IB/rect.LC = $(AI/BL)^3$, while rect.IB = $AO^2$ and rect.LC = $BP^2$ ;
hence $(AO/BP)^2 = (AI/BL)^3$].

MGI/ΔFAC

[155]

# PROGRESSIONUM GEOMETRICARUM

### PARS QUARTA

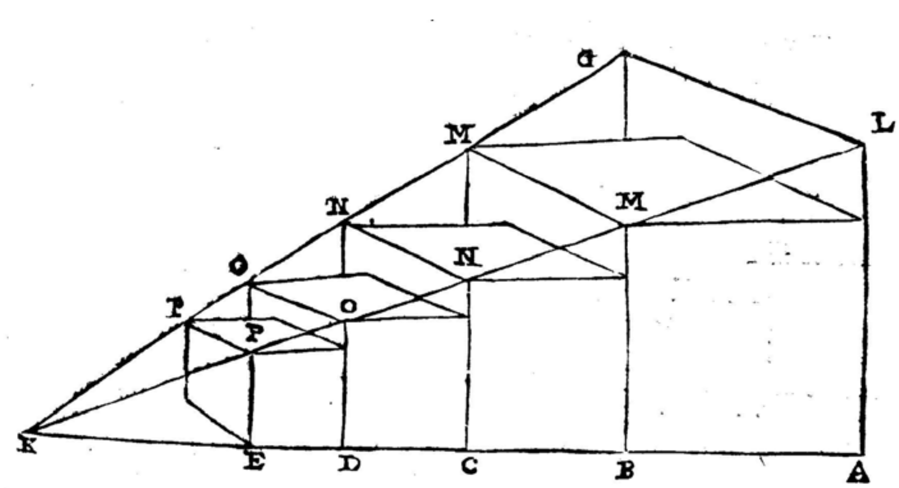
***Part Four of Geometrical Progressions.***

***Doctrinam praecedenti parte in planis demonstratam, corporibus, solidisque, applicat.***
***Having demonstrated the principles in the preceding part for plane figures, it is now applied to bodies and solid figures.***

***PROPOSITIO CLXIII.***

Data sit quadratorum continue proportionalium series, basibus in directum positis, cuius longitudinis terminus sit K; super singulis autem quadratis cubi construantur.
Dico constitui seriem cuborum continue proportionalium, quae eumdem quoque habeant terminum longitudinis K.

***Demonstratio.***




**Prop. 163. Fig. 1**.

Ratio cubi AM ad cubum BN, triplicata est rationis [a] AB ad BC, item ratio cubi BN ad cubum CO, triplicata est rationis [b] BC ad CD; id est rationis AB ad BC; (ponuntur enim quadrata AM, BN, CO &c. in continua analogia) quare cum rationes utraeque, cubi AM ad cubum BN, & cubi BN ad cubum CO, triplicatae sint rationis AB ad BC, eadem erunt. Sunt igitur cubi AM, BN, CO in continua analogia; eodem modo erunt & reliqui omnes continue proportionales. Quod erat primum, ex quo patet etiam secundum : Cum enim quadratorum & cuborum series, pariter semper procedant, idem utriusque terminus sit longitudinis necesse est. Quod erant demonstranda. *a 33 undicimi; b 20 sexti ?*

***L2.§ 4.*** ***PROPOSITION 163.***

A series of squares is given in continued proportion, with the bases ordered along a line, and the terminus of the series is at a length K ; moreover, cubes are constructed on the individual squares.
I say that a series of cubes in continued proportion has been constructed, which have the same terminus of length K.

***Demonstration.***

The ratio of the cube [in the volume sense] AM to the cube BN, is the cube [in the power sense: the text calls this the triplicate ratio, which is confusing for us as we thing of the triplicate rather as × 3] of the ratio [a] AB to BC, likewise the ratio of the cube BN to the cube CO, is the cube of the ratio [b] BC to CD; or of the ratio AB to BC; (for the squares AM, BN, CO, etc. are placed in an analogous continued proportion) since each of the ratios of the cube AM to the cube BN, and of the cube BN to the cube CO, are the cubes of the same ratio AB to BC. Therefore the cubes AM, BN, CO are in continued analogous proportion [in the sense that the common ratio is derived from another simpler ratio]; and in the same manner the rest of the cubes are in continued proportion. Which proves the first part of the proposition, from which the second part is also apparent : For indeed the series of squares and cubes always proceed equally, and likewise the terminus of each by necessity is of the same length. Q.e.d. *a 33 undicimi; b 20 sexti ?*

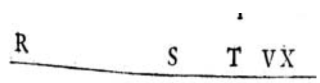
### *PROPOSITIO CLXIV.*

Iisdem positis; primae AB, & quartae DE, aequales fiant RS, ST : continueturque ratio RS ad ST, per plures semper terminos TV, VX, &c.

Dico cubum primum AM, esse ad quemlibet cubum seriei propositis, verbi gratia ad quartum DP, ut est linea RS ad quartam VX.

[p. 156]

### *Demonstratio.*



**Prop. 164. Fig. 1**

Cubus AM ad cubum DP, est in triplicata [a] rationis AB ad DE, hoc est per constructionem rationis RS ad ST. Atque etiam RS ad quartam VX, est in triplicata ratione eius, quam habet RS ad ST : ergo ut RS ad VX, sic cubus AM ad cubum DP. Simili ratiocinatione ostendemus cubum primum, ad quemvis seriei cubum, eandem habere rationem, quam habet RS ad lineam quae aeque distabit a prima RS, atque cubus a cubo primo AM. Quod erat demonstrandum. *a 33 undecimi*.

### **Corollarium.**

Duo haec theoremata eadem servata demonstratione, ad omnia similium corporum genere licebit extendere.

### *L2.§4.* *PROPOSITION 164.*

With the same figure in place ; RS and ST are made equal to the first and fourth lengths AB and DE: and the ratio RS to ST is continued through many terms TV, VX, etc., in the same manner.

I say that the first cube AM, is to any cube of the proposed series, for argument's sake to the fourth DP, as the line RS is to the line VX.

[p. 156]

### *Demonstration.*

The cube AM to the cube DP, is in the ratio a of the cube of AB to DE, that is by the construction as RS to ST. And indeed RS to the fourth term VX, is in the cubic ratio that RS has to ST : hence as RS to VX, thus the cube AM to the cube DP. By similar reasoning, we can show that the first cube, to any cube of the series you wish, has the same ratio as RS has to the line which is at the same distance from the first RS, and the cube from the first cube AM. Q.e.d. *a 33 undecimi.*

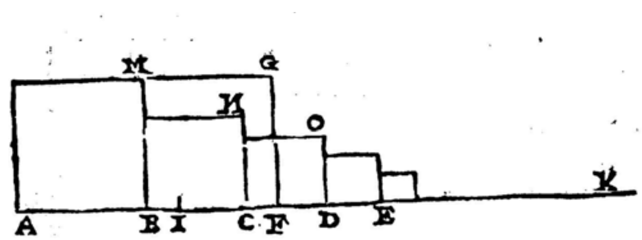
### *Corollarium.*

These two theorems established by the same demonstration can be extended to kinds of similar bodies.

## *PROPOSITIO CLXV.*

Data sit quadratorum series, habens bases in directum, & terminum longitudinis K. super quadratis autem singulis, extructi sint cubi : Petitur seriei cubicae aequale parallelipipedum exhiberi.

### *Constructio &Demonstratio.*




**Prop. 165. Fig. 1**

Seriei rationis primae baseos AB, ad quartam DE, fac [b] aequalem AF; & super AF, in altitudinem AB, rectangulum AG : deinde super rectangulo AG, in altitudine AB, construe parallelepipedum rectangulum. Dico hoc seriei cubiae aequari. Vel super quadrato AB in altitudine lineae, aequalis seriei rationis AB ad DE, fac parallelepipedum. Dico hoc esse quaesitum. Fiat enim BI aequalis DE. Quoniam igitur ex constructione seriei rationis AB ad BI, hoc est ut AB ad DE, aequalis est AF, erit [c] AF ad BF, ut AB ad BI, hoc est ut AB ad DE : quia autem quadrata ex hypothesi sunt continua, erunt [d] lineae AB, BC, CE, DE, &c. in continua analogia : unde & cubus [e] AM ad cubum BN, ut AB ad DE, hoc est (sicut ostendi) ut AF ad BF : Quare cum parallelepipedum AG, sit ad parallelepipedum BG, [f] ut basis AG ad basim BG, hoc est ut [g] AF ad BF, erit cubus AM ad cubum BN, ut parallelepipedum AG, ad parallelepipedum BG : Atqui tota [h] series cubica MK, est ad seriem cubicam, NK, ut cubus AM ad cubum BN, ergo parallelepipedum AG, est ad parallelepipedum BG, ut series cubica MK, ad seriem cubicam NK : ergo dividendo cubus AM, est ad parallelepipedum BG, ut cubus idem AM, ad seriem cubicam NK; (cum enim parallelepipeda AG, BG constructa sint supra bases AG, BG, in communi altitudine AB, patet cubum AM esse excessum parallelepipedum BG super parallelepipedum BG.) Itaque series cubica NK & parallelepipedum, erunt aequalia; communique addito cubo AM, tota series cubica, & parallelepipedum AG aequalia erunt. Factum igitur est quod petebatur. *b 80 huius; c 82 huius; d 22 sexti; e 33 undecimi; f 25 undecimi; g 1 sexti; h 82 huius.*

[p.157]

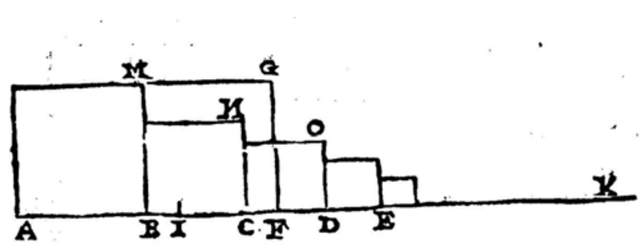
***Corollarium*.**

Itaque si fuerint propositae binae, vel plures cuborum progressiones, etiam rationum dissimilium, cognoscetur earum proportio inter se, si per hanc propositionem singulis cuborum progressionibus aequalia parallelepipeda constituantur.

## *L2.§4.* *PROPOSITION 165.*

A series of squares is given, having the bases ordered along a line, and having the terminus at a length K. Moreover, upon the individual squares, cubes are to be set up : It is required to exhibit a parallelepiped equal in volume to the series of cubes.

### *Construction &Demonstration.*




**Prop. 165. Fig. 1**

Make AF equal to the sum of the series of ratios of the first base AB to the fourth base DE b; and on AF with height AB, make the rectangle AG : then on the rectangle AG, with height AB, construct a rectangular parallelepiped [ppd.]. I say that this ppd. has a volume equal to the series of cubes. Or alternatively, on the square AB, construct a parallelepiped with a height equal to the sum of the series of ratios AB to DE. I say that this is the volume sought. For if BI is made equal to DE, then from the construction of a series of ratios AB to BI, that is in the same ratio as AB to DE, for which the sum is equal to AF, then c AF is to BF as AB to BI, or as AB to DE : moreover since the squares are in a continued progression by hypothesis, the lines d AB, BC, CE, DE, &c. are in an analogous continuous progression: from which the cube e AM is to the cube BN, as [the cube of] AB is to [the cube of] DE, or (as was shown) as AF to BF : Whereby, since the parallelepiped AG is to the parallelepiped BG, f as the base AG is to the base BG, or as g AF to BF, the cube AM is to the cube BN, as the ppd. AG is to the ppd. BG : But h the sum of the whole series of cubes MK is to the series of cubes NK, as the cube AM is to the cube BN, hence the ppd. AG is to the ppd. BG, as the sum of the series of cubes MK is to the series of cubes NK : hence by division, the cube AM is to ppd. BG, as likewise the cube AM is to the series of cubes NK; (since indeed the ppd's AG and BG are constructed on the bases AG and BG, with the common altitude AB, it is apparent that the cube AM is the difference of the ppd's AG and BG.) Thus the series of cubes NK and the given ppd BG are equal; and by adding the common cube AM, the sum of the series of cubes and the ppd. AG are equal to each other. Therefore what was sought has been done. *b 80 huius; c 82 huius; d 22 sexti; e 33 undecimi; f 25 undecimi; g 1 sexti; h 82 huius.*

[For BI = DE, then AB/BI = AB/ DE, and the sum of the cubes is AF, then AF/BF = AB to BI, or AB/ DE : also AB, BC, CE, DE, &c. are in an analogous continuous progression: and cubeAM/cube BN = cube $AB^3/DE^3$, or AF/BF : Whereby, as ppd.AG/ppd. BG = rect.AG/rect.BG = AF/BF,
AM/ BN = ppd. AG /ppd.BG : But the sum MK/sum NK = AM/BN, hence ppd.AG/ppd.BG = sum MK/NK : ( ppd.AG/ppd.BG – 1) = (sum MK/sumNK – 1); hence cube AM/ppd. BG = cube AM/sum NK; Thus the series of cubes NK = ppd BG ; In some respects this proof resembles a modern inductive proof, and up to this stage it shows the self – consistency of the argument, rather than an actual formula for the sum.

In modern terms, the sum of the series of cubes is $\frac{a^3}{1-r^3}$ ; and the sum of the series of lengths of the sides of the cubes, AF in the text, is $\frac{a}{1-r^3}$ , while the series BF, with the first term missing, is $\frac{ar^3}{1-r^3}$ ; hence the ratio AF/BF = $1/r^3$. This is the method used to evaluate the infinite sums of linear, square, or cubic terms, where the sum is assumed for the whole series from the first term, and likewise from the second term, and the ratio taken, which is then set equal to either $1/r$,$1/ r^2$, or $1/r^3$. Thus, in this case,
AF/BF = $1/r^3$ = AB / DE. Hence, AF can be determined as above to be $\frac{a}{1-r^3}$ , from which the sum for the series of cubes $\frac{a^3}{1-r^3}$ follows.]

[p.157]

### *Corollarium.*

Thus if two or more progressions of cubes are proposed, also having different ratios, the proportion is known between these, if the parallelepipeds are constructed for the individual progressions of the sums of the cubes.

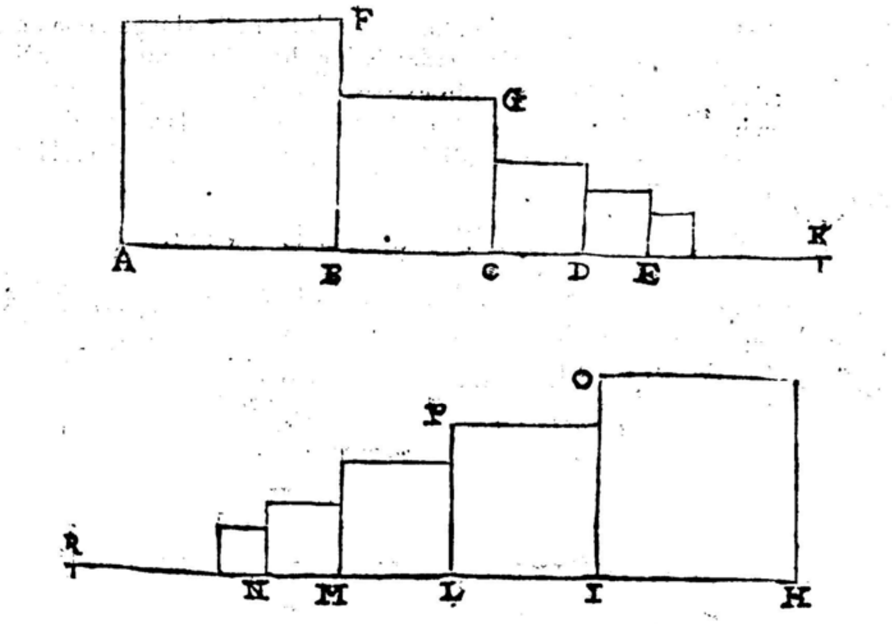
### *PROPOSITIO CLXVI.*

Dentur binae, sed aequales cuborum progressiones rationum dissimilium.
Dico seriem cubicam AK, ad seriem cubicam HR, rationem habere compositam, ex ratione primi quadrati AF, ad primum quadratum HO, & ratione seriei rationis AB primae, ad quartam DE, ad seriem rationis HI, primae, ad quartam MN.

### *Demonstratio.*

Parallelepipedum factum super quadrato AB, in altitudine lineae seriei rationis AB ad DE, per praecedentem seriei cubicae AK, erit aequale : similiter parallelepipedum super quadrato HI, in altitudine lineae rationis HI ad MN, seriei cubicae HR aequale est. Quare cum series cubicae ponantur aequales, dicta quoque parallelepipida aequalia erunt; ergo reciprocam habent basium & [a] altitudinem rationem, hoc est habent rationem compositam ex rationibus basium & altitudinum rationem : Quare & series cubicae AK & HR illis aequales, rationem habent compositam ex ratione dictarum altitudinem, hoc est ex ratione seriei

AB, DE, &c. ad seriem HI, MN, &c. & ex ratione basium, hoc est ex ratione quadrati AF ad quadratum HO. Quod erat demonstrandum. *a 34 undecimi.*




**Prop.166. Fig. 1.**

***L2.§4.*** ***PROPOSITION 166.***

Two series of cubes in progressions with dissimilar ratios but with the same sum are given.

I say that the series of cubes AK, to the series of cubes HR, has a ratio composed from the ratio of the first square AF to the first square HO, and from the ratio [composed from two ratios, the first of which is the sum] of the series with a ratio of the first term AB to fourth term DE, etc., [to the second which is ] the sum of the series with the ratio of first term HI to fourth MN, etc.

***Demonstration.***

A parallelepiped is constructed on the square AB, with the height in the ratio of AB to DE, which is equal to the sum of the cubes AK by the preceding proposition : similarly a parallelepiped constructed on the square HI, with a height in the ratio HI to MN, is equal to the series of cubes HR. Whereby since the series of cubes are put equal, then the said parallelepipeds are also equal ; hence they have a reciprocal ratio of bases and heights [a], that is they have a ratio composed from the ratios of the bases and the ratio of the heights : Whereby the series of cubes AK and HR from these are equal, they have a ratio composed from the ratio of the given heights, that is from the ratio of the series AB and DE, &c. to the series HI and MN, &c. and from the ratio of the bases, that is from the ratio of the square AF to the square HO. Q.e.d.

*a 34 undecimi.*

[This proposition follows directly from the previous proposition. ]

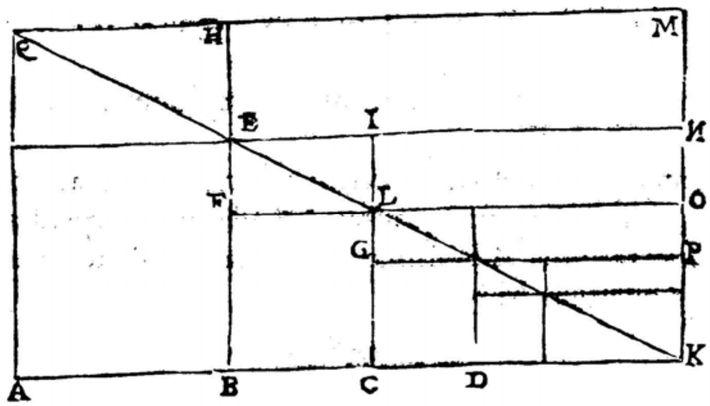
***PROPOSITIO CLXVII.***

Data sit quadratorum progressio, basibus in directum positis, quae terminum longitudinis habeat K, & iuxta 131, huius inscripta sit triangulo AQK; completo autem rectangulo AM, producantur latera quadratorum in N, O, P, &c. & in H, I, L, &c. [p. 158]

Dico seriem parallelepipedorum, super rectangulis EM, FN, GO, &c, in altitudine linearum BE, CF, DG, &c. aequalem esse seriei cubicae, super quadratis exstructae.

***Demonstratio.***

Dicta enim EM, FN, &c. rectangula, sunt complementa rectangulorum, quae sunt circa diametrum, ergo a singula quadratis singulis ordine sunt aequalia. Quare Parallelepipeda super complementis illis exstructa, cum easdem quoque cum cubis quadratorum habeant altitudines BE, CF, &c. patet singula b parallelepipeda singulis cubis aequalea esse : ergo tota parallelepipidorum series, toti cubicae aequatur : quod erat demonstrandum. *a 45 primi; b 7 duodecimi.*




**Prop. 167, Fig. 1.**

***L2.§4.*** ***PROPOSITION 167.***

A series of squares is given, with bases arranged in order along a line, and which has a terminus of length K, and just as in Prop. 131 of this book, the series is inscribed in the triangle AQK; moreover with the rectangle AM completed, the sides of the squares are extended to N, O, P, &c. & to H, I, L, &c. [p. 158]

I say that the sum of the series of parallelepipeds, erected on the rectangles EM, FN, GO, &c, with the heights of the vertical lines BE, CF, DG, &c., is equal to the sum of the of the cubes built up on the squares.

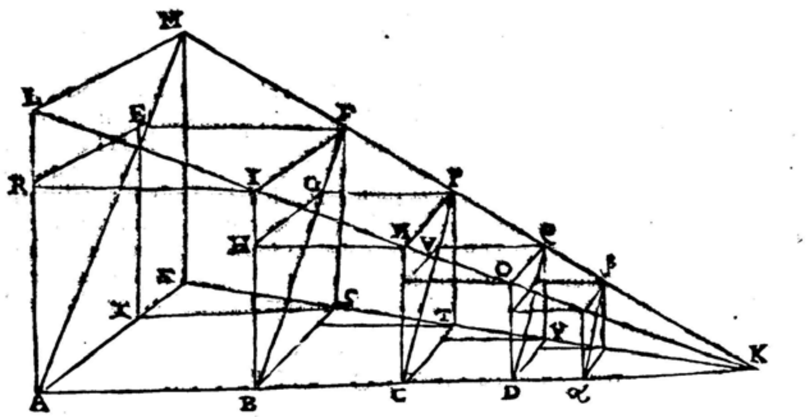
***Demonstration.***

For the said rectangles EM, LN, &c., are the complements of the rectangles which lie around the diagonal, hence [a] the individual rectangles are equal to the squares in order. [Thus, the rectangle EM has the same area as the square AB, rect. LN = sq. BL, next rect. ?O = sq. on CD, etc; ] Whereby the Parallelepipeds constructed on these complements, with the same heights also with the cubes of the squares BE, CL, etc. it is apparent that the individual [b] parallelepipeds are equal to the individual cubes : Hence the whole series of parallelepipeds is equal to the whole series of cubes: q.e.d. *a 45 primi; b 7 duodecimi.*
[ See note to Prop. 155 of section 3.]

### *PROPOSITIO CLXVIII.*

Data sit progressio AB, BC, CD, &c. terminata in K, & super lineis quadrata, super quadratis autem cubis. Petitur cuborum series inscribi pyramidi, quadratam basim habenti.

### *Constructio &Demonstratio.*



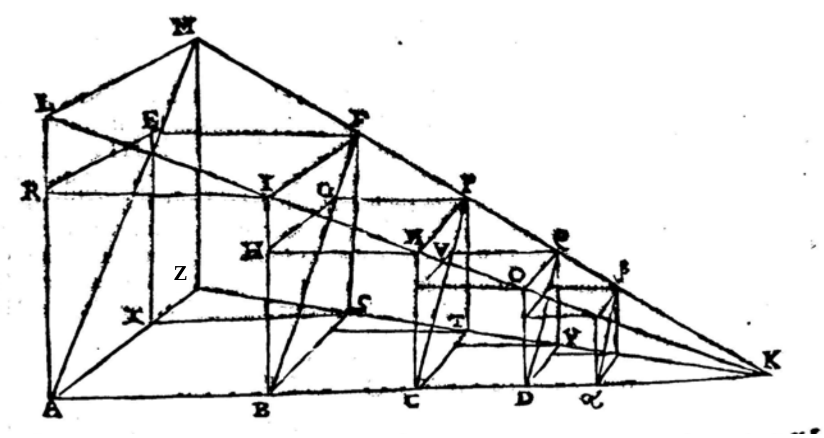

**Prop. 168, Fig. 1.**

Ex puncto K, per I, S, F, ducantur recte KI, KS, KF, quarum duae primae KI, KS, occurrant lineis AR, AX productis in L & Z : producto deinde plane AE, occurrat linea KF in M, iunganturque LM, ZM. Dico factum quod petabatur. Ducantur enim in adversus cuborum planis diametri AE, BF, CP, DQ, &c. primo igitur ex hypothesi manifestum est utriusque; seriei quadrata AI, BN, CO, &c. AS, BT, &c. [p.159]esse in eodem plano. quare lineae [a]: KIL, KSZ transeunt per omnia puncta N, O, &c. T, Y, &c. hoc est tangunt totam cuborum seriem. Superest ergo ut demonstremus lineam KFM transire etiam per omnia puncta P, Q. &c. quod sic praestabimus. IF est ad HG, ut IB ad HB, id est ut IK ad NK, hoc est ut BK ad CK; Atqui cum seriei AB, BC, CD terminus sit K, [b] AK, BK, CK sunt continuae proportionales; ergo BK est ad CK, ut AB ad BC : & IF ad HG ut AB ad BC : quare cum AB, IF aequales sint, etiam BC & intercepta HG, aequales erunt; ergo BF transit per verticem anguli G, quadrati SBHG, cum intercipiat parallelam HG aequalem lateri dicti quadrati; unde B, G, F sunt in directum. Itaque cum ex elementis constet diametros adversas CP, BG esse parallelas, etiam CP, BF erunt parallelae. Quia igitur linea BF est in [c] plano BFK, etiam CP in eodem [d] plano BFK erit. Similiter ostendemus lineas DQ, CP, esse parallelas, & proinde cum CP sit in plano BFK, etiam DQ esse in plano BFK. eodem discursu demontrabimus omnes BF, CP, DQ, Xβ, &c. esse in eodem plano BFK, sive AMK : deinde BF, CP, &c. cum sint in oppositis planis parallelis, productae nunquam convenient. quare cum sint omnes in plano BFC, erunt omnes inter [e] se parallelae. Praeterea ex elementis & ex datis patet diametros BF, CP, &c. esse lateribus IB, NC, &c. hoc est AB, BC, CE, &c. proportionales: quare [f] KFM transit per omnia puncta P, Q, , &c. Quod autem etiam basis ZM quadrata sit, sic ostendo: ZA est ad XA, ut ZK ad SK, hoc est ut AK ad BK, hoc est ut LK ad IK, hoc est denique ut LA ad RA: Quia ergo XA, RA aequales sunt, etiam ZA, LA [g] aequales erunt. Praeterae LM est ad IF ut LK ad hoc est ut AK ad BK, hoc est ut AZ ad BS, atqui IF, BS aequales sunt, ergo etiam LM, AZ aequales erunt. Deinde cum LK sit ad IK, ut [h] MK ad FK, erit LM parallela ad IF, quae cum ad AXZ parallela sit, etiam LM ad AXZ parallela erit. Quia igitur MZ & AL aequales & parallelas LM, AZ connectunt, ipse quoque [i] aequales & parallelae erunt; est autem angulus LAZ rectus, ac proinde etiam angulus MZA, & consequenter anguli illis oppositi sunt recti, basis igitur ZL est quadrata : Factum ergo est quod petebatur. quod erat demonstrandum. *a 131huius; b 82 huius;c 2 undecimi; d Defin. 34 primi;e ibid;f Lemms ad 131 huius; g 14 quinti; h 17 undecimi; i 33 primi .*

***L2.§4.*** ***PROPOSITION 168.***

A progression is given AB, BC, CD, &c. with the terminus in K, and upon the lines squares, and again upon the squares, cubes are constructed. It is required to inscribe the series of cubes in a pyramid having a square base.

***Construction &Demonstration.***



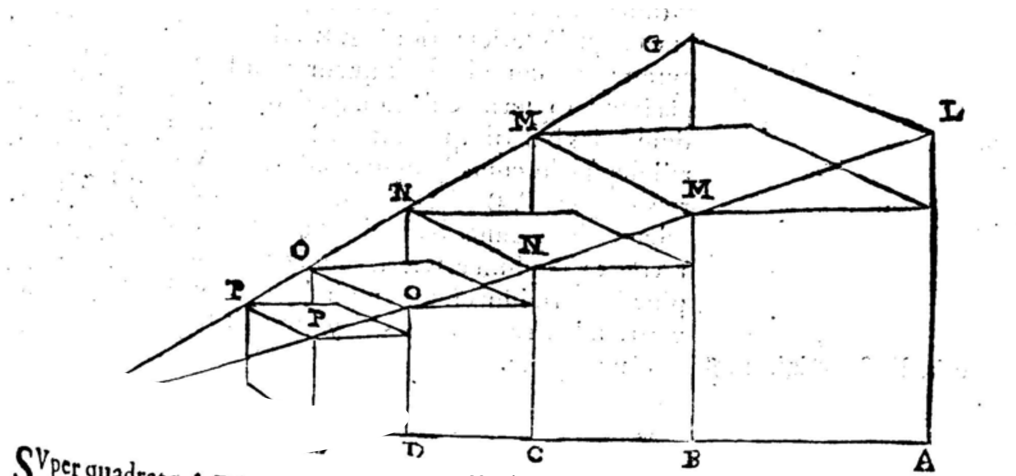

**Prop. 169, Fig. 1.**

From the point K, the lines KI, KS, and KF are drawn through I, S, F, of which the first two lines cross with the lines AR and AX produced in L and Z : then by producing the plane AE, this crosses the line KF in M, and LM, ZM are joined. I say that what was desired has been done. For the transverse diagonals AE, BF, CP, DQ, &c are drawn in the planes of the cubes. Therefore in the first place by hypothesis, it is clear that each series AI, BN, CO, &c. and AS, BT, &c. lie in the same planes. [p.159] Whereby the lines [a]: KIL, KSZ pass through all the points N, O, &c. T, Y, &c. that is they are tangents to all the series of the cubes. Therefore it remains that we must show the line KFM also cuts through all the points P, Q. &c. , which thus we will establish. IF is to HG, as IB to HB, that is as IK to NK, that is as BK ad CK; But since the terminus of the series AB, BC, CD is K, then [b] AK, BK, CK are in continued proportion; hence BK is to CK, as AB to BC : & IF to HG is as AB to BC : whereby since AB and IF are equal, also BC and the intercept HG are equal ; hence BF passes through the vertex of the angle G of the square SBHG, since it intercepts the parallel line HG of the said square; hence B, G, F are on a line. Thus since from the Elements it is agreed that the transverse diagonals CP and BG are parallel, also CP and BF are parallel. Therefore since the line BF is in [c] the plane BFK, also CP is in the same plane [d]BFK. Similarly we can show that the lines DQ and CP are parallel, and hence since CP is in plane BFK, also DQ is in the plane BFK. By the same argument we can demonstrate that all of BF, CP, DQ, Xβ, &c. are in the same plane BFK, or AMK : then it follows that BF, CP, &c. , since they are in opposite parallel planes, their productions never meet. Whereby since they are all in the plane BFC, they are all parallel amongst themselves [e]. In addition, from the Elements and from what is given, it is apparent that the diagonals BF, CP, &c. are with the sides IB, NC, &c. ; that is AB, BC, CE, &c. are in proportion: whereby [f] KFM crosses all the points P, Q, *β*, etc. Moreover, since also the base is the square of ZM, I show thus : ZA is to XA, as ZK to SK, that is as AK to BK, or as LK to IK, and that is hence as LA to RA: Therefore, since XA and RA are equal, also ZA and LA [g] are equal. In addition LM is to IF as LK to IK, that is as AK to BK, that is as AZ ad BS, but IF and BS are equal, hence LM and AZ are equal. Then since LK is to IK, as [h] MK to FK, LM is parallel to IF, which since it is parallel to AXZ, also LM is parallel to AXZ. Therefore since MZ and AL are equal and parallel; LM, AZ taken together, are themselves equal and parallel also; moreover the angle LAZ is right, and hence also the angle MZA, and consequently the opposite angles are right, therefore the base ZL is a square : What was asked has been done. Q.e.d. *a 131huius; b 82 huius;c 2 undecimi; d Defin. 34 primi;e ibid;f Lemms ad 131 huius; g 14 quinti; h 17 undecimi; i 33 primi .*

***PROPOSITIO CLXIX.***

Series seu pyramis cubica inscripta sit pyramidi ALGK. Oporteat pyramidis includentis, & inculsae differentiam exhibere.

***Constructio &Demonstratio.***




**Prop. 169, Fig. 1.**

Super quadrato AB in altitudine lineae aequalis seriei rationis AB ad DE fac parallelepipedum, hoc est seriei [k] cubicae aequale erit. Deinde fiat ut quadratum AB ad quadratum ALG, ita pyramidis GLK altitudo AK ad aliquam Q : denique super quadrato AB in altitudine lineae, quae contineat unam tertiam rectae Q fiat parallelipedum, erit hoc pyramidi LK aequale; nam parallelipedum super quadratio [p. 160]

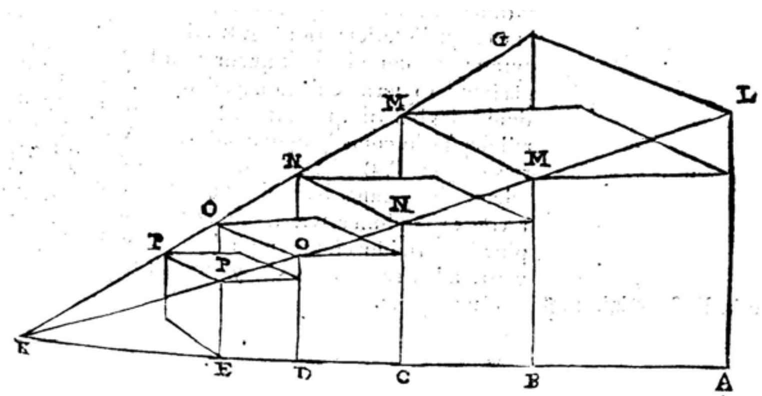
AB in altitudine lineae Q, aequale est [a] parallelipedo super quadrato AG in altitudine AK, cum habeant bases ex constr. & altitudines reciprocas. Quare cum pyramis [b] LK sit una tertia parallelipedi super AG in altitudine AK (est enim AK altitudo, quia AK ut ex construct. praecenentis propositionis patet, est normalis ad AL) itemque parallelipedum super quadrato AG in altitudine tertiae partis rectae Q, sit eiusdem parallelipedui tertia pars, erunt pyramis & dictum parallelipedum aequalia. Quare cum pyramis maior sit inscripta cubica pyramide, etiam dictum parallelipedum nempe super quadrato AB in altitudine tertiae partis rectae Q, erit maius parallelipedo quod pyramidi cubicae aequale feceramus. Eadem igitur erit pyramidis includentis & inclusa quae horum parallelipedorum differentia. exhibuimus ergo, &c. quod petebatur.

*k 165 huius*; *a 34 undecimi; b 7 undicemi.*

***L2.§4.*** ***PROPOSITION 169.***

A progression is given AB, BC, CD, &c. with the terminus in K, and upon the lines squares, and again upon the squares, cubes are constructed. It is required to inscribe the series of cubes in a pyramid having a square base.

***Construction &Demonstration.***




Make a parallelepiped [ppd. $AB^2$.LG] on the square AB with a height LG, equal to the sum of the cubes in the series, in the ratio AB to DE, that is, the ppd. is equal [in volume] to the sum of the series of cubes [k]. Then make the ratio of the square AB to the square ALG to be the same as the height AK of the pyramid GLK to some other length Q : and then upon the square AB [p. 160] with a line which has a length equal to one third of the line Q make a ppd.: this is equal to the pyramid LK; for the ppd. upon the square AB with the height of the line Q, is equal to the ppd. [a] on the square AG with height AK [these extra shapes are not shown on the diagram], since from the construction, they have their bases and heights in reciprocal proportions. Whereby since the pyramid [b] LK is one third of the ppd. on the square AG with height AK (for the altitude is AK, since AK is normal to AL, as is apparent from the construction of the preceding proposition), and likewise the ppd. on the square AG with height one third of Q is a third part of the same ppd., then the [greatest] pyramid and the said ppd. are equal in volume. Whereby, since the largest pyramid is greater than the inscribed series of cubes, and also truly equal to the said ppd. erected on the square AB with height equal to one third of the line Q, then the ppd. is greater than the series of cubes we have constructed. Therefore the series of included cubes is also less, and includes these for which the ppd's. are different; hence we have shown what was required.

*k 165 huius*; *a 34 undecimi; b 7 undicemi.*

[Thus, the volume of the series of cubes is $AB^3$ (1 + ED/AB + ..........), or in modern terms,

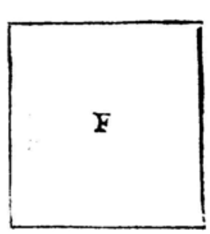
$V = a^3(1 + r^3 + ..........) = \frac{a^3}{1-r^3}$, where as before $a$ is the length AB, and $r$ = CB/AB, etc; Initially, construct a ppd. on the square AB with height GL, equal to the sum of the cubes V; then GL = $a/(1 - r^3)$. Subsequently, put the ratio of the end squares $AB^2/GL^2$ = KA/Q, where AK is the height of the largest pyramid, and Q is some other length; then $AB^2 \times Q/3$ is the volume of the ppd. on the square AM of height Q/3 is equal to volume of the largest pyramid $GL^2 \times AK/3$, since $GL^2 \times KA = AB^2 \times Q$. Consequently, since the volume of the pyramid LK is greater than V, also the volume of the ppd. on the square AM is greater than V; thus the series of cubes is enclosed. ]

## *PROPOSITIO CLXX.*

Datae seriei sive pyramidi cubicae, pyramidem super F quadrato dato, aequalem exhibere.

### *Constructio &Demonstratio.*

Sit series cuborum AM, BN, CO, &c. & datum quadratum sit F, & fiat ut quadratum F ad quadratum AB, ita linea aequalis seriei rationis AB ad DE ad lineam Q. Dico pyramidem cuius basis sit quadratum F, altitudino autem tripla ipsius Q seriei cubicae aequalem esse. Nam parallelepipedum cuius basis sit quadratum F altitudo Q aequatur [c] parallelepipedo, cuius




**Prop. 170. Fig. 1.**

basis sit quadratum AB, altitudo vero series rationis AB ad DE, hoc est [d] seriei cubicad. Ergo parallelepipedum cuius basis sit quadratum F & altitudo tripla rectae Q triplum erit seriei cubicae. atqui [e] idem parallelepipedum triplum est pyramidis habentis basim F, altitudinem vero triplicata rectae Q, ergo pyramis illi seriei cubicae aequalis erit. Factum igitur est quod petebatur. *c 34 undecimi;d 165 huius;e 7 duodecimi.*

[p. 161]

***L2.§4.*** ***PROPOSITION 170.***

A progression is given AB, BC, CD, &c. with the terminus in K, and upon the lines squared are formed, and again upon the squares, cubes are formed. It is required to inscribe the series of cubes in a pyramid having a certain square base.

***Construction &Demonstration.***


**Prop. 170. Fig. 1.**

Let AM, BN, CO, etc. be a series of cubes, and F is a given square, and as the square F to the square AB, thus a line equal to the sum of the series of ratios AB to DE is to some line Q. I say that the pyramid the base of which is the square F, and moreover with a height equal to three times Q, is equal to the sum of the series of cubes. For the ppd. the base of which is the square F and with height Q is equal to the volume of the ppd. [c], the base of which is the square AB, with the height truly equal to the sum of series of ratio AB to DE, that is, to the sum of the cubes [d]. Hence the ppd. with the base equal to the square F, and with height equal to the triple of the length of the line Q is three times the sum of series of cubes. But [e] the same ppd. is three times the volume of the pyramid having the base F, and with height three times the length Q, and hence the pyramid is equal to that series of cubes. Therefore what was required has been done. *c 34 undecimi;d 165 huius;e 7 duodecimi.*

[Let $f^2$ be the size of the given square F, and $a$ the length of the first side AB of the series in geometric proportion with a common ratio $r$. Let S be the sum of the series $a(1 + r^3 + r^6 + .....)$, *i.e.* $S = \frac{a}{1-r^3}$ ; then $f^2/a^2 = S/Q$. According to the proposition, the sum of the required cubes is equal to the volume of the pyramid with base $f^2$ and height $3Q$: For, the volume of such a pyramid is $\frac{1}{3}f^2.3Q = f^2 \times a^2 S/f^2 = \frac{a^3}{1-r^3}$ as required. The demonstration considers the ppd. with base F and height 3Q, which is equal to the length 3S of the series, and the volume of the pyramid is 1/3 of this amount as required.]

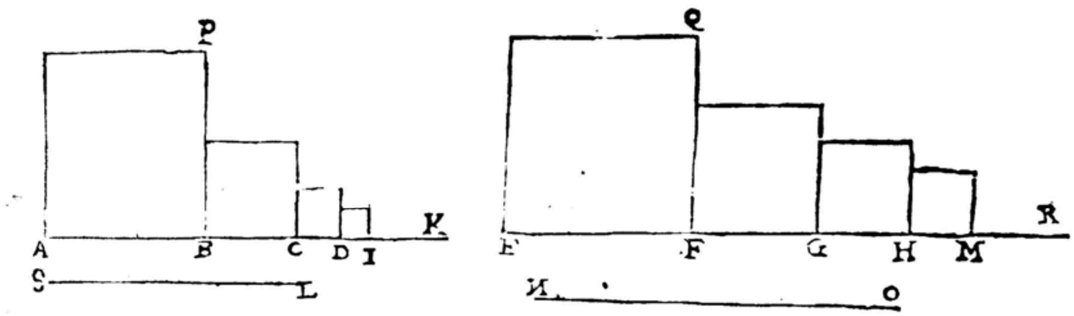
[p. 161]

***PROPOSITIO CLXXI.***

Binae cuborum series quarumvis rationum ab aequalibus cubis AD, EQ incipiant.

Dico cubicas series eamdem habere ad invicem proportionem, quam habent lineae SL, NO aequales seriebus rationum AB ad DI, & EF ad HM.

***Constructio &Demonstratio.***



**Prop. 171, Fig. 1.**

Parallelepipedum super quadrato AB in altitudine SL, aequatur [a] seriei cubicae PK. Item parallelepipedum super quadratio EF in altitudine NO, aequatur seriei cubicae QR : cum autem cubi AP, EQ ponatur aequales, etiam quadrata AB, EF aequalia erunt. Quare dicta parallelepipeda easdem bases habebunt; itaque dicta parallelepipeda, hoc est series cubicae eamdem habebunt rationem, quam altitudines SL, NO, hoc est quam habent series rationis AB ad DI, & rationis EF ad HM : quod erat demonstrandum.

*a 165 huius*.

***L2.§4. PROPOSITION 171.***

Two series of cubes of any ratios start from the equal cubes AD and EQ.

I say that the sums of the series of cubes are in the inverse proportion of the lines SL and NO, equal to the ratios AB to DI and EF to HM of the series.

***Construction &Demonstration.***

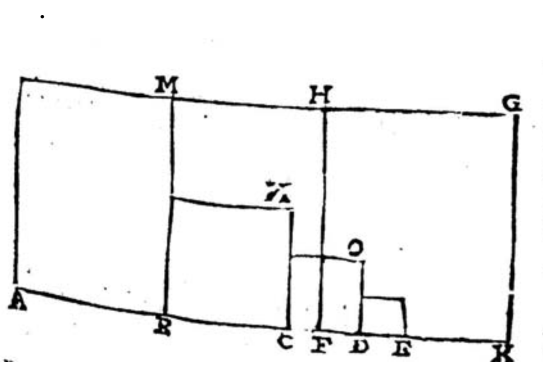
The ppd. on the square AB with height SL, is equal to [a] the series of cubes PK. Likewise the ppd. on the square EF with height NO, is equal to the series of cubes QR : moreover since the cubes AP and EQ are placed equal, also the squares AB and EF are equal. Whereby the said ppd's have the same bases; thus the said ppds, that is the series of cubes have the same ratio, as the heights SL and NO, that is as the series of ratio AB to DI, and of the ratio EF to HM : q.e.d. *a 165 huius.*

***PROPOSITIO CLXXII.***

Data sit quadratorum progressio solita, superque illis exstructa series cuborum, & inscripta parallelepipedo AG, cuius basis sit quadratum AB, altitudo AK, eadem nempe quae longitudino seriei cubicae.

Dico parallelepipedi ad seriem cubicam, eamdem esse rationem, quae est DA trium primorum laterum, ad latus primum AB.

***Demonstratio.***




**Prop. 172, Fig. 1.**

Linea AF aequalis fiat seriei rationis AB ad DE; erit parallelepipedum super quadrato AB in altitudine AF (quod vocemus parallelepipedum AH) aequale [b] seriei cubicae, sed parallelepipedum AG est ad parallelepipedum AH, (cum eadem sit basis utriusque) ut AK ad AF, hoc est [c] DA ad BA. Ergo parallelepipedum AG etiam erit ad seriem cubicam ut DA ad BA. Quod erat demonstrandum.

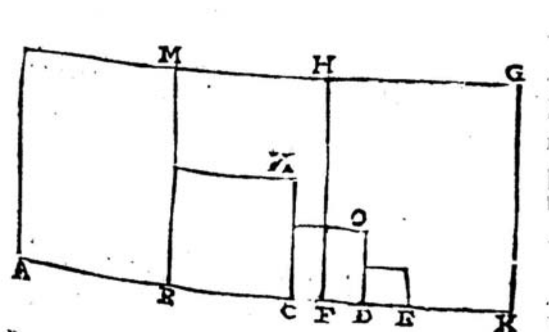
*b ibid; c 103 huius.*

***L2.§4. PROPOSITION 172.***

A single progression of squares is given, and upon these a series of cubes are formed, and inscribed in a ppd. AG, the base of which is the square AB, with height AK which is the same as the length of the series of cubes.

I say that the ratio of the ppd. to the sum of the series of cubes is the same as that which the length of the first three sides DA has to the first side AB.

***Construction &Demonstration.***




**Prop. 172, Fig. 1.**

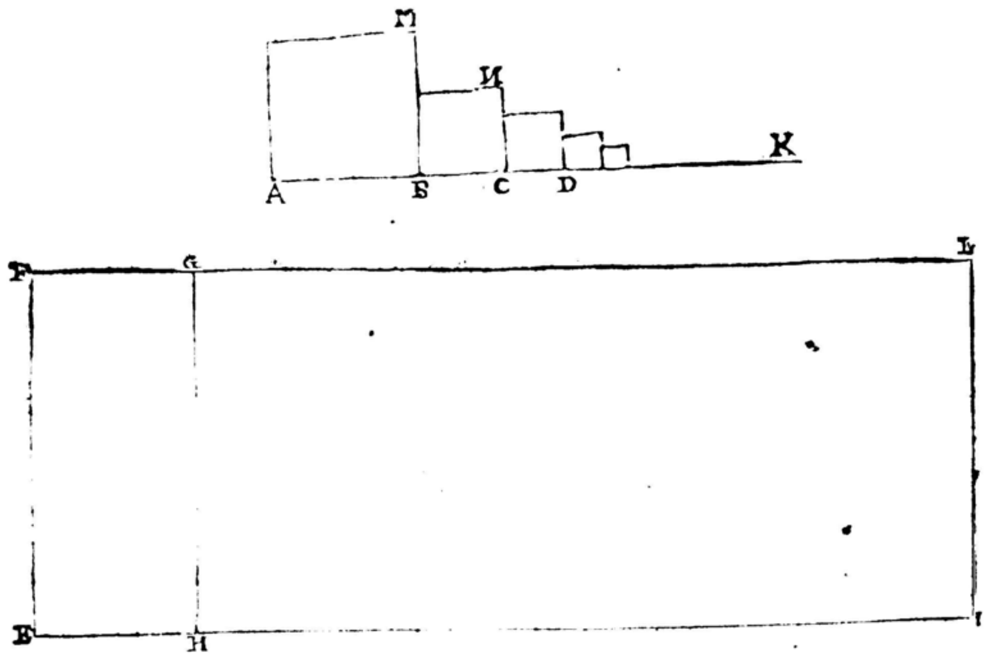
The line AF is made equal to the sum of the series of ratios AB to DE; the ppd. on the square AB with height AF (that we call the ppd. AH) is equal [b] to the series of cubes, but the ppd. AG is to the ppd. AH, (since both have the same base) as AK to AF, that is [c] DA to BA. Hence the ppd. AG also is to the series of cubes as DA to BA. Q.e.d.

*b ibid; c 103 huius.*

***PROPOSITIO CLXVIII.***

Data sit vt supra cuborum series. Oportet exhibere superficiem omnibus superficiebus omnium cuborum progressione datae aequale.

***Constructio &Demonstratio.***




**Prop. 173, Fig. 1.**

Flat [a] rectangulum EHGF aequale progressioni quadratorum AM, BN, &c. tum E I sextupla fiat lineae E H. Dico rectangulum F I esse id quod quaeritur. Cum superficies singulorum cuborum constet sex quadratis aequalibus, manifestum est omnes feriei cubicae superficies consitui ex sex seriebus quadratorum AM, BN, &c. atqui rectangulum H F ex constructione seriei quadratorum AM, BN. est aequale. Ergo sex rectangula F H, hoc eft ex construct. rectangulam FI constituet omnes seriei cubicae datae fuperficies. Fecimus ergo quod petebatur.

***L2.§4. PROPOSITION 173.***

A series of squares is given as above. It is necessary to show a surface equal to the sum of the progression of the surfaces of all the given cubes.

***Construction &Demonstration.***

The rectangle [a] EHGF is made equal to the sum of the progression of the squares AM, BN, &c. then E I is made six times the length of the line E H. I say that the rectangle FI is that which is requires.

Since it is agreed that the surfaces of the individual cubes consist of six equal squares, it is seen that the series of the surfaces of all the cubes are constituted from six series of squares AM, BN, etc, but the rectangle H F is equal to the construction of the series of squares AM, BN. Hence six rectangles F H, that is FI from the construction, make up all the given surfaces of the given cubes. Therefore we have done what was required.

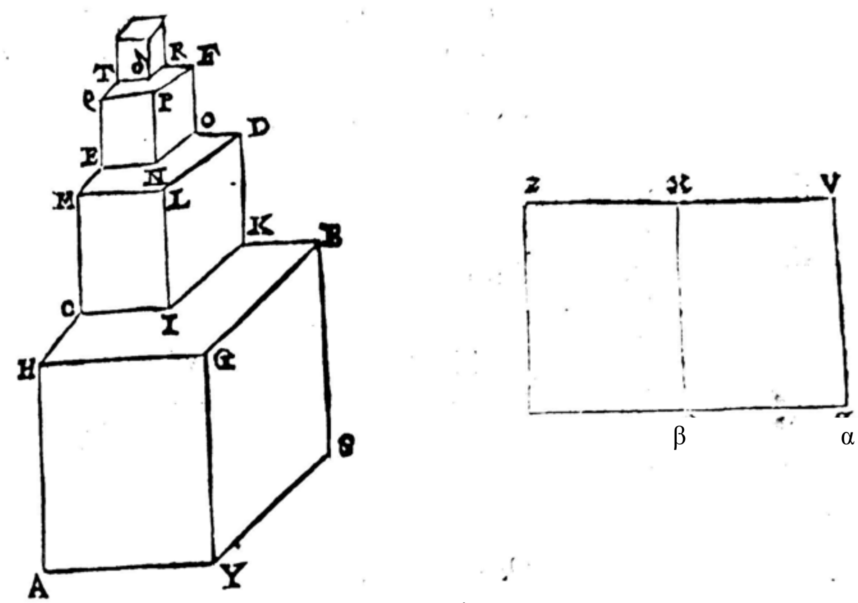
***PROPOSITIO C L X X I V.***

Data sit cuborum progressio sibi mutuo insistentium, constituens pyramidem cubicam.

Dico residuas basium superficies, nempc BKICHG , DONEML
& reliquas omnes in infinitum simul sumptas, quadrato primi cubi aequales esse.

***Demonstratio.***

Fiat enim seriei rationis primae A H ad tertiam EQ aequalis VZ; super quavis altitudine AH fiat rectangulum Zα, sumptaque V X aequali, A H ducutur ad
V$\alpha$ parallela X$\beta$: quae abscindat quadratum Xα aequale quadrato AG seu HB.
rectangulum $\alpha$Z per 79 huius aequatur seriei quadratorum A G, C L, EP, &c.
hoc est: (quonium cuborum plana omnia sunt quadrata aequalia) seriei quadratorum
HB, MD, QF.&c.ergo cum αX ex constr. quadrato AB aequale sit, erit reliquum
β Z rcliquae quadratorum seriei MD, QF. &c. aequa1e. Atqui series quadratorum [p. 163]




**Prop. 174. Fig. 1.**

MD, QF, etc. eadem est cum serie quadratorum KC, OE, &c. rectangulum
igitur $\beta$Z seriei quadratorum CK, OE, &c. aequatur. Quare cum rectangulum
$\alpha$Z & series quadratorum HB, MD, QF, &c. itemque rectangulum $\beta$Z
& series quadratorum CK, EO, &c., aequalia sint, etiam excessus rectanguli $\alpha$Z
super $\beta$Z, & excessus seriei HB, MD, *&c.* super seriem CK, EO, &c. aequales
erunt. Atqui excessus $\alpha$Z super $\beta$Z est $\alpha$X, id est ex construct. quadratum HB:
excessus vcro seriei quadratorum BH, MD, &c. super seriem quadratorum CK,
EO, &c. sunt figurae BKICHG, DONEML,FRδTQP, &c. ergo figurae
illae omnes simul sumptae aequantur quadrato HB. Quod erat demonstrandum.

***L2.§4.*** ***PROPOSITION 174.***

A progression of stacked cubes is given as shown, making a cubic pyramid.

I say that the remaining surfaces of the bases, truly BKICHG, DONEML, etc., and the rest of the like surfaces of the bases of all the cubes summed together to infinity, is equal to the square of the base of the first cube.

***Demonstration.***

For VZ is made equal to the sum of the series with the ratio of the third term AH of the given series to the first term EQ; upon which line the rectangle Zα is constructed with height AH; and with VX taken equal to AH, V$\alpha$ is drawn parallel to X$\beta$: which cuts off the square Xα equal to the square AG or HB. By prop. 79 of this book, the rectangle αZ is equal to the sum of the series of squares AG, CL, EP, &c.,

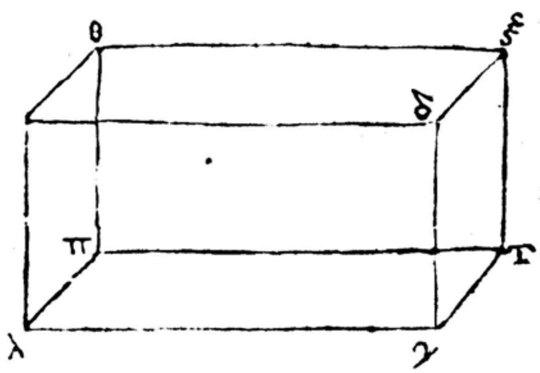
that is: (since the squares in all the planes of the cubes are equal) to the series of squares HB, MD, QF, etc. Hence, since by construction αX is equal to the square AB, the remainder *β* Z is equal to the rest of the series of cubes [p. 163] MD, QF, etc. But the series of squares MD, QF, etc., is the same with the series of squares KC, OE, etc. Therefore the rectangle *β*Z is equal to the series of squares CK, OE, etc. Whereby since the rectangle *α*Z is equal to the series of squares HB, MD, QF, etc. and likewise the rectangle *β*Z is equal to the series of squares C K, EO, etc., then also the differences of the rectangle *α*Z over *β*Z, and the series H B, M D, etc. over the series C K, EO, etc. are equal. But the difference of *α*Z over *β*Z is *α*X, that is the square H B from the construction: truly the figures BKICHG, DONEML, FRδTQP, etc are the difference of the series of squares BH, MD, etc. over the series of squares CK, EO, etc. are. Hence the sum of all these figures is equal to the square HB. Q.e.d.

[We can set AH = $a$; EQ = $ar^2$ ; then VZ is the sum of the series $a(1+r^2+r^4+....)=\frac{a}{1-r^2}$ . Thus, the rect. Zα $=a^2(1+r^2+r^4+....)=\frac{a^2}{1-r^2}$ ; consequently $\beta$Z = $a^2(r^2+r^4+....)=\frac{a^2r^2}{1-r^2}$ . Hence, $\alpha$Z - $\beta$Z = $a^2$; but this difference can be expressed as the differences of the squares BKICHG = $a^2(1-r^2)$; DONEML = $a^2r^2(1-r^2)$; etc.; the infinite sum of which is $a^2$.]

### *PROPOSITIO CLXXV.*

Iisdem positis,

Dico supcrficiem cubicae pyramidis, aequalem esse fuperficiei parallelepipedi γθ, cuius basis *γξ,* sit primi cubi quadratum, altitudo vero, γλqua aequalis seriei rationis primae AH ad tertiam EQ. Per supeficiem autem pyramidis cubicae intelligo hic superficies omnium cuborum, exceptis quadratis CK, EO, T R, &c.




Prop. 175. Fig.1.

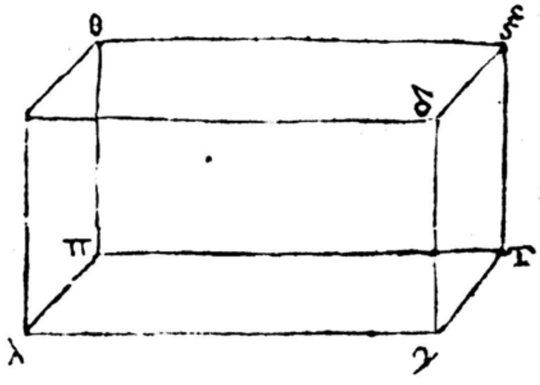
***Demonstratio.***

Rectangulum λδ continetur linea γλ, aequali seriei rationis AH ad EQ, & altitudine γδ, quae aequalis est AH; est enim quadratum γξ aequale quadrato AG : igitur [a] rectangulum λδ omnibus quadratis AG, CL, &c. aequale est : reliquae igitur hederae τθ, δθ, γπ aequales sunt seriei quadratorum oppositae seriei AG,CL, &c. & series quadratorum BY, DI, &c. nec non illi quae infra huius opposita est : est autem & quadratum γξ basis parallelepipedi aequalis quadrato [p.164] AS basi pyramidis cubicae, & per praecedentem; quadratum HB, id est λθ aequale est omnium basium residuis. Ergo tota superficies parallelepipedi, toti pyramidis cubicae superficiei aequalis est. Quod erat demonstrandum. *a 128 huius;*

### *L2.§4.* *PROPOSITION 175.*

With the same figures in place,

I say that the surface of the pyramid of cubes is equal to the surface of the parallelepiped γθ, the base of which is *γξ,* the square of the first cube, with height γλ, which is equal to the sum of the series with the ratio of the third term EQ to the first term AH. Moreover I understand that the surface of the cubic pyramid is the surface of all the cubes of the series with the exception of the squares CK, EO, TR, &c.




***Demonstration.***

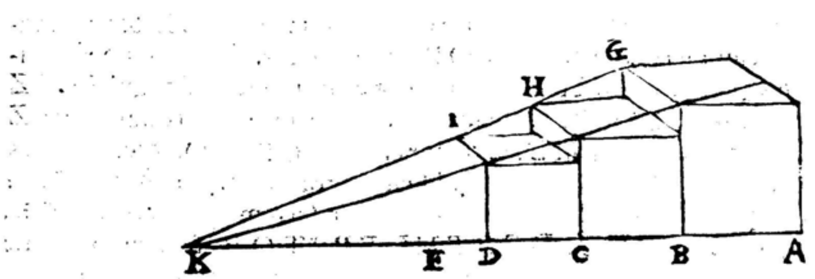
The rectangle λδ consists of the line γλ, equal to the sum of the series with the ratio EQ to AH, and with the height γδ, which is equal to AH; for indeed the square γξ is equal to the square AG : hence [a] the rectangle λδ is equal to the sum of all the squares AG, CL, etc.: therefore the rest of the rectangles on the sides τθ, δθ, γπ are equal to the series of squares opposite the series AG,CL, &c. & the series of squares BY, DI, &c. all these which are on opposite sides : moreover the square γξ is equal to the base of the ppd. which is equal to the base of the cubic pyramid [p.164] AS; and by the preceding theorem, the square HB, that is λθ, is equal to the sum of all the remaining bases. Therefore the whole surface of the ppd. is equal to the total surface of the cubic pyramid. Q.e.d. *a 128 huius.*

Prop. 175. Fig.1.

### *PROPOSITIO CLXXVI.*

Data sit quadratorum progressio cui terminus Iongitudinis sit K; fuper quadratis autem exstructa fit cuborum series. Deinde per 165 huius factum fit parallelepipedum MP, aequale seriei cubicae. Dico superficiem huius parallelepipedi , ad superficiem pyramidis cubicae (sumendo hic superficiem pyramidis cubicae , vt in propositione prescedenti sumpsimus) eam habere rationem, quam linea aequalis seriei rationis A B prima: ad DE quartam, vna cum dimidia ipsius AB, habet ad aequalem seriei rationis primae A B ad CD tertiam, vna cum dimidia AB.

***Demonstratio.***




A progression is given AB, BC, CD, &c. with the terminus in K, and upon the lines squares, and again upon the squares, cubes are constructed. It is required to inscribe the series of cubes in a pyramid having a square base.

***Demonstratio.***

Parallelepipedum M P factum eft aequale seriei cubicae , igitur latus MN aequale est [a] seriei rationis AB ad DE, & NO,OP aequales sunt singulae ipsi AB. Fiat iam super quadrato R T quod sit aequale quadrato A B , paral1elep[illegible], aequali seriei rationis A B ad CD.

**Prop. 176, Fig. 1.**

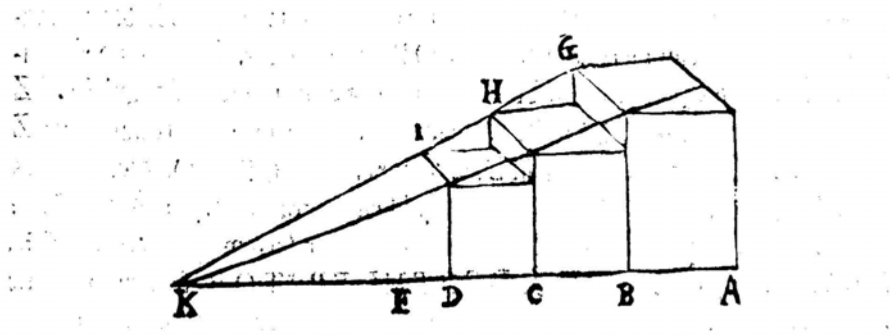
Ergo per praecedentem superficies parallelepipedi QT [p.165] aequalis erit superficiei cubicae pyramidis. Deinde superficies parallelepipedi M P aequalis est rectangulo [b] quod linea composita ex quadrupla M N & dupla NO,& altitudine NO sive OP continetur. Similiter supcrficies parallelepipedi Q T aequalis eft rectangulo cuius basis sit composita ex quadrupla QR & dupla RS ; altitudo vero RS sive ST (sunt enim R S, ST aequa1es) quia latera aumt quadrati R T. Quare cum dicta rectangula sint vt bases (altitudines enim NO, R S aequalcs habent eidem AB, ideoque aequales inter se) etiam erit parallelepipedi MP superficies, ad superficiem paralle1epipedi QT ut basis ad basim ; nempe ut composita ex quadrupla MN & dup1a NO, ad compositam ex quadrupla QR & dupla R S. Atqui ut quadrupla MN cum dupla NO, ad quadruplam Q R cum dupla R S, sic M N cum dimidia NO ad QR cum dimidia R S; ergo superficies parallelepipedi MP, est ad superficiem parallelepipedem QT, hoc est ad superficiem pyramidis cubicae, ut MN cum dimidia NO, hoc est ut series rationis A B ad D E cum dimidia AB, ad QR cum dimidia RS , hoc eft ad seriem rationis A B ad CD cum dimidia A B: quod erat demonstrandum. *a 165 huius; b 8 sexti;*

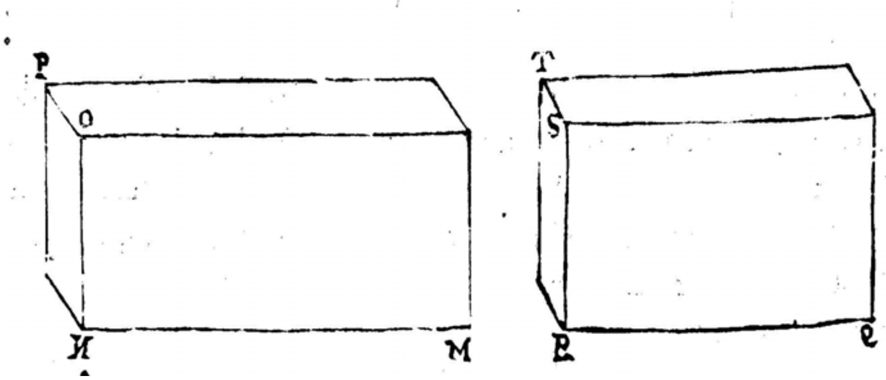
***L2.§4.*** ***PROPOSITION 176.***

A progression of squares is given the terminus of which is at a distance K; moreover on the squares a series of cubes is set up. Then by Prop. 165 of this book, a parallelepiped MP is made equal to the sum of this series of cubes.

I say that the surface of this parallelepiped is in a ratio to the surface of the pyramid of cubes (by taking this surface of a cubic pyramid, as we assumed in the previous proposition) equal to the ratio a line equal in length to the series in the ratio of the fourth term DE to the first term A B, together with half of AB, has to an equal series taken in the ratio of the third term CD to the first term AB, together with half of AB.

***Demonstration.***



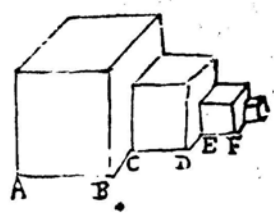



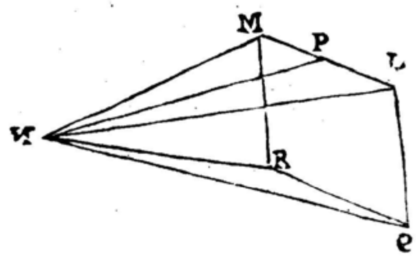

The parallelepiped M P is made equal to the sum of the series of cubes, therefore the side MN is equal to a series in the ratio DE to AB [a], and NO and OP are equal to the particular term AB. Now on the square RT that is equal to the square AB, a paral1elepiped QT is made, with height QR equal to the series in the ratio CD to AB. Hence by the preceding Proposition, the surface of the parallelepiped QT [p.165] is equal to the surface of the cubic pyramid. Then the surface of the parallelepiped M P is equal to a rectangle [b] composed from a line four times the length of M N and twice NO, and with height NO or OP. Similarly the surface of the ppd. Q T is equal to a rectangle of which the base is four times QR and twice the height RS, and with height RS or ST (for R S and ST are equal) since the sides are the squares R T. Whereby since the said rectangles are as the bases (for the heights NO and R S are equal to the same length AB, and so are equal to each other) also the surface of the ppd. MP, to the surface of the ppd. QT are as base to base ; truly as they are composed from the quadruple of MN and double NO, to that composed from the quadruple of QR and the double of R S. But as four times MN with twice NO, to four times QR with twice R S, thus MN with half of NO to QR with half of R S; hence the surface of the ppd. MP is to the surface of the ppd. QT, that is, the surface of the pyramid of cubes, as MN with half NO, that is as the sum of the series of ratios D E to AB as half AB, to QR with half RS , that is as the sum of the series of ratios CD to A B with half A B: Q.e.d. *a 165 huius; b 8 sexti;*

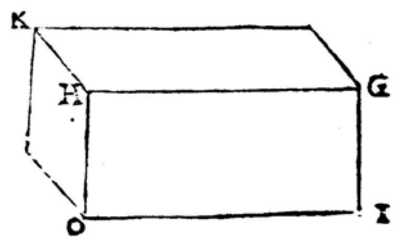
### *PROPOSITIO CLXXVII.*

Proportionem exhibere quam superficies pyramidis habet ad inscriptae sibi pyramidis cubicae superficiem : eo modo intelligendo
superficiem seriei cubicae, quo in pracedenti propositione.

### *Constructio &Demonstratio.*



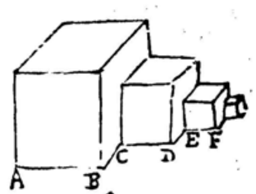



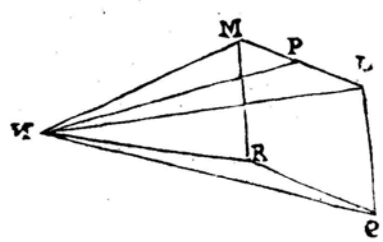

Assumemus hoc loco pyramidem isosceliam, facilitatis gratia. fit ergo pyramis QLMRN isoscelis, cuius basis sit quadratum QM, cui pyramis cubica AB, CD, &c. inscripta intelligatur, factoque quadrato OK aequali quadrato AB reperiatur linea GH aequalis seriei rationis AB primae ad tertium EF; & super quadrato in altitudine GH, fac parallelepipedum, cuius superficies aequebitur [a] pyramidis cubicae. Dico ut rectangulum super dupla LN & L M tamquam una recta, [p.166] in altitudine LM, est ad rectangulum super quadrupla GH & dupla AO tamquam una recta, in altitudine HO, sic pyramidis includentis superficies, ad superficiem inclusae pyramidis cubicae, Ducatur enim ex vertice pyramidis N ad LM normalis NP, quae ut ex datis facile colliges, bisecat L M in P: rectangulum igitur NLP duplum est trianguli LPN, ut patet ex elementis; ergo rectangulum NLP aequale est triangulo LMN. & rectangulum NLP, aequale eft triangulo LMN. & rectangulum NLM duplum est triangulum LMN. ergo rectangulum super dupla LN, in altitudine LM, est quadruplum triangulum LMN, hoc est aequatur toti superficiei pyramidis praeter basim; quare rectangulum super dupla L N & LJ tamquam una recta in altitudine L M, aequatur toti [a] superficies pyramidis. simili discursu demonftrabimus rectangulum super quadrupla GH & dupla HO tamquam una recta, in altitudine HO aequari superficiei parallelepipedi , ergo superficies pyramidis N, est ad superficiem parallelepipedi, hoc eft ex conftruct. ad
superficiem pyramidis cubicae , vt funt dicta rectangula inter se, Exhibuimus ergo,&c. quod petebatur

**Pro**

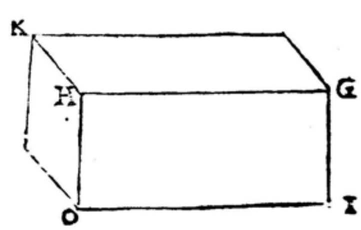
*a 1. secundi.*

*Libri secundi finis.*




ubes inscribed in the
previous proposition.



Here we will assume an isosceles pyramid in place, for the sake of simplicity. This is therefore the isosceles pyramid QLMRN, the base of which is the square QM, and the cubes AB, CD, etc. are understood to be inscribed, and with the square OK made equal to the square AB the line GH is found to be equal to the sum of the series of ratios of the first AB to the third EF [the actual ratio in the progression being EF to AB of course]; and upon the square with height GH, construct a ppd., the surface of which is equal to the pyramid of cubes[a]. I say that as the rectangle on twice LN and LM as one line, [p.166] with height LM, is to the rectangle upon the quadruple of GH and with twice AO as one line, with height HO, thus the surface of the enclosing pyramid to the surface of the enclosed pyramid of cubes. For from the vertex of the pyramid N to LM a normal NP is drawn, which is easily understood from what is given, bisects LM in P: therefore the rectangle NLP is twice the triangle LPN, as is apparent from the Elements; hence the rectangle NLP is equal to the triangle LMN; and the rectangle NLP, is equal to the triangle LMN; and the rectangle NLM is twice the triangle LMN. Hence the rectangle of twice LN, with height LM, is four times the triangle LMN, that is equal to the whole surface of the pyramid except the base; whereby the rectangle on twice LN and LJ taken as one line with height L M, is equal to the total surface of the pyramid[a]. By a similar argument we can show that the rectangle on the quadruple of GH and twice HO considered as one line, with height HO is equal to the surface of the ppd. Hence the surface of the pyramid N, is to the surface of the ppd., that is from construction to the surface of the pyramid of cubes, as the said rectangles are to each other. Therefore we have shown what was required. *a 1. secundi.*

*End of the Second Book.*