HOROLOGII OSCILLATORII

Part TWO (continued).

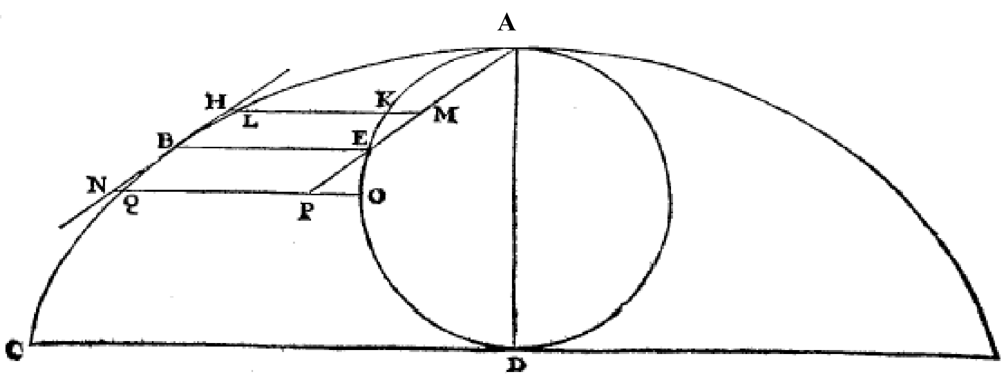
Concerning the descent of bodies under gravity and the motion of these on a cycloid.

PROPOSITION XV.

For a given point on a cycloid, to draw a line through that point tangent to the cycloid.

Let ABC be the cycloid, and B some given point on the curve, through which it is required to draw the tangent.

Around the axis AD of the cycloid is drawn the generating circle AED, and BE is drawn parallel to the base cycloid, which crosses the said circle at E, and AE is joined, and to this finally HBN is constructed parallel through B. I say that this line is a tangent to the circle at B.

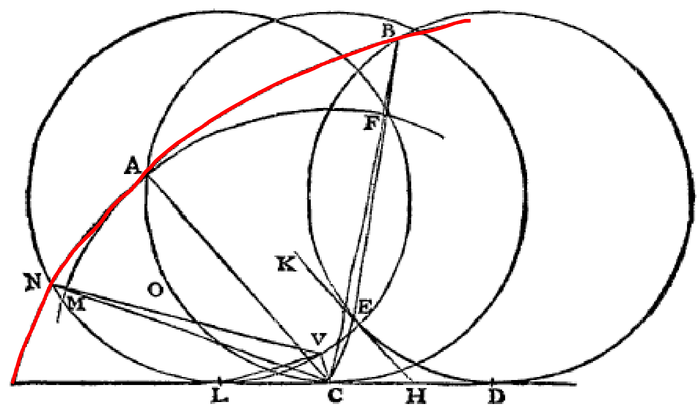


Indeed some point on the cycloid different from B is considered, and in the first place some higher point such as H is taken, and through H is drawn a line parallel to the base of the cycloid, which crosses the cycloid in L, the circle AED in K, and the line AE in M. Therefore, since KL is equal to the arc KA, and since the line KM is less than the arc KE, the line ML is less than the arc AE, that is to say, to the line EB, or MH; hence it is apparent that the point H lies outside the cycloid.

In the next situation a point N is taken on the line HN below B, and through N is sent forth, as before, a line parallel to the base, which crosses the cycloid in Q, with the circle AED in O, and with the line AE produced in P. Therefore since OQ is equal to the arc OA; moreover OP is greater than the arc OE; and PQ is less than the arc EA, that is, the line EB, or PN. Thus it is again apparent that the point N lies outside the cycloid. Therefore when some point besides B is taken on the line HBN, it shall be outside the cycloid, and it is agreed that the line NG through the that point B is a tangent; QED.

I have doubted whether I should relinquish this demonstration here in place of that found by the most distinguished Wren and published in Wallis's book *de Cycloide,* since my demonstration is not greatly different from that. Moreover, it is possible to absolve the proposition to be agreed upon from a general construction, which can be applied not only to the cycloid but for other curves arising from the rotation of some figure ; provided the figure is concave in the same part [as the generating circle], and arises from these that are called geometrical. [p. 40] [Huygens now proceeds to show that the line CA is normal to the tangent at A, and the neighbouring points N and B on the cycloid lie outside this circle.]

If indeed the curve NAB, arisen from the rotation of a figure along the straight edge LD; truly by describing the point N taken on the circumference of the figure OL. And it is required that the tangent to the curve is drawn at the point A. The line CA is drawn from the point C, where the figure is a tangent to the line at the point describing the line at A : since the point of contact can always be found, if indeed the problem can be reduced to that in order that two lines parallel to each other can be drawn, of which the one may cross through a point describing the revolution of the given figure and the other is a tangent to the figure, and which stand apart from each other as much as the given point A from the line LD : I say that CA crosses the curve at right angles, or the circumference MAF described with centre C and with radius CA, touches the curve at the

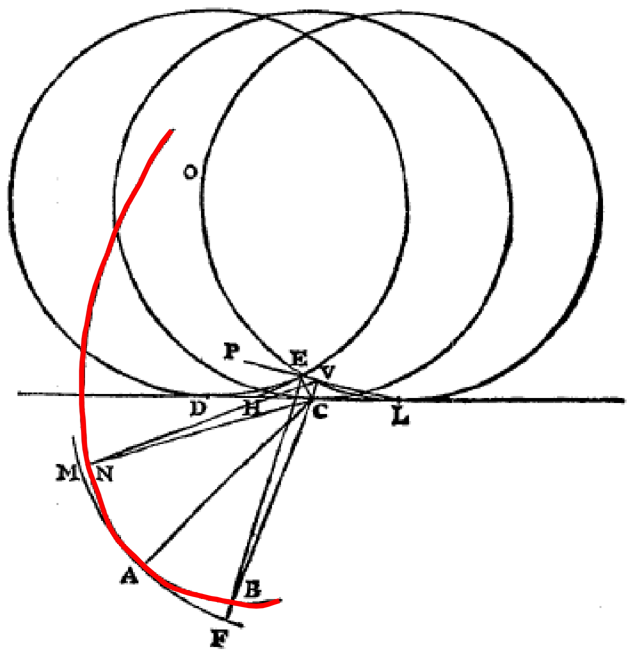


point A, thus the perpendicular to AC drawn through A is a tangent to the curve at that point.

Indeed CB is drawn first to the point of the curve B, which is more distant than the point A from the level line LD, and it is understood that the rotating figure is in the position BED, with the describing point at B, with the contact to the ruler line at D, than with the point of the curve at C, when the point describing the curve was at A, and C is now raised to the point E ; and EC and EB can be joined, and KH is a tangent to the figure at E, crossing the ruler at H.

Therefore, since the line CD is equal to the arc of the curve ED; truly the sum of EH and HD is greater than the length of the arc ED ; from which EH will be greater than CH. Hence the angle ECH is greater than CEH, and hence the angle ECL is smaller than the angle CEK. And on adding the angle KEB (equal to LCA) to KEC, there is the angle CEB: and on taking away from ECL the angle LCB, there is the angle ECB. Therefore the angle CEB is always greater than the angle ECB. And thus in the triangle CEB, the side CB is greater than EB, but EB appears to be equal to CA, since it arises from the same figure moved. [p. 41]. Therefore CB is also greater than CA, or CF. From which it is understood that the point B lies outside the circumference MAF.

Again let a point N on the curve be taken between the line LD and the point A. Since the describing point is at N, the position of the [rolling] figure can be put at VL, and the point of contact L, truly the point which before was a tangent at C, shall now be raised to V : and CN, NV, VC, and VL are joined. Therefore VN is equal to CA [chords of equal length]; indeed it is CA itself translated to VN. Since now the line LC is equal to the arc of the curve LV, and hence is greater than the straight line LV, in the triangle CLV, the angle LVC is greater than LCBV. Whereby by adding the angle LVN to LVC, the whole angle NVC shall certainly be made greater than LCV, and hence generally greater than the angle NCV, which is a part of LCV. Therefore in the triangle CVN, the side CN is greater than the side VN, which is equal to CA, and hence CN is greater than CA also, or CM. Thus it is apparent that the point N falls outside the circle MAF, which hence is the tangent at the point A. Q.e.d.



Moreover, it is the same construction and demonstration too, for a curve arising by describing a point, which is taken either inside or outside the curve of the rolling figure. Except that, in this latter case, a certain part of the curve falls below the ruler, thus there is some difference arising from the demonstration

Indeed let A be a point, through which the tangent is drawn, in a given part of the curve NAB, which has dropped below the ruler CL, obviously described by a point N taken outside the rolling figure, but having a particular position in the same plane of this figure. [p.42]. Therefore by finding a point C, where the rotating figure is tangential to the ruler CD when the describing point is at A, the straight line CA is drawn. I say that this line crosses the NAB at right angles, or the circumference of the circle described with radius CA, and with centre C, is a tangent to the curve NAB at the point A. Moreover, it will be shown the touch the circle externally, since in the part of the curve above the ruler CD it will touch from within.

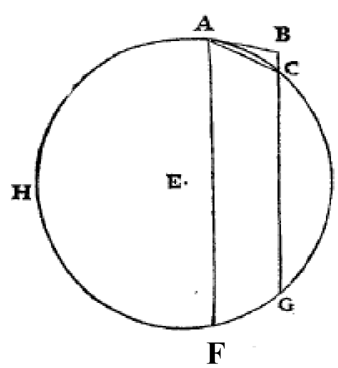
Indeed with the same position and description for everything from before, the angle ECH is again shown to be greater than the angle CEH; and with the angle HCB added to the angle ECH, the angle is ECB; and from the angle CEH is taken HEB, which is equal to DCA, and the angle shall be CEB. Therefore the angle ECB is generally greater than CEB. Hence in triangle ECB, the side EB is greater than CB. But EB is equal to CA, or CF. And hence CF is greater than CB: and thus the point F on the circumference is beyond the curve NAB and more distant from the centre.

Likewise it will be shown again that the angle LVC is greater than the angle LCV. Whereby CVP, which with LVC is equal to two right angles, is less than VCD. And by adding to VCD the angle DCN, the angle is equal to VCN; and by taking the angle PVN from the angle CVP, the angle is CVN. Therefore the angle VCN is generally greater than CVN. Thus in the triangle CVN, the side VN is greater than CN. Moreover, VN is itself equal to CA or CM. And hence CM is greater than CN, and thus the point on the circumference M lies more distant from the centre C, beyond the curve NAB. And thus it can be agreed that the circumference MAF is a tangent to the curve at the point A. Q.e.d.

Concerning which, if the tangent drawn through that point of the curve where the ruler also cuts the curve, then the tangent sought will always be perpendicular to the ruler; as is easily shown.

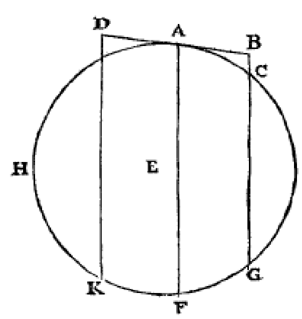
PROPOSITION XVI.

If two parallel lines AF and BG cut the circumference of a circle, of which the centre is E, and each of which crosses at the same side from the centre, or to the other AF passes through the centre itself, and from the point A which cuts the circumference closer to the centre, is drawn a tangent line : I say that the part of this AB, between the parallel intercepts, is less than the arc AC.



Indeed the subtended chord of the arc AC is drawn. Therefore since the angle BAF is equal to that which is caught by the part of the circle AHF, which is either greater than the semicircle or is the semicircle; hence the angle BAF, [p. 43] is either less than a right angle or is a right angle; and thus the angle ABC is either greater than a right angle or is a right angle. Whereby in triangle ABC, the side AC subtended by the angle B will be greater than the side AB. But the same side AC is less than the arc AC. Therefore in general, AB is less than the arc AC.

PROPOSITION XVII.

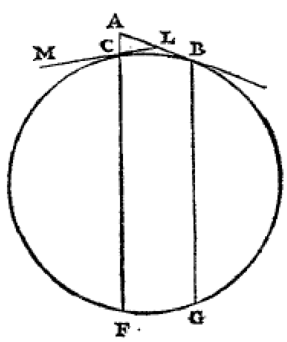


With the same points and lines in place, if a third line DK parallel to the previous lines has cut the circle, from which AF is closer to the centre, it will stand at just as large a distance as the remaining BG : I say that the part of the tangent at A, from the final parallel line added, and the mean intercept, surely AD, is less than the arc intercepted by the first two parallel lines.

This indeed is apparent when AD is equal to AB, since before we have shown that the arc AC is smaller.

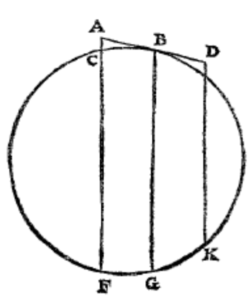
PROPOSITION XVIII.

If two parallel straight lines AF and BG cut the circle with centre E; and from the point B, which is either the more distant from the centre, or equidistant when the other line is also the same distance from the centre, a line is drawn running to the circumference* [at G]*, a line is drawn tangent to the circumference at B* [p. 44] *: the part BA of this line, between the parallel intercepts, will be greater than the arc BC arising from the same parallel intercepts.



For indeed through the point C, a line MCL is drawn tangent to the circumference, which crosses the tangent BA in L. Therefore in the triangle ACL, the angle C is equal to the angle MCF, that is, to that which is fixed by the part CBF of the circle. Moreover, the angle A is equal to that which is fixed by the portion of the circle BCG, which portion shall be greater or equal to the portion CBF, obviously when BG is either at a greater distance from the centre than CF, or just as far: hence the angle A of the triangle ACL is less or equal to the angle C : and consequently the side CL is either less than or equal to the side AL. And CL together with LB are greater than the arc CB. Therefore AL together with LB, that is, the tangent AB, is greater than the same arc CB. Q.e.d.

PROPOSITIO XIX.

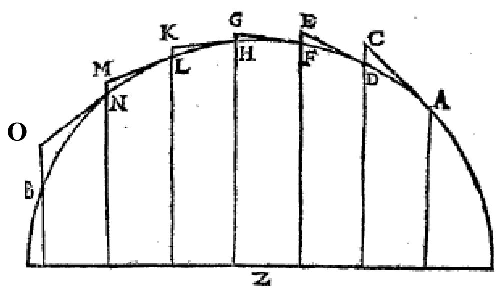


With the same situation, if a third line DK parallel to the previous lines cuts the circle, which is just the same distance from that which is further from the centre, as this line is from the remaining AF : There is a section of the tangent at B, [p. 45] ***from the mean of the parallel lines, which intercepts the final line added DK, which truly is BD, and which is greater than the arc BC.***

This indeed can be shown since BD can be made equal to BA itself, which we have shown to be greater than the arc BC.

PROPOSITION XX.

If the arc of a circle AB, less than half the circumference, is cut by parallel lines into a number of parts, such that regular intervals are constituted between the arcs themselves, and parallel lines are drawn from the end other than A, etc, such as CD, EF, GH, KL, &c, and to all the remaining points of the sections. Lines are drawn tangent to the circumference, each in its own section, and so that each and every one crosses the closest of the said parallel lines; in this manner the tangents are AC, DE, FG, HK, &c. I say that the sum of these tangents, with the first AC taken away, is less than the given arc AB. Truly the sum of all of these, without omitting AC, is greater than the arc AB diminished by the extreme part NB, that is, the sum is greater than the arc AN.



Indeed for the first part of the proposition, we can put some parallel lines in place to cross from one side of the centre Z, and let GH be one of these which is from the side B, which is nearest the centre, or which cuts across the centre itself. Thus all the tangents between GH & BO are taken, as HK, LM, and NO are each smaller than their own arcs (Prop.16). Again moreover , the tangent GF, is less than the following arc FD (Prop.17), & similarly the tangent ED with the arc DA. Hence all the tangents between BO & CD added together are less than the arcs BH & FA, and hence generally less than the arcs BH and HA, which is the arc BA, which was to be shown first.

Now again, we will show that the sum of the tangents between BO and A is greater than the arc AN. Indeed the parallel line GH, either passes closer to the centre Z than the parallel line EF, when I put the nearer to be from those which crosses from the side away from A [p. 46], or it can be further away from the centre, or equidistant.

For if EF is further from the centre or at the same distance as GH, the tangent FG is greater than its own arc FH, and the rest of the tangents towards A, obviously ED and CA are greater than their own individual arcs (by P.18) ; thus the sum of GF, ED and CA is greater than the arc HA. But the tangent LM is greater than the arc HL (by P.19), and the tangent NO with the arc LN ; and hence with more tangents, with HK itself added, the sum of all the tangents taken between A and B are greater than the arc AN.

If indeed GH is placed further from the centre than EF, the tangent KH is greater than the arc HF (by P.19), and the tangent ML as before is greater than the arc LH, and the tangent ON is greater than the arc NL, and hence the sum of all the tangents ON, ML, and KH is greater than the arc NF. And moreover, the tangent ED is greater than its own arc FD (by P.18), and similarly the tangent CA is greater than its own arc DA. Hence all the tangents added together between BO and A, are greater than the arc NA.
[Thus : ON + ML + LH + GF + FD < arc AB < ON + ML + LH + GF + EF + AC + arc NB]

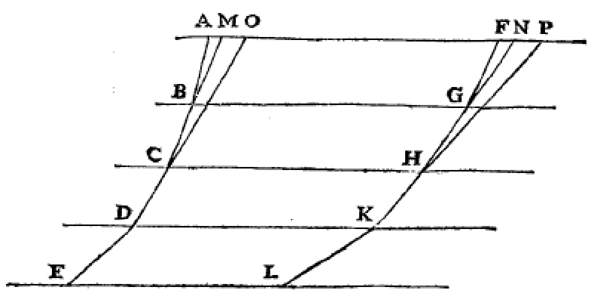
From these truly the demonstration can be shown for the other cases, whatever the arc of the semicircle is taken, since obviously it will either be the same everywhere, or just a part of the preceding demonstration.

PROPOSITION XXI.

If a body descents by moving down continually along some adjoining inclined planes, and again falls from an equal height through the same number of contiguous inclined planes, thus by comparison with the individual heights, they correspond with the former individual planes, but with the inclination greater than these. I say that the time of descent for those with the greater inclination is greater than the time of descent for those with the smaller inclination.

[Note the use of the word inclination, the greater the leaning or inclination, the *smaller* is the angle of the inclined plane to the horizontal.]

Two series of planes ABCDE and FGHKL are taken between the same horizontal parallel lines, and thus in order that each two corresponding planes of each series are included between the same horizontal parlor lines ; truly each one of the series FGHKL has a greater inclination to the horizontal than the plane corresponding to its own altitude of the series ABCDE. I say that the descent along ABCDE is to be completed in a shorter time than along FGHKL.

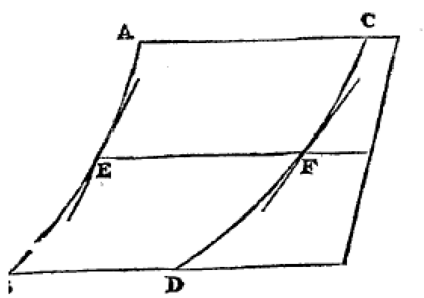


For indeed in the first descent along AB, it is agreed that the time is shorter for the descent along FG, when the ratio of the times shall be the same as the ratio of the lines AB to FG (by P.7), and AB is less than FG, on account of the smaller inclination. Now the lines CB and HG are produced upwards, and cross the horizontal AF in M and N. Thus the time to pass through BC after AB, is equal to the time along the same BC after MB, since it reaches the same speed at B, either by passing along AB, or by descending along MB (by P.6). And likewise, the time to pass along GH after FG, is equal to the time to pass along the same GH after NG. Moreover the time to travel along BC after MB to the time to pass along GH after NG, shall be as the length BC to GH, or as CM to HN, since they have this ratio for the times both for the total times MC, NH, and for the parts MB, NG (by P.7), and hence also for the remaining times; and BC is less than GH on account of the smaller inclination. Therefore it is apparent that the time to pass along BC after AB, is less than the time to pass along GH after NG or after FG.

Similarly it can be shown, with DC and KH produced upwards, in order that they cross the horizontal line AF in O & P, the time to pass along CD after ABC, or after OC, is shorter than the time to pass along HK after FGH or after PH. And finally the time to pass along DE after ABCD, is shorter than the time to pass along KL after FGHK. Whereby the total time of descent along ABCDE, will be shorter than the time to descend along FGHKL. Q.e.d.

Hence it is established, by considering curved lines as composed from innumerable straight lines, if there are two surfaces, following curved lines inclined at the same height, and of which for any points of the same height, the one is always inclined more than the remaining one, also the time for a body to descent along the more inclined surface is greater than along the less inclined surface.

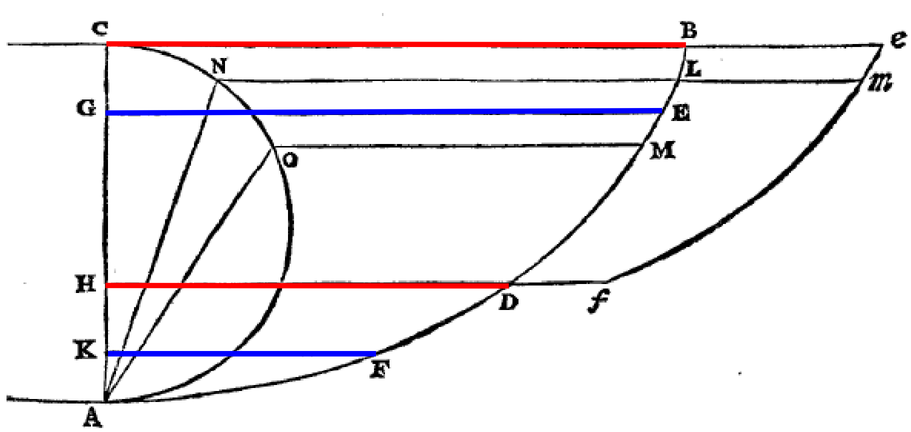
Just as if there were two inclined surfaces following the curves AB and CD, of equal height, of which for some points of equal height E and F, the inclination of CD shall be greater than the inclination of AB [in the sense that it leans more, though of course the angle is smaller; a possible point of confusion for English speaking people, where the greater the angle of inclination, the more upright the curve is.], that is, [p. 48] as the line tangent to the curve CE at F, shall have a greater inclination to the horizontal, than that which the tangent to the curve AB has at the point E. The time for the descent along AB is shorter than the descent along CD.



Likewise it follows that if one of the lines were straight : since the inclination of the line, which is straight everywhere, will be either greater or less than the inclination of the curve at some point.

PROPOSITION XXII.

If with a Cycloid of which axis stands upright to the vertical, with the vertex seen to be pointing downwards, two parts of the curve with the same height are taken, but one of which is closer to the vertex ; the time of descent along the upper will be less than the time of descent along the lower.



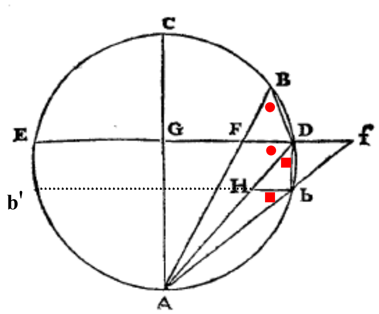
Let AB be the Cycloid, the axis of which AC is along the vertical, and the vertex A is observed to point downwards ; and the portions BD & EF of this curve are taken of the same height, that is, of such a nature that the parallel horizontal lines BC and DH, which include the upper part BD, are the same distance between each other [p. 49] as EG and

FK, which include the lower part EF. I say that the time of descent along the curve BD to be shorter than the time along EF.

For on BD some point L is taken, & on EF some point M, thus in order that the height of E above M and of B is above L shall be the same. And with a semicircle described on the axis AC, the horizontal lines LN and MO cross this at N & O, & NA and OA are joined. Thus, with the point N to be higher than the point O. Moreover, NA is parallel to the tangent to the curve at the point L (by P.15), and OA is parallel to the tangent to the curve at M. Therefore the curve BD at the point L has a smaller inclination [to the vertical] than the curve EF at the point M. Therefore since if the portion EF [between the blue lines], without changing the inclination, is understood to be taken higher just as at *ef*, thus in order that for the portion BD taken between the same parallels [coloured red here, and not so in the original], the point M at *m* can be found to be equal in height to the point L. Also the inclination if the curve *ef* at the point *m*, which is the same as the inclination of the curve EF at M, is greater than the inclination of the curve BD at L. Similarly, indeed, for any other point of the curve *ef*, the inclination can be shown to be greater than for the point at the same height in the curve BD. Thus the time of descent along BD from some point at the same height is shorter than for the time along *ef*, (by P.21), or, which is the same, along EF. Q.e.d.

LEMMA

AC shall be the circle with diameter AC, that DE cuts at right angles, and from the end of the diameter A a line AB can be drawn crossing the circumference at B, and DE in F. I say that these three lines AB, AD and AF are in proportion.



In the first place, the point of intersection F is within the circle ; and the subtended line BD is drawn to the arc. Therefore, since the arcs AE and AD are equal the angles to the circumference standing on the arcs, EDA and ABD, are themselves equal. [p. 50] Thus in the triangles ABD and ADF, the angles ABD and ADF are equal. Moreover the angle A is common to each. Hence, the said triangles are similar, and thus BA is to AD as AD is to AF. [from the similar triangles ABD and ADF : BA/**AD** = **AD**/AF are in proportion as required.]

Now in second place, the point of intersection f shall be outside the circle, and bH is drawn parallel to DE, which crosses the line AD in H. Thus following what has been shown already: as DA is to A*b*, thus A*b* is to AH that is, thus A*f* to AD : and thus again A*f*, AD and A*b* are in proportion. Whereby the proposition is agreed upon.
[from the similar triangles AbH and ADf : Hb/Df = Ab/Af = AH/AD or Af/AD = Ab/AH ; and it follows by applying the same argument to the arcs Ab' and Ab, that DA/**Ab** = **Ab**/AH from the similar triangles ADb and AbH. Note that the coloured marks and the dotted line have been added to the original diagram for clarity.]

PROPOSITION XXIII.

Let ABC be a Cycloid, the vertex of which points downwards, with the axis AD along the perpendicular; and by taking some point B on this curve, a line BI is drawn downwards tangent to the cycloid, and which is terminated by the horizontal line AI. The line BF is drawn perpendicular to the axis, and FA is cut in two equal parts by X, upon which the semicircle FHA is described. Then through some point G taken on the cycloid curve BA, the line ΣG is drawn parallel to BF, which crossed the circumference FHA in H, and the axis AD in Σ. It is understood that through the points G and H there is a tangent to each curve, and the intercepts of these tangents shall be the parts MN and ST within the same two horizontal lines MS and NT. Between these lines MS and NT are included the part OP of the tangent, and the part QR of the axis DA .

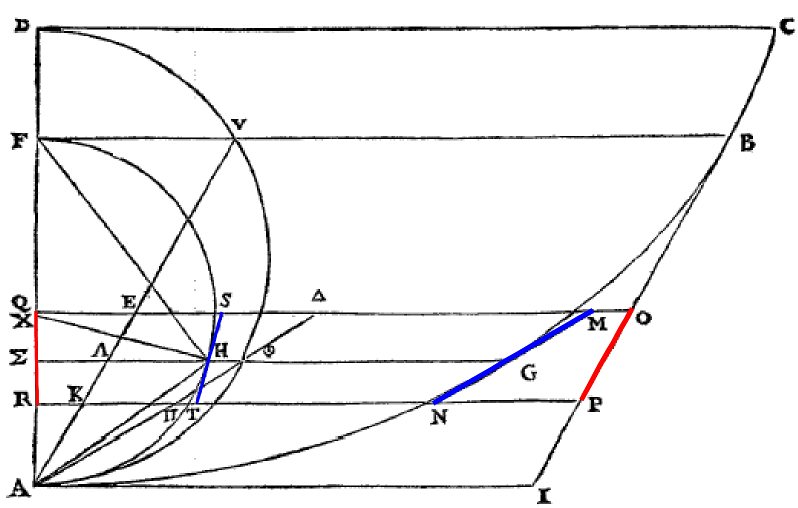
Thus from these properties established, I say that the time by which a body passes along the line MN, with a constant speed equal to that acquired by descending along the arc of the cycloid BG, is to the time by which it crosses the line OP, with a constant speed equal to half of that which it acquired falling along the whole tangent BI, as the tangent ST is to the part of the axis QR.

For the semicircle DVA is described on the axis AD and cutting the line BF in V, ΣG in Φ, & AV is joined, cutting the lines OQ, PR, GΣ in EK & Λ. Likewise HF, HA, HX & AΦ are joined; and this last line cuts the lines OQ and PR in the points Δ & π.

Therefore the said time to pass along MN to the time to pass along OP can be found from the ratio composed from the lines MN to OP, and from the inverse ratio of the [average] speeds by which these distances are traversed, (Prop. 5. Galil. *Concerning Uniform Motion*.), that is, from the ratio of half the [final] speed from BI or from FA, to the speed from BG, or from FΣ (Prop.8).

[Thus, in an obvious notation, $\frac{T_{MN}}{T_{OP}}=\frac{d_{MN}}{d_{OP}}/\frac{\bar{v}_{MN}}{\bar{v}_{OP}}=\frac{d_{MN}}{d_{OP}}/\frac{v_{MN}}{v_{OP}}$; for a body released from rest at B, and accelerating down the inclined plane of the tangent at B, (or falling from rest vertically), the final speed v is twice the average speed $\bar{v}$.]

And the ratio of the final speed from falling FA to the final speed from falling FΣ, [p. 51]



is equal to the ratio of the square root of the length FA to FΣ, and hence is the same as FA to FH.
[For by similar triangles, FA/FΣ = FA/FH × FH/FΣ = $(FA/FH)^2$; and $\frac{v_{FA}}{v_{F\Sigma}} = \sqrt{\frac{FA}{F\Sigma}} = \frac{FA}{FH}$.]
Hence half the final speed from falling a distance FA from rest to the final speed from falling FΣ will be as FX to FH. And thus the said time for passage through MN to the time through OP has its ratio composed from the ratios MN to OP, and from FX to FH.
[*i. e.* $\frac{v_{FA}/2}{v_{F\Sigma}} = \frac{\bar{v}_{OP}}{v_{MN}} = \frac{FX}{FH}$; and $\frac{T_{MN}}{T_{OP}} = \frac{d_{MN}}{d_{OP}} / \frac{\bar{v}_{MN}}{\bar{v}_{OP}} = \frac{MN}{OP} / \frac{FH}{FX} = \frac{MN}{OP} \times \frac{FX}{FH}$. The actual speed at the point G is taken as the average speed over the distance MN.]
Of these, the first ratio, truly MN to OP, can be shown to be the same as FH to HΣ, as follows.

For the tangent to the cycloid BI is parallel to the line VA, and similarly the tangent MGN is parallel to the line ΦΛ; and hence the line MN is equal to Δπ, and OP is equal to EK. Therefore the said ratio of the lines MN to OP is the same as Δπ to EK; that is, ΔA to EA ; that is, ΦA to ΛA; that is VA to ΦA (Lemma preced.). Again, as VA is to ΦA, thus FA is to AH: for the square VA is equal to the rectangle DAF, and the square AΦ is equal to the rectangle DAΣ, which rectangles are between themselves as FA to ΣA, that is as the square FA to the square AH, and hence the square is VA to the square ΦA as the square FA to the square AH; and also with the length VA to AΦ, as FA to AH. Thus the ratio of MN to OP, will be the same that FA has to AH, that is, because the triangles FAH and FHΣ are similar, the same as FH to HΣ, as was said.
[From the similar triangles ADV and FAV in the semicircle,
$AV^2 = AD.AF$ and $A\Phi^2 = AD.A\Sigma$; and from the similar triangles FAH and AΣH, FA/ΣA = $(FA/AH)^2$, from which $AF/A\Sigma = AV^2/A\Phi^2 = (FA/AH)^2$, from which the result follows, *i. e.* AV/AΦ = FA/AH as required. From above, $\frac{MN}{OP} = \frac{\Delta\Pi}{EK} = \frac{\Delta A}{EA} = \frac{\Phi A}{\Lambda A} = \frac{VA}{\Phi A}$ (by the lemma); and hence : AV/AΦ = FA/AH = **FH/HΣ = MN/OP**.]

And thus the said ratio of the time per MN to the time per OP, is composed from the ratios FX to FH and FH to HΣ, and thus this will be the same as FX or XH to HΣ. Moreover, thus as the radius XH to HΣ, thus is the tangent ST to the line QR; for this is easily seen [similar right-angled triangles.] Thus agreed to be as ST to QR. Q.e.d. [p. 52]
[Thus, finally, the ratio of the times defined on MN and OP is equal to the ratio ST to QR, as required.]

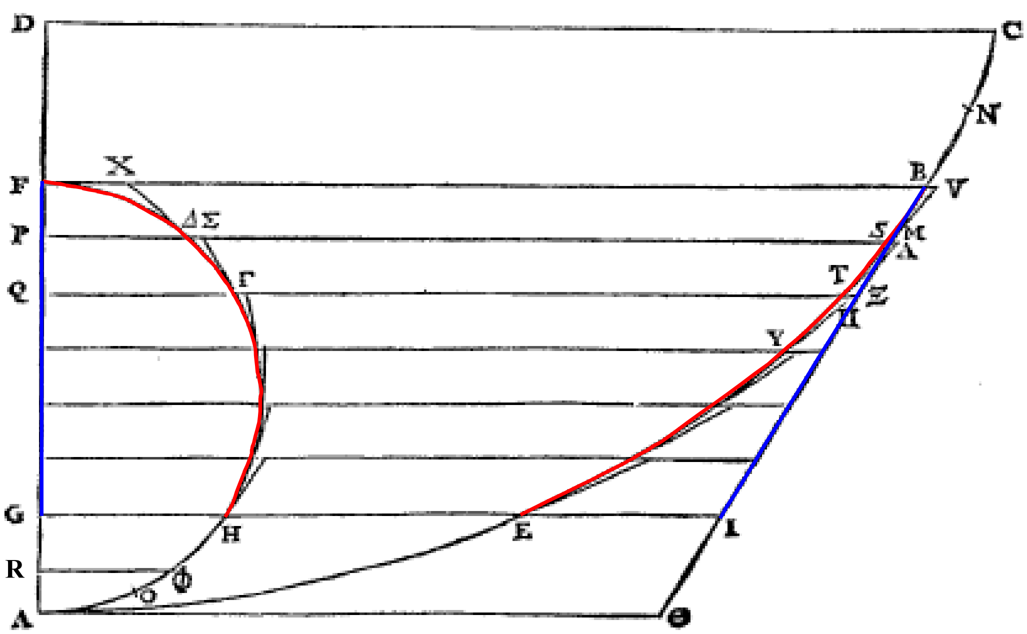
PROPOSITION XXIV.

Again, as in the preceding proposition, ABC is a cycloid, the vertex A of which is seen to point downwards, and with the axis AD perpendicular to the horizontal ; and by taking some point B on the curve, thus the line BΘ is drawn downwards tangent to the cycloid, and which crosses the horizontal line AΘ in Θ. The line BF is constructed perpendicular to the axis, and on FA the semicircle FHA is drawn; another line GE, parallel to FB, cuts the cycloid in E, the line BΘ in I, the circumference FHA in H, and finally the axis DA in G.

I say that the time of descent along the arc of the cycloid BE, to the time of descent along the tangent BI with a speed equal to half the final speed from passing down BΘ, to be as the arc FH to the line FG.

[This is a long *reductio ad absurdum* proof running to 5 pages; how Huygens came upon an actual analytic style proof is not stated. Thus, the ratio of the times for a body to traverse the red arcs with the stated speeds is in the same ratio as lengths of the blue lines ; to prove the theorem, essentially, an arc of the curve is tweaked up or down to restore the equality, assumed false by Huygens, from which an absurdity eventually results.]

If indeed this assertion is not true, then the time to cross the arc BE to the said time to pass along BI, is either greater or less than the ratio of the length of the arc FH to the line



FG. In the first case, if it were possible, let the ratio be considered greater.

Thus the actual time to pass along BE is assumed to be some shorter time (this shall be the time Z) than to the said time for BI as with the arc FH to the line FG.

[Thus, the theorem asserts that $\frac{T_{BE}}{T_{BI}} = \frac{\text{arc } FH}{\text{line } FG}$.

We now assert that T_{BE} = Z and $Z > T_{BI} \times \frac{\text{arc } FH}{\text{line } FG}$; in which case a shorter time should be considered than that to fall along the curve BE, for equality of the ratios.]

For if now on the cycloid some other point N is taken above B, the time to pass through BE after descending the distance NB shall be shorter than the time to traverse BE [falling from rest at B]. Moreover the point N can be taken in the vicinity of B itself, so that the difference of these times can be as small as you please, and hence it can lie within the time by which Z is greater than the time along BE [following the descent along NB]. Thus the point N is so chosen [p.53]; hence a certain time for the fall BE after NB is smaller than the time Z, and this time T'_{BE} still has a greater ratio to the said time through BI with half the speed from BΘ, than the arc FHO has to the line FG.

[Thus, $Z > T'_{BE} > T_{BI} \times \frac{\text{arc}\, FH}{\text{line}\, FG}$, while the inequality still holds, for a suitable choice of N.]

FG is now divided into equal parts FP, PQ, &c., each one of which shall be less than the height of the line NB, and likewise to the height of the arc HO; it can be shown that this is indeed possible to be done; and the lines PΛ, QΞ, &c can be set out from the points of division, parallel to the base DC, and made to end on the tangent line BΘ. Each line cuts the circumference FH, and from these points, as far as the point H, tangents are drawn to the next parallel line, such as ΔX, ΓΣ, &c. Similarly, from the points, in which the said parallel lines cross the cycloid, tangents are drawn such as SV, TM, &c. Indeed together with a part GR added to the line FG, equal to these from the division, and RΦ is similarly drawn parallel to DC, it is apparent that it crosses the circumference FHA between H & O, since GR is less than the height of the point H above O. For now we can again turn to the original argument.

The time taken to traverse the tangent VS [T_{VS}]with a constant speed equal to that acquired from BS [$\bar{v}_{BS}$], is greater than the time for the motion with continued acceleration along the arc BS [T'_{BS}] after NB. Now the speed from BS is less than the speed from NB [$\bar{v}_{BS} < \bar{v}_{NB}$], since the height BS is less than NB. For the speed from BS along the tangent VS is supposed to remain constant, while the speed acquired from NB again is to accelerate continually along the arc BS, which arc is smaller than the above tangent VS, and with all of its parts more upright than any part of the tangent VS; thus, in order that the time for the body to traverse the tangent VS, with a constant speed from BS [equal to the exact speed of the body falling from rest at B], shall be entirely greater than the time to traverse the arc BS after falling along NB
[*i. e.* $T_{VS} > T'_{BS}$ as VS/$\bar{v}_{BS}$ > BS/$\bar{v}_{NB}$].
Similarly, the time for the tangent MT, with the speed from BT, will be greater than the time to traverse the arc ST after NS, and the time after the tangent πY with the speed from BY, greater than the time for the arc TY. And thus for the times of the equal motions along all the tangents as far as the end, which touches the cycloid at E, with the speeds along the individual lengths acquired by falling from B as far as the point of contact of these, taken simultaneously will be greater than the time to traverse the arc BE after falling NB.
[Hence, sum of times like T_{VS} along tangents > sum of times like T'_{BS} along arc BE, starting at rest from N.]
But the same sum of the times can be smaller, as we now will show. For the times of the same constant motions are to be considered anew along the tangents of the cycloid. Indeed the time to pass along the tangent VS [T_{BS}] with the speed acquired from the arc BS, to the time along the line BΛ [$T^*_{BΛ}$] (with the speed equal to half that acquired from FA), is as the tangent section to the circumference ΔX is to the part of the axis FP (Prec. Prop. :
[(time to cross tangent section of cycloid at a point with final section speed)/(time to cross tangent section at B with speed equal to half final speed along BΘ) = section of tangent to circle/corresponding section of diameter); or $T_{BS} / T^*_{BΛ} = ΔX / FP$, etc. Thus, T and S are points on the cycloid; M is the intersection of the tangent at T with PS produced; and Λ and Π are corresponding points on the tangent line BΘ]. [p. 54]

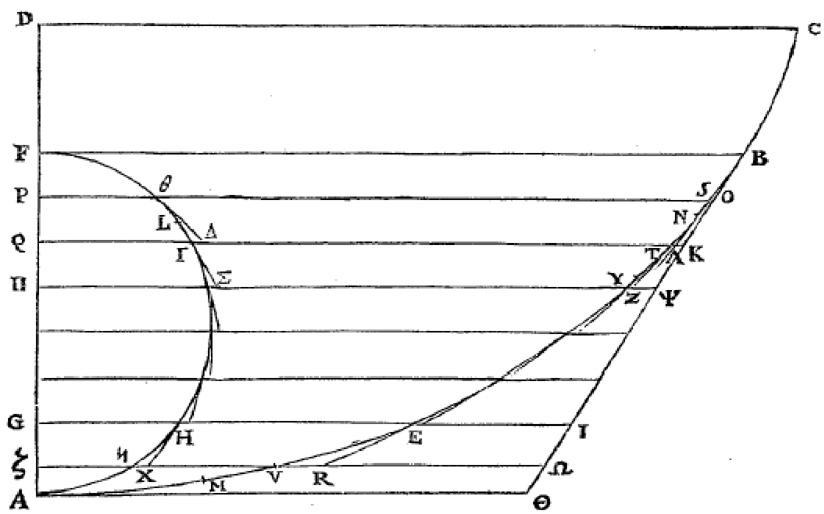
Similarly, the time along the tangent MT (with the speed from the final speed along BT), to the time along the line ΛΞ (with half the same final speed from FA), are as the tangent ΓΣ to the line PQ. And thus the individual times along the tangents of the cycloid, which are the same as along the tangents of the cycloid mentioned above, will be to the times of the constant motion along the parts corresponding to the initial tangent line BI with the speed half of that acquired from BΘ, thus as the tangents to the circumference FH, taken with the same parallel lines, to the corresponding parts of the line FG. [This is as statement of Prop. 23 applied to the present circumstances. Note the the ratio of the **times** along two distinct tangents on the right are compared to the ratio of the **lengths** on the left for the circle.]

There are therefore lines of a certain length FP, PQ, &c. and the same number of others, the lines BΛ, ΛΞ, &c, and for both series the times are known to be traversed with a steady motion equal to half the final speed from BΘ; and each quantity in the first series in turn refers to the same proportion, and by which each of the latter follows according to their own ratio ; and indeed the two ratios are equal to each other in both places. [That is, the lines FA and BΘ are each divided by the same equally spaced parallel lines FB, PS, etc.] Moreover, the first quantities refer to the same proportions of certain others, surely to the tangents of the circle ΔX, ΓΣ, &c, with the same ratio and in the same order too as the latter refer to certain others, surely to the times of a certain motion we have talked about along the tangents of the cycloid VS, MT &c. Therefore, since the sum has the same ratio to which the first ratio itself refers to, or as the whole lengh FG is to the sum of all the tangents XΔ, ΓΣ, &c., thus too the time for the whole length of the tangent line BI (in the time for which the body traverses the whole length BI with a speed equal to half from BΘ) is to the sum of the times of the motion such as we have discussed along the tangents of the cycloid VS, MT, &c (Prop. 2. Archimedis: *Concerning the Sphere and Conoid*).
[Thus, the line FG : arc FH = time for line BI : time for arc BE ; there is hence a correspondence between these ratios;]
And thus on inverting, the times for the said motion along the tangents of the cycloid, to the time along the line BI with the speed equal to half the final speed from BΘ, has the same ratio as the sum of the said tangents of the circumference FH to the line FG; and hence is less than the arc FO to the same line FG; [Thus, the arc length FO corresponds to an extra length of arc BN, while FG and BI remain the same, leading to a greater ratio] since the arc FΦ, and hence all of the arc FO is greater than all the tangents arcs of FH (P.20). But we have said that the time along BE after NB, to the time along BI with the speed from half the speed BΘ, is put equal to the arc FO to the line FG. Hence, the sum of all the said times of the tangents along the cycloid is less than the time along BE after NB, which has already been shown to be greater; ***which is absurd***. Thus, the time to pass along the arc of the cycloid BE, to the time to pass along the tangent BI, with a speed equal to half the final speed acquired from BΘ or from FA, does not have a greater ratio than the ratio or the arc of the circumference FH to the line FG. [The motion on the circle is irrefutable; an extra speed implies a greater time; the time cannot be shortened in this way; hence the time for the motion down the cycloid cannot be greater than the stated amount. In the same manner, it can be shown not to be less.]

Now it shall be the case, if possible, that the first ratio of the times shall be smaller than the second ratio. Therefore some time greater than the time assumed to traverse the

arc BE will be required, (this is the time Z) in order that the ratio will be to the given time along BI, as the arc FH to the line FG.
[In this case, $Z < T_{BI} \times \frac{\text{arc } FH}{\text{line } FG}$, and some greater time is required for Z in order to obtain equality in the ratio.]

For if now the arc NM is taken equal to the height of the arc BE, (p. 55), but the upper end of which N is lower than the point B, the time to traverse the arc NM will be greater than the time to cross the arc BE (P.22). Moreover, it is seen that the point N can be taken so near to the point B, in order that the difference of the said times shall be as small as you please and hence less than that by which the time Z is greater than the time to travel along the arc BE. Thus, the point N is taken in this way. Thus a certain time to



cross the arc NM will be less than the time Z, and hence it will have to the given time along BI, with half the final speed from BΘ, a smaller ratio than the arc FH to the line FG. It can therefore have that ratio which the arc LH has to the line FG.

Now FG is divided into equal parts FP,PQ, &c. each of which is less than the arc BN of the cycloid in height, and likewise less than the arc of the circumference FL; and by adding FG to one of these parts Gζ, lines PQ, QK, &c, are drawn parallel to the base DC from the points of division, and to the ends of the tangent BΘ ; likewise from the point ζ the line ζΩ is drawn which cuts the cycloid in v, the circumference in η, and from each point parallel lines are drawn that cut the circumference FH, from these tangents are drawn downwards as far as the next parallel line, such as θΔ, ΓΣ: From the smallest of these drawn from the point H to cross the line ζΩ in X. Similarly indeed from the points in which the said parallel lines cross the cycloid, all the tangents are drawn downwards , such as SΛ, TΞ, &c. and from the smallest of which, the tangent is drawn from the point E, that crosses the line ζΩ in R.

Therefore since Pζ is equal to the the height FG of the arc BE, to which it is equal to the height of the arc NM from the construction, and Pζ is equal to the height of the arc NM. Moreover, by the construction PO is higher than the end N (p. 56). Therefore ζΩ, and in that point V, is higher than the end M. Whereby, since the arc SV is equal to the height if the arc NM, but with the end S lower than N, the time to cross SV is shorter than the time to cross NM. (P. 22).

And the time to pass along the tangent SΛ, with the speed equal to that from BS, is shorter than the time of the descent by accelerating along the arc ST, beginning at S. For the speed from BS, which is gained from the transmission across the whole length SΛ, is equal to the speed from ST (P.2), which in the end is acquired from the motion along the arc ST; and thus SΛ is less than ST. Similarly the time along the tangent TΞ, with a speed equal to that from BT, is less than the time of descent for the accelerated motion along the arc TY after ST. Similarly the time along the tangent TΞ, with a speed equal to that from BT, is less than the time for the accelerated motion of descent along the arc TY after ST; when the speed from BT, which is put for the whole length crossed TΞ , is equal to the speed from SY, which in the end is acquired at last, is equal to the speed from SY; and TΞ is itself less than TY. And thus the time for all of the constant motions along the tangents to the cycloid, with the speeds along the individual lenths to be acquired by falling from B as far as to the point of contanct of these, are less taken together than for the time of descent by accelerating along the arc SV. Truly the same are to be longer, as we now can show.

Indeed the said time along the tangent SA, with a speed equal to that from BS, to the time along the line OK with the speed equal to half that from BΘ, thus shall be as the tangent of the semicircle θΔ to the line PQ (Prop. preced.) and likewise the time along the tangent TZ, with a speed equal to that from BT, is to the time along KΨ with the constant speed equal to half that from BΘ, as the tangent ΓΣ to the line Qπ. And thus henceforth the individual times for the tangents of the cycloid, which are the same as those mintioned above, are to the timee of the equal motion along the parts themselves corresponding to the line OΩ, with the speed equal to half of that from BΘ, as the tangents of the circumference θπ, between the same parallels, corresponding to the parts of the line Pζ themselves. Hence, as in the first part of the demonstration, it is concluded that all the lines PQ, Qπ &c., taken together, that is, the whole length Pζ is to all the tangents taken together θΔ, ΓΣ, &c. thus as the time for which the whole length OΩ is traversed, with a speed equal to half that from BΘ, to the time of the whole motion such as we have said along the tangents of the cycloid OΛ, TΞ,&c. Whereby on inverting the ratio, the whole time along the tangents of the cycloid, has that ratio to the said time for the constant motion along the line OΩ, or along BI, as all the said tangents of the arc θη to the line Pζ or FG, and hence the ratio shall be greater than the arc LH to the line FG; indeed the arc θH, and thus also the arc LH everywhere, shall be less than the stated tangents of the arc θη (Prop. 20). But the time along NM from the beginning we have put [p.57] to the same time along BI to have the same ratio as the arc LH to the line FG. Therefore the time along NM, much greater than the time along SV, is less than for the time along the tangents to the cycloid. ***Which is absurd***, since this time, from that along the arc SB, was shown to be less previously. It is apparent, therefore, that the time along the arc of the cycloid BE to the time along the tangent BI with a constant speed equal to half of that from BΘ, does not have a smaller ratio than the arc FH to the line FG. But is was shown not to have a larger ratio. Therefore by necessity, it must have the same ratio . Q.e.d.

PROPOSITION XXV.

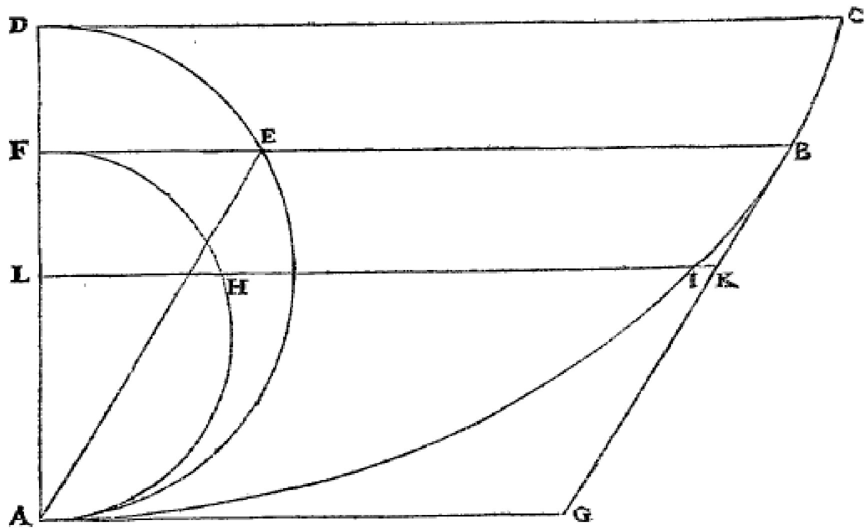
In a Cycloid with a vertical axis, and with the vertex seen to be the lowest point, the times of descent for some body, on leaving any point on the cycloid from rest until it reaches the lowest point at the vertex, are equal to each other; and this time has the same ratio to the time of fall along the whole axis of the cycloid as the semi-circumference of the circle to the diameter.

The cycloid shall be ABC the vertex of which is seen to be A pointing downwards, with the vertical axis truly AD, and by taking some point on the cycloid, such as B, a body is free to fall from natural forces along the arc BA, or along a surface thus curved into this shape. I say that the time of descent of this body to be as to the time of fall along the axis DA, thus as the semi-circumference of the circle to the diameter. From which demonstration, the time of descent along any arc of the cycloid to finish at A also, are agreed to be equal to each other.

A semicircle is described on the axis DA, the circumference of which cuts the line BF in E, parallel to the base DC; EA is joined, and BG is drawn parallel to that line, which indeed is a tangent to the cycloid at B. Truly the same line crosses the line drawn through the horizontal from A in G : and also on FA the semicircle FHA is drawn.

Therefore, by the preceding, the time of the descent along the arc of the cycloid BA, to the time of steady motion along the line BG with a speed equal to half of that from BG, thus as the arc of the semicircle FHA to the straight line FA. Truly the time said to be equal to the time for the constant motion along BG, is equal to the time of descent by accelerating naturally along the same BG, or along EA, to which itself it is parallel and equal, that is, to the time of descent for the acceleration along the axis DA (Prop. 6. Galileo. *de motu Acc*.). Thus the time along the arc BA, also shall be to the time of descent along the axis DA, as the circumference of the semicircle FHA to the diameter FA. Q.e.d. [p.58].

However, if the whole cavity of the cycloid constructed is put in place, it is agreed that the moving body, after it has descended along the arc BA, thus by continuing the motion will rise to an equal height along the arc (P.9), and to that height in the same time as was taken in the descent. (P.11). Then

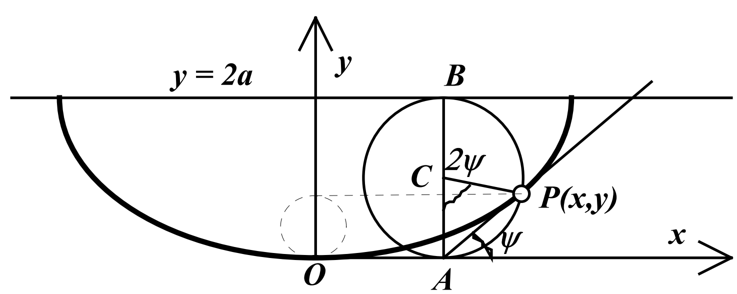


again it will return to B passing through A ad B, and in the individual oscillations of this kind, in larger or smaller arcs of the cycloid set in motion, the time for the swing to be to the time to fall vertically along the axis DA, thus as the circumference of the whole circle to the diameter of the same.
[A swing was considered to be a motion from one side to the other.]

PROPOSITION XXVI.

With the same positions, if some line HI is drawn above which cuts the arc BA in I, and the circumference FHA in H : I say that the time to pass through the arc BI, to the time to cross the arc IA after BI, has the ratio which the arc of the circumference FH has to HA.

Indeed the line HI crosses the tangent BG in K, from the axis DA in L. Thus the time to cross the arc BA, to the time of the steady motion along BG with the speed equal to half that from BG, thus is as the arc FHA to the line FA (P. 24). Moreover, the said time for the constant motion along BG, is to the time of the steady motion along BK, with that speed half that from BG, thus as BG to the length BK, that is, as FA to FL. And againm the time for the steady motion, with the said speed along BK, to the time to pass along the arc BI, as FL to the arc FH (P.24). Therefore from the equality the time along the arc BA to the time along BI, is thus as the arc FHA to FH. And on dividing and inverting, the time along BI, to the time along IA after BI, thus as the arc FH to HA. Q.e.d.
[A modern equivalent of Huygens' great labour:
We give a brief outline of the relevant equations substituted into the final theorem : The point P on the cycloid shown is generated by the circle of radius a



rolling without slipping on the horizontal line y = 2a; the coordinates are :
$x = a(2\psi + \sin 2\psi); y = a(1 - \cos 2\psi) = 2a\sin^2\psi.$
The angle ψ is the angle formed by the tangent to the curve at P, given by $dy/dx = \tan\psi$.
It is readily shown that the arc length s = 4asinψ, measured from O.
It is also easy to show that a bead of unit mass sliding on this curve executes s.h.m., and that $d^2s/dt^2 = -(g/4a)s = -\omega^2 s.$ A first integration of this equation gives the energy integral : $\dot{s}d\dot{s}/ds = -\omega^2 s$, i.e. $\frac{1}{2}\dot{s}^2 + \frac{1}{2}\omega^2 s^2 = \frac{1}{2}\omega^2 s_0^2$, where $s_0 \leq 4a$. Hence, the first term gives the kinetic energy, while the second term becomes $2ag\sin^2\psi$ or gy, representing the potential energy (This derivation is given in the notes to E001, Euler's first paper, in this series of translations). From energy considerations, $v_A^2 = 4ag\sin^2\psi$ and hence the average speed down the slope is $\bar{v}_A = \sqrt{ag}\sin\psi$; also, the length of the inclined plane L is 2asinψ, so that the time to descent this line is $L/\bar{v}_A = \sqrt{\frac{4a}{g}}$; note that this time is independent of the angle, while the arc length /diameter = 1/π, leading to $T = \frac{1}{\pi}\sqrt{\frac{4a}{g}}$ as required for a single swing in Huygens' final theorem.]

HOROLOGII OSCILLATORII

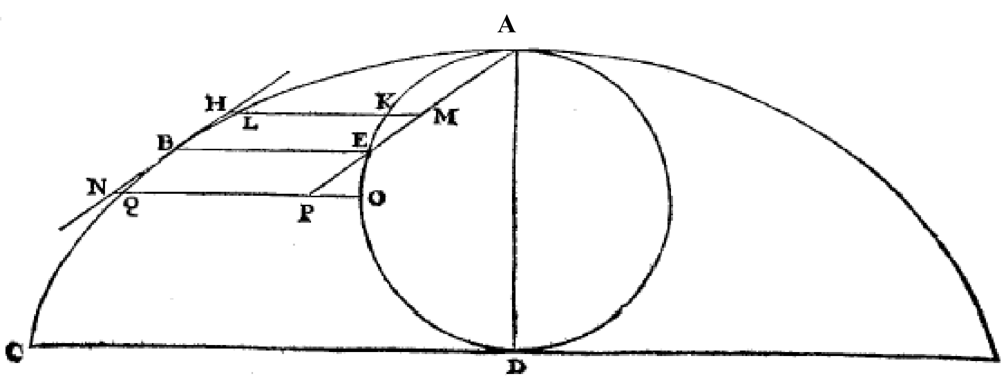
PARS SECUNDA.

De descensu Gravium & motu eorum in Cycloide.

PROPOSITIO XV.

Dato in Cycloide puncto , rectam per illud ducere quae Cycloidem tangat.

Sit cyclois ABC, & punctum in ea datum B, per quod tangentem ducere oporteat.
Circa axem cycloidis AD describatur circulus genitor AED, & ducatur BE parallela basi cycloidis, quae dicto circulo occurrat in E, & jungatur AE, cui denique parallela per B agatur HBN. Dico hanc cycloidem in B contingere.



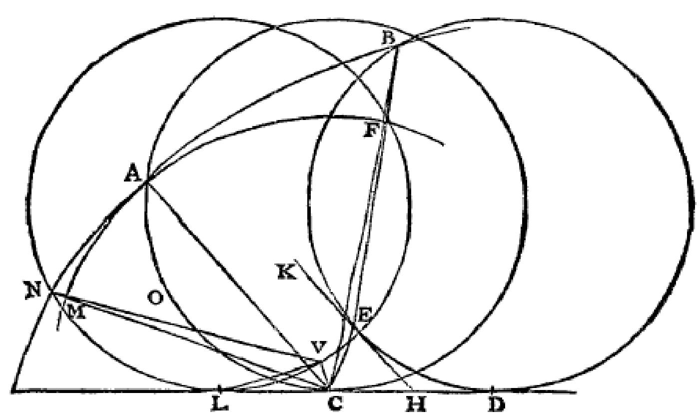
Sumatur enim in ea punctum quodlibet, à B diversum, ac primo versus superiora velut H, & per H ducatur recta basi cycloidis parallela. quae occurrat cycloidi in L, circulo

AED in K, rectae AE in M. Quia ergo KL est aequalis arcui KA, recta autem KM minor arcu KE, erit recta ML minor arcu AE, hoc est, recta EB, sive MH; unde apparet punctum H esse extra cycloidem.

Deinde in recta HN sumatur punctum N inferius B, & per N agatur, ut ante, base parallela, quae occurat cycloidi in Q. circulo AED I O, rectae AE productae in P. Quia ergo OQ , aequalis est arcui OA; OP autem major arcu OE; erit PQ minor arcu EA, hoc est, recta EB, sive PN. Unde apparet rursus punctum N esse extra cycloidem. Cum igitur quodlibet punctum praeter B, in recta HBN sumptum, sit extra cycloidem, constat illam in puncto B cycloidem contingere; qed.

Huic demonstrationi an locum suum hic relinquerem dubitavi, quod non multum ei absimiem à clarissimo Wrennio editam inveniam in libro Wallisii de Cycloide. Potest autem & universali constructione propositum absolve, quae non cycloidi tantum sed & aliis curvis, ex cujuslibet figurae circumvolutione genitis, conveniat; dummodo sit figura in eandem partem cava, & ex iis quae geometricae vocantur. [p. 40]

Sit enim curva NAB, orta ex circumvolutione figurae OL super regula LD ; describente nempe puncto N, in circumferentia figurae OL sumto. Et oporteat ad punctum curvae A tangente ducere. Ducatur recta CA à puncto C, ubi figura regulam tangebat cum punctum describens esset in A : quod punctum contactus semper inveriri potest, siquidem eo reducitur problema ut duae rectae inter se parallelae ducendae sint, quarum altera transeat per punctum describens in figurae ambitudatum, altera figuram tangat, quaeque inter se distent quantum distat punctum datum A ab regula LD : dico ipsam CA occurrere curvae ad angulos rectos, sive circumferentiam MAF descriptam centro C radio CA, tangere curvam in puncto A, unde perpendicularis ad AC per punctum A ducta curvam ibidem continget.



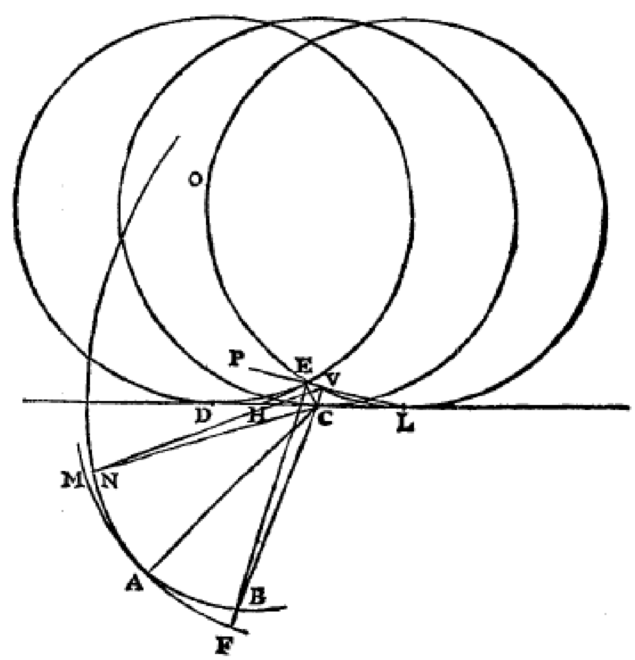
Ducatur enim CB primum ad punctum curvae B, quod distet ultra punctum A ab regula LD, intelligaturque figurae positus in BED, cum punctum describens esset in B, contactus regulae in D & punctum curvae quod erat in C, cum punctum describens esset in A, hic jam sublatum sit in E; & jungantur EC, EB, tangatque figuram in E recta KH, occurrens regulae in H.

Quia ergo recta CD aequalis est curvae ED; eadem veo curva major est utraque simul EH, HD; erit EH major quam CH. Vnde angulus ECH major quam CEH, & proinde ECL minor quam CEK. Atqui addendo angulum KEB, qui aequalis est LCA, ad LEC, sit angulus CEB: & auferendo ab ECL angulum LCB, sit ECB. Ergo angulus CEB major omnino angulo ECB. Itaque in triangulo CEB, latus CB majus erit quam EB, sed EB aequale patet esse CA, cum sit idemmet ipsum unà cum figura transpositum [p. 41]. Ergo CB etiam major quam CA, hoc est , quam CF. unde constat punctum B esse extra circumferentiam MAF.

Sit rursus punctum N in curva sumptum inter regulam LD & punctum A. Cumque punctum describens esset in N, ponatur situs figurae fuisse in VL, punctumque contactus L, punctum vero quod tangebat prius regulam in C, sit jam sublatum in V : & jungantur CN, NV, VC, VL. Erit ergo VN aequalis CA; imo erit ipsa CA translata in VN. Iam quia recta LC aequatur curvae LV, ac proinde major est recta LV, erit in triangulo CLV angulus LVC major LCBV. Quare addito insuper angulo LVN ad LVC, fiet totus NVC major utique quam LCV, ac proinde omnino major qngulo NCV, qui pars es LCV. Ergo in triangulo CVN latus CN majus erit latere VN, cui aequatur CA, ideoque CN major quoque quam CA,hoc est quam CM. Unde apparet punctum N cadere extra circulum MAF, qui proinde tanget curvam in puncto A. qed.

Est autem eadem quoque tum constructio tum demonstratio, si curva genita sit à puncto describente , vel intra vel extra ambitum figurae circumvolutae sumpto. Nisi quod, hoc posteriori casu, pars quaedam curvae infra regulam descendit, unde nonnulla in demonstratione oritur diversitas.

Sit enim punctum A, per quod tangens ducenda est, datum in



parte curvae NAB, quae infra regulam CL descendit, descripta nimirum à puncta N extra figuram revolutam sumpto, sed certam [p.42] positionem in eodem ipsius plano habente. Invento igitur puncto C, ubi figura revoluta tangit regulam CD quum punctum describens esset in A, ducatur recta CA. Dico hanc curvae NAB occurrere ad rectos angulos, sive circumferentiam radio CA centro C descriptam tangere curvam NAB in puncto A. Ostendetur autem exterius ipsam contingere, cum in curvae parte supra regulam CD posita interius contingat.

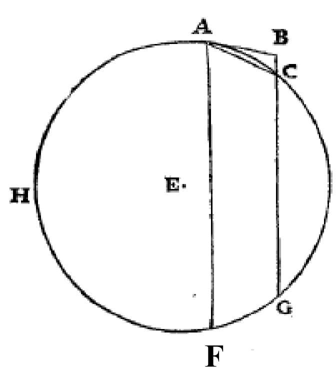
Positis enim & descriptis iisdem omnibus quae prius, ostenditur rursus angulus ECH major quam CEH. atque ad ECH addito HCB sit angulus ECB. & à CEH auferendo HEB, qui aequalis est DCA, sit angulus CEB. Ergo ECB major omnino quam CEB. unde in triangulo ECB latus EB majus quam CB. sed ipsi EB aequalis est CA, sive CF. Ergo & CF major quam CB: ideoque punctum circumferentiae F est ultra curvam NAB à centro remotum.

Item rursus ostenditur angulus LVC major LCV. Quare CVP, qui cum LVC duos rectos aequat, minor erit quam VCD. Atqui addendo ad VCD angulam DCN, sit VCN; & auferendo ab CVP angulum PVN, sit CVN. Ergo angulus VCN omnino major quam CVN. In triangulo itaque CVN, latus VN majus erit quam CN. Est autem ipsi VN aequalis CA sive CM. Ergo & CM major quam CN, ideoque punctum circumferentiae M erit ultra curvam NAB à centro C remotum. Itaque constat circumferentiam MAF tangere curvam in puncto A. qed.

Quod si punctum curvae per quod tangens ducenda est, sit illud ipsum ubi regula curvam secat, erit tangens quaesita semper regulae perpendicularis; ut facile esset ostendere.

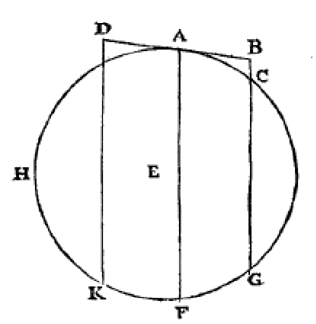
PROPOSITIO XVI.

Si circuli circumferentiam, cujus centrum E, secent rectae duae parallelae AF, BG, quarum utraque ad eandem partem centri transeat, vel altera AF per centrum ipsum, & à puncto A, quo centro propior circumferentiam secat, ducatur recta ipsam contingens : dico partem hujus AB, à parallela utraque interceptam, minorem esse arcu AC, ab utraque eadem parallela intercepto.



Ducatur enim arcui AC subtensa recta AC. Quia ergo angulus BAF est aequalis ei quem capit portio circuli AHF, quae vel major est semicirculo vel semicirculus, erit proinde angulus BAF, [p. 43] vel minor recto vel rectus; ideoque angulus ABC vel major recto vel rectus. Quare in triangulo ABC latus AC, angulo B subtensum, majus erit latere AB. sed idem latus AC minus est arcu AC. Ergo omnino & AB arcu AC minor erit.

PROPOSITIO XVII.

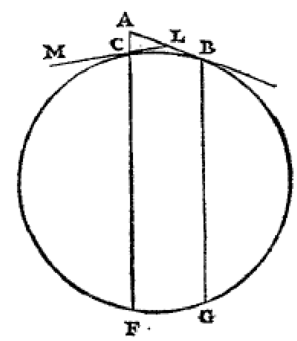


Iisdem positis, si tertia recta prioribus parallela DK, circulum secuerit, quae ab ea quae centro propior est AF, tantumdem distet quantum hac à reliqua BG : dico partem tangentis in A, à parallela ultimo adjecta, & media interceptam, nempe AD, arcu AC à primis duabus parallelis intercepto minorem esse.

Hoc enim patet quum AD ipsi AB aequalis sit, quam antea ostendimus arcu AC minorem esse.

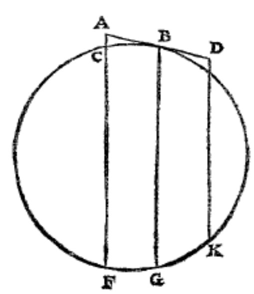
PROPOSITIO XVIII.

Si circulum, cujus centrum E, duae rectae parallelae secuerint AF, BG; & à puncto B, ubi quae à centro remotior est, vel tantundem atque altera distat, circumferentiae occurrit, [p. 44] ***ducatur recta circumferentiam tangens : erit pars hujus BA, à parallelis intercepta, major arcu ab iisdem parallelis intercepto BC.***



Ducatur enim in puncto C, recta MCL circumferentiam tangens, quae occurrat tangenti BA in L. In triangulo igitur ACL, angulus C aequalis est angulo MCF, hoc est, ei quem capit portio circuli CBF. angulus autem A aequatur angulo quem capit portio circuli BCG, quae portio quum sit major vel aequalis portioni CBF, quippe quum BG vel ulterius distet à centro quam CF, vel tantundem: erit proinde trianguli ACL angulus A minor vel aequalis angulo C : & consequenter latus CL vel minus vel aequale lateri AL. Atque CL una cum LB majores sunt arcu CB. Ergo & AL una cum LB, hoc est, tangens AB, eadem arcu CB major erit. qed.

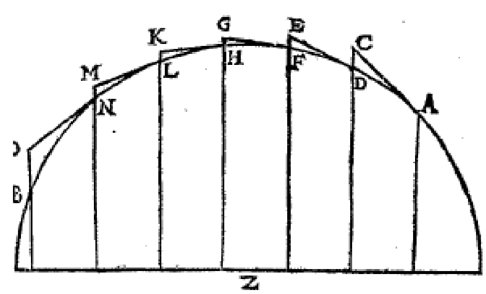
PROPOSITIO XIX.



Iisdem positis, si tertia recta prioribus parallela DK circulum secet, quae tantundem distet ab ea quae remotior est à centro quantum haec à reliqua AF : Erit pars tangentis in B, [p. 45] ***à parallela media, & ultimo addita DK, intercepta, nimirum BD, major arcu BC.***

Hoc enim manefestum est cum BD fiat ipsi BD fiat ipsi BA aequalis, quam ostendimus arcu BC majorem esse.

PROPOSITIO XX.



Si arcus circuli, semicircumferentia minor, AB, in partes quodlibet secetur lineis rectis parallelis, quae & inter se, & cum rectis sibi parallelis per terminos arcus ductis, aequalia intervalla constituant, quales sunt CD, EF, GH, KL, &c. ducanturque ad terminum arcus alterutrum A, & ad reliqua omnia sectionum puncta rectae circumferentiam tangentes, omnes in eandem partem, & ut unaquaque occurrat

proximae dictarum parallelarum; cujusmodi sunt tangentes AC, DE, FG, HK, &c. Dico has tangentes, dempta prima AC, simul sumptas, minores esse arcu proposito AB. Easdem vero omnes, non omissae AC, majores esse arcu AB diminuto parte extrema NB, hoc est, majores arcu AN.

Ponamus enim primo parallelarum aliquas transire ab utraque parte centri Z, & sit GH, earum quae sunt à parte B, centro proxima, vel per ipsum centrum transeat. Itaque tangentes omnes inter GH & BO comprehensae, ut HK, LM, NO, singuale suis arcubus minores sunt (Prop. 16). Porro autem & tangens GF, arcu sequente FD minor est, & similiter tangens ED arcu DA. Itaque tangentes omnes inter BO & CD interjectae, minores sunt arcubus BH & FA, ac proinde omnino minores arcubus BH, HA, sive arcu BA, quod erat primo ostendendum.

Porro jam demonstrabimus tangentes omnes inter BO & A majores esse arcu AN. Enim vero parallela GH, vel propius centrum Z transit quam parallela EF, quam pono proximam esse [p. 46] earum quae à parte A transeunt, vel erit remotior, vel aeque distabit.

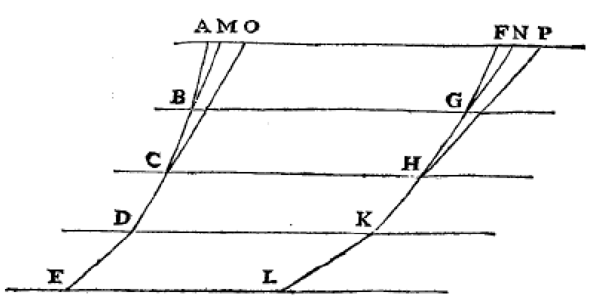
Quod si EF longius à centro vel aeque remota est GH, erit tangens FG major arcu suo FH, & reliquae tangentes versus A, nimirum ED, CA majores singulae arcubus suis (P. 18) ; adeo ut omnes simul GF, ED, CA majores sint arcu HA. sed & arcu HL major erit tangens LM (P. 19), & arcu LN tangens NO; itaque tangentes magis, accedente ipsa HK, tangentes omnes inter A & B comprehensae arcu eodem AN majores erunt.

Si vero GH à centro longius distat quam EF, erit tangens KH major arcu HF (P. 19), & tangens ML ut ante major arcu LH, & tangens ON major arcu NL, & omnes proinde tangentes ON, ML, KH majores arcu NF. Sed & tangens ED major est arcu suo FD (P. 18), & tangens CA major similiter arcu suo DA. Itaque tangentes omnes inter BO & A interjiciuntur, eodem arcu NA majores erunt.

Ex his vero etiam demonstratio manifesta est in casibus aliis, qualiscunque semicircumferentiae arcus accipiatur, quippe cum vel ieadem sit ubique, vel pars tantum praecedentis demonstrationis.

PROPOSITIO XXI.

Si mobile descendat continuato motu per qualibet plana inclinatata contingua, ac rursus ex pari altitudine descendat per plana totidem contigua, ita comparata ut singula altitudine respondeant singulis priorum planorum, sed majori quam illa sint inclinatione. Dico tempus descensus per minus inclinata, brevius esse tempore descensus per magis inclinata.



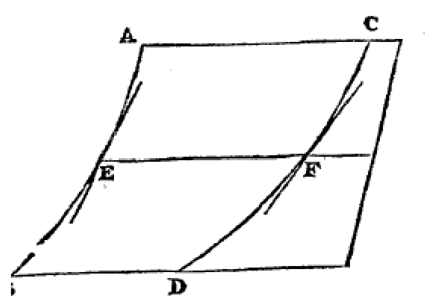
Sint series duae planorum inter easdem parallelas horizontales comprensae ABCDE, FGHKL, atque ita ut bina quaeque sibi correspondentia plana utruisque seriei iisdem parallelis horizontalibus includantur; unumquodque vero seriei FGHKH magis inclinatum sit ad horizontem quam planum sibi altitudine respondens seriei ABCDE. Dico breviori tempore absolvi descensum per ABCDE, quam per FGHKL.

Nam primo quidem tempus descensus per AB, brevius esse constat tempore descensus per FG, quum sit eadem ratio horum temporum quae rectarum AB ad FG (P.7), sitque AB minor quam FG, propter minorem inclinationem. Producantur jam sursum rectae CB, HG, ocurranctque horizontali AF in M & N. Itaque tempus per BC post AB, aequale est tempori per eandem BC post MB, cum in puncto B eadem celeritas contingat, sive per AB, sive per MB descendenti (P.6). Similerque tempus per GH post FG, aequale erit tempori per eadem GH post NG. Est autem tempus per BC post MB ad tempus per GH post NG, ut BC ad GH longitudine, sive ut CM ad HN, cum hanc ratonem habeant & tempora per totas MC, NH, & partes MB, NG (P.7), ideoque etiam tempora reliqua. Estque BC, minor quam GH propter minorem inclinationem. Patet igitur tempus per BC post AB, brevius esse tempore per GH post NG sive post FG.

Similiter ostenditur, productis DC, KH sursum, donec occurrant horizontali AF in O & P, tempus per CD post ABC, sive post OC, brevius esse tempore per HK post FGH sive post PH. Ac denique tempus per DE post ABCD, brevius esse tempore per KL post FGHK. Quare totum tempus descensus per ABCDE, brevius erit tempore per FGHKL. qed.

Hinc vero manifestum est, considerando curvas lineas tanquam ex innumeris rectis compositas, si fuerint duae superficies, secundum lineas curvas ejusdem altitudinis inclinatae, quarum in punctis quibuslibet aeque altis major semper sit inclinatio unius quam reliquae, etiam tempore breviori per minus inclinationam grave descensurum quam per magis inclinationam.

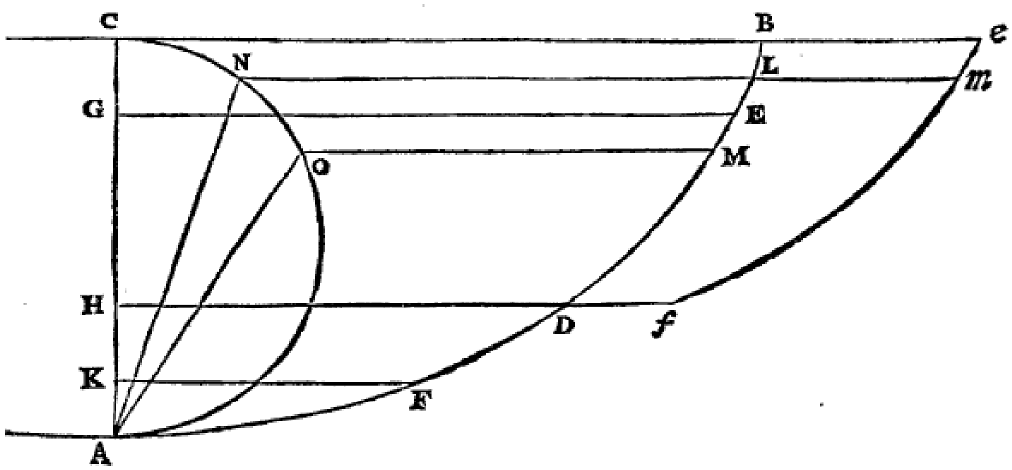
Velut si sint duae superficies inclinatae secundum curvas AB, CD, aequalis altitudinis, quarum in punctis aeque altis quibuislibet E, F, major sit inclinatio ipsius CD quam AB, hoc est, ut [p. 48] recta tangens curvam CE in F, magis inclinata sit ad horizontem, quam quae curvam AB tangit in puncto E. erit tempus descensus per AB brevius quam per CD.



Idemque continget si altera linearum rectae fuerit: dummodo inclinatio rectae, quae ubique est eadem, major minorve fuerit inclinatione rectae, quae ubique est eadem, major minorve fuerit inclinatione curvae in quolibet sui puncto.

PROPOSITIO XXII.

Si in Cycloide cujus axis ad perpendiculum erectus stat, vertice deorsum spectante, duae portiones curva aequalis altitudinis accipiantur, sed quarum alter a propior sit vertici; erit tempus descensus per superiorem, brevius tempore per inferiorem.

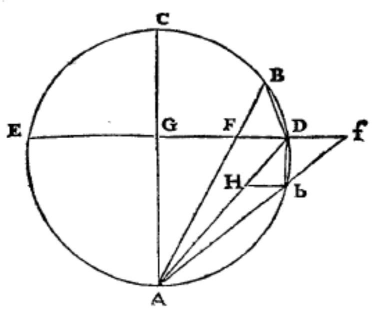


Sit Cycloid AB, cujus axis AC ad perpendiculum erectus, vertex A deorsum spectet; & accipiantur in ea portiones BD & EF, aequalis altitudinis, hoc est, ejusmodi ut parallelae horizontales BC, DH, quae superiorem portionem BD includunt, aeque inter se [p. 49] distent ac EG, FK, inferiorem portionem EF includentes. Dico tempus descensus per curvam BD brevius fore tempore per EF.

Sumatur enim in BD punctum quodlibet L, & in EF punctum M, ita ut eadem sit altitudo E supra M quae B supra L. Et descripto super axe AC semicirculo, occurrant ei rectae horizontales LN,MO, in N & O, & jungantur NA, OA. Itaque quum punctum N sit altius puncto OA. Est autem ipsi NA parallela tangens curvae in L puncto (P.15), & ipsi OA parallel tangens curvae in M. Ergo curva BD in puncto L minus inclinata est quam curva EF in puncto M. Quod si igitur portio EF, invariata inclinatione, altius extolli intelligatur velut in *ef*, ita ut inter easdem parallelas cum portione BD comprehendatur,invenietur punctum M and *m*, aequali altitudine cum puncto L. eritque etiam inclinatio curvae ef in puncto *m*, quae eadem est inclinationi curvae EF in M, major inclinatione curvae BD in L. Similiter vero, & in quolibet alio puncto curvae *ef*, major ostendetur inclinatio quam curvae BD in puncto aeque alto. Itaque tempus descensus per BD in puncto aeque alto. Itaque tempus descensus per BD brevius erit tempore per *ef*, (P.21), sive, quod idem est, per EF. qed.

LEMMA

Esto circulus diametro AC, quem secet ad angulos rectos DE, & à termino diametri A educta recta AB occurrat circumferentiae in B, ipsi vero DE in F. Dico tres hasce, AB, AD, AF, proportionales esse.



Sit enim primo intersectio F intra circulum; & arcui BD recta subtensa ducatur. Quia igitur arcus aequales sunt AE, AD, erunt anguli ad circumferentiam ipsis insistentes, EDA, ABD aequales. [p. 50] Itaque in triangulis ABD, ADF, aequales ABD, ADF. Communis autem utrique est angulus ad A. Ergo dicti trianguli similes erunt, ideoque BA ad AD ut AD ad AF.

Sit jam punctum intersectionis f extra circulum, & ducatur bH parallela DE, quae occurrat rectae AD in H. Itaque secundum jam demonstrata erit ut DA ad A*b*, ita A *b* ad AH, hoc est, ita A*f* ad AD : Ideoque rursus proportionales erunt A*f*, AD, A*b*. Quare constat propositum.

PROPOSITIO XXIII.

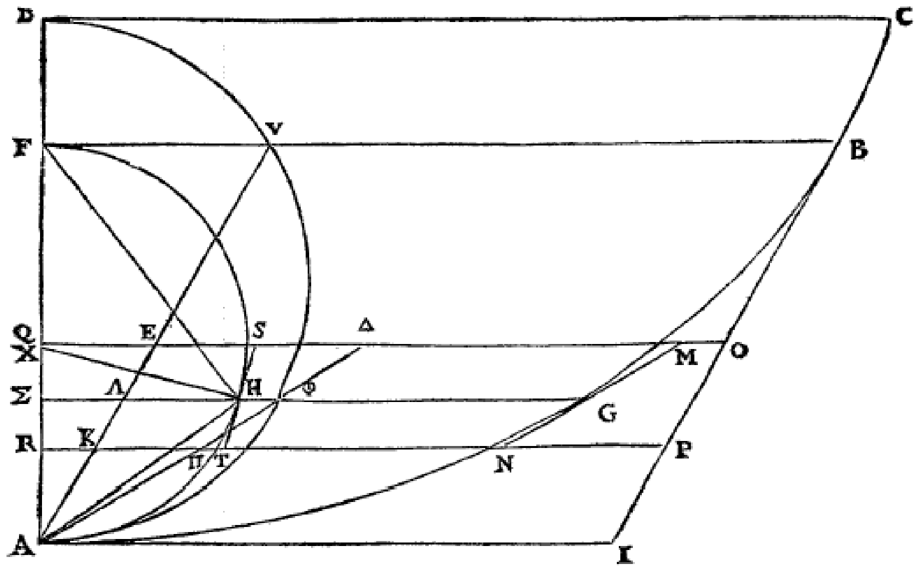
Sit Cycloïs ABC, cujus vertex A deorsum conversus sit, axe AD ad perpendiculum erecto; sumptoque in ea quolibet puncto B, ducatur inde deorsum recta BI quae Cycloidem tangat, termineturque recta horizontali AI. recta vero BF ad axem perpendicularis agatur, & divisa bifariam FA in X, super ea describatur semicirculus FHA. Ducta deinde per punctum quodlibet G in curva BA sumptum, recta ΣG parallela BF, quae circumferentiae FHA occurrat in H, axi AD in Σ, intelligantur per puncta G & H rectae tangentes utriusque curva, earumque tangentium partes iisdem duabus horizontalibus MS, NT intercepta sint MN, ST. Iisdemque rectis MS, NT includantur tangentis BI pars OP, & axis DA pars QR.

Quibus ita se habentibus, dico tempus quo grave percurret rectam MN, celeritate aequabili quanta acquiritur descendendo per arcum Cycloidis BG, fore ad tempus quo percurretur recta OP, celeritate aequabili dimidia ejus quae acquiritur descendendo per totam tangentem BI, sicut est tangens ST ad partem axis QR.

Describatur enim super axe AD semicirculus DVA secans rectam BF in V, & ΣG in Φ, & jungatur AV secans rectas OQ, PR, GΣ in EK & Δ. Iungatur item HF, HA, HX & AΦ; quae postrema secet rectas OQ, PR in punctis Δ & π.

Habet ergo dictum tempus per MN ad tempus per OP, rationem eam quae componitur ex ratione ipsarum linearum MN ad OP, & ex ratione celeritatum quibus ipsae percurruntur, contrarie sumpta (Prop. 5. Galil. de motu aequab.), hoc est, & ex ratione dimidiae celeritatis ex BI sive ex FA, ad celeritatem ex BG, sive ex FΣ (Prop.8). Atqui tota celeritas ex [p. 51] FA ad celeritatem ex FΣ, est in subduplicata ratione longitudinum FA ad FΣ, ac proinde eadem quae FA ad FH. Ergo dimidia celeritas ex FA ad celeritas ex FA ad celeritatem ex FΣ erit ut FX ad FH. Itaque tempus dictum per MN ad tempus per

OP habebit rationem compositam ex rationibus MN, ad OP, & FX ad FH. Harum vero prior ratio, nempe MN ad OP, eadem ostendetur quae FH ad HΣ.



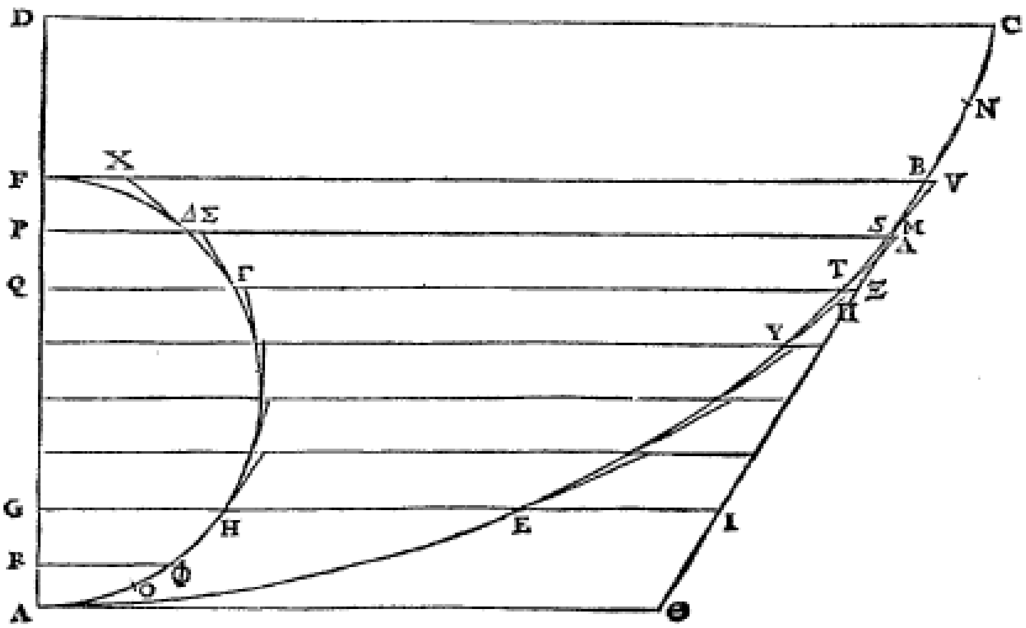
Est enim tangens Cycloidis BI parallela rectae VA, similiterque tangens MGN parallela rectae ΦA; ac proinde recta MN aequalis Δπ, & OP aequalis EK. Ergo dicta ratio rectae MN ad OP eadem est quae Δπ ad EK; hoc est, ΔA ad EA ; hoc est, ΦA ad ΛA; hoc est VA ad ΦA (Lemma praeced.) ita FA ad AH; nam quia quadratum VA aequale est rectangulo DAF, & quadratum AΦ aequale rectangulo DAΣ, quae rectangula sunt inter se ut FA ad ΣA, hoc est ut quadratum FA ad quadratum AH, erit proinde & quadratum VA ad quadratum ΦA ut quadratum FA ad quadratum AH; atque etiam VA ad AΦ longitudine, ut FA ad AH. Ratio itaque MN ad OP, eadem erit quae FA ad AH, hoc est, propter triangula similia FAH, FHΣ, eadem quae FH ad HΣ, ut dictum fuit. Itaque dicta ratio temporis per MN ad tempus per OP, componitur ex rationibus FX ad FH & FH ad HΣ, ideoque eadem erit quae FX sive XH ad HΣ. Sicut autem radius XH ad HΣ, ita est tangens ST ad rectam QR; hoc enim facile perspicitur. Igitur tempus motus qualem diximus per MN, ad tempus per OP constat esse sicut ST ad QR. qed. [p. 52]

PROPOSITIO XXIV.

Sit rursus ut in praecedenti propositione Cyclois ABC, cujus vertex A deorsum spectet, axis AD ad horizontalem erectus sit; & sumpto in ea quovis puncto B, ducatur inde deorsum recta BΘ quae Cycloidem tangat, occurratque rectae horizontali AΘ in Θ: recta vero BF ad axem perpendicularis agatur, & super FA describatur semicirculus FHA. Deinde alia recta GE, parallela FB, secet Cycloidem in E, rectam BΘ in I, circumferentiam FHA in H, & denique axem DA in G.

Dico tempus descensus per arcum Cycloidis BE, esse ad tempus per tangentem BI cum celeritate dimidia ex BΘ, sicut arcus FH ad rectam FG.

Si enim hoc verum non est, habebit tempus per arcum BE ad dictum tempus per BI, vel majorem rationem quam arcus FH ad rectam FG vel minorem. Habeat primo, si fieri potest, majorem.



Itaque tempus aliquod brevius tempore per BE (sit hoc tempus z) erit ad dictum tempus per BI ut arcu FH ad rectam FG. Quod si jam in Cycloide supra punctum B sumatur punctum aliud N, erit tempus per BE post NB. brevius tempore per BE. Manifestum est autem punctum N tam propinquum sumi posse ipsi B, ut differentia eorum temporum sit quamlibet exigau, ac proinde ut minor sit ea qua tempus z superatur à tempore pe BE. Sit itaque [p.53] punctum N ita sumptum. unde quidem tempus per BE post NB majus erit tempore Z, majoremque proinde rationem habebit ad tempus dictum per BI cum dimidia celeritate ex BΘ, quam arcus FH ad rectam FG. Habeat itaque eam arcus FHO ad rectam FG.

Dividatur FG in partes aequales FP, PQ, &c. quarum unaquaeque minor sit altitudine linea NB, atque item altitudine arcus HO; hoc enim fieri posse manifestum est; & à punctis divisionum agantur rectae, basi DC parallelae, & ad tangentem BΘ terminatae agantur rectae, basi DC parallelae, & ad tangentem BΘ terminatae PΛ, QΞ, &c. Quibusque in punctis hae secant circumferentiam FH, ab iis, itemque à punctis H, tangentes sursum ducantur usque ad proximam quaeque parallelam, velut ΔX, ΓΣ, &c. Similiter vero & à punctis, in quibus dictae parallelae Cycloidi occurrunt, tangentes sursum ducantur velut SV, TM, &c. addita vero ad rectam FG parte una GR aequali iis quae ex divisione, ductaque RΦ parallela similiter ipsi DC, patet eam occurrere circumferentiae FHA inter H & O, quia GR minor est altitudine puncti H supra O. Iam vero sic porro argumentabimur.

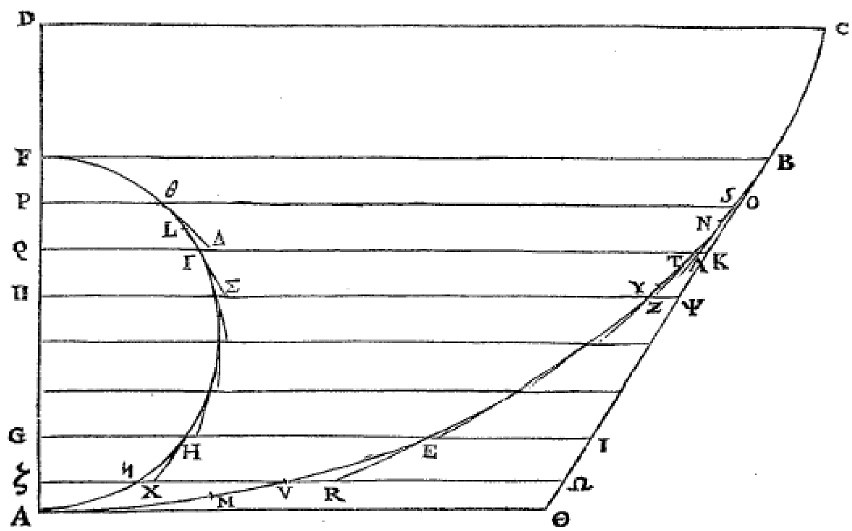
Tempus per tangentem VS cum celeritate aequabili quae acquireretur ex BS, majus est tempore motus continue accelerati per arcum BS post NB. Nam celeritas ex BS minor est celeritate ex NB, propterea quod minor altitudo BS quam NB. At celeritas ex BS aequabiliter continuari ponitur per tangentem VS, cum celeritas acquisita ex NB continue porro acceleretur per arcum BS, qui arcus minor insuper est tangere VS, omnibusque partibus suis magis erectus quam ulla pars tangentis VS. Adeo ut omnino majus sit futurum tempus per tangentem VS cum celeritate ex BS, tempore per arcum BS post NB. Similiter tempus per tangentem MT, cum celeritate ex BT, majus erit tempore per arcum ST post NS, & tempus post tangentem πY cum celeritate ex BY, majus tempore per arcum TY. Atque ita tempore motuum aequabilium per tangentes omnes usque ad infimam quae tangit cycloidem in E, cum celeritatibus per singulas quantae acquiruntur cadendo ex B adusque punctum ipsarum contactus, majora simul erunt tempore per arcum BE post NB. Eadem vero & minora essent, ut nunc ostendemus.

Considerentur enim denuo tempora eadem motuum aequabilium per tangentes cycloidis. Et est quidem tempus per tangentem VS cum celeritate ex BS, ad tempus per rectam BΛ cum celeritate dimidia ex FA, ut tangens circumferentiae ΔX ad partem axis FP (Praec. Prop.). [p. 54] Similiterque tempus per tangentem MT, cum celeritate ex BT, ad tempus per rectam ΛΞ cum eadem dimidia celeritate ex FA, ut tangens ΓΣ ad rectam PQ. Atque ita deinceps singula tempora per tangentes cycloidis, quae sunt eadem supradictis, erunt ad tempora motus aequabilis per partes sibi respondentes rectae BI cum celeritate dimidia ex BΘ, sicut tangentes circumferentiae FH, iisdem parallelis comprehensae, ad partes rectae FG ipsis respondentes.

Sunt igitur quantitates quaedam rectae FP, PQ, &c. & totidem aliae, tempora scilicet quibus percurruntur rectae BΛ, ΛΞ, &c, motu aequabili cum celeritate dimidia ex BΘ; Et unaquaeque quantitatas in prioribus ad sequentem eadem proportione refertur, qua unaquaeque posteriorum ad suam sequentem; sunt enim utrobique inter se aequales. Quibus autem proportionibus priores quantitates ad alias quasdam, nempe ad tangentes circuli ΔX, ΓΣ, &c, referuntur, iisdem proportionibus & eodem ordine posteriores quoque referuntur ad alias quasdam, nempe ad tempora motus qualem diximus per tangentes cycloidis VS, MT &c. Ergo, sicut se habent omnes dimul priores ad omnes eas ad quas ipsae referuntur, hoc est, sicut tota FG ad tangentes omnes XΔ, ΓΣ, &c. ita tempus quo percurritur tota BI cum celeritate dimidia ex BΘ, ad tempora omnia motuum quales diximus per tangentes cycloidis VS, MT, &c (Prop. 2. Archimedis de Sphaeroid. & Conoid). Et invertendo itaque, tempora motuum dictorum per tangentes cycloidis, ad tempus per rectam BI cum celeritate dimidia ex BΘ, eandem rationem habebunt quam dictae tangentes omnes circumferentiae FH ad rectam FG; ac minorem proinde quam arcus FO ad rectam eandem FG; quia arcus FΦ, ideoque omnino & arcus FO major est dictis omnibus arcus FH tangentibus (P.20). Atqui tempus per BE post NB, ad tempus per BI cum celeritate dimidia ex BΘ, posuimus esse ut arcus FO ad rectam FG. Ergo dicta tempora omnia per tangentes cycloidis minora simul erunt tempore per BE post NB, cum antea majora esse ostensum sit; quod est absurdum. Itaque tempus per arcum cycloidis BE, ad tempus per tangentem BI, cum celeritate dimidia ex BΘ vel ex FA, non habet majorem rationen quam arcus circumferentiae FH ad rectam FG.

Habeat jam, si potest, minorem. Ergo tempus aliquod majus tempore per arcuum BE, (sit hoc tempus Z) erit ad tempus dictum per BI, ut arcus FH ad rectam FG.

Quod si jam sumatur arcus NM aequalis altitudine cum arcu BE, (p. 55), sed cujus terminus superior N sit humilior puncto B, erit tempus per arcum NM majus tempore per arcum BE (P.22). Manifestum autem quod punctum N tam propinquum sumi potest puncto B, ut differentia dictorum temporum sit quamlibet exigua ac proinde minor ea qua tempus Z superat tempus per arcum BE. Sit itaque punctum N ita sumptum. Unde quidem tempus per NM minus erit tempore Z, habebitque proinde ad dictum tempus per BI,cum dimidia celeritate ex BΘ, minorem rationem quam arcus FH ad rectam FG. Habeat ergo eam quam arcus LH ad rectam FG.



Dividatur jam FG in parte aequales FP,PQ, &c. quarum unaquaeque minor sit arcus cycloidis BN altitudine, itemque minor altitudine arcus circumferentiae FL; & addita FG una earum partium Gζ, ducantur à punctis divisionum recta basi DC parallelae, & ad tangentem BΘ terminatae, PQ, QK, &c; itemque à puncto ζ recta ζΩ quae secet cycloidem in V, circumferentiam in η, quibusque in punctis ductae parallelae secant circumferentiam FH, ab iis tangentes deorsum ducantur usque ad proximam quaeque parallelam, velut θΔ, ΓΣ: Quarum infima à puncto H ducta occurrat rectae ζΩ in X. Similiter vero & à punctis, in quibus dictae parallelae occurrunt cycloidi, ducantur totidem tangentes deorsum,velut SΛ, TΞ, &c. quarum infima, tangens nempe à puncto E ducto, occurrat rectae ζΩ in R.

Quia igitur Pζ aequalis est FG altitudini arcus BE, cui aequalis est ex constructione altitudo arcus NM, erit & Pζ aequalis altitudini arcus NM. Est autem recta PO ex constructione superior termino N. (p. 56). Ergo & ζΩ, & in ea punctum V, superius termino M. Quare, cum arcus SV aequalis sit altitudinis cum arcu NM, sed termino S sublimiore quam N, erit tempus per SV brevius tempore per NM. (P. 22).

Atque tempus per tangente SΛ, cum celeritate aequabili ex BS, brevius est tempore descensus accererati per arcum ST, incipientis in S. Nam celeritas ex BS, qua tota SΛ transmissa ponitur, aequalis est celeritati ex ST (P.2), quae motui per arcum ST in fine demum acquiritur; ipsaque SΛ minor est quam ST. Similiter tempus per tangentem TΞ, cum celeritate aequabili ex BT, brevius est tempore descensus accelerati per arcum TY post ST. Similiter tempus per tangentem TΞ, cum celeritate aequabili ex BT, brevius est tempore descensus accelerati per arcum TY post ST; quum celeritas ex BT, qua tota TΞ transmissa ponitur, sit aequalis celeritati ex SY, quae in fine demum acquiritur, sit aequalis celeritati ex SY; ipsaque TΞ minor sit arcu TY. Atque ita tempora omnia motuum aequabilium per tangentes cycloidis, cum celeritatibus per singulas quantae acquiruntur descendendo ex B usque ad punctum ipsarum contactus, breviora simul erunt tempore descensus accelerati per arcum SV. Eadem vero & longiora essent, ut nunc ostendemus.

Est enim tempus dictam per tangentem SA, cum celeritate aequabili ex BS, ad tempus per rectam OK cum celeritate aequabili dimidia ex BΘ, sicut tangens semicirculi θΔ ad rectam PQ (Prop. praeced.) similiterque tempus per tangentem TZ, cum celeritate aequabili ex BT, est ad tempus per rectam KΨ cum celeritate aequabili dimidia ex BΘ, ut tangens ΓΣ ad rectam Qπ. Atque ita deinceps singula tempora per tangentes cycloidis, quae sunt eadem supra dictis, erunt ad tempora motus aequabilis per partes sibi repondentes rectae OΩ, cum celeritate dimidia ex BΘ, ut tangentes circumferentiae θπ, iisdem parallelis inclusae, ad partes reectae Pζ ipsis respondentes. Unde, ut in priori parte demonstrationis, concludetur omnes simul rectas PQ, Qπ &c. hoc est, totam Pζ esse ad omnes simul tangentes θΔ, ΓΣ, &c. sicut tempus quo percurritur tota OΩ, cum celeritate dimidia ex BΘ, ad tempora omnia motuum quales diximus per tangentes cycloidis OΛ, TΞ, &c. Quare & convertendo, tempora omnia per tangentes cycloidis, eam rationem habebunt ad tempus dictum motus aequabilis per rectam OΩ, sive per BI, quam dictae tangentes omnes arcus θη ad rectam Pζ vel FG, ac proinde majorem quam arcus LH ad rectam FG; est enim arcus θH, adeoque etiam omnino arcus LH, minor dictis tangentibus arcus θη (Prop. 20). Sed tempus per NM posuimus [p.57] ab initio ad idem tempus per BI se habere ut arcus LH ad rectam FG. Ergo tempus per NM, multoque magis tempus per SV, minus erit tempore per tangentes cycloidis. Quod est absurdum, cum hoc tempus, illo per arcum SB, antea minus ostensum fuerit. Patet igitur tempus per arcum cycloidis BE ad tempus per tangentem BI cum celeritate aequabili dimidia ex BΘ, non minorem rationem habere quam arcus FH ad rectam FG. Sed nec majorem habere ostensum fuit. Ergo eandem habeat necesse est. qed.

PROPOSITIO XXV.

In Cycloide cujus axis ad perpendiculum erectus est, vertice deorsum spectante, tempora descensus quibus mobile, à quocunque in ea puncto dimissum, ad punctum imum verticis pervenit, sunt inter se aequalia; habentque ad tempus casus perpendicularis per totam axem cycloides ead rationem, quam semicircumferentia circuli ad diametrum.

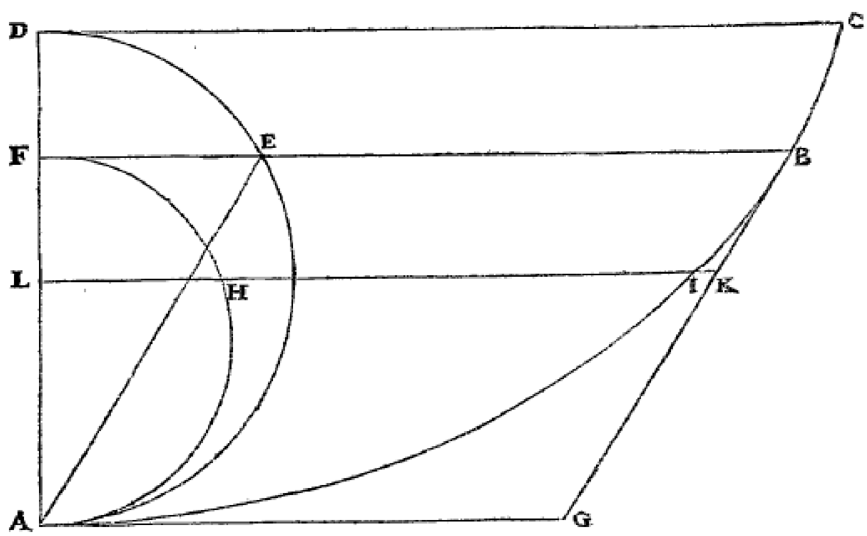
Esto cyclois ABC cujus vertex A deosrum spectet, axis vero AD ad perpendiculum erectus sit, & à puncto quovis in cycloide sumpto, velut B, descendat mobile impetu naturali per arcum BA, sive pe superficiem ita inflexam. Dico tempus descensus hujus esse ad tempus casus per axem DA, sicut semicircumferentia circuli ad diametrum. Quo demonstrato, etiam tempora descensus, per quoslibet cycloidis arcus ad A terminatos, inter se aequalia esse constabit.

Describatur super axe DA semicirculus, cujus circumferentiam secet recta BF, basi DC parallela, in E; junctaque EA, ducatur ei parallela BG, quae quidem cycloidem tanget in B. Describatur super axe DA semicirculus, cujus circumferentiam secet recta BF, basi DC parallela, in E; junctaque EA, ducatur ei parallela BG, quae quidem cycloidem tanget

in B. Eadem vero occurrat rectae horizontali per A ductae in G : sitque etiam super FA descriptus semicirculus FHA.

Est igitur, per praecedentem, tempus descensus per arcum cycloidis BA, ad tempus motus aequabilis per rectam BG cum celeritate dimidia ex BG, sicut arcus semicirculi FHA ad rectam FA. Tempus vero dicti motus aequabilis per BG, aequatur tempori descensus naturaliter accelerati per eandem BG, sive per EA, quae ipsi parallela est & aequalis, hoc est, tempori descensus accelerati per axem DA (Prop. 6. Galil. de motu Acc.). Itaque tempus per arcum BA, erit quoque ad tempus descensus per axem DA, ut semicirculi circumferentia FHA ad diametrum FA. qed. [p.58].

Quod si tota cycloidis cavitas perfecta ponatur, constat mobile, postquam per arcum BA descenderit, inde continuato motu per alterum ipsi aequalem arcum ascensurum (P.9) , atque in eo tantundem temporis atque descendendo consumpturum (P.11). Deinde



rursus per A ad B perventurum, ac singularum ejusmodi reciprocationum, in magnis parvisve cycloidis arcubus peractarum, tempora fore ad tempus casus perpendicularis per axem DA, sicut circumferentia circuli tota ad diametrum suam.

PROPOSITIO XXVI.

Iisdem positis, si ducatur insuper recta horizontalis HI quae arcum BA secet in I, circumferentiam vero FHA in H : dico tempus per arcum BI, ad tempus per arcum IA post BI, eam rationem habere quam arcus circumferentiae FH ad HA.

Occurrat enim recta HI tangenti BG in K, axi DA in L. Est itaque tempus per arcum BA, ad tempus motus aequabilis per BG cum celeritate dimidia ex BG, sicut arcus FHA ad rectam FA (P.24). Tempus autem dicti motus aequabilis per BG, est ad tempus motus aequabilis per BK, cum eaem celeritate dimidia ex BG, sicut BG ad BK longitudine, hoc est, sicut FA ad FL. Et rursus tempus motus aequabilis, cum dictaceleritate, per BK, ad tempus per arcum BI, sicut FL ad arcum FH (P.24). Igitur ex aequo erit tempus per arcum BA ad tempus per BI, ut arcus FHA ad FH. Et dividendo, & convertendo, tempus per BI, ad tempus per IA post BI, ut arcus FH ad HA. qed.