SECTION V.

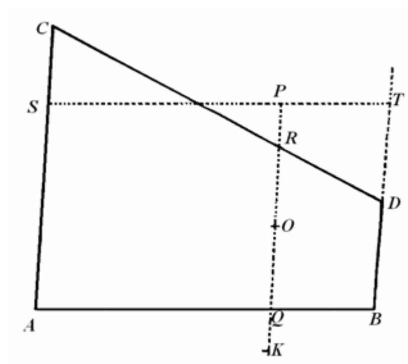
Finding the orbits where neither focus is given.

[A thorough investigation of the origin and use of these Lemmas is given by D.T.Whiteside in Vol. VI of his *Mathematical Papers of Isaac Newton*, CUP, p.238 onwards. In addition, a work can be reconstructed from Newton's Waste Book on the Solid Locus of the ancient Greek mathematicians, which was lightly modified for these Lemmas of the *Principia* (See Vol. IV Whiteside, p. 274 onwards for an account of this, which bears a close resemblance to the version in the *Principia*.) Mention should also be made of J. L. Coolidge's little book : *A History of Conic Sections and Quartic Surfaces*, available as a Dover reprint, especially Ch.'s 3 & 4. This book gives a modern impression on some of Newton's trail-blazing work, as he was unaware of the work done already by others into the projective nature of conics. Newton clearly had an eye towards an exhaustive survey of the construction of conic sections dating from antiquity, to which he added significantly, with regard to possible applications to the orbits of planets and comets; for in addition to the conventional treatment, he investigated the construction and properties of conic sections from points on the curve only; for the directrix and focus, relating to such curves given at a few points only, are unknown initially.]

LEMMA XVII.

If from some point P of a given conic section to the four sides AB, CD, AC, DB of some trapezium ABDC produced indefinitely, and inscribed in that conic section, just as many right lines PQ, PR, PS, PT may be drawn at given angles, one line to each side : the rectangle $PQ \times PR$ drawn to the two opposite sides, will be in a given ratio to the rectangle $PS \times PT$ drawn to the other two opposite sides.

Case 1. In the first place we may put the lines drawn to the opposite sides to be parallel to one of the remaining sides, e.g. *P*Q and *PR* [are parallel] to the side *AC, PS* and *PT* to the side *AB.* And in addition the two opposite sides [of the trapezium], e.g. *AC* and *BD,* themselves in turn shall be parallel. A right line, which may bisect those parallel sides, will be one of the diameters of the conic section, and it also will bisect *RQ*. Let *O* be the point in which *R*Q may be bisected, and *PO* will be the applied ordinate for that diameter. Produce *PO* to *K,* so that *OK* shall be equal to *PO,* and *OK* will be the applied ordinate for the other part of the diameter

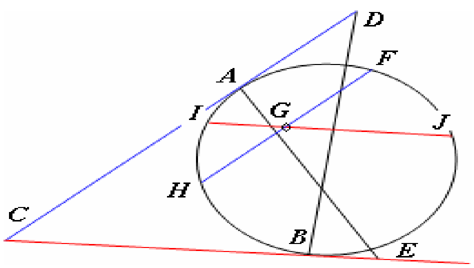


[Note: The use of the term *applied ordinate* by Apollonius for the distance from the centre of the conic along an oblique axis to the curve was a forerunner of the idea of a coordinate, developed by De Cartes some 1800 years later.]

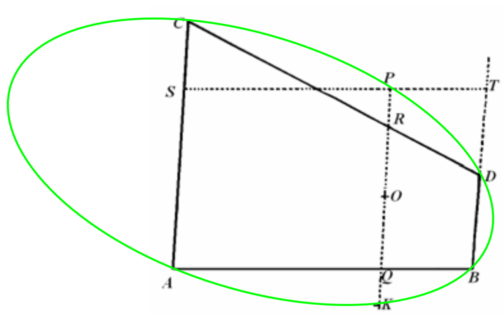
Therefore since the points *A*, *B*, *P* and *K* shall be on the conic section, and *PK* may cut *AB* in a given angle, the rectangle *PQ.QK* will be (by Prop.17,19, 21 & 23. Book III. *Apollonius* Conics) in a given ratio to the rectangle *AQ.QB*. But *QK* & *PR* are equal, as from the equality of *OK*, *OP*, and their difference from *OQ*, *OR*, and thence also the rectangles *PQ.QK* and $PQ \times PR$ are equal; and thus the rectangle $PQ \times PR$ is to the rectangle *AQ.QB*, that is in the given ratio to the rectangle $PS \times PT$.

Q.E.D.

[The initial theorems referring to Apollonius relate to the rectangles formed by chords of a conic section *IJ* and *HF* intersecting at the point *G*, drawn through two random points on the section *I* and *H*, to the ratio of the tangents squared *CA* and *CB* from an external point *C*, which are parallel to the given chords and vice versa. Thus, in the diagram added, the letters of which bear no relation to those above, the red and blue chords are parallel to the tangents from some external point *C*. The normals *AE* and *BD* also have been drawn and are part of a proof, which we do not give here, but the proposition shown by Apollonius is that

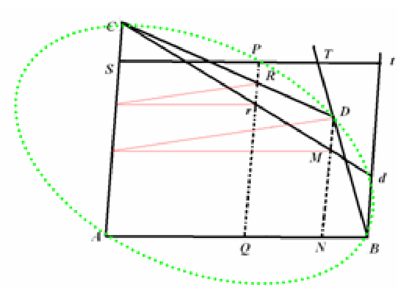


$\frac{FG \times GH}{JG \times GI} = \frac{CA^2}{CB^2}$. We indicate here the ellipse drawn for these five points :



This Lemma can be extended to hyperbolic, circular and parabolic sections, and is further generalised below. In the following, we shall include the ellipse that the reader had to imagine drawn around the trapezium or quadrilateral; in general coloured lines have been added by this translator; I am sorry if they cause offense; the purpose is to improve the readability of the work.]

Case 2. Now we may consider the opposite sides of the figure [*trapezium*] *AC* and *BD* not to be parallel. *Bd* acts parallel to *AC* and then crosses to the right line *ST* at *t*, and to the section of the cone at *d*. Join *Cd* cutting *PQ* in *r*, and *PQ* itself acts parallel to *DM*, cutting *Cd* in *M* and *AB* in *N*. Now on account of the similar triangles *BTt*, *DBN*; *Bt* or *PQ* is to *Tt* as *DN* to *NB*. Thus *Rr* is to *AQ* or *PS* as *DM* to *AN*.

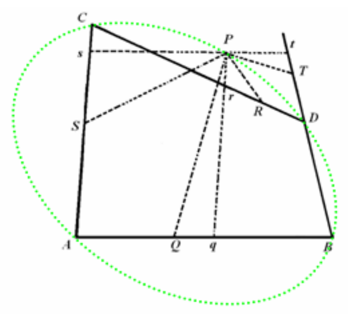


[*i.e.* $\frac{Bt}{Tt}=\frac{PQ}{Tt}=\frac{DN}{NB}$ and $\frac{Rr}{AQ}=\frac{Rr}{PS}=\frac{DM}{AN}$.]

Hence, by taking antecedents multiplied into antecedents and consequents into consequents, so that the rectangle $PQ\times Rr$ is to the rectangle $PS\times Tt$, thus as the rectangle *ND.DM* it to the rectangle *AN.NB*, and (by case I.) thus the rectangle $PQ\times Pr$ is to the rectangle $PS\times Pt$, and dividing thus the rectangle $PQ\times PR$ is to the rectangle $PS\times PT$.

Q.E.D.

Case 3. And then we may put the four lines *PQ, PR, PS, PT* not to be parallel to the sides *AC, AB* but at some inclination to that. Of these in turn *Pq, Pr* act parallel to *AC* itself; *Ps, Pt* parallel to *AB* itself ; and therefore the given angles of the triangles *PQq, PRr, PSs, PTt,* will give the ratios *PQ* to *Pq, PR* to *Pr, PS* to *Ps,* and *PT* to *Pt* ;



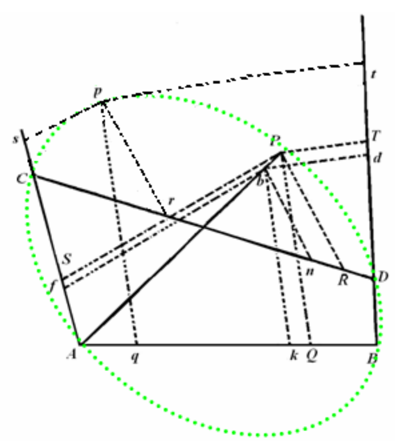
[*i.e.* $\frac{PQ}{Pq}$, $\frac{PR}{Pr}$, $\frac{PS}{Ps}$ and $\frac{PT}{Pt}$.]

and thus the composite ratios $PQ\times PR$ to $Pq\times Pr$, and $PS\times PT$ to $Ps\times Pt$. But, by the above demonstrations, the ratio $Pq\times Pr$ to $Ps\times Pt$ has been given : and therefore the ratio $PQ\times PR$ to $PS\times PT$ also is given. *Q.E.D.*

LEMMA XVIII.

With the same in place ; if the rectangle drawn to the two opposite sides of the trapezium $PQ\times PR$ ***shall be in a given ratio to the rectangle drawn to the remaining two sides*** $PS\times PT$ ***; the point P, from which the lines are drawn, will lie on the conic section described about the trapezium***.

Consider a conic section to be described through the points *A, B, C, D,* and any of the infinitude of points *P*, for example *p*: I say that the point *P* always lies on this section. If you deny this, join *AP* cutting this conical section elsewhere than at *P,* if it were possible, for example at *b.* Therefore if the lines *pq, pr, ps, pt* & *bk, bn, bf, bd* may be drawn from these points *p* & *b* at given angles to the sides of in the right trapezium ; so that $bk\times bn$ will be to $bf\times bd$ as (by Lem. XVII.) $pq\times pr$ to $ps\times pt$, and thus (by hypothesis) $PQ\times PR$ to $PS\times PT$. And on account of the similitude of the trapeziums *bkAf, PQAS,* so that *bk* is to *bf* thus as *PQ* to *PS.* Whereby, on applying the terms of the first proportions to the corresponding terms of this, there will be *bn* to *bd* as *PR* to *PT.* Therefore the equal angled trapeziums *Dnbd* and *DRPT* are similar, and the diagonals of these, *Db* and *DP* are similar on that account. And thus *b* lies at the



intersection of the lines *AP*,*DP* and thus it coincides with the point *P*. Whereby the point *P*, where ever it is taken, to be inscribed on the designated conic section.

Q *E.D.*

Corol. Hence if the three right lines *PQ*, *PR*, *PS* are drawn at given angles from a common point *P* to just as many given right lines in position *AB*, *CD*, *AC*, each to each in turn, and let the rectangle under the two drawn $PQ \times PR$ to the square of the third *PS* be in a given ratio: the point *P*, from which the right lines are drawn, will be located in the section of a cone which touches the lines *AB*, *CD* in *A* and *C;* and vice versa. For the line *BD* may fit together with the line *AC*, with the position of the three lines *AB*, *CD*, *AC* remaining in place*;* then also the line *PT* fits with the line *PS:* and the rectangle $PS \times PT$ becomes *PS* squared and the right lines *AB*, *CD*, which cut the curve in the points *A* and *B*, *C* and *D*, now are no longer able to cut the curve in these points taken together, but only touch.
[Thus, the lines *AB* and *CD* are now tangents to the conic. Apollonius derived the classical three-line locus as a special case of the four-line locus for generating a conic : See *Conics* III, Prop. 54-56.]

Scholium.

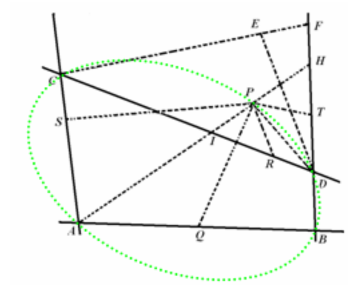
The name of the conic section in this lemma is taken generally, thus so that both a section passing through a vertex of the cone as well as a circle parallel to the base may be included . For if the point *p* falls on the line, by which the points *A* and *D* or *C* and *B* are joined together, the conic section is changed into two right lines, of which one is that right line on which the point *p* falls, and the other is a right line from which the two others from the four points are joined together. If the two opposite angles of the trapezium likewise may be taken as two right angles, and the four lines *PQ*, *PR*, *PS*, *PT* may be drawn to the sides of this either perpendicularly or at some equal angles, and let the rectangle drawn under the two $PQ \times PR$ be equal to the rectangle under the other two $PS \times PT$, so that the rectangle under the sines of the angles *S*, *T*, in which the two final *PS*, *PT* are drawn, to the rectangle under the sines of the angles *Q*, *R*, in which the first two *PQ*, *PR* are drawn. In the rest of the cases the position of the point *P* will be from the other three figures, which commonly are called conic sections. But in place of the trapezium *ABCD* it is possible to substitute a quadrilateral, the two opposite sides of which cross each other mutually like diagonals. But from the four points *A*, *B*, *C*, *D* one or two are able to go off to infinity, and in that case the sides of the figure, which converge to these points, emerge parallel: in which case the section of the cone will be crossed by the other points, and will go off to infinity as parallel lines.

[A full solution of this problem can be found as a note in Whiteside, Vol. VI, p. 275.]

LEMMA XIX.

To find a point P, from which if four right lines PQ, PR, PS, PT may be drawn to just as many other right lines AB, CD, AC, BD, given in position, from one to the other in turn, at given angles, the rectangle drawn under the two, $PQ \times PR$, will be in a given ratio to the rectangle under the other two, $PS \times PT$.

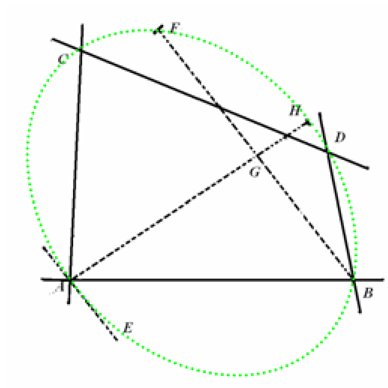
The lines *AB*, CD, to which the two right lines *PQ*, *PR* are drawn containing one of rectangles, come together with the other two lines given at the points *A*, *B*, *C*, *D*. From any of these points *A* some right line *AH* may be drawn, in which the point you wish *P* may be found. That line cuts the opposite lines *BD*, *CD*, without doubt *BD* in *H* and *CD* in I, and on account of all the given angles of the figure, the ratios *PQ* to *PA* and *PA* to *PS* are given, and thus the ratio *PQ* to *PS* is given. By taking [*i.e.* by dividing] this ratio from the given ratio $PQ \times PR$ to $PS \times PT$, the ratio *PR* to *PT* will be given, and by adding [*i.e.* multiplying by] the given ratios *PI* to *PR*, and *PT* to *PH* the ratio *PI* to *PH* will be given, and thus the point *P*.



Q.*E.1.*

Corol. I. Hence also it is possible to draw the tangent at some point *D* of the infinite numbers of locations of the points *P*. For the chord *PD*, when the points *P* and *D* meet, that is, where *AH* is drawn through the point *D*, becomes the tangent. In which case, the final vanishing ratio of the lines *IP* and *PH* may be found as above. Therefore draw *CF* parallel to *AD* itself, crossing *BD* in *F*, and cut at *E* in the same final ratio, and *DE* will be the tangent, because therefore *CF* and the vanishing *IH* are parallel, and similarly cut in *E* and *P*.

Corol. 2. Hence it is apparent also that the position of all the points *P* can be defined. Through any of the points *A*, *B*, C, *D*, e.g. *A*, draw the tangent *AE* of the locus and through some other point *B* draw the parallel of the tangent *BF* meeting the curve [or locus] at the position *F*. But the point *F* may be found by Lem. XIX. With *BF* bisected in *G*, and *AG* produced indefinitely, this will be the position of the diameter to which the ordinates *BG* and *FG* may be applied. This line A*G* may meet the curve in *H*, and *AH* will be a diameter or a transverse width to which the latus rectum will be as BG^2 to $AG \times GH$. If *AG* never meets the curve, the *AH* proves to be infinite, the locus will be a parabola, and the latus rectum of this pertaining to the diameter *AG* will be $\frac{BG^2}{AG}$. But if that meets somewhere, the locus will be a hyperbola, where the points *A* and *H* are placed on the same side of *G:* and an ellipse, when *G* lies between, unless perhaps the angle *AGB* shall



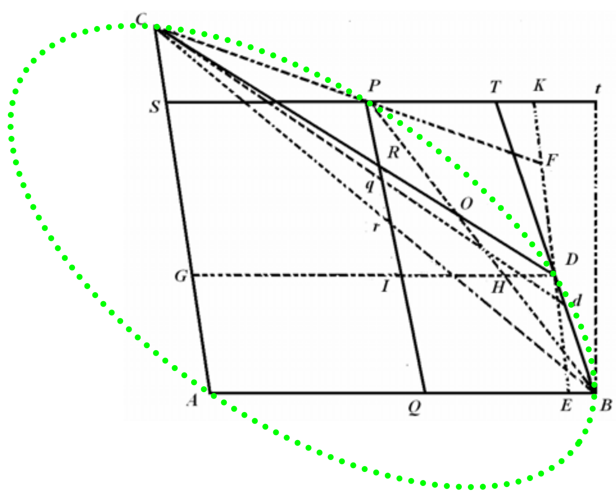
be right, and the above BG^2 is equal to the rectangle *AGH*, in which case a circle will be had.

And thus [a solution] of the problem of the ancients concerning the four lines, started by *Euclid* and continued by *Apollonius* and such as the ancients sought, not from a calculation but composed geometrically, is shown in this corollary.

[There is next presented an important Lemma that is fundamental to the applications that follow.**]**

LEMMA XX.

If in some parallelogram ASPQ, the two opposite angles A and P touch the section of a cone at the points A and P ; and with the sides of one of the angles AQ and AS produced indefinitely, meeting the same section of the cone at B and C ; moreover from the meeting points B and C to some fifth point D of the conic section, the two right lines BD and CD are drawn meeting the other two sides of the parallelogram PS and PQ produced indefinitely at T & R: the parts PR and PT of the sides [*of the parallelogram*] *will always be cut in turn in a given ratio. And conversely, if these cut parts are in turn in a given ratio, the point D touches the section of the cone passing through the four points A, B, C, P.*



Case I. *BP* and *CP* are joined together and from the point *D* the two right lines *DG* and *DE* are acting , the first of which *DG* shall be parallel to *AB* itself and meets *PB* and *PQ* and *CA* in *H, I* and *G;* the other shall be *DE* parallel to *AC* itself and meeting *PC* and *PS* and *AB* in *F, K* and *E:* and the rectangle $DE \times DF$ will be (by Lem. XVII.) in a given ratio to the rectangle $DG \times DH$. But *PQ* to *DE* (or *IQ*) shall be as *PB* to *HB*, and thus as *PT* to *DH;* and in turn *PQ* to *PT* as *DE* to *DH.* And there is *PR* to *DF* as *RC* to *DC*, thus as (*IG* or) *PS* to *DG,* and in turn *PR* to *PS* as *VF* to *DG;* and with the ratios joined the rectangle $PQ \times PR$ shall be to the rectangle $PS \times PT$ as the rectangle $DE \times DF$ to the rectangle $DG \times DH$, and thus in a given ratio. But *PQ* and *PS* are given, and therefore the ratio *PR* to *PT* is given. Q *E.D.*

Case 2. Because if *PR* and *PT* may be put in place in a given ratio in turn, then by retracing the reasoning, it follows that the rectangle $DE \times DF$ to be in a given ratio to the rectangle $DG \times DH$, and thus the point *D* (by Lem. XVIII.) touches the conic section passing through the points *A*, *B*, *C* and *P*.

Q. E. D.

Corol. I. Hence if *BC* acts cutting *PQ* in *r*, & on *PT* there may be taken *Pt* in the ratio to *Pr* that *PT* has to *PR*, *Bt* will be a tangent of the conic section at the point *B*. For consider the point *D* to coalesce with the point *B*, thus so that, as with the chord *BD* vanishing, *BT* may become a tangent; and *CD* and *BT* coincide with *CB* and *Bt*.

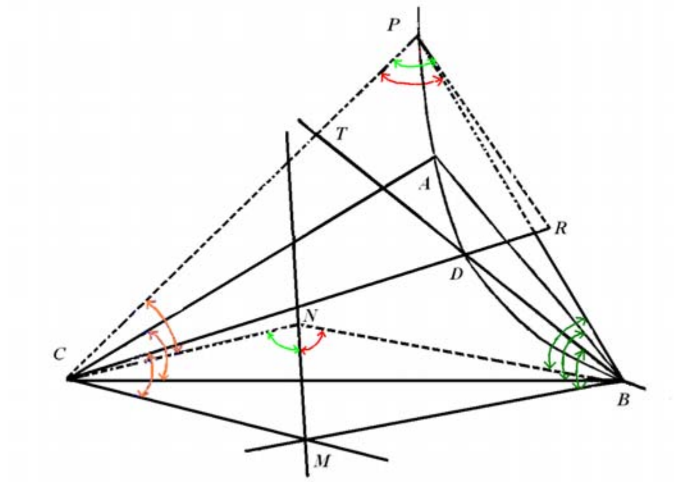
Corol. 2. And in turn if *Bt* shall be a tangent, and at some point *D* of the conic section *BD* and *CD* may come together; *R* will be to *PT* as *Pr* to *Pt*. And counter wise, if there shall be *PR* to *PT* as *Pr* to *Pt*: *BD* and *CD* may come together at some point *D* of the conic section.

Corol. 3. A conic section does not cut a conic section in more than four points. For, it were possible to happen, the two conic sections may pass through each other in the five points *A*, *B*, *C*, *P*, *O*; and these may cut the right line *BD* in the points *D*, *d*, and *PQ* itself may cut the right line *Cd* in *q*. Hence *PR* is to *PT* as *Pq* to *PT*; from which *PR* and *Pq* in turn themselves may be equal, contrary to the hypothesis.

[The following lemma, related to the above, shows how to describe a branch of a hyperbola without making use of the focus, using points on the curve only, as well as a reference line on which related points and angles may be defined. Note the positions of the points *A*, *B*, *C*, *D* and *P* in the diagrams relating to these lemmas, where the hyperbola in the latter can be viewed as an inverted form of the ellipse in the former. Newton has not followed with a like proof, but has introduced a new way of drawing a conic section.]

LEMMA XXI.

If two moveable and indefinite right lines BM and CM drawn through the given points or poles B and C, a given line MN may be described from their meeting position M;



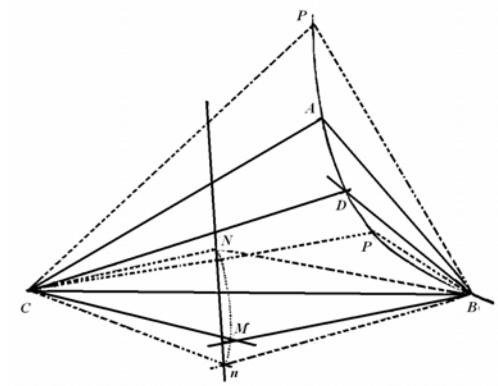
and two other indefinite right lines BD and CD may be drawn making given angles MBD and MCD with the first two lines at these given points B and C : I say that these two lines BD and CD, by their meeting at D, describe the section of a cone passing through the points B and C. And vice versa, if the right lines BD and CD by their meeting at D describe the section of a cone passing through B, C, and A, and the angle DBM shall always be equal to the given angle ABC, and the angle DCM always shall be equal to the given angle ACB: then the point M remains in place on the given line.

For a [fixed] point *N* may be given on the line *MN*, and when the mobile point *M* falls on the motionless point *N*, the mobile point *D* may fall on the motionless [*i.e.* fixed] point *P*. Join *CN*, *BN*, *CP*, *BP*, and from the point *P* direct the lines *PT* and *PR* crossing with *BD* and *CD* themselves in *T* and *R*, and making the angle *BPT* equal to the given angle *BNM*, and the angle *CPR* equal to the given angle *CNM*. Therefore since (from the hypothesis) the angles *MBD* and *NBP* shall be equal, and also the angles *MCD* and *NCP;* take away the common angles *NBD* and *NCD,* and the equal angles *NBM* and *PBT* , *NCM* and *PCR* remain*:* and thus the triangles *NBM* and *PBT* are similar, and also the triangles *NCM*, *PCR.* Whereby *PT* is to *NM* as *PB* to *NB,* and *PR* to NM as *PC* to NC. But the points *B, C, N, P* are fixed. Therefore *PT* and *PR* have a given ratio to *NM,* and therefore a given ratio between themselves; and thus (by Lem. XX.) the point *D,* always the meeting point of the mobile right lines *BT* and *CR* , lies on a conic section passing through the points *B, C*, *P*.

[The triangles *NBM* , *PBT* , and *NCM*, *PCR* are similar ;
$\therefore \frac{NM}{PT}=\frac{NB}{PB}=\frac{MB}{TB}$ and $\frac{NM}{PR}=\frac{NC}{PC}=\frac{MC}{CR}$; hence a definite ratio is formed for the lines *PT* and *PR* , as in the above lemma.]

Q E.D.

And conversely, if the moveable point *D* may lie on a conic section passing through the given points *B, C, A,* and the angle *DBM* always shall be equal to the given angle *ABC,* and the angle *DCM* always equal to the given angle *ACB,* and when the point *V* falls successively on some two immoveable points of the section *p, P,* the moveable point *M* falls successively on two immoveable points *n, N:* through the same *n* and *N* the right line *nN* acts, and this will be the perpetual locus of that mobile point *M.* For, if it

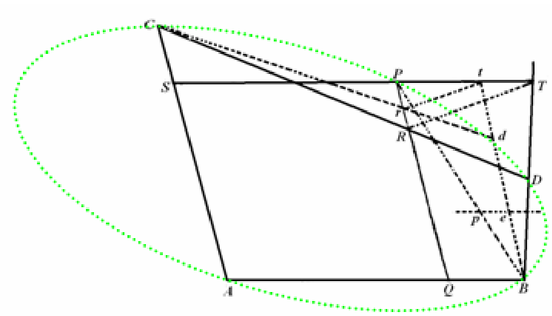


should happen that the point *M* can move along some curved line. Therefore, the point *D* will touch the conic section passing through the five points *B, C, A, p, P*, where the point *M* always lies on a curved line. But also, from the demonstration now made, the point *D* also lies on the conic section passing through the five points *B, C, A, p, P,* where the point *M* always lies on a right line. Therefore the two conic sections will pass through the same five points, contrary to Corol. 3, Lemma. XX. Therefore is absurd for the point *M* to be moving on some curved line. *Q. E. D.*

PROPOSITION XXII. PROBLEM XIV.

To describe a trajectory through five given points.

Five points *A, B, C, P* and *D* may be given. From any one of these points *A* to some other two, which may be called the poles *B* and *C*, draw the right lines *AB* and *AC,* and from these draw the parallel lines *TPS, PRQ* through the fourth point *P*. Then from the two poles *B* and *C,* draw the two indefinite lines *BDT, CRD* through the fifth point *D,* crossing the most recently drawn lines *TPS* and *PRQ* at *T* and *R* (the first to the first and the second to the second). And then from the right lines *PT* and *PR,* with the right line drawn *tr* parallel to *TR* itself, cut some proportion *Pt* and *Pr* of *PT* and *PR*; and if through the ends *t* and *r* of these and the poles *B* and *C*, *Bt* and *Cr* are drawn concurrent in *d*, that point *d* will be located in the trajectory sought. For that point *d* (by Lem. XX) may be placed in a conic section crossed over by the four points *A, B, C, P* ; and with the lines *Rr* and *Tt*

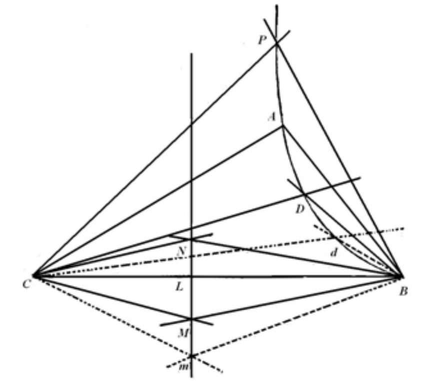


vanishing, the point *d* coincides with the point *D.* Therefore the five points *A, B, C, P, D* will pass through the conic section.

Q.E.D.

The same otherwise.

From the given points join any three *A, B,* C; and around two of these *B, C,* or the poles, by rotating the given angles with magnitude *ABC* and *ACB,* the sides *BA* and *CA* may be applied first to the point *D*, then to the point *P,* and the points *M* and *N* may be noted with which the other sides *BL* and *CL,* themselves cross over in each case. The indefinite line *MN* may be drawn and these mobile angles may be rotated around their poles *B, C,* from that rule so that the intersection of the legs *BL, CL* or *BM,* C*M*, which now shall be *m,* always lies on that infinite line *MN;* and the intersection of the legs *BA, CA,* or *BD, CD,* which now shall be *d,* will delineate the trajectory sought *PADdB.* For the point *d* (by Lem. XXI.) contains the section of the cone passing through the points *B* and *C;* and when the point *m* approaches towards the points *L, M, N,* the point *d* (by construction) will approach towards the points *A, D, P*. And therefore the conical section passes through the five points *A, B, C, P, D.*



Q.E.F.

Corol. 1. Hence the right line can be drawn readily, which touches the trajectory at some given point *B*. The point *d* may approach the point *B,* and the line *Bd* emerges as the tangent sought.

Corol.2. From which also the centres of the trajectories, the diameters and the latera recta can be found, as in the second corollary of Lemma XIX.

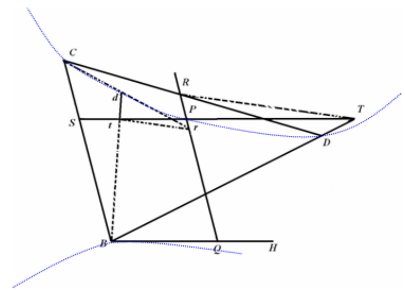
Scholium.

The first construction arose a little simpler by joining *BP*, and in that, if there was a need, produced by requiring that *Bp* to *BP* is as *PR* ad *PT;* and by drawing an infinite right line *pe* through *p* parallel to *SPT* itself, and on that always by taking *pe* equal to *Pr;* and with the right lines *Be* , *Cr* drawn concurrent in *d.* For since there shall be *Pr* to *Pt, PR* to *PT, pB* to *PB, pe* to *Pt* in the same ratio ; *pe* and *Pr* always will be in the same ratio. By this method the points of a trajectory can be found most expeditiously, unless you prefer a curve, as in the following construction, to be described mechanically.

[More information on this and related topics can be found in the book by J.L. Coolidge : *A History of Conic and Quartic Sections*, originally published by OUP (1945), and later as a paperback by Dover Books. The connection to Newton's ongoing research activities can be found in Vol. IV of Whiteside's *Mathematical Papers*......, p.299, and in Vol. VI, p.258 of the same. The entire writings of Greek geometry and many other things can be found at the wilbourhall.org website, in Greek and Latin; these are corrected versions of the ham-fisted efforts of Google in scanning old texts. Of particular interest is the monumental translation of the works of Apollonius by Edward Halley in 1712 from Greek and Arabic sources into Latin; this was published about the same time as the second edition of the *Principia*.]

PROPOSITION XXIII. PROBLEM XV.

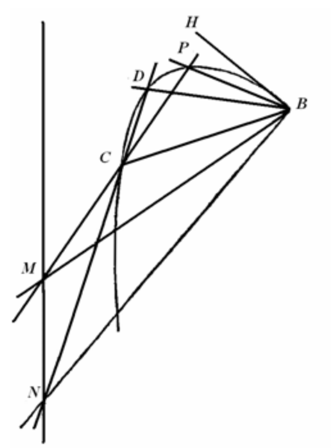
To describe the trajectory, which will pass through four given points, and which will touch a given right line in place.



Case 1. The tangent *HB* may be given, the point of contact *B,* and three other points *C, D, P*. Join *BC,* and with *PS* acting parallel to the right line *BH,* and *PQ* parallel to the right line *BC,* complete the parallelogram *BSP*Q. Draw *BD* cutting *SP* in *T*; and *CD* cutting *PQ* in *R.* And then, with some line *tr* parallel to *TR* , from *PQ, PS* cut *Pr, Pt* proportional to *PR, PT* themselves respectively; and the meeting point *d* of the lines drawn *Cr, Bt* (by Lem. XX.) always lies on the described trajectory.

[Thus, the two methods of defining the conic section are shown, the first above using the parallelogram method, while the second below uses the idea of poles with an angle rotating about one pole and chords passing through the other pole from a variable point on a line.]

The same otherwise.

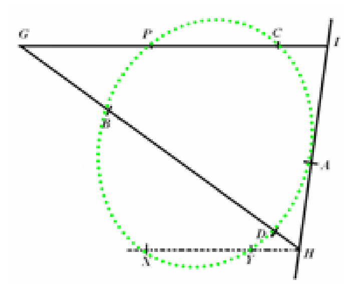


While the angle with given magnitude *CBH* may rotate about the pole *B,* then also some rectilinear radius *DC* has been produced at both ends about the pole *C*. The points *M, N* may be noted, in which the leg *BC* of the angle may cut that radius, when the other leg *BH* meets the same radius at the points *P* and *D.* Then for *MN* drawn indefinitely always meeting that radius *CP* or *CD,* and the leg *BC* of the angle, the join of the other leg *BH* with the radius will delineate the trajectory sought.

For if in the constructions of the above problems the point *A* may fall on the point *B,* the lines *CA* and *CB* coincide, and the line *AB* in its ultimate position becomes the tangent *BH;* and thus the constructions put in place there become the same as the constructions described here. Therefore the meeting of the leg *BH* with the radius passing through the points *C, D, P* will delineate the section of the cone, and the right line *BH* tangent at the point *B.*

Q *E.F.*

Case 2. Four points may be given *B, C, D, P* , the tangent *HI* placed outside. With the two lines *BD, CP* joined meeting in *G,* and with these lines crossing the tangent line in *H* and *I*. The tangent may be cut at *A*, thus so that *HA* shall be to *IA*, as the rectangle under the mean proportion between *CG* and *GP* and the mean proportion between *BH* and *HD*, to the rectangle under the mean proportion between *DG* and *GB* and the mean proportion between *PI* and *IC;* and *A* will be the point of contact.



[Thus, $\frac{HA^2}{AI^2} = \frac{CG \times PG}{DG \times BG} \times \frac{BH \times DH}{PI \times CI}$, which is a constant ratio. This can be viewed expediently as an application in analytic geometry relating to the Rectangle Theorem mentioned earlier; Whiteside has related this and the following lemmas to invariant cross ratios in projective geometry, which of course did not exist as a theory at the time; we have to take the Proposition of Apollonius (Book III, Prop.17) mentioned above as the basis of this lemma and the following.]

For if *HX* parallel to the right line *PI* may cut the trajectory at some points *X* and *T:* the point *A* thus will be located (from the theory of conics [Apollonius *Conics* III, 17&18.]), so that HA^2 will be to AI^2 in the ratio composed from the ratio of the rectangle *XHT* to the rectangle *BHD*, or of the rectangle *CGP* to the rectangle *DGB*, and from the ratio of the rectangle *BHD* to the rectangle *PIC*. Moreover with the point of contact found *A*, the trajectory may be described as in the first case.

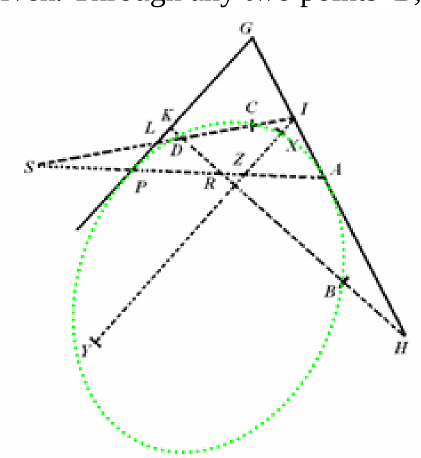
Q.E.F.

But the point *A* can be taken either between the points *H* & *I*, or beyond ; and likewise a twofold trajectory can be described.

PROPOSITION XXIV PROBLEM XVI.

To described a trajectory, which will pass through three given points and which may touch two given right lines in place.

The tangents *HI*, *KL* and the points *B*, *C*, *D* may be given. Through any two points *B*, *D* draw the indefinite right line *BD* meeting the tangents in the points *H*, *K*. Then also through any two of the other points *C*, *D* draw the indefinite line *CD* crossing the tangent lines at the points *I*, *L*. Thus with the drawn lines cut these in *R* and *S*, so that *HR* shall be to *KR* as the mean proportional between *BH* and *HD* is to the mean proportional between *BK* and *KD;* and *IS* to *LS* as the mean proportional is between *CI* and *ID* to the mean proportional between *CL* and *LD*. Moreover cut as it pleases either between the points *K* and *H*, *I* and *L*, or beyond the same ; then draw *RS* cutting the tangents at *A* and *P*, and *A* and *P* will be the points of contact. For if *A* and *P* may be supposed to be the points of contact situated somewhere on the tangents ; and through



some of the points *H*, *I*, *K*, *L* some *I*, placed in either tangent *HI*, the right line *IT* is drawn parallel to the other tangent *KL*, which meet the curve at *X* and *Y*, and on that *IZ* may be taken the mean proportional between *IX* and *IY:* there will be, from the theory of conics, the rectangle *XIY* or IZ^2 to LP^2 as the rectangle *CID* to the rectangle *CTD*, that is (by the construction) as SI^2. ad SL^2 and thus *IZ* to *LP* as *SI* to *SL*. Therefore the points *S*, *P*, *Z* lie on one right line. Again with the tangents meeting at *G*, there will be (from the theory of conics), the rectangle *XIY* or IZ^2 to IA^2 as GP^2 to GA^2 and thus *IZ* to *IA* as *GP* to *GA*. Therefore the points *P*, *Z* and *A* lie on a right line, and thus the points *S*, *P* and *A* are on one right line. And by the same argument it will be approved that the points *R*, *P* and *A* are on one right line. Therefore the points of contact *A* and *P* lie on the right line *RS*. But with these found, the trajectory may be described as in the first case of the above problem.

Q.E.F.

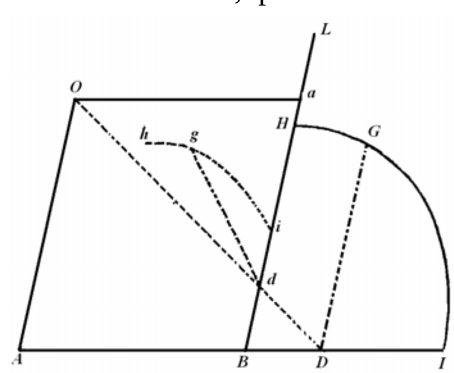
In this proposition, and in the following case of the above proposition the constructions are the same, whither or not the right line *XY* may cut the trajectory at *X* and *Y* ; and these may not depend on that section. But from the demonstrated constructions where that right line may cut the trajectory, the constructions may be known, where it is not cut ; I shall not linger with further demonstrations for the sake of brevity.

[According to Whiteside, Pemberton, the editor of the 3rd and final edition, tried to induce Newton to make some corrections to indicate the existence of two real solutions : see note 60 p.243, Vol.6 *Math. Papers*......]

LEMMA XXII.

To change figures into others of the same kind.

Some figure *HGI* shall be required to be changed. Two parallel lines may be drawn in some manner *AO*, *BL* cutting some third given line *AB* in place at *A* and *B*, and from some point *G* of the figure, some line *GD* may be drawn to the line *AB*, parallel to *OA* itself. [The initial skew axis can be taken as *BDI* of the abscissa or ordinate *x* with origin *A*, and *AO* as the applied line or coordinate *y;* this general one to one degree preserving transformation, the product of a simple affine transformation and a plane perspectivity, or a simple translation and rotation, and rescaling, had been published originally by de la Hire in his *Conic Sections*, (Note 67 Whiteside); subsequently used here to convert converging lines into parallel lines.].Then from some point *O*, given on the line *OA*, the right line *OD* is drawn to the point *D*, crossing *BL* itself in *d*, and from the crossing point there is raised the given line *dg* containing some angle with the right line *BL*, and having that



ratio to *Od* which *DG* has to *OD*; and *g* will be the point in the new figure *hgi* corresponding to the point *G*. [Thus the first axis are translated and rotated and rescaled to become th new axis; the new abscissa of the original point *G* is *ad*.] By the same method the individual points of the first figure will give just as many points in the new figure. Therefore consider the point *G* by moving continually to run through all the points of the first figure, and likewise the point *g* by moving continually will run through all the points of the new figure and describe the same. For the sake of distinction we may call the *DG* the first order, and *dg* the new order; *AD* the first abscissa, *ad* the new abscissa; *O* the pole, *OD* the cutting radius, *OA* the first order radius, and *Oa* (from which the parallelogram *OABa* is completed) the new order radius.

Now I say that, if the point *G* touches the right line in the given position, the point *g* will also touch the same right line in the position given. If the point *G* touches a conic section, the point *g* will also touch a conic section. Here I count the circle with the conic sections. Again if the point *G* touches a line of the third analytical order, the point *g* touches a line of the third analytical order ; and thus with curved lines of higher order. The two lines which the points *G, g* touch will always be of the same analytical order. And indeed as *ad* is to *OA* thus are *Od* to *OD, dg* to *DG,* and *AB* to *AD;* and thus *AD* is equal to $\frac{OA\times AB}{ad}$, and *DG* is equal to $\frac{OA\times dg}{ad}$. Now if the point *G* touches a right line, and thus in some equation, in which a relation may be had between the abscissa *AD* and the ordinate *DG*, these indeterminate lines *AD* and *DG* rise to a single dimension only, by writing $\frac{OA\times AB}{ad}$ for *AD* in this equation, and $\frac{OA\times dg}{ad}$ for *DG*, a new equation will be produced, in which the new abscissa *ad* and the new ordinate *dg* rise to single dimension only, and thus which designate a right line. But if *AD* and *DG*, or either of these, will rise to two dimensions in the first equation, likewise *ad* and *dg* will rise to two in the second equation. And thus with three or more dimensions. The indeterminates *ad, d*g in the second equation, and *AD, DG* in the first always rise to the same number of the dimensions, and therefore the lines, which touch the points *G, g,* are of the same analytical order.

I say besides, that if some right line may touch a curved line in the first figure ; this right line in the same manner with the transposed curve in the new figure will touch that curved line in the new figure ; and conversely. For if some points of the curve approach to two and join in the first figure, the same transposed points will approach in turn and unite in the new figure; and thus the right lines, by which these points are joined, at the same time emerge as tangents in tangents of the curves in each figure.

The demonstrations of these assertions may be put together in a more customary manner by geometry. But I counsel brevity.

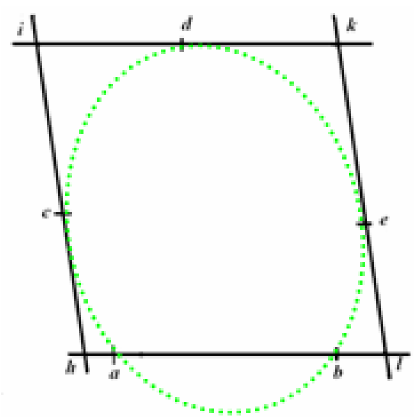
Therefore if a rectilinear figure is to be transformed into another, if it is constructed from right lines, it will suffice to transfer intersections, and through the same to draw right lines in the new figures. But if it may be required to transform curvilinear figures, points, tangents and other right lines are to be transferred, with the aid of which a curved line may be defined. But this lemma is of assistance in the solution of more difficult problems, by transforming the proposed figures into simpler ones. For any converging right lines are transformed into parallel lines, by requiring to take some right line for the first order radius, which passes through the meeting point of convergent lines; and thus because that meeting point with this agreed upon will go to infinity ; since they are parallel lines, which never meet. But after the problem is solved in the new figure; if by inverse operations this figure may be changed into the first figure, the solution sought will be had.

Also this lemma is useful in the solution of solid problems. For as often as two sections of cones are come upon, of which a problem is required to be solved by the intersection, it is possible to change either of these, if it shall be either a hyperbola or a parabola into an ellipse: then it may be easily changed into a circle. Likewise a right line and a conic section, in the construction of plane problems, may be turned into a right line and a circle.

[The interested reader may like to know that Edmond Halley the first editor, according to Note 71 on p. 272 of Vol. VI of the *Math. Works*....., wished further explanation from Newton on this Lemma; this he was given in a letter from Newton, but which never made it translated into Latin into the *Principia*. Coolidge, using coordinates, derives a transformation of the kind indicated on p. 46 of his history of sections.]

PROPOSITION XXV. PROBLEM XVII.

To describe a trajectory, which will pass through two given points, and touch three given lines in place.



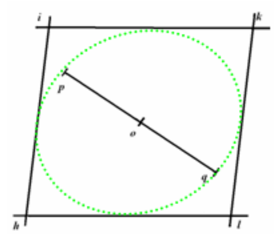
Through the meeting of any two tangents with each other in turn, and the meeting of the third tangent with that line, which passes through the two given points, draw an indefinite line; and with that taken for the first order radius, the figure may be changed by the above lemma, into a new figure. In that new figure these two tangents themselves emerge parallel in turn to each other, and the third tangent becomes parallel to the right line passing through the two given points. Let *hi, kl* be these two parallel tangents, *ik* the third tangent, and *hl* a right line parallel to this passing through these points *a*, *b*, through which the conic section in this new figure must pass through, and completing the parallelogram *hikl*. The right lines *hi, ik, kl* may be cut in *c, d, e,* thus so that the side *hc* is to the square root of the rectangle *ah.hb,* [*i.e. hc* squared shall be to the rectangle *ah.hb*], as *ic* to *id,* and *ke* to *kd* as the sum of the rectangles *hi* and *kl* is to the sum of the three lines, the first of which is the right line *ik,* and the other two are as the squared sides of the rectangles *ah.hb* and *al.lb:* and *c, d, e* will be the points of contact. For indeed; from the conics, hc^2 is to the rectangle a*h.hb,* as ic^2 to id^2, and ke^2 to kd^2, and in the same ratio el^2 to the rectangle *al.lb*; and therefore *hc* is to the square root of *ahb*, *ic* to *id, ke* to *kd,* and *el* to the square root of *al.lb* are in that square root ratio, and on adding, in the given ratio of all the preceding *hi* and *kt* to all the following, which are to the square root of the rectangle *ah.hb*, and to the rectangle *ik*, and the square root of the rectangle *al.lb*. Therefore the points of contact *c, d, e* may be had from that given ratio in the new figure. By the inverse operations of the newest lemmas these points may be transferred to the first figure, and there (by Prob. XIV.) the trajectory may be described. Q. *E. F.*

[From the Rectangle Theorem : $\frac{hc^2}{ha\times hb}=\frac{ic^2}{id^2}$ and hence $\frac{ke^2}{kd^2}=\frac{ic^2}{id^2}=\frac{le^2}{lb\times la}=\frac{hc^2}{ha\times hb}$; or $\frac{hc}{\sqrt{ha\times hb}}=\frac{ic}{id}=\frac{ke}{kd}=\frac{le}{\sqrt{lb\times la}}$. Hence $\frac{hc+ci+ke+el}{id+dk+\sqrt{ha\times hb}+\sqrt{lb\times la}}=\frac{hi+kl}{ik+\sqrt{ha\times hb}+\sqrt{lb\times la}}=$ given ratio.]

Moreover thence so that the points *a,b* lie either between the points *h, l,* or beyond, the points *c, d, e* must lie between the points taken *h, i, k, l* , or beyond. If either of the points *a, b* fall between the points *h, l,* and the other beyond, the problem is impossible.

PROPOSITION XXVI. PROBLEM XVIII.

To describe a trajectory, which will pass through a given point, and will touch four given lines in place.



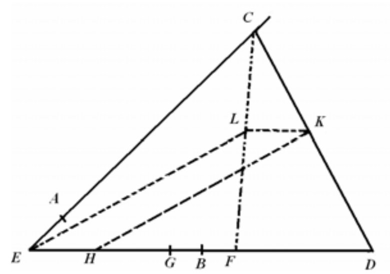
From the common intersection of any two tangents to the common intersection of the remaining two an indefinite right line is drawn, and the same taken for the first order radius, the figure may be transformed (by Lem. XXII.) into a new figure, and the two tangents, which met at the first order radius now emerge parallel. Let these be *hi* and *kl* ; *ik* and *hl* containing the parallelogram *hikl*. And let *p* be the point in this new figure corresponding to a given point in the first figure. Through the centre of the figure *Opq* is drawn, and putting *Oq* to equal *Op, q* will be another point through which the conic section in this new figure must pass. By the operation of the inverse of Lemma XXII this point may be transferred into the first figure, and here two points will be had through which the trajectory is to be described. Truly that same trajectory can be described by Problem XVII.

Q.E.F.

[On p. 272 of Vol. VI of the *Math. Papers....*, Whiteside has drawn a hyperbola for the external case, where the curve and the points *p* and *q* lie outside the parallelogram.]

LEMMA XXIII.

If two given right lines in place AC, BD may be terminated in two given points A, B, and they may have a given ratio in turn, and the line CD, by which the indeterminate points C, D are joined together, may be cut in the given ratio at K : I say that the point K will be located on a given fixed right line.



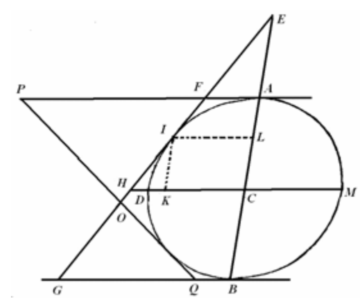
For the right lines *AC* and *BD* meet in *E,* and on *BE* there may be taken *BG* to *AE* as *BD* is to *AC*, and *FD* always shall be equal to the given *EG ;* and from the construction there will be *EC* to *GD,* that is, to *EF* as *AC* to *BD,* and thus in the given ratio, and therefore a kind of triangle *EFC* is given. *CF* may be cut in *L* so that *CL* to *CF* shall be in the ratio *CK* to *CD;* and on account of that given ratio, a kind of triangle *EFL* will also be given *;* and thence the point *L* will be place in a given position on the line *EL*. Join *LK,* and *CLK, CFD* will be similar triangles*;* and on account of *FD* given and the given ratio *LK* to *FD*, *LK* will be given. This may be taken equal to *EH,* and *ELKH* always will be a parallelogram. Therefore the point *K* is located on the side *HK* of this parallelogram in place.

Q.E.D.

Corol. On account of the given kind of figure *EFLC*, the three right lines *EF*, *EL* and *EC,* that is, *GD, HK* and *EC,* in turn have given ratios.

LEMMA XXIV.

If three right lines may touch some conic section, two of which shall be parallel and may be given in position; I say that the semi-diameter of the section parallel to these two lines, shall be the mean proportional between the segments of these, from the points of contact and to the third interposed tangent.



Let *AF* and *GB* be the two parallel tangents of the conic section *ADB* touching at *A* and *B; EF* the third tangent of the conic section touching at *I,* and crossing with the first tangents at *F* and *G*; and let *CD* be the semi-diameter of the figure parallel to the tangents : I say that *AF*, *CD*, *BG* are continued proportionals.

For if the conjugate diameters *AB* and *DM* cross the tangent *FG* at *E* and *H,* and mutually cut each other at *C* and the parallelogram *IKCL* may be completed*;* from the nature of the conic section as *EC* is to *CA* thus *CA* is to *CL,* [by Apoll. Book III, Prop. 42] and thus by division *EC–CA* to C*A–CL,* or *EA* to *AL,* and from adding *EA* to *EA*+*AL* or *EL* as *E*C to $EC + CA$ or *EB;* and thus, on account of the similar triangles *EAF*, *ELI*, *ECH*, *EBG*, *AF* to *LI* as *CH* to *BG*. Likewise, from the nature of conic sections, *LI* or *CK* is to *CD* as *CD* is to *C*H; and thus from the rearranged equation, *AF* to *CH* as *CD* to *BG*.

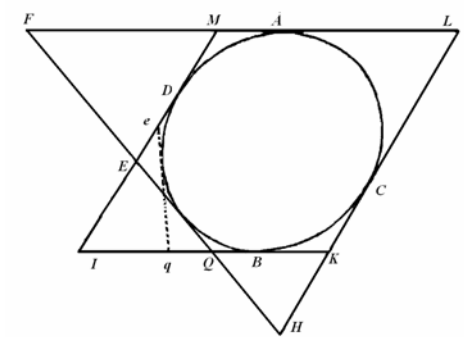
Q.E.D.

Corol. I. Hence if the two tangents *FG, PQ* with the parallel tangents *AF, BG* cross at *F* and *G, P* and *Q* and mutually cut each other at *O*; from the rearranged equation there will be *AF* to *BQ* as *AP* to *BG*, and on dividing as *FP* to *GQ*, and thus as *FO* to *OG.*

Corol. 2. From which also the two right lines *PG, FQ*, drawn through the points *P* and *G, F* and *Q*, concur at the right line *ACB* through the centre of the figure and passing through the points of contact *A, B* .

LEMMA XXV.

If the four sides of a parallelogram produced indefinitely may touch some conic section, and are cut by some fifth tangent; moreover the ends of any two neighbouring sides* [*sections*] *cut off opposite the angles of the parallelogram may be taken* : *I say that each section shall be to its side, as the part of the other neighbouring side between the point of contact and the third side, is to the section of this other side.



The four sides *ML, IK, KL, MI* of the parallelogram *MLIK* may touch the conic section at *A*, *B*, *C*, *D*, and the fifth tangent *FQ* cuts these sides at *F, Q, H* and *E;* moreover the sections of the sides *MI*, *KI* may be taken *ME* and *KQ*, or of the sides *KL, ML* the sections *KH, MF* : I say that *ME* to *MI* shall be as *BK* to *KQ*; and *KH* to *KL* as *AM* to *MF.* For by the first corollary of the above lemma *ME* is to *EI* as *AM* or *BK* to *BQ*, and by taking *ME* to *MI* as *BK* to *KQ*. Q. *E. D.*

Likewise *KH* to *HL* as *BK* or *M* to *AF*, and on dividing *KH* to *KL* as *AM* to *MF*. Q *E.D.*

Corol. I. Hence if the parallelogram *IKLM* is given, described about some given conic section, the rectangle $KQ\times ME$ will be given, and also as equally to that the rectangle $KH\times MF$. For the rectangles are equal on account of the similarity of the triangles *KQH* and *MFE.*

Corol. 2. And if a sixth tangent *eq* is drawn crossing with the tangents *KI, MI* at *q* and *e;* the rectangle $KQ\times ME$ will be equal to the rectangle $Kq\times Me$ *;* and there will be *KQ* to *Me* as *Kq* to *ME*, and by division as *Qq* to *Ee.*

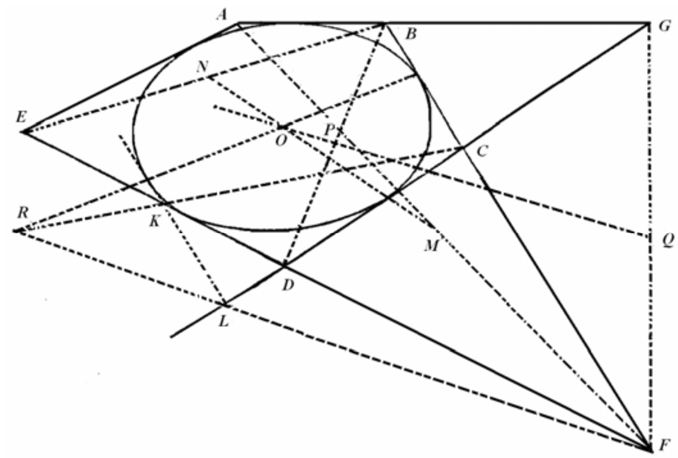
Corol. 3. From which also if *Eq*, *eQ* may be joined and bisected, and a right line is drawn through the point of bisection, this will pass through the centre of the conic section. For since there shall be *Qq* to *Ee* as *KQ* to *Me*, the same right line will pass through the midpoints of every *EQ, eQ, M K* (by Lem. XXIII.), and the midpoint of the line *MK* is the centre of the section.

[See Whiteside note 80, Vol. VI p.280 *Math. Papers*... for some of the interesting history on this Lemma]

PROPOSITION XXVII. PROBLEM XIX.

To describe a trajectory, which touches five given lines in position.

The tangents *ABG*, *BCF*, *GCD*, *FDE*, *EA* may be given in position. For the quadrilateral figure *ABFE* contained by any four, bisect the diagonals *AF*, *BE* in *M* and *N*, and (by Corol.3. Lem. XXV.) the right line *MN* drawn through the point of bisection will pass through the centre of the trajectory . Again for the figure of the quadrilateral *BGDF*, contained by any other four tangents, the diagonals (as thus I may say) *BD*, *GF* bisected in *P* and *Q*: and the right line *PQ* drawn through the point of bisection will pass

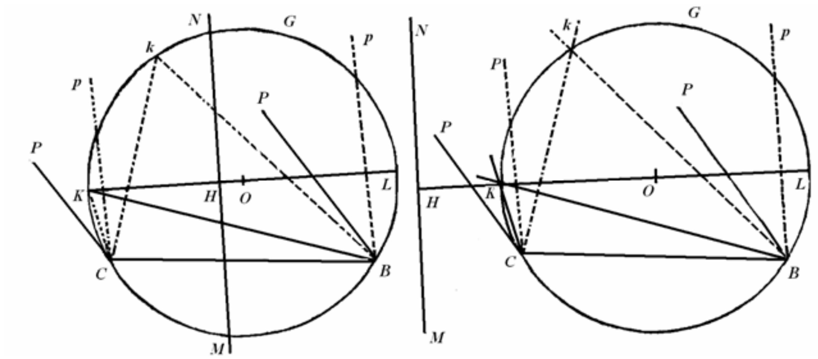


through the centre of the trajectory. Therefore the centre will be given at the meeting point of the bisectors. Let that be *O*. Draw *KL* parallel to any tangent *BC* of this [trajectory], to that distance so that the centre *O* may be located at the mid-point between the parallel lines ; and the line *KL* drawn will touch the trajectory to be described [*note* : the diagram is misleading: *K* does not lie on the ellipse] . It will cut these other two tangents in *GCD* and *FDE* in *L* and *K*. Through the meeting points of the non-parallel tangents *CL*, *FK* with the parallel tangents *CF*, *KL*, *C* and *K*, *F* and *L* draw *CK*, *FL* meeting in *R*, and the right line *OR* drawn and produced will cut the parallel tangents *CF*, *KL* in the points of contact. This is apparent by Corol.2, Lem. XXIV. By the same method it will be possible to find the other points of contact, and then finally by the construction of Prob. XIV. to describe the trajectory. *Q. E .F.*

Scholium.

The problems, where either the centres or asymptotes of the trajectory are given, are included in the proceeding. For with the points and tangents given together with the centre, just as many other points and tangents are given from the other parts of this trajectory equally distant from the centre. But an asymptote may be taken as a tangent, and the end of this will be considered for the point of contact at an infinite distance (if thus it shall be spoken of). Consider the contact point of any tangent to go off to infinity, and the tangent will change into an asymptote, and the constructions of the preceding problems [XIV and Case 1 of XV] will be changed into constructions where the asymptote is given.

After the trajectory has been described, we are free to find the axes and the foci of this curve by this following method. In the construction and in the figure of Lemma XXI,



made so that the legs *BP*, *CP* of the moveable angles *PBN*, *PCN*, by the meeting of which the trajectory was described, in turn themselves shall become parallel, and then maintaining that position [the angles] may be rotated about their poles *B* and C in that figure. Meanwhile truly the other legs *CN*, *BN* of these angles may describe the circle *BGKC*, by their meeting at *K* or *k*,. Let *O* be the centre of this circle. From this centre to the ruler *MN*, to which these other legs *CN* and *BN* meanwhile will concur, while the trajectory may be described, send the normal *OH* from the centre crossing at *K* and *L*. And where these other legs *CK* and *BK* meet at that point *K* which is closer to the ruler, the first legs *CP* and *BP* will be parallel to the major axis, and perpendicular to the minor; and the opposite comes about, if the same legs meet at the more distant point *L*. From which if the centre of the trajectory may be given, the axes are given. But from these given, the foci are evident.
[A detailed explanation of these results is given by Whiteside in the notes 92 – 95 Vol. VI of the *Math. Papers*....p.285 ; Whiteside also considers the hyperbolic case with a splendid diagram.]

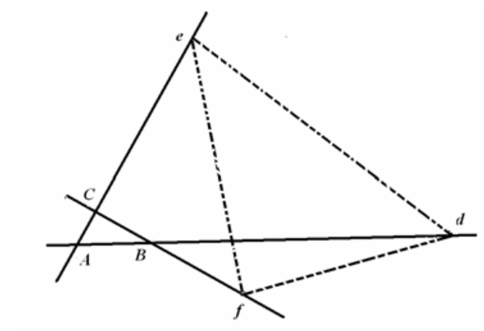
Truly the squares of the axes are one to the other as *KH* to *LH*, and from this the kind of the trajectory given by the given four points is easily described. For if from the two points given [on the curve], the poles *C* and *B* may be put in place, the third will give the mobile angles *PCK* and *PBK*; moreover with these given the circle *BGKC* can be described. Then on account of the given kind of trajectory, the ratio *OH* to *OK* will be given, and thus *OH* itself, With centre *O* and with the interval *OH* describe a circle, and the right line, which touches this circle, and passes through the meeting point of the legs *CK* and *BK*, where the first legs *CP* and *BP* concur at the fourth given point, will be that ruler *MN* with the aid of which the trajectory may be described. From which also in turn the kind of trapezium [*i.e.* quadrilateral] given (if indeed certain impossible cases are excepted) in which some given conic section can be described.

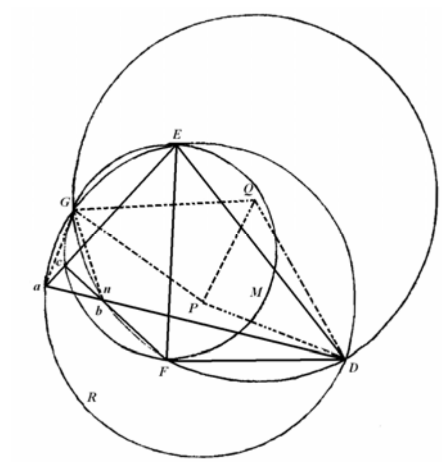
Also there are other lemmas with the aid of which given kinds of trajectories, with given points and tangents, are able to be described. That is of this kind, if a right line may be drawn through some given point in place, which may intersect the given conic section in two points, and the interval of the intersection may be bisected, the point of bisection may touch another conic section of the same kind as the first, and having the axes parallel with those of the former. But I will hurry on to more useful matters. [This question is examined by Whiteside in note 98 of the above.]

LEMMA XXVI.

The three angles of a triangle given in kind and magnitude are put in place one to one to as many given right lines in place, which are not all parallel.

Three right lines *AB*, *AC*, *BC* are given in place and it is required thus to locate the triangle *DEF*, so that the angle of this *D* may touch the line *AB*, likewise the angle *E* the line A*C*, and the angle *F* the line *BC*. Upon the sides *DE, DF* & *EF* describe three segments of circles *DRE, DGF, EMF,* which take angles equal to the angles *BAC*, *ABC*, *ACB* respectively. But these segments may be described to these parts of the lines *DE, DF, EF,* so that the letters *DRED* may be returned in the same cyclic order with the letters *BACB*, the letters *DGFD* with the same letters *ABCA*, and the letters *EMFE* with the letters *ACBA*; then these segments may be completed in whole circles. The first two circles cut each other mutually in *G*, and let *P* and *Q* be the centres of these. With *GP* and *PQ* joined, take *Ga* to *AB* as *GP* is to *PQ* , and with the centre *G*, with the interval *Ga* describe the circle, which will cut the first circle *DGE* in *a*. Then *aD* is joined cutting the second circle *DFG* in *b*, then *aE* cutting the third circle *EMF* in *c*. And now the figure *ABCdef* can be set up similar and equal to the figure *abcDEF*. With which done the problem is completed.





For *Fc* itself may be drawn crossing *aD* in *n*, and *aG*, *bG*, *QG*, *QD*, *PD* may be joined. From the construction the angle *EaD* is equal to the angle *CAB*, and the angle *acF* is equal to the angle *ACB*, and thus the triangle *anc* is equiangular to the triangle *ABC* . Hence the angle *anc* or *FnD* is equal to the angle *ABC*, and thus is equal to the angle *FbD* ; and therefore the point *n* falls on the point *b*. Again the angle *GPQ*, which is half of the angle at the centre *GPD*, is equal to the angle at the circumference *GaD*; and the angle *GQP*, which is half the angle at the centre *GQD*, is equal to the complement of two right angles at the circumference *GbD*, and thus equal to the angle *Gba*; and thus the two triangles *GPQ* and *Gab* are similar; and *Ga* is to *ab* as *GP* to *PQ* ; that is (from the construction) as *Ga* to *AB*. And thus *ab* and *A B* are equal; and therefore the triangles *abc* and *ABC*, which we have approved in a similar manner, are also equal. From which, since the sides *ab, ac, bc* respectively may touch the above angles *D, E, F* of the triangle *DEF* , the angles of the triangle *abc*, the figure *ABCdef* can be completed similar to the similar and equal figure *abcDEF*, and that on completion solves the problem. *Q E F*.

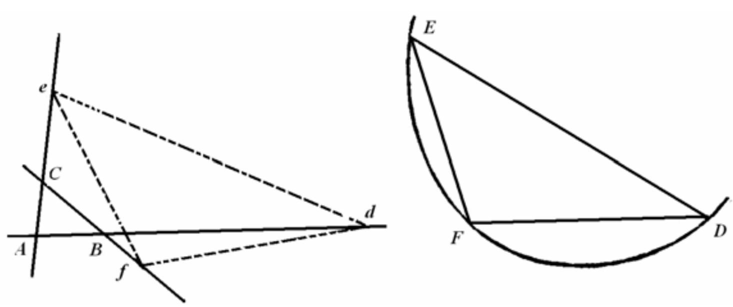
Corol. Hence a right line can be drawn the parts of which given in length will lie between three given right lines in place. Consider the triangle *DEF*, with the point *D* approaching the side *EF*, and with the sides *DE, DF* placed along a line, to be changed into a right line, the given part of which *DE* must be placed between the right lines given in place *AB, AC*, and the preceding part *DF* between the given right lines *AB, BC* in place ; and by applying the preceding construction to this case the problem may be solved.

PROPOSITION XXVIII. PROBLEM XX.

To describe a trajectory given in kind and magnitude, the given parts of which will lie in position between three given lines.

The trajectory shall be required to be described, which shall be similar and equal to the curved line *DEF*, and which with the three right lines *AB, AC, BC* given in position, will be cut into the given parts of this by the similar and equal parts of this *DE* and *EF*.

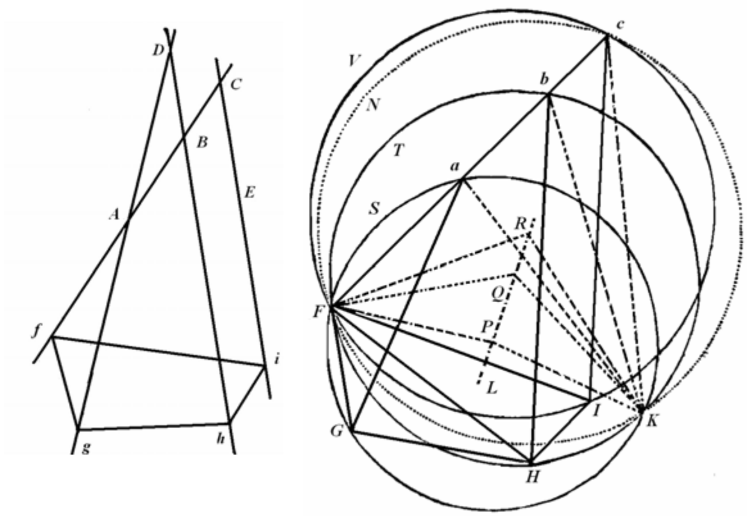
Draw the right lines *DE, EF, DF*, and to the triangle of this *DEF* place the angles *D, E, F* to these given right lines in place (by Lem. XXVI), then describe a similar and equal trajectory of the curve *DEF* about the triangle. *Q.E.F.*



LEMMA XXVII.

To describe a given kind of trapezium* [*i.e. quadrilateral*]*, the angles of which are given to four right lines in position, one to one in position, and which are not all parallel, nor converge to a common point,.

The four right lines may be given in position *ABC, AD, BD, CE;* the first of which may cut the second at *A,* the third at *B,* and the fourth at *C:* and the trapezium *fghi* shall be required to be described, which shall be similar to the trapezium *FGHI* ; and the angle *f* of which shall be equal to the given angle *F*, may touch the right line *ABC*; and the other angles *g, h, i,* shall be equal to the other given angles *G, H, I* , may touch the lines *AD*, *BD*, *CE* respectively. *FH* and the above *FG* may be joined, *FH* and *F1* may describe as many sections of the circle *FSG, FTH, FVI;* of which the first *FSG* may take an angle equal to the angle *BAD,* the second *FTH* may take an angle equal to the angle *CBD,* and the third *FVI* may take an angle equal to the angle *ACB.* But the segments must be described according to these parts of the lines *FG, FH, F1*, so that the letters *FSGF* shall be in the same cyclic order as the letters *BADB,* and so that the letters *FTHF* shall be returned in the same circle with the letters *CBDC,* and the letters *FVIF* with the letters *ACEA.* The segments may be completed into whole circles, and let *P* be the centre of the first circle *FSG,* and *Q* the centre of the second *FTH.* Also *PQ* may be joined and produced in each direction and in that *QR* may be taken in the same ratio to *PQ* as *BC* has

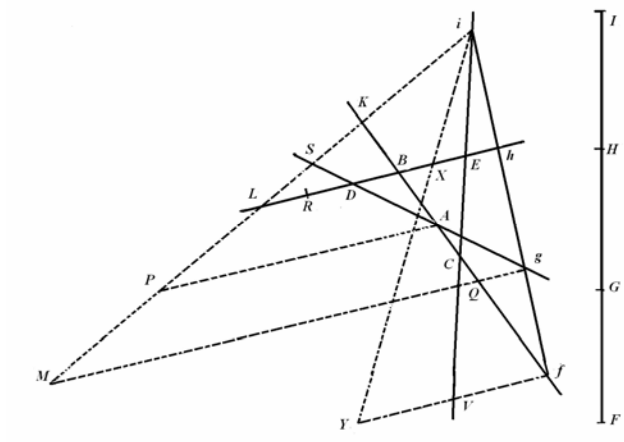


to *AB.* But *QR* may be taken on the side of the point *Q* so that the order *P, Q, R* of the letters shall be the same as of the letters *A*, *B*, *C:* and with centre *R* and with the interval *RP* the fourth circle may be described *FNc* cutting the third circle *FVI* in *c. Fc* may be joined cutting the first circle *a,* and the second in *b. aG*, *bH,* and *cI* are constructed and the figure *abcFGHI* can be put in place similar to the figure *ABCfghi.* With which done the trapezium *fghi* will be that itself, which it was required to construct.

For the two first circles *FSG* and *FTH* mutually cut each other in *K. PK, QK, RK, aK, bK,* and *cK* may be joined and *QP* may be produced to *L.* The angles to the circumferences *FaK, FbK, FcK* are half of the angles *FPK, FQK, FRK* at the centres, and thus equal to the halves of these angles *LPK, LQK, LRK*. Therefore the figure *PQRK* is equiangular and similar to the figure *abcK,* and therefore *ab* is to *bc* as *PQ* to *QR,* that is, as *AB* to *BC.* By construction, to the above *FaG, FbH, FcI* the angles *fAg, fBh, fCi* are equal. Therefore to the figure *abcFGHI* the similar figure *ABCfghi* is able to be completed. With which done the trapezium *fghi* may be constructed similar to the trapezium *FGHI,* and with its angles *f, g, h, i* touching the right lines *ABC, AD, BD, CE.*

Q. E. F.

Corol. Hence a right line can be drawn whose parts, with four given lines in position intercepted in a given order, will have a given proportion to each other. The angles *FGH* and *GH1* may be augmented as far as that, so that the right lines *FG, GH, HI* may be placed in a direction, and these in this case by constructing the problem will lead to the right line *fghi*, the parts of which *fg, gh, hi,* intersected by the four right lines given in position *AB* and *AD, AD* and *BD, BD* and *CE,* are to one another as the lines *FG, GH, HI*, and they will maintain the same order among themselves. Truly the same shall be expedited thus more readily.



AB may be produced to *K,* and *BD* to *L,* so that *B K* shall be to *AB* as *HI* to *GH;* and *DL* to *BD* as *GI* to *FG;* and *KL* may be joined crossing the right line *CE* in *i.* There may be produced *iL* to *M,* so that there shall be *LM* to *iL* as *GH* to *HI,* and then there may be drawn *MQ* parallel to *LB* itself, and crossing the right line *AD* in *g*, then *gi* cuts *AB, BD* in *f, h.* I say that it has been done. [See note 110 of Whiteside for an explanation.]

For *Mg* may cut the right line *AB* in *Q,* and *AD* the right line *KL* in *S,* and *AP* is drawn which shall be parallel to *BD* itself and may cross *iL* in *P,* and there will be *gM* to *Lh* (*gi* to *bi, Mi* to *Li, GI* to *HI, AX* to *BK*) and *AP* to *BL* in the same ratio. *DL* may be cut in *R* so that *DL* shall be to *RL* in that same ratio, and on account of the proportionals *gS* to *gM, AS* to *AP*, and *DS* to *DL;* there will be, from the equation, as *gS* to *Lh* thus *AS* to *BL* and *DS* to *RL;* and on mixing [the ratios], *BL–RL* to *Lh–BL* as *AS–DS* to *gS–AS.* That is, as *BR* to *Bh* so *AD* to *Ag,* and thus as *BD* to *gQ.* And in turn *BR* to *BD* as *Bh* to *gQ,* or *fh* to *fg.* But by construction the line *BL* will be cut in the same ratio in *D* and *R* and the line *FI* in *G* and *H:* and thus *BR* is to *BD* as *FH* to *FG.* Hence *fh* is to *fg* as *FH* to *FG.* Therefore since also there shall be *gi* to *hi* as *Mi* to *Li,* that is, as *GI* to *HI,* it is apparent the lines *FI, fi* similarly are cut in *g* and *h, G* and *H.*

Q. E.F.

In the construction of this corollary after *LK* is drawn cutting *CE* in *i,* it is allowed to produce *iE* to *V,* so that there shall be *EV* to *Ei* as *FH* to *HI,* and to draw *Vf* parallel to *BD* itself. The same is returned if from the centre *i,* with the radius *IH,* a circle may be described cutting *BD* in *X,* and *iX* may be produced to *r,* so that *iT* shall be equal to *IF,* and *Tf* may be drawn parallel to *BD.*

Other solutions of this problem were devised formerly by *Wren and Wallis*.

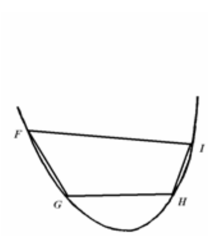
PROPOSITION XXIX PROBLEM XXI.

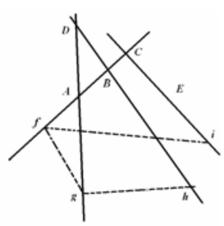
To describe a trajectory of a given kind, which from four given right lines will be cut into parts, in order, in given kind and proportion.

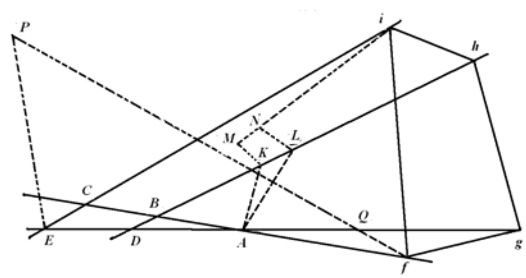
A trajectory shall be required to be described, which shall be similar to the curved line *FGHI,* and the parts of which, similar and proporional to the parts of this *FG, GH, HI,* with the right lines given in position *AB* and *AD, AD* and *BD, BD* and *CE,* the first may lie between the first, the second with the second, and the third with the third. With the right lines drawn *FG, GH, HI, FI,* the trapezium [read as quadrilateral] *fghi* may be described (by Lem. XXVII.) which shall be similar to the trapezium *FGHI,* and the angles of which *f, g, h, i* may touch these right lines given in position *AB, AD, BD, CE,* one to one in the said order. Then about this trapezium the trajectory of the like curved line *FGHI* may be described.

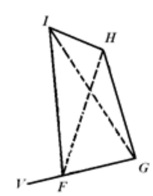
Scholium.

It is possible for this problem to be constructed as follows. With *FG, GH, HI, FI* joined produce *GF* to *P,* and join *FH, 1G,* and with the angles *FGH, PFH* make the angles *CAK, DAL* equal. *A K a*nd *AL* are concurrent with the right line *BD* in *K* and *L,* and thence *KM* and *LN* are drawn, of which *KM* may make the angle *AKM* equal to the angle *GHI,* and it shall be to *AK* as *HI* is to *GH* ; and *LN* may make the angle *ALN* equal









to the angle *FHI,* and it shall be to *AL* as *HI* to *FH.* Moreover *AK, KM, KM, AL, LN* may be drawn to these parts of the lines *AD, AK, AL,* so that the letters *CAKMC, ALKA, DALND* may be returned in the same order with the letters *FGHIF* in the orbit*;* and with *MN* drawn it may cross the right line *CE* in *i.* Make the angle *iEP* equal to the angle *1GF*, and *PE* shall be to *Ei* as *FG* to *G1;* and through *P* there is drawn *PQf,* which with the right line *ADE* may contain the angle *PQE* equal to the angle *FIG,* and crosses the right line *AB* in *h* and *fi* may be joined. But *PE* and *PQ* may be drawn to these sections of the lines *CE* and *PE,* so that the cyclic order of the lines *PEiP* and *PEQp* shall be the same as of the letters *FGHIF,* and if above on the line *fi* also the same order of the letters may be put in place, the trapezium *fghi* will be similar to the trapezium *FGHI,* and the given kind of trajectory may be circumscribed, the problem may be solved.

Up to this point concerned with the finding of orbits. It remains that we may determine the motion of bodies in the orbits found.

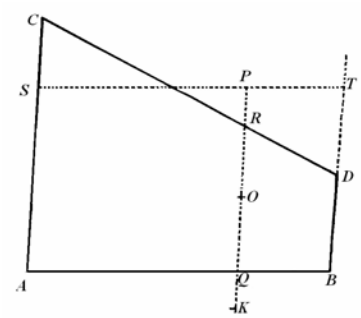
SECTIO V.

Inventio orbium ubi umbilicus neuter datur.

LEMMA XVII.

Si a datae conicae sectionis puncto quovis P ad trapezii alicuius ABDC, in conica illa sectione inscripti, latera quatuor infinite producta AB, CD, AC, DB totidem rectae PQ, PR, PS, PT in datis angulis ducantur, singulae ad singula: rectangulum ductarum ad opposita duo latera $PQ \times PR$***, erit ad rectangulum ductarum ad alia duo latera opposita*** $PS \times PT$ ***in data ratione.***

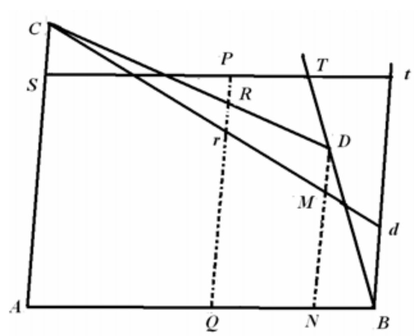
Cas: I. Ponamus primo lineas ad opposita latera ductas parallelas esse alterutri reliquorum laterum, puta *PQ* & *PR* lateri *AC, & PS* ac *PT* lateri *AB.* Sintque insuper latera duo ex oppositis, puta *AC* & *BD*, sibi invicem parallela. Et recta, quae bisecat parallela illa latera, erit una ex diametris conicae sectionis, & bisecabit etiam *RQ*, Sit *O* punctum in quo *R*Q bisecatur, & erit *PO* ordinatim applicata ad diametrum illam. Produc *PO* ad *K*, ut sit *OK* aequalis *PO*, & erit *OK* ordinatim applicata ad contrarias partes diametri.



Cum igitur puncta *A, B, P* & *K* sint ad conicam sectionem, & *PK* secet *AB* in dato angulo, erit (per Prop. 17, 19, 21 & 2.3. Lib.III. conicorum *Apollonii*) rectangulum *PQK* ad rectangulum *AQB* in data ratione. Sed *QK* & *PR* aequales sunt, utpote aequalium *OK*, *OP*, & *OQ*, *OR* differentiae, & inde etiam rectangula *PQK* & $PQ \times PR$ aequalia sunt; atque ideo rectangulum $PQ \times PR$ est ad rectangulum *AQB*, hoc est ad rectangulum $PS \times PT$ in data ratione.

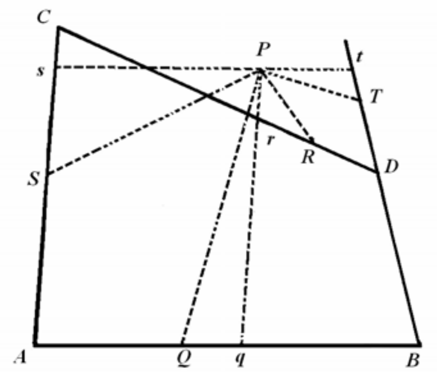
Q.E.D.

Cas: 2. Ponamus iam trapezii latera opposita *AC* & *BD* non esse parallela. Age *Bd* parallelam *AC* & occurrentem tum rectae *ST* in *t*, tum conicae sectioni in *d*. Iunge *Cd* secantem *PQ* in *r*, & ipsi *PQ* parallelam age *DM* secantem *Cd* in *M* & *AB* in *N*. Iam ob similia triangula *BTt, DBN;* est *Bt* seu *PQ* ad *Tt* ut *DN* ad *NB.* Sic & *Rr* est ad *AQ* seu *PS* ut *DM* ad *AN*. Ergo, ducendo antecedentes in antecedentes & consequentes in consequentes, ut rectangulum *PQ* in *Rr* est ad rectangulum *PS* in *Tt*, ita rectangulum *NDM* est ad rectangulum *ANB*, & (per cas. I.) ita rectangulum *PQ* in *Pr* est ad rectangulum *PS* in *Pt*, ac divisim ita rectangulum $PQ \times PR$ est ad rectangulum $PS \times PT$.



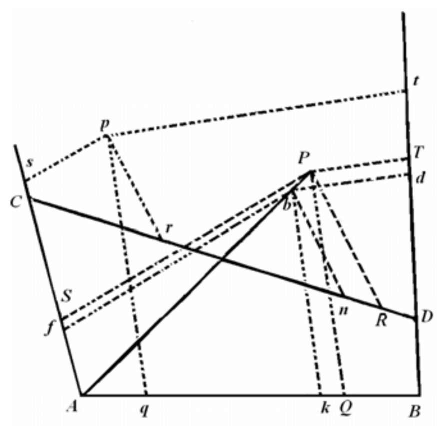
Q.E.D.

Cas. 3. Ponamus denique quatuor lineas *PQ, PR, PS, PT* non esse parallelas lateribus *A C, AB* sed ad ea uncunque inclinatas. Earum vice age *Pq, Pr* parallelas ipsi *AC* ; & *Ps, Pt* parallelas ipsi *AB*; & propter datos angulos triangulorum *PQq, PRr, PSs, PTt,* dabuntur rationes *PQ* ad *Pq*, *PR* ad *Pr*, *PS* ad *Ps*, & *PT* ad *Pt* ; atque ideo rationes compositae; $PQ\times PR$ ad $Pq\times Pr$, & $PS\times PT$ ad $Ps\times Pt$. Sed, per superius demonstrata, ratio $Pq\times Pr$ ad $Ps\times Pt$ data est : ergo & ratio $PQ\times PR$ ad $PS\times PT$. *Q.E.D.*



LEMMA XVIII.

Iisdem positis ;si rectangulumm ductarum ad opposita duo latera trapezii $PQ\times PR$ ***sit ad rectangulum ductarum ad reliqua duo latera*** $PS\times PT$ ***in data ratione; punctum P, a quo lineae ducuntur, tanget conicam sectionem circa trapezium descriptam***.



Per puncta *A, B, C, D* & aliquod infinitorum punctorum *P,* putata *p,* concipe conicam sectionem describi: dico punctum *P* hanc semper tangere. Si negas, junge *AP* secantem hanc conicam sectionem alibi quam in *P,* si fieri potet, puta in *b*. Ergo si ab his punctis *p* & *b* ducantur in datis angulis ad latera trapezii rectae *pq, pr, ps, pt* & *bk, bn, bf, bd* ; erit ut $bk\times bn$ ad $bf\times bd$ ita (per Lem. XVII.) $pq\times pr$ ad $ps\times pt$, & ita (per hypoth.) $PQ\times PR$ ad $PS\times PT$. Est & propter similitudinem trapeziorum *bkAf; PQAS,* ut *bk* ad *bf* ita *PQ* ad *PS.* Quare, applicando terminos prioris proportionis ad terminos correspondentes huius, erit *bn* ad *bd* ut *PR* ad *PT.* Ergo trapezia aequiangula *Dnbd, DRPT* similia sunt, & eorum diagonales *Db, DP* propterea coincidunt. Incidit itaque *b* in intersection rectarum *AP,DP* ideoque coincidit cum puncto *P.* Quare punctum *P,* ubicunque sumatur, incidit in assignatam conicam sectionem.

Q *E.D.*

Corol. Hinc si rectae tres *PQ, PR, PS* a puncto communi *P* ad alias totidem positione datas rectas *AB, CD, AC,* singulae ad singulas, in datis angulis ducantur, sitque rectangulum sub duabus ductis $PQ\times PR$ ad quadratum tertiae *PS* in data ratione: punctum *P,* a quibus rectae ducuntur, locabitur in sectione conica quae tangit lineas *AB, CD* in *A* & *C;* & contra. Nam coeat linea *BD* cum linea *AC,* manente positione trium *AB, CD, AC;* dein coeat etiam linea *PT* cum linea *PS:* & rectangulum $PS\times PT$

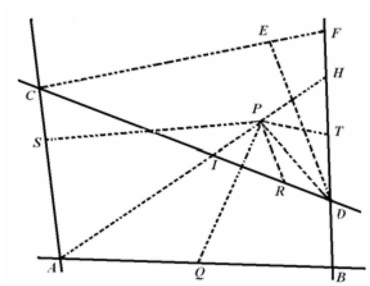
evadet *PS quad*. rectaeque *AB, CD,* quae curvam in punctis *A* & *B, C* & *D* secabant, iam curvam in punctis ills coeuntibus non amplius secare possunt, sed tantum tangent.

Scholium.

Nomen conicae sectionis in hoc lemmate late sumitur, ita ut sectio tam rectilinea per verticem coni transiens, quam circularis basi parallela includatur. Nam si punctum *p* incidit in rectam, qua puncta *A* & *D* vel *C* & *B* iunguntur, conica sectio vertetur in geminas rectas, quarum una est recta illa in quam punctum *p* incidit, & altera est recta qua alia duo ex punctis quatuor iunguntur. Si trapezii anguli duo oppositi simul sumpti aequentur duobus rectis, & lineae quatuor *PQ*, *PR*, *P S*, *PT* ducantur ad latera eius vel perpendiculariter vel in angulis quibusvis aequalibus, sitque rectangulum sub duabus ductis $PQ \times PR$ aequate rectangulo sub aliis duabus $PS \times PT$, ut rectangulum sub sinubus angulorum *S*, *T*, in quibus duae ultimae *PS*, *PT* ducuntur, ad rectangulum sub Sinubus angulorum *Q*, *R*, in quibus duae primae *PQ*, *PR* ducuntur. Caeteris in casibus locus puncti *P* erit aliqua trium figurarum, quae vulgo nominantur sectiones conicae. Vice autem trapezii *ABCD* substitui potest quadrilaterum, cuius latera duo opposita se mutuo instar diagonalium decussant. Sed & e punctis quatuor A, *B, C, D* possunt unum vel duo abire ad infinitum, eoque pacto latera figurae. quae ad puncta illa convergunt, evadere parallela: quo in casu sectio conica transibit per caetera puncta, & in plagas parallelarum abibit in infinitum.

LEMMA XIX.

Invenire punctum P, a quo si rectae quatuor PQ, PR, PS, PT ad alias totidem positione datas rectas AB, CD, AC, BD, singulae ad singulas, in datis angulis ducantur, rectangulum sub duabus ductis, $PQ \times PR$***, sit ad rectangulum sub aliis duabus,*** $PS \times PT$ ***, in data ratione.***

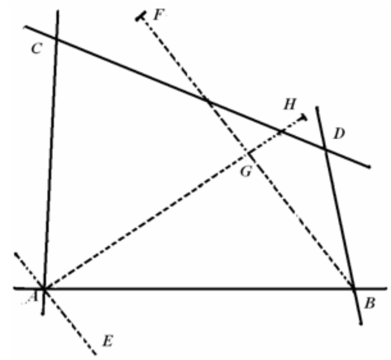


Lineae *AB*, CD, ad quas rectae duae *PQ*, *PR* unum rectangulorum continentes ducuntur, conveniant cum allis duabus positione datis lineis in punctis *A, B, C, D.* Ab eorum aliquo *A* age rectam quamlibet *AH*, in qua velis punctum *P* reperiri. Secet ea lineas oppositas *BD*, *CD*, nimirum *BD* in *H* & *CD* in I, & ob datos omnes angulos figurae, dabuntur rationes *PQ* ad *P*A & *PA* ad *PS*, ideoque ratio *PQ* ad *PS.* Auferendo hanc a data ratione $PQ \times PR$ ad $PS \times PT$, dabitur ratio *PR* ad *PT*, & addenda datas rationes *PI* ad *PR*, & *PT* ad *PH* dabitur ratio *PI* ad *PH*, atque ideo punctum *P*.

Q.*E.1.*

Corol. I. Hinc etiam ad loci punctorum infinitorum *P* punctum quodvis *D* tangens duci potest. Nam chorda *PD*, ubi puncta *P* ac *D* conveniunt, hoc est, ubi *AH* ducitur per punctum *D*, tangens evadit. Quo in casu, ultima ratio evanescentium *IP* & *PH* invenietur ut supra. Ipsi igitur *AD* duc parallelam *CF*, occurrentem *BD* in *F*, & in ea ultima ratione sectam in *E*, & *DE* tangens erit, propterea quod *CF* & evanescens *IH* parallelae sunt, & in *E & P* similiter sectae.

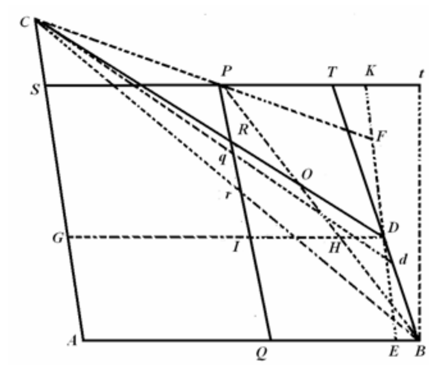
Corol. 2. Hinc etiam locus punctorum omnium *P* definiri patet. Per quodvis punctorum *A*, *B*, C, *D*, puta *A*, duc loci tangentem *AE* & per aliud quodvis punctum *B* duc tangenti parallelam *BF* occurrentem loco in *F*. Invenietur autem punctum *F* per Lem. XIX. Biseca *BF* in *G*, & acta indefinita *AG* erit positio diametri ad quam *BG* & *FG* ordinatim applicantur. Haec A*G* occurrat loco in *H*, & erit *AH* diameter sive latus transversum, ad quod latus rectum erit ut *B*G*q* ad $AG \times GH$. Si *AG* nusquam occurrit loco, linea *AH* existente infinita, locus erit parabola, & latus rectum eius ad diametrum *AG* pertinens erit $\frac{BGq}{AG}$. Sin ea alicubi occurrit, locus hyperbola erit, ubi puncta *A* & *H* sita sunt ad easdem partes ipsius *G:* & ellipsis, ubi *G* intermedium est, nisi forte angulus *AGB* rectus sit, & insuper *BGquad.* aequale rectangulo *AGH*, quo in casu circulus habebitur.



Atque ita problematis veterum de quatuor lineis ab *Euclide* incoepti & ab *Apollonio* continuati non calculus, sed compositio geometrica, qualem veteres quaerebant, in hoc corollario exhibetur.

LEMMA XX.

Si parallelogrammum quodvis ASPQ angulis duobus oppositis A & *P tangit sectionem quamvis conicam in punctis A &* P; & *lateribus unius angulorum illorum infinite productis AQ, AS occurrit eidem sectioni conicae in B* & *C*; *a punctis autem occursuum B* & *C ad quintum quodvis si sectionis conicae punctum D agantur rectae duae BD, CD occurrentes alteris duobus infinite productis parallelogrammi lateribus PS, PQ in T & R*: *erunt simper abscissae laterum partes PR & PT ad invicem in data ratione. Et contra, si partes illae abscissae sunt ad invicem in data ratione, punctum D tanget sectionem conicam per puncta quatuor A, B, C, P transeuntem.*



Cas. I. Iungantur *BP*, *CP* & a puncto *D* agantur rectae duae *DG, DE,* quarum prior *DG* ipsi *AB* parallela sit & occurrat *PB, PQ, CA* in *H, I, G;* altera est *DE* parallela sit ipsi *AC* & occurrat *PC, PS, AB* in *F, K, E:* & erit (per Lem. XVII.) rectangulum $DE \times DF$ ad rectangulum $DG \times DH$ in ratione data. Sed est *PQ* ad *DE* (seu *IQ*) ut *PB* ad *HB,* ideoque ut *PT* ad *DH;* & vicissim *PQ* ad *PT* ut *DE* ad *DH*. Est & *PR* ad *DF* ut *RC* ad *DC,* ideo ut (*IG* vel) *PS* ad *DG,* & vicissim *PR* ad *PS* ut *VF* ad D*G;* & conjunctis rationibus sit rectangulum $PQ \times PR$ ad rectangulum $PS \times PT$ ut rectangulum $DE \times DF$ ad rectangulum $DG \times DH$, atque ideo in data ratione. Sed dantur *PQ* & *PS*, & propterea ratio *PR* ad *PT* datur.

Q *E.D.*

Cas. 2. Quod si *PR* & *P T* ponantur in data ratione ad invicem tum simili ratiocinio regrediendo, sequetur esse rectangulum $DE \times DF$ ad rectangulum $DG \times DH$ in ratione data, ideoque punctum *D* (per Lem. XVIII.) contingere conicam sectionem transeuntem per puncta *A, B, C, P*.

Q. E. D.

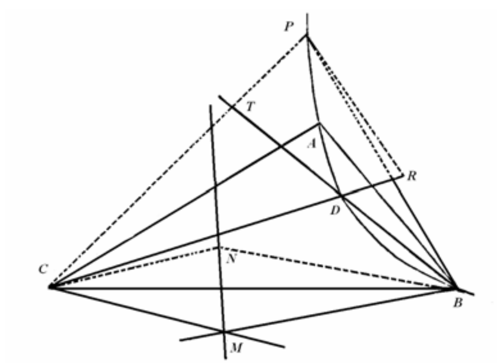
Corol. I. Hinc si agatur *BC* secans *PQ* in *r,* & in *PT* capiatur *Pt* in ratione ad *Pr* quam habet *PT* ad *PR:* erit *Bt* tangens conicae sectionis ad punctum *B.* Nam concipe punctum *D* coire cum puncto *B,* ita ut, chorda *BD* evanescente, *B T* tangens evadat; & *CD* ac *BT* coincident cum *CB* & *Bt.*

Corol. 2. Et vice versa si *Bt* sit tangens, & ad quodvis conicae sectionis punctum *D* conveniant *BD., CD;* erit *R* ad *PT* ut *Pr* ad *Pt.* Et contra, si sit *PR* ad *PT* ut *Pr* ad *Pt:* convenient, *BD, CD* ad conicae sectionis punctum aliquod *D.*

Corol. 3. Conica sectio non secat conicam sectionem in punctis pluribus quam quatuor. Nam, si fieri potest, transeant duae conicae sectiones per quinque puncta *A, B, C, P,O*; easque secet recta *BD* in punctis *D, d,* & ipsam *PQ* secet recta *Cd* in *q*. Ergo *PR* est ad *PT* ut *Pq* ad *PT;* unde *PR* & *Pq* sibi invicem aequantur, contra hypothesin.

LEMMA XXI.

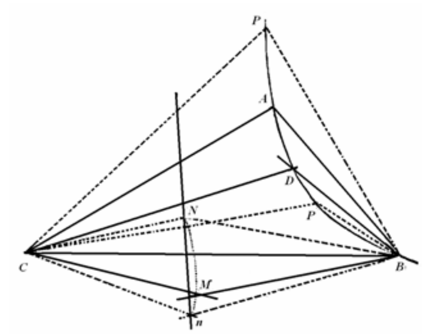
Si rectae duae mobiles & infinitae BM, CM per data puncta B, C ceu polos ductae, concursu suo M describant tertiam positione datam rectam MN; & aliae duae infinitae *rectae BD, CD cum prioribus duabus ad puncta illa data B, C datos angulos MBD, MCD efficientes ducantur* : *dico quod hae duae BD, CD concursu suo D describent sectionem* coninic*am per puncta B, C transeuntem. Et vice versa, si rectae BD, CD concursu suo D*



describant sectionem conicam per data puncta B,C, A transeuntem, & sit angulus DBM femper aequalis angulo dato ABC, angulusque DCM semper aequalis angulo dato ACB: *punctum M continget rectam postione datam.*

Nam in recta *MN* detur punctum *N,* & ubi punctum mobile *M* incidit in immotum *N,* incidat punctum mobile *D* in immotum *P*. Junge *CN, BN, CP, B P,* & a puncto P age rectas *PT, PR* occurrentes ipsis *BD, CD* in *T* & *R,* & facientes angulum *BPT* anguIo dato *BNM,* & angulum *CPR* aequalem angulo dato *CNM.* Cum ergo (ex hypothesi) aequates sint anguli *MBD, NBP,* ut & anguli *MCD, NCP;* aufer communes *NBD & NCD,* & restabunt aequaIes *NBM* & *PBT; NCM* & *PCR:* ideoque triangula *NBM,* *PBT* similia sunt, ut & triangula *NCM, PCR.* Quare *PT* est ad *NM* ut *PB* ad *NB,* & *PR* ad NM ut *PC* ad NC. Sunt autem puncta *B, C, N, P* immobilla. Ergo *PT & PR* datam habent rationem ad *NM*, proindeque datam rationem inter se; atque ideo (per Lem. XX.) punctum *D,* perpetuus rectarum mobilium *BT* & *CR* concursus, contingit sectionem conicam, per puncta *B, C, P* transeuntem.

Q E.D.

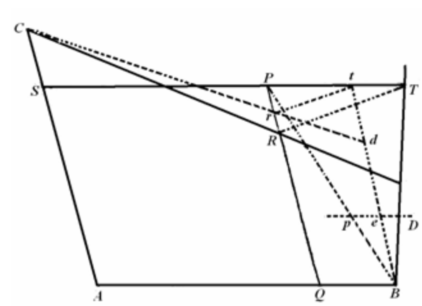


Et contra, si punctum mobile *D* contingat sectionem conicam transeuntem per data puncta *B, C, A,* & sit angulus *DBM* semper aequalis angulo dato *ABC,* & angulus *DCM* semper aequalis angulo dato *ACB,* & ubi punctum *V* incidit successive in duo quaevis sectionis puncta immobilla *p, P,* punctum mobile *M* incidat successive in puncta duo immobilia *n, N:* per eadem *n, N* agatur. recta *nN,* & haec erit locus perpetuus puncti illius mobilis *M.* Nam, si fieri potest, versetur punctum *M* in linea aliqua curva. Tanget ergo punctum *D* sectionem conicam per puncta quinque *B, C, A, p, P* transeuntem, ubi punctum *M* perpetuo tangit lineam curvam. Sed & ex jam demonstratis tanget etiam punctum *D* sectionem conicam per eadem quinque puncta *B, C, A, p, P*, transeuntem, ubi *M* punctum *M* perpetuo tangit lineam rectam. Ergo duae sectiones conicae transibunt per eadem quinque puncta, contra Corol. 3, Lemma. XX. Igitur punctum *M* versari in linea curva absurdum est.

Q. E. D.

OPOSITIO XXII. PROBLEMA XIV.

Trajectoriam per data quinque puncta describere.

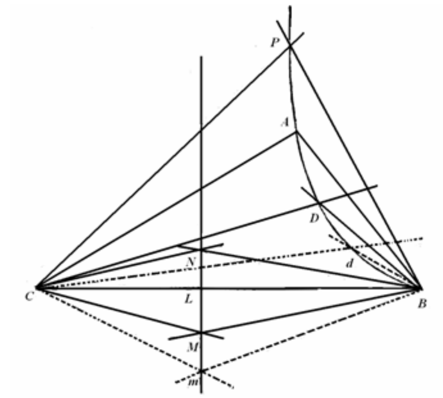


Dentur puncta quinque *A, B, C, P, D.* Ab eorum aliquo *A* ad alia duo quevis *B, C* quae poli nominentur, age rectas *AB, AC* hisque parallelas *TPS, PR Q* per punctuun quartum *P.* Deinde a polis duobus *B, C* age per punctum quintum *D* infinitas duas *BDT, CRD,* novissime ductis *TPS, PRQ* (priorem priori & posteriorem posteriori) occurrentes in *T* & *R.* Denique de rectis *PT* ; *PR,* acta recta *tr* ipsi *TR* parallela, abscinde quasvis *Pt*, *Pr* ipsis *PT*, *PR* proportionales; & si per earum terminos *t, r* & polos *B, C* actae *Bt, Cr* concurrant in *d,* locabitur punctum illud *d* in traiectoria quaesita. Nam punctum illud d (per Lem. XX) versatur in conica sectione per puncta quatuor *A, B, C, P* transeunte; & lineis *Rr, Tt* evanescentibus, coit punctum *d* cum puncto *D.* Transit ergo sectio conica per puncta quinque *A, B, C, P, D.*

Q.E.D.

Idem aliter.

E puntis datis iunge tria quaevis *A*, *B*, C; & circum duo eorum *B*, *C*, ceu polos, rotando angulos magnitudine datos *ABC*, *ACB*, applicentur crura *BA*, *CA* primo ad punctum *D*, deinde ad punctum *P*, & notentur puncta *M*, *N* in quibus altera crura *BL*, *CL*, casu utroque se decussant. Agatur recta infinita *MN*, & rotentur anguli illi mobiles circum polos suos *B*, *C*, ea lege ut crurum *BL*, *CT* vel *BM*, C*M* intersectio, quae iam sit *m*, incidat semper in rectam illam infinitam *MN;* & crurum *BA*, *CA*, vel *BD*, *CD* intersectio, quae iam sit *d*, traiectoriam quaesitam *PADdB* delineabit. Nam punctum *d* (per Lem. XXI.) continget sectionem conicam per puncta *B*, *C* transeuntem; & ubi punctum *m* accedit ad puncta *L*, *M*, *N*, punctum *d* (per constructionem) accedet ad puncta *ADP*.



Describetur itaque sectio conica transiens per puncta quinque *A*, *B*, *C*, *P*, *D*.

Q.E.F.

Corol. 1. Hinc recta expedite duci potest, quae traiectoriam quaesitam in puncto quovis dato *B* continget. Accedat punctum *d* ad punctum *B*, & recta *Bd* evadet tangens quesita.

Corol.2. Unde etiam trajectoriarum centra, diametri & latera recta inveniri possunt, ut in corollario secundo lemmatis XIX.

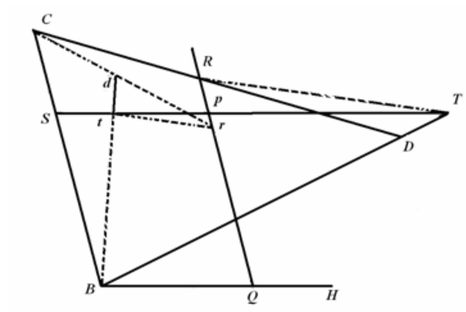
Scholium.

Constructio prior evadet paulo simplicior iungendo *BP*, & in ea, si opus est, producta capiendo *Bp* ad *B P* ut est *PR* ad *PT; &* per *p* agendo rectam infinitam *pe* ipsi *SPT* parallelam, & in ea capiendo semper *pe* aequalem *Pr;* & agendo rectas *Be*, *Cr* concurrentes in *d*. Nam cum sint *Pr* ad *Pt*, *PR* ad *PT*, *PB* ad *PB*, *Pe* ad *Pt* in eadem ratione; erunt *pe* & *Pr* semper aequales. Hac methodo puncta trajectoriae inveniuntur expeditissime, nisi mavis curvam, ut in construtione secunda, describere mechanice.

PROPOSITIO XXIII. PROBLEMA XV.

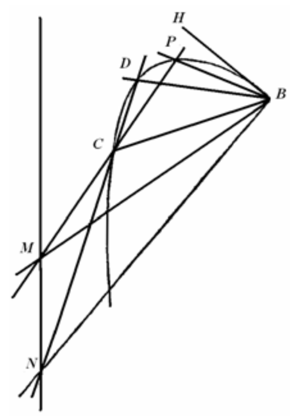
Traiectoriam describere, quae per data quatuor puncta transibit, & rectam continget positione datam.

Cas. 1. Dentur tangens *HB*, punctum contactus *B*, & alia tria puncta *C, D, P*. Iunge *BC*, & agendo *PS* parallelam rectae *BH, & PQ* parallelam rectae *BC*, comple parallelogrammum *BSP*Q. Age *BD* secantem *SP* in *T*; & *CD* secantem *PQ* in *R*. Denique, agenda quamvis *tr* ipsi *TR* parallelam, de *PQ*, *PS* abscinde *Pr*, *Pt* ipsis *PR, PT* proportionales respective; & actarum *Cr, Bt c*oncursus *d* (per Lem. XX.) incidet semper in traiectoriam describendam.



Idem aliter.

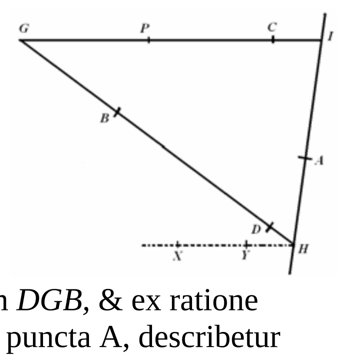
Revolvatur tum angulus magnitudine datus *CBH* circa polum *B*, tum radius quilibet rectilineus & utrinque productus est *DC* circa polum *C*. Notentur puncta *M, N*, in quibus anguli crus *BC* secat radium illum, ubi crus alterum *BH* concurrit cum eodem radio in punctis *P* & *D*. Deinde ad actam infinitam *MN* concurrant perpetuo radius ille *CP* vel *CD* anguli crus *BC,* & cruris alterius *BH* concursus cum radio delineabit traiectoriam quaesitam.



Nam si in constructionibus problematis superioris accedat punctum *A* ad punctum *B*, lineae *CA* & *CB* coincident, & linea *AB* in ultimo suo situ fiet tangens *BH;* atque ideo constructiones ibi positae evadent eaedem cum constructionibus hic descriptis. Delineabit igitur cruris *BH* concursus cum radio sectionem conicam per puncta *C, D, P* transeuntem, & rectam *BH* tangentem in puncto *B.*

Q *E.F.*

Cas. 2. Dentur puncta quatuor *B, C, D, P* extra tangentem *HI* sita. Iunge bina lineis *BD, CP* concurrentibus in *G*, tangentique occurrentibus in *H* & *I.* Secetur tangens in *A*, ita ut sit *HA* ad *IA*, ut est rectangulum sub media proportionali inter *CG & GP* & media proportionali inter *BH* & *HD,* ad rectangulum sub media proportionali inter *DG* & *GB* & media proportionali inter *PI* & *IC; &* erit *A* punctum contactus. Nam si rectae *PI* parallela *HX* traiectoriam secet in punctis quibusvis *X* & *T:* erit (ex conicis) punctum *A* ita locandum, ut fuerit *HA quad.* ad *AI quad.* in ratione composita ex ratione rectanguli *XHT*ad rectangulum *BHD*, seu rectanguli *CGP* ad rectangulum *DGB,* & ex ratione rectanguli *BHD* ad rectangulum *PIC.* lnvento autem contactus puncta A, describetur traiectoria ut in casu primo.



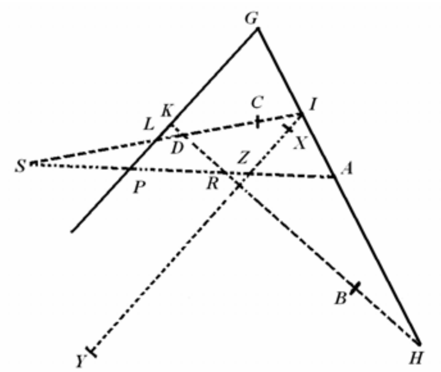
Q.E.F.

Capi autem potest punctum *A* vel inter puncta *H* & *I,* vel extra ; & perinde traiectoria dupliciter describi.

PROPOSITIO XXIV PROBLEMA XVI.

Traiectoriam describere, quae transibit per data tria puncta, & rectas duas positione datas contingent.

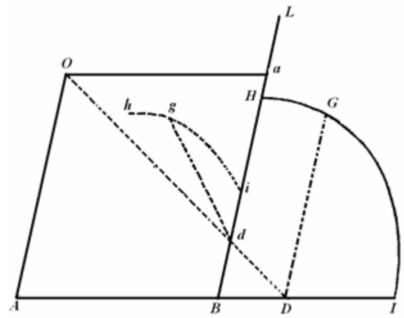
Dentur tangentes *HI*, *KL* & puncta *B*, *C*, *D*. Per punctorum duo quaevis *B*, *D* age rectam infinitam *BD* tangentibus occurrentem in punctis *H*, *K*. Deinde etiam per alia duo quaevis *C*, *D* age infinitam *CD* tangentibus occurrentem in punctis *I*, *L*. Actas ita seca in *R* & *S*, ut sit *HR* ad *KR* ut est media proportionalis inter *BH* & *HD* ad mediam proportionalem inter *BK* & *KD; & IS* ad *LS* ut est media proportionalis inter *CI* & *ID* ad mediam proportionalem inter *CL* & *LD*. Seca autem pro lubitu vel inter puncta *K* & *H*, *I* & *L*, vel extra eadem; dein age *RS* secantem tangentes in *A* & *P*, & erunt *A* & *P* puncta contactuum. Nam si *A* & *P* supponantur esse puncta contactuum alicubi in tangentibus sita; & per punctorum *H*, *I*, *K*, *L* quodvis *I*, in tangente alterutra *HI* situm, agatur recta *IT* tangenti alteri *KL* parallela, quae occurrat curvae in *X* & *Y*, & in ea sumatur *IZ* media proportionatis inter *IX* & *IY:* erit, ex conicis, rectangulum *XIY* seu *IZ quad.* ad *LP quad.* ut rectangulum *CID* ad rectangulum *CTD*, id est (per constructionem) ut *SI quad.* ad *SL quad.* atque ideo *IZ* ad *LP* ut *SI* ad *SL*. Iacent ergo puncta *S*, *P*, *Z* in una recta. Porro tangentibus concurrentibus in *G*, erit (ex conics) rectangulum *XIY* seu *IZquad.* ad *IAquad.* ut *GP quad.* ad *GAquad.* ideoque *IZ* ad *IA* ut *GP* ad *GA*. Iacent ergo puncta *P*, *Z* & *A* in una recta, ideoque puncta *S*, *P* & *A* sunt in una recta. Et eodem argumento probabitur quod puncta *R*, *P* & *A* sunt in una recta. Iacent igitur punta contactuum *A* & *P* in recta *R S*. Hisce autem inventis, traiectoria describetur ut in casu primo problematis superioris. *Q.E.F.*



In hac propositione, & casu secundo propositionis superioris constructiones eaedem sunt, sive recta *XY* trajectoriatn secet in *X* & *Y*; sive non secet; eaeque non pendent ab hac sectione. Sed demonstratis constructionibus ubi recta illa traiectoriam secat, innotescunt constructiones, ubi non secat; iisque ultra demonstrandis brevitatis gratia non immoror.

LEMMA XXII.

Figuras in alias eiusdem generis figuras mutare.



Transmutanda sit figura quaevis *HGI*. Ducantur pro lubitu rectae duae parallelae *AO*, *BL* tertiam quamvis positione datam *AB* secantes in *A* & *B*, & a figurae puncto quovis *G*, ad rectam *AB* ducatur quaevis *GD*, ipsi *OA* parallela. Deinde a puncto aliquo *O*, in linea *OA* dato, ad punctum *D* ducatur recta *OD*, ipsi *BL* occurrens in *d*, & a puncto occursus erigatur recta *dg* datum quemvis angulum cum recta *BL* continens, atque eam habens rationem ad *Od* quam habet *DG* ad *OD*; & erit *g* punctum in figura nova *hgi* puncta *G* respondens. Eadem ratione puncta singula figurae primae dabunt puncta totidem figurae novae. Concipe igitur punctum *G* motu continuo percurrere puncta omnia figurae primae, & punctum *g* motu itidem continuo percurret puncta omnia figurae novae & eandem describet. Distinctionis gratia nominemus *DG* ordinatam primam, *dg* ordinatam novam; *AD* abscissam primam, a*d* abscissam novam; *O* polum, *OD* radium abscindentem, *OA* radium ordinatum primum, & *Oa* (quo parallelogrammum *OABa* completur) radium ordinatum novum.

Dico iam quod, si punctum *G* tangit rectam lineam positione datam, punctum *g* tanget etiam lineam rectam positione datam. Si punctum *G* tangit conicam sectionem, punctum *g* tanget etiam conicam sectionem. Conicis sectionibus hic circulum annumero. Porro si punctum *G* tangit lineam tertii ordinis analytici, punctum *g* tanget lineam tertii itidem ordinis; & sic de curvis lineis superiorum ordinum. Linere duae erunt eiusdem semper ordinis analytici quas puncta *G*, *g* tangunt. Etenim ut est *ad* ad *OA* ita sunt *Od* ad *OD*, *dg* ad *DG*, & *AB* ad *AD;* ideoque *AD* aequalis est $\frac{OA \times AB}{ad}$, & *DG* aequalis est $\frac{OA \times dg}{ad}$. Iam si punctum *G* tangit rectam lineam, atque ideo in aequatione quavis, qua relatio inter abscissam *AD* & ordinatam *DG* habetur, indeterminatae illae *AD* & *DG* ad unicam tantum dimensionem ascendunr, scribendo in hac aequatione $\frac{OA \times AB}{ad}$ pro *AD*, & $\frac{OA \times dg}{ad}$ pro *DG*, producetur aequatio nova, in qua abscissa nova *ad* & ordinata nova *dg* ad unicam tantum dimensionem ascendent, atque ideo quae designat lineam rectam. Sin *AD* & *DG*, vel earum alterutra, ascendebant ad duas dimensionibus in aequatione prima, ascendent itidem *ad* & *dg* ad duas in aequatione secunda. Et sic de tribus vel pluribus dimensionibus. Indeterminatae *ad*, *d*g in aequatione secunda, & *AD*, *DG* in prima ascendent semper ad eundem dimensionum numerum, & propterea lineae, quas puncta *G*, *g* tangunt, sunt eiusdem ordinis analytici.

Dico praeterea, quod si recta aliqua tangat lineam curvam in figura prima; haec recta eadem modo cum curva in figuram novam translata tanget lineam illam curvam in figura nova; & contra. Nam si curvae puncta quaevis duo accedunt ad invicem & coeunt in figura prima, puncta eadem translata accedent ad invicem & coibunt in figura nova; atque ideo rectae, quibus haec puncta iunguntur, simul evadent curvarum tangentes in figura utraque.

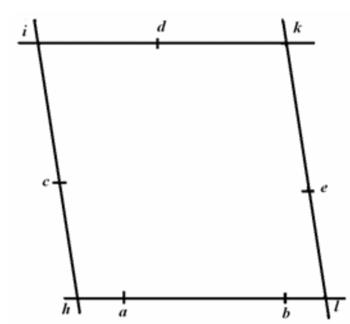
Componi possent harum assertionum demonstrationes more magis geometrico. Sed brevitati consulo.

Igitur si figura rectilinea in aliam transmutanda est, sufficit rectarum, a quibus conflatur, intersectiones transferre, & per easdem in figura nova lineas rectas ducere. Sin curvilineam transmutare oportet, transferenda sunt puncta, tangentes, & aliae rectae, quarum ope curva linea definitur. Inservit autem hoc lemma solutioni difficiliorum problematum, transmutando figuras propositas in simpliciores. Nam rectae quaevis convergentes transmutantur in parallelas, adhibendo pro radio ordinato primo lineam quamvis rectam, quae per concursum convergentium transit; idque quia concursus ille hoc pacto abit in infinitum; linem autem parallelae sunt, quae nusquam concurrunt. Postquam autem problema solvitur in figura nova; si per inversas operationcs transmutetur haec figura in figuram primam, habebitur solutio quaesita.

Utile est etiam hoc lemma in solutione solidorum problematum. Nam quoties duae sectiones conicae obvenerint, quarum intersectione problema solvi potest, transmutare licet earum alterutram, si hyperbola sit vel parabola, in ellipsin: deinde est ipsis facile mutatur in circulum. Recta item & sectio conica, in constructione planorum problematum, vertuntur in rectam & circulum.

PROPOSITIO XXV. PROBLEMA XVII.

Traiectoriam describere, quae per data duo puncta transibit, & rectas tres continget positione datas.



Per concursum tangentium quarumvis duarum cum se invicem, & concursum tangentis tertiae cum recta illa, quae per puncta duo data transit, age rectam infinitam; eaque adhibita pro radio ordinato primo, transmutetur figura, per lemma superius, in figuram novam. In hac figura tangentes illae duae evadent sibi invicem parallelae, & tangens tertia fiet parallela rectae per puncta duo data transeunti. Sunto *hi, kl* tangentes illae duae parallelae, *ik* tangens tertia, & *hl* recta huic parallela transiens per puncta illa *a*, *b,* per quae conica sectio in hac figura nova transire debet, & parallelogrammum *hikl.* complens. Secentur rectae *hi, ik, kl* in *c, d, e,* ita ut sit *hc* ad latus quadratum rectanguli *ahb, ic* ad *id,* & *ke* ad *kd* ut est summa rectarum *hi* & *kl* ad summam trium linearum, quarum prima est recta *ik, &*alterae duae sunt latera quadrata rectangulorum *ahb* & *alb: &* erunt *c, d, e* puncta contactuum. Etenim; ex conicis, sunt *hc* quadratum ad rectangulum a*hb,* & *ic* quadratum ad *id* quadratum, & *ke* quadratum ad *kd* quadratum, & *el* quadratum ad rectangulum *alb* in eadem ratione; & propterea *hc* ad latus quadratum ipsius *ahb, ic* ad *id, ke* ad *kd,* & *el* ad latus quadratum ipsius *alb* sunt in subduplicata illa ratione, & compolite, in data ratione omnium antecedentium *hi* & *kt* ad omnes consequentes, quae sunt latus quadratum rectanguli *ahb,* & recta *ik,* & latus quadratum rectanguli *alb*. Habentur igitur ex data illa ratione puncta contactuum *c, d, e,* in figura nova. Per inversas operationes lemmatis novissimi transferantur haec puncta in figuram primam, & ibi (per Prob. XIV.) describetur traiectoria.

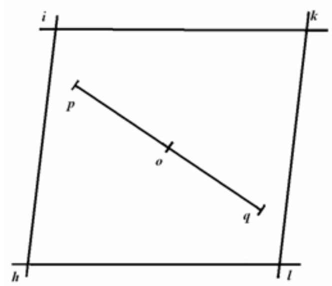
Q. *E. F.*

Caeterum perinde ut puncta *a,b* iacent vel inter puncta *h, l,* vel extra, debent puncta *c, d, e* vel inter puncta *h, i, k, l* capi, vel extra. Si punctorum *a, b* alterutrum cadit inter puncta *h, l,* & alterum extra, problema impossibile est.

PROPOSITIO XXVI. PROBLEMA XVIII.

Traiectoriam describere, quae transibit per punctum datum, & rectas quatuor positione datas continget.

Ab intersectione communi duarum quarumlibet tangentium ad intersectionem communem reliquarum duarum agatur recta infinita, & eadem pro radio ordinato primo adhibita, transmutetur figura (per Lem. XXII.) in figuram novam, & tangentes binae, quae ad radium ordinatum primum concurrebant, iam evadent parallelae. Sunto illae *hi* & *kl*, ; *ik* & *hl* continentes parallelogrammum *hikl*. Sitque *p* punctum in hac nova figura puncto in figura prima dato respondens. Per figurae centrum *O* agatur *pq*, & existente *Oq* aequali *Op*,erit *q* punctum alterum per quod sectio conica in hac figura nova transire debet. Per Lemmatis XXII operationem inversam transferatur hoc punctum in figuram primam, & ibi habebuntur puncta duo per quae traiectoria describenda est. Per eadem vero describi potest traiectoria illa per Problema XVII.

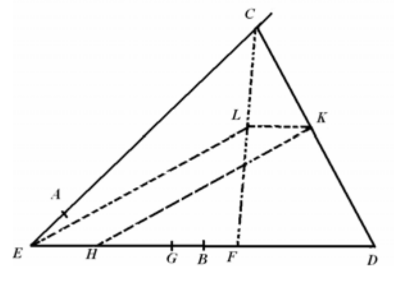


Q.E.F.

LEMMA XXIII.

Si rectae duae positione datae AC, BD ad data puncta A, B, terminentur, datamque habeant rationem ad invicem, & recta CD, qua puncta indeterminata C, D iunguntur, secatur in ratione data K : dico quod punctum K locabitur in recta positione data.

Concurrant enim rectae *AC, BD* in *E,* & in *BE* capiatur *BG* ad *AE* ut est *BD* ad *AC,* sitque *FD* semper aequalis datae *EG* ;& erit ex constructione *EC* ad *GV,* hoc est, ad *EF* ut *AC* ad *BD,* ideoque in ratione data, & propterea dabitur specie triangulum *EFC.* Secetur *CF* in *L* ut sit *CL* ad *CF* in ratione *CK* ad *CD;* & ob datam illam rationem, dabitur etiam specie triangulum *EFL;* proindeque punctum *L* locabitur in recta *EL* positione data. Iunge *LK,* & similia erunt triangula *CLK, CFD;* & ob datam *FD* & datam rationem

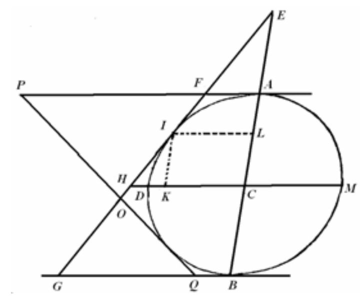


LK ad *FD*, dabitur *LK*. Huic aequaIis capiatur *EH*, & erit semper *ELKH* parallelogrammum. Locatur igitur punctum *K* in parallelogrami illius latere positione *HK*. *Q.E.D.*

Corol. Ob datam specie figuram *EFLC*, rectae tres *EF*, *EL* & *EC*, id est, *GD*, *HK* & *EC*, datas habent rationes ad invicem.

LEMMA XXIV.

Si rectae tres tangant quamcunque coni sectionem, quarum duae parallelae sint ac dentur positione; dico quod sectionis simidiameter hisce duabus parallela, sit media proportionalis inter harum segmenta, punctis contactuum & tangenti tertiae interiecta.



Sunto *AF*, *GB* parallelae duae coni sectionem *ADB* tangentes in *A* & *B; EF* recta tertia coni sectionem tangens in *I*, & occurrens prioribus tangentibus in *F* & *G*; sitque *CD* semidiameter figurae tangentibus parallela : dico quod *AF*, *CD*, *BG* sunt continue proportionales.

Nam si diametri conjugatae *AB, DM* tangenti *FG* occurrant in *E* & *H*, seque mutuo secent in *C* & compleatur parallelogrammum *IKCL;* erit ex natura sectionum conicarum ut *EC* ad *CA* ita *CA* ad *CL*, & ita divisim *EC–CA* ad C*A–CL*, seu *EA* ad *AL*, & composite *EA* ad *EA+AL* seu *EL* ut *E*C ad $EC+CA$ seu *EB;* ideoque, ob similitudmem triangulorum *EAF*, *ELI*, *ECH*, *EBG*, *AF* ad *LI* ut *CH* ad *BG*. Est itidem, ex natura sectionum conicarum, *LI* seu *CK* ad *CD* ut *CD* ad *C*H; atque ideo ex aequo perturbate *AF* ad *CH* ut *CD* ad *BG*. Q.E.D.

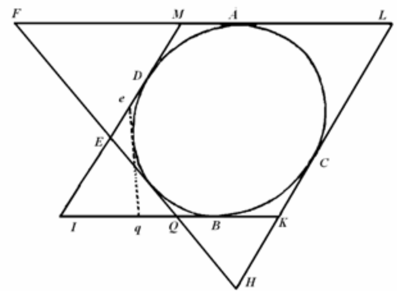
Corol. I. Hinc si tangentes duae *FG, P* Q tangentibus parallelis *AF, BG* occurrant in *F* & *G, P* & *Q* seque mutua secent in *O*; erit ex aequo perturbate *AF* ad *BQ* ut *AP* ad *BG*, & divisim ut *FP* ad *GQ*, atque ideo ut *FO* ad *OG*.

Corol. 2. Unde etiam rectae duae *PG, FQ*, per puncta *P* & *G, F* & *Q* ductae, concurrent ad rectam *ACB* per centrum figuae & puncta contactuum *A, B* transeuntem.

LEMMA XXV.

Si parallelogrammi latera quatuor infinite producta tangant sectonem quamcunque eonicam, & abstindantur ad tangentem quamvis quintam; sumantur autem laterum quorumvis duorum conterminorum abscissae terminatae ad angulos oppositos parallelogrammi : dico quod asscissa alterutra sit ad latus illud a quo est abscissa, ut pars lateris alterius contermini inter punctum contactus & latus tertium est ad abscissarum alteram.

Tangant parallelogrammi *MLIK* latera quatuor *ML, IK, KL, MI* sectionem conicam in *A, B, C, D,* & secet tangens quinta *FQ* haec latera in *F, Q, H* & *E;* sumantur autem laterum *MI, KI* abscissae *ME, KQ,* vel laterum *KL, ML* abscissae *KH, MF:* dico quod sit *ME* ad *MI* ut *BK* ad *KQ*; & *KH* ad *KL* ut *AM* ad *MF*. Nam per corollarium primum lemmatis superioris est *ME* ad *EI* ut *AM* seu *BK* ad *BQ,* & componendo *ME* ad *MI* ut *BK* ad *KQ*. Q. *E. D.*
Item *KH* ad *HL* ut *BK* seu *M* ad *AF,* & dividendo *KH* ad *KL* ut *AM* ad *MF.* Q *E.D.*



Corol. I. Hinc si datur parallelogrammum *IKLM,* circa datam sectionem conicam descriptum, dabitur rectangulum $KQ \times ME$, ut & huic aequale rectangulum $KH \times MF$. Aequantur enim rectagula illa ob similitudinem triangulorum *KQH, MFE.*

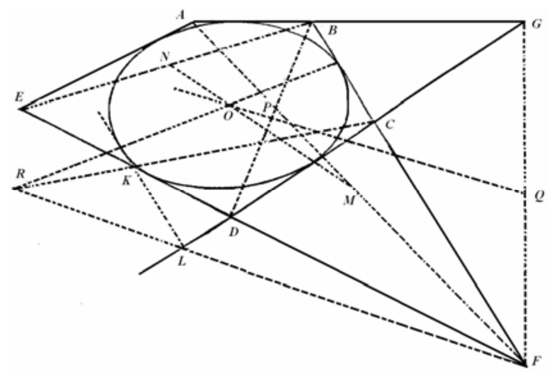
Corol. 2. Et si sexta ducatur tangens *eq* tangentibus *KI, MI* occurrens in *q* & *e;* rectangulum $KQ \times ME$ aequabitur rectangulo $Kq \times Me$; eritque *KQ* ad *Me* ut *Kq* ad *ME,* & divisim ut *Qq* ad *Ee.*

Corol. 3. Unde etiam si *Eq,* eQ iungantur & bisecentur, & recta per puncta bisectionum agatur, transibit haec per centrum sectionis conicae. Nam cum sit *Qq* ad *Ee* ut *KQ* ad *Me,* transibit eadem recta per medium omnium *EQ, eQ, M K* (per Lem. XXIII.) & medium rectae *MK* est centrum sectionis.

PROPOSITIO XXVII. PROBLEMA XIX.

Traiectoriam describere, quae rectas quinque positione datas continget.

Dentur positione tangentes *ABG*, *BCF*, *GCD*, *FDE*, *EA*. Figurae quadrilaterae sub quatuor quibusvis contentre *ABFE* diagonales *AF*, *BE* biseca in *M* & *N*, & (per Corol.3. Lem. XXV.) recta *MN* per puncta bisectionum acta transibit per centrum traiectoria. Rursus figurae quadrilaterae *BGDF*, sub aliis quibusvis quatuor tangentibus contentae, diagonales (ut ita dicam) *BD*, *GF* biseca in *P* & *Q*: & recta *PQ* per puncta bisectionum acta transibit per centrum traiectorire. Dabitur ergo centrum in concursu bisecantium. Sit illud *O*. Tangenti cuivis *BC* parallelam age *KL*, ad eam distantiam ut centrum *O* in media inter parallelas locetur, & acta *KL* tanget trajectoria *Q* describendam. Secet haec tangentes alias quasvis duas *GCD*, *FDE* in *L & K*. Per harum tangentium non parallelarum *CL*, *FK* cum parallelis *CF*, *KL* concursus *C* & *K*, *F* & *L* age *CK*, *FL* concurrentes in *R*, & recta *OR* ducta & producta secabit tangentes parallelas *CF*, *KL* in punctis contactuum. Patet hoc per Corol.2, Lem. XXIV. Eadem methodo invenire licet alia contactuum puncta, &

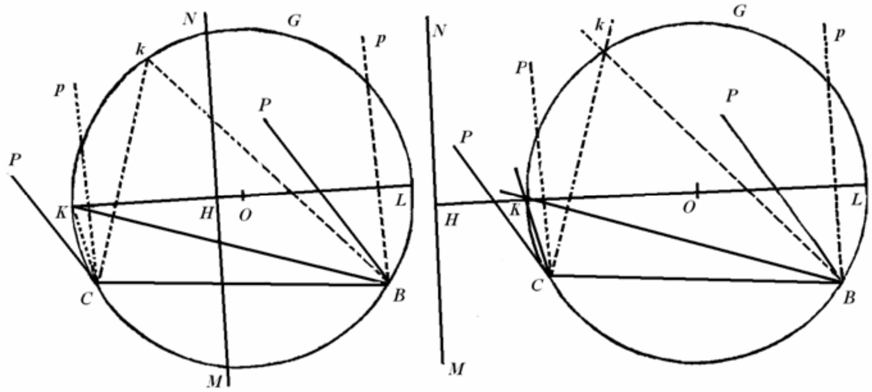


tum demum per construct Prob. XIV. traiectoriam describere. *Q. E .F.*

Scholium.

Problemata, ubi dantur traiectoriarum vel centra vel asymptoti, includuntur in praecedentibus. Nam datis punctis & tangentibus una cum centro, dantur alia totidem puncta aliaeque tangentes a centro ex altera ejus parte aequaliter distiantes. Asymptotos autem pro tangente habenda est, & eius terminus infinite distans (si ita loqui fas sit) pro puncto contactus. Concipe tangentis cuiusvis punctum contactus abire in infinitum, & tangens vertetur Asymptotos, atque constructiones problematum praecedentium vertentur in constructiones ubi Asymptotos datur.

Postquam traiectoria descripta est, invenire licet axes & umbilicos eius hac methodo. In constructione & figura Lemmatis XXI, fac ut angulorum mobilium *PBN*, *PCN* crura *BP*, *CP*, quorum concursu traiectoria describebatur, sint sibi invicem parallela, eumque servantia situm revolvantur circa polos suos *B*, C in figura illa. Interea vero describant altera angulorum illorum crura *CN*, *B N*, concursu suo *K* vel *k*, circulum *BGKC*. Sit circuli hujus centrum *O*. Ab hoc centro ad regulam *MN*, ad quam altera illa crura *CN*, *BN* interea concurrebant, dum traiectoria describebatur, demitte normalem *OH* circulo

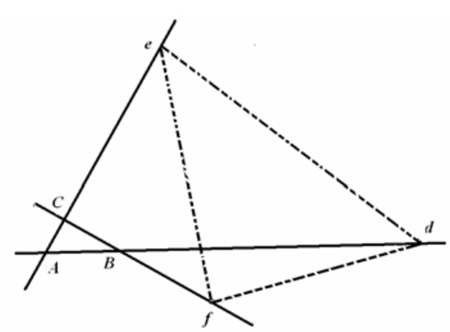


occurrentem in *K* & *L*. Et ubi crura illa altera *CK*, *BK* concurrunt ad punctum illud *K* quod regulae propius est, crura prima *CP*, *RP* parallela erunt axi majori, & perpendicularia minori; & contrarium eveniet, si crura eadem concurrunt ad punctum remotius *L*. Unde si detur traiectoriae centrum, dabuntur axes. Hisce antem datis, umbilici sunt in pomptu.

Axium vero quadrata sunt ad invicem ut *KH* ad *LH*, & inde facile est traiectoriam specie datam per data quatuor puncta describere. Nam si duo ex punctis datis constituantur poli *C*, *B*, tertium dabit angulos mobiles, *PCK*, *PBK*; his autem datis describi potest circulus *BGKC*. Tum ob datam specie traiectoriam, dabitur ratio *OH* ad *OK*, ideoque ipsa *OH*. Centro *O* & intervallo *OH* . Centro *O* & intervallo *OH* describe alium circulum, & recta, quae tangit hunc circulum, & transit per concursum crurum *CK*, *BK*, ubi crura prima *CP*, *BP* concurrunt ad quartum datum punctum, erit regula illa *MN* cujus ope traictoria describetur. Unde etiam vicissim trapezium specie datum (si casus quidam impossibiles excipiantur) in data quavis sectione conica in scribi potest.

Sunt & alia lemmata quorum ope traiectoriae specie datae, datis punctis & tangentibus, describi possunt. Eius generis est quod, si recta linea per punctum quodvis positione datum ducatur, quae datam coni sectionem in punctis duobus intersecet, & intersectionum intervallum bisecetur, punctum bisectionis tanget aliam coni sectionem eiusdem speciei cum priore, atque axes habentem prioris axibus parallelos. Sed propero ad magis utilia.

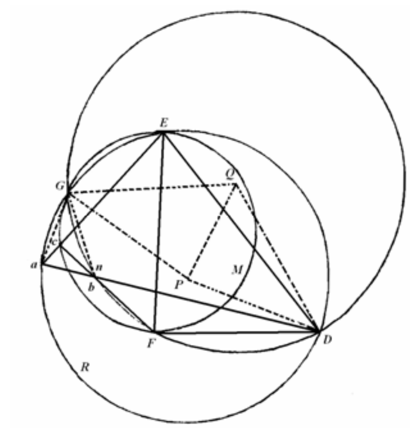
LEMMA XXVI.



Trianguli specie & magnitudine dati tres angulos ad rectas totidem positione datas, quae non sunt omnes parallelae, singulos ad singulas ponere.

Dantur positione tres rectae infinitae *AB*, *AC*, *BC*, & oportet triangulum *DEF* ita locare, ut angulus ejus *D* lineam *AB*, anguIus *E* lineam A*C*, & angulus *F* lineam *BC* tangat. Super *DE*, *DF* & *EF* describe tria circulorum segmenta *DRE*, *DGF*,*EMF*, quae capiant angulos angulis *BAC*, *ABC*, *ACB* aequales respective. Describantur autem haec segmenta ad eas partes linearum *DE, DF, EF,* ut literae *DRED* eodem ordine cum literis *BACB,* literae *DGFD* eodem cum literis *ABCA*, & literae *EMFE* eodem cum literis *ACBA* in orbem redeant; deinde compleantur haec segmenta in circulos integros. Secent circuli duo priores se mutuo in *G,* sintque centra eorum *P* & *Q*. Iunctis *GP, PQ,* cape *Ga* ad *AB* ut est *GP* ad *PQ* & centro *G,* intervallo *Ga* describe circulum, qui secet circulum primum *DGE* in *a*. Iungatur tum *aD* secans circulum secundum *DFG* in *b*, tum *aE* secans circulum tertium *EMF* in *c*. Et iam licet figuram *ABCdef* constituere similem & aequalem figurae *abcDEF*. Quo facto perficitur problema.

Agatur enim *Fc* ipsi *aD* occurrens in *n*, & iungantur *aG, bG, QG, QD, PD*. Ex constructione est angulus *EaD* aequalis angulo *CAB*, & angulus *acF* aequalis angulo *ACB,* ideoque triangulum *anc* triangulo *ABC* aequiangulum. Ergo angulus *anc* seu *FnD* angulo *ABC,* ideoque angulo *FbD* aequalis est; & propterea punctum *n* incidit in punctum *b*. Porro angulus *GPQ*, qui dimidius est anguli ad centrum *GPD*, aequalis est angelo ad circumferentium GaD; & angulus GQP, qui dimidius est anguli ad centrum GQD, aequalis est complemento ad duos rectos anguli ad circumferentiam *GbD,* ideoque aequalis angulo *Gba*; suntque ideo triangula *GPQ*, *Gab* similia; & *Ga* est ad *ab* ut *GP* ad PQ ; id est (ex constructione) ut *Ga* ad *AB*. Aequantur itaque *ab* & *A B; &* propterea triangula *ab c, ABC,* quae modo similia esse probavimus, sunt etiam aequalia. Unde, cum tangant insuper trianguli *DEF* anguli *D, E, F* trianguli *abc* latera *ab, ac, bc* respective, compleri potest figura *ABCdef* figurae *abcDEF* similis & aequalis, atque eam complendo solvetur problema. *Q E F.*

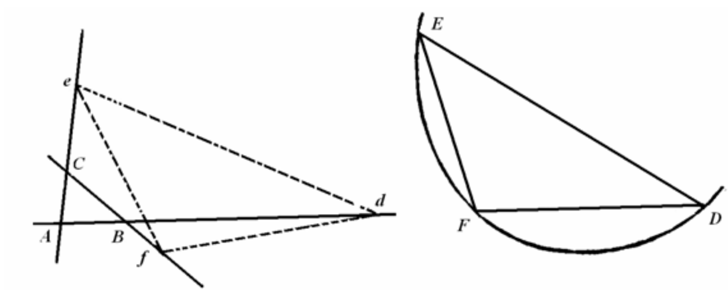


Corol. Hinc recta duci potest cuius partes longitudine datae rectis tribus positione datis interiacebunt. Concipe triangulum *DEF,* puncto *D* ad latus *EF* accedente, & lateribus *DE, DF* in directum positis, mutari in lineam rectam, cujus pars data *DE* rectis positione datis *AB, AC*, & pars data *DF* rectis positione datis *AB, BC* interponi debet; & applicando constructionem praecedentem ad hunc casum solvetur problema.

PROPOSITIO XXVIII. PROBLEMA XX.

Traiectoriam specie & magnitudine datam describere, cuius partes datae rectis tribus positione datis interiacebunt.

Describenda sit traiectoria, quae sit similis & aequalis lineae curvae *DEF*, quaeque a rectis tribus *AB, AC, BC* positione datis, in partes huius partibus *DE* & *EF* similis & aequales secabitur.

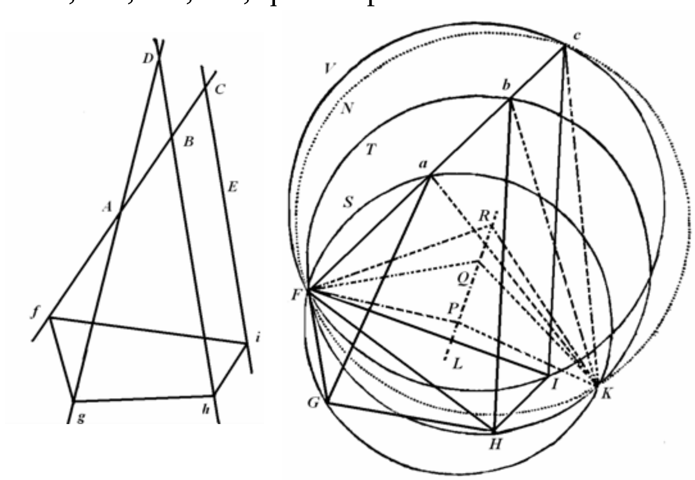


Age rectas *DE, EF, DF*, & trianguli hujus *DEF* pone angulos *D, E, F* ad rectas illas positione datas (per Lem.XXVI) dein circa triangulum describe traiectoriam curvre *DEF* similem & aequalem. *Q.E.F.*

LEMMA XXVII.

Trapezium specie datum describere, cuius anguli ad rectas quatuor positione datas, quae neque omnes parallelae sunt, neque ad commune punctum convergunt, singuli ad singulas consistent.

Dentur positione rectae quatuor *ABC, AD, BD, CE;* quarum prima secet secundam in *A*, tertiam in *B*, & quartam in *C:* & describendum sit trapezium *fghi*, quod sit trapezio *FGHI* simile ; & cuius angulus *f*, angulo dato *F* aequalis, tangat rectam *ABC*; caeterique anguli *g, h, i*, ceteris angulis datis *G, H, I* aequales, tangant caeteras lineas *AD, BD, CE* respective. Iungatur *FH* & super *FG, FH, F1* describantur totidem circulorum segmenta *FSG, FTH, FVI;* quorum primum *FSG* capiat angulum equalem angolo *BAD,* secundum *FTH* capiat angulum aequalem angulo *CBD,* ac tertium *FVI* capiat angulum aequalem angulo *ACB.* Describi autem debent

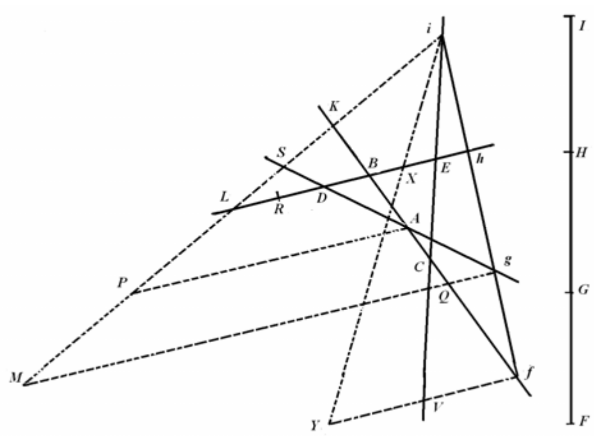


segmenta ad eas partes linearum *FG, FH, F1,* ut literarum *FSGF* idem sit ordo circularis qui literarum *BADB,* utque literae *FTHF* eodem ordine cum literis *CBDC,* & literae *FVIF* eodem cum literis *ACEA* in orbem redeant. Compleantur segmenta in circulos integros, sitque *P* centrum circuli primi *FSG, & Q* centrum secundi *FTH.* Iungatur & utrinque producatur *PQ* & in ea capiatur *QR* in ratione ad *PQ* quam habet *BC* ad *AB.* Capiatur autem *QR* ad eas partes puncti *Q* ut literarum *P, Q, R* idem sit ordo atque literarum *A, B, C:* centroque *R* & intervallo *RP* describatur circulus quartus *FNc* secans circulum tertium *FVI* in *c.* Iungatur *Fc* secans circulum primum in *a,* & secundum in *b.* Agantur *aG, bH, cI,* & figurae *abcFGHI* similis constitui potest figura *ABCfghi.* Quo facto erit trapezium *fghi* illud ipsum, quod constituere oportebat.

Secent enim circuli duo primi *FSG, FTH* se mutuo in *K.* Iungantur *PK, QK, RK, aK, bK, cK,* & producatur *QP* ad *L.* Anguli ad circumferentias *FaK, FbK, FcK* sunt semisses angulorum *FPK, FQK, FRK* ad centra, ideoque angulorum illorum dimidiis *LPK, LQK, LRK* aequales. Est ergo figura *PQRK* figurae *abcK* aequiangula & similis, & propterea *ab* est ad *bc* ut *PQ* ad *QR,* id est, ut *AB* ad *BC.* Angulis insuper *FaG, FbH, FcI* aequantur *fAg, fBh, fCi* per constructionem. Ergo figure *abcFGHI* figura similis *ABCfghi* compleri potest. Quo facto trapezium *fghi* constituetur simile trapezio *FGHI,* & angulis suis *f, g, h, i* tanget rectas *ABC, AD, BD, CE.*

Q. E. F.

Corol. Hinc recta duci potest cuius partes, rectis quatuor positione datis dato ordine interiectae, datam habebunt proportionem ad invicem. Augeantur anguli *FGH, GH1* usque eo, ut rectae *FG, GH, HI* in dirctam iaceant, & hi hoc casu construendo problema ducetur recta *fghi*, cuius partes *fg, gh, hi*, rectis quatuor positione datis *AB & AD*, *AD & BD*, *BD & CE* interiectae, erunt ad invicem ut lineae *FG, GH, HI*, eundemque servabunt ordinem inter se. Idem vero sic sit expeditius.



Producantur *AB* ad *K,* & *BD* ad *L,* ut sit *BK* ad *AB* ut *HI* ad *GH;* & *D L* ad *BD* ut *GI* ad *FG;* & iungatur *KL* occurrens rectae *CE* in *i.* Producatur *iL* ad *M*, ut sit *LM* ad *iL* ut *GH* ad *H I*, & agatur tum *MQ* ipsi *LB* parallela, rectaeque *AD* occurrens in *g*, tum *gi* secans *AB, BD* in *f, h.* Dico factum.

Secet enim *Mg* rectam *AB* in *Q, & AD* rectam *KL* in *S,* & agatur *AP* quae sit ipsi *BD* parallela & occurrat *iL* in *P,* & erunt *gM* ad *Lh* (*gi* ad *bi*, *Mi* ad *Li*, *GI* ad *HI*, *AX* ad *BK*) & *AP* ad *BL* in eadem ratione. Secetur *DL* in *R* ut sit *DL* ad *RL* in eadem illa ratione, & ob proportionales *gS* ad *gM, AS* ad *AP,* & *DS* ad *DL;* erit, ex aequo, ut *gS* ad *Lh* ita *AS* ad *BL* & *D S* ad *R L; &* mixtim, *BL–RL* ad *Lh–BL* ut *AS–DS* ad *gS–AS.* Id est *BR* ad *Bh* ut *AD* ad *Ag,* ideoque ut *BD* ad *gQ.* Et vicissim *BR* ad *BD* ut *Bh* ad *gQ,* seu *fh* ad *fg.* Sed ex constructione linea *BL* eadem ratione secta fuit in *D & R* atque linea *FI* in *G & H:* ideoque est *BR* ad *BD* ut *FH* ad *FG.* Ergo *fh* est ad *fg* ut *FH* ad *FG.* Cum igitur sit etiam *gi* ad *hi* ut *Mi* ad *Li*, id est, ut *GI* ad *HI*, patet lineas *FI*, *fi* in *g* & *h*, *G*& *H* similiter sectas esse.

Q. E.F.

In constructione corollarii hujus postquam ducitur *L K* secans *CE* in *i*, producere licet *iE* ad *V*, ut sit *EV*ad *Ei* ut *FH* ad *HI*,& agere *Vf* parallelam ipsi *BD*. Eodem recidit si centro *i*, intervallo *IH*, describatur circulus secans *BD* in *X*, & producatur *iX* ad *r*, ut sit *iT* aequalis *IF*, & agatur *Tf* ipsi *BD* parallela.

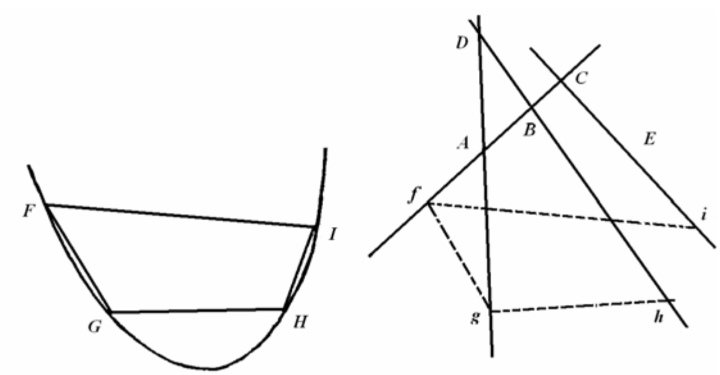
Problematis hujus solutiones alias *Wrennus* & *Wallisius* olim excogitarunt.

PROPOSITIO XXIX PROBLEMA XXI.

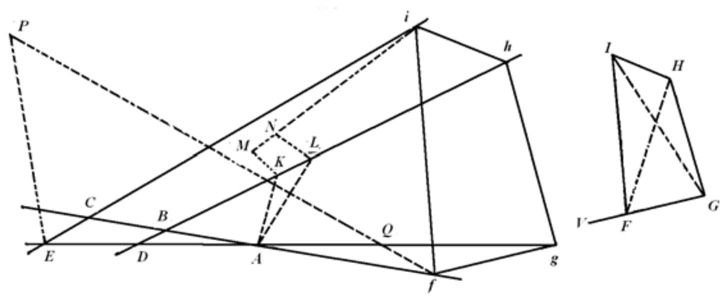
Traiectoriam specie datam describere, quae a rectis quatuor positione datis in partes secabitur, ordine, specie & proportione datas.

Describenda sit traiectoria,quae similis sit lineae curvae *FGHI*, & cuius partes, illius partibus *FG*, *GH*, *HI* similes & proportionales, rectis *AB* & *AD*, *AD* & *BD*, *BD* & *CE* positione datis, prima primis, secunda secundis, tertia tertiis interiaceant. Actis rectis *FG*, *GH*, *HI*, *FI*, describatur (per Lem. XXVII.) Trapezium *fghi* quod sit trapezio *FGHI* simile, & cuius anguli *f*, *g*, *h*, *i* tangant rectas illas positione datas *AB*, *AD*, *BD*, *CE*, singuli singulas dicto ordine. Dein circa hoc trapezium describatur traiectoria curvae linea *FGHI* consimilis.

Scholium.



Construi etiam potest hoc problema ut sequitur. Iunctis *FG, GH, HI, FI* produc *GF* ad *P,* iungeque *FH, 1G,* & angulis *FGH, PFH* fac angulos *CAK, DAL* aequales. Concurrant *AK, AL* cum recta *BD* in *K* & *L,* & inde agantur *KM, LN,* quarum *KM* constituat angulum *AKM* aequaliam angulo *GHI,* sitque ad *AK* ut est *HI* ad *GH*; & *LN* constituat angulum *ALN* aequalem angulo *FHI,* sitque ad *AL* ut *HI* ad *FH.* Ducantur autem *AK, KM, KM, AL, LN* ad eas partes linearum *AD, AK, AL,* ut literae *CAKMC, ALKA, DALND* eodem ordine cum literis *FGHIF* in orbem redeant*;* & acta MN occurrat rectae *CE* in *i.* Fac



angulum *iEP* aequalem angulo *1GF,* sitque *PE* ad *Ei* ut *FG* ad *G 1;* & per *P* agatur *PQf,* quae cum recta *ADE* contineat angulum *PQE* aequalem angulo *FIG,* rectaeque *AB* occurrat in *h* & iungatur *fi.* Agantur autem *PE & PQ* ad eas partes linearum *CE, PE,* ut literarum *PEiP* & *PEQp* idem sit ordo circularis qui literarum *FGHIF,* & si super linea *fi* eodem quoque literarum ordine constituatur trapezium *fghi* trapezio *FGHI* simile, & circumscribatur traiectoria specie data, solvetur problema.

Hactenus de orbibus inveniendis. Superest ut motus corporum in orbibus inventis determinemus.