Specifying the roots of equations of the first & second canons.

PROPOSITION 1.

The root of the equation : $aa - ba$
$+ ca = + bc$ is b, equal to the root of a sought;
i. e. $a = b$.
For if in the equation: $aa - ba$
$+ ca = + bc$, b is put equal to the root a, by changing a into b,
then the equation becomes : $bb - bb$
$+ cb = + cb$,
for which the equality is apparent.
Therefore, as has been stated, put b equal to a for equality.
Moreover, there is no other given root of the equation equal to a, except b, as established in the following Lemma.

Lemma.

If it is possible to be giving another root (*i.e.* positive) of the equation equal to a, which is unequal to the root b, let this root be c, without any other.
Therefore by placing $c = a$, the equation becomes . . . . . $cc - bc$
$+ cc = + bc$.
Therefore . . . . $cc + cc = + bc + bc$.
as . . . . $c + c \,| = c + c \,|$
$\underline{\quad c} \,| \quad \underline{\quad b} \,|$
Therefore . . . . $c = b$. Which is contrary to the hypothesis.
Therefore c cannot be set equal to a, which is the case for any value of a except b, which can be similarly demonstrated.
[Note for Prop. 1 : The first equation solved is $a^2 - (b+c)a - bc = (a-b)(a+c) = 0$.
The contemporary thinking (due to Vieta) that only positive roots of equations were to be considered is applied; in this case the root $a = -c$ is not allowed. The factored form of the quadratic does not appear in the proof.]

PROPOSITION 2.

The roots of the equation: $aa - ba$
$- ca = - bc$, are b and c, equal to the roots of a sought;
i. e. $a = b$ and $a = c$.

For if in the equality $aa - ba$
$- ca = - bc$ for the root a, b is put equal to a, by changing a
into b, it becomes $bb - bb$
$- cb = - bc$. But this equality is itself apparent.
Therefore, Therefore, $b = a$, satisfies the equation.

The same is the case if $c = a$, and changing a into c, then the equation becomes :

$$\begin{array}{l} cc - bc \\ \quad - cc = - bc. \end{array}$$

This same equality has itself become apparent.
Therefore, by placing $c = a$, it too is [seen to be] equal.
Therefore $b = a$ or $c = a$ are the roots sought after, as has been stated.
[Note for Prop. 2 : The second equation solved is $a^2 - (b+c)a - bc = (a-b)(a-c) = 0$. Note that these equations have been verified by direct substitution rather than factorising.]

Moreover another root equal to a cannot be given in addition to b or c, and this is shown in the following Lemma.

Lemma.

If it should be possible to give another root equal to a, which is unequal to either of the roots b or c, then let it be d, or any other.
Therefore by placing $d = a$, the equation become

$$\begin{array}{l} dd - bd \\ \quad - cd = -bc. \end{array}$$

Therefore . . . . $dd - cd = + bd - bc$.

Therefore . . . . $\left.\dfrac{+d - c}{d}\right| = \left.\dfrac{+d - c}{b}\right|$

Therefore $d = b$, which is contrary to the hypothesis.
or it shall be . . . . $dd - bd = + cd - bc$.

Therefore . . . . $\left.\dfrac{-d - b}{d}\right| = \left.\dfrac{-d - b}{c}\right|$

Therefore $d = c$, which is again contrary to the hypothesis.
Therefore it is shown that there is no possible value $d = a$ that can be put in place, except b or c.

PROPOSITION 3.

The root of the equation :

$$\begin{array}{l} aaa + baa + bca \\ \quad + caa - bda \\ \quad - daa - cda = + bcd \end{array}$$

is d, equal to the root of a sought; *i. e.* $a = d$

For if d is put equal a in the equation :

$$\begin{array}{l} aaa + baa + bca \\ \quad + caa - bda \\ \quad - daa - cda = + bcd \end{array}$$

for the root a, by changing a into d, the equation becomes :

$$\begin{array}{l} ddd + bdd + bcd \\ \quad + cdd - bdd \\ - ddd - cdd \qquad = + bcd \end{array}$$

. But this equality is apparent from the rejection of contradictory parts.
Therefore, $d = a$, satisfies the equation.

Moreover, another root equal to *a* cannot be given in addition to *d,* and this is shown in the following Lemma.

Lemma.

If it should be possible to give another root equal to *a*, not equal to the root *d*, then let that root be *b* or *c,* or some other.

Therefore by placing $c = a$:

$$\begin{array}{l} ccc + bcc + bcc \\ \quad + ccc - bdc \\ \quad\; - dcc - cdc = + bcd \end{array}$$

Therefore with the particular order $2.ccc + 2.bcc = + 2.ccd + 2.bcd.$

Therefore . . . . $\dfrac{+cc + bc}{c} = \dfrac{+cc + bc}{d}$

Therefore $c = d$, which is contrary to the hypothesis.
or it shall be . . . $.dd - bd = + cd - bc.$
Therefore . . . . $\dfrac{-d - b}{d} = \dfrac{-d - b}{c}$

Therefore $d = c$, which is contrary to the hypothesis.
Therefore the equation is not satisfied by setting $c = a$. Similarly, b or any other value except *d* is excluded by the same reasoning.
[Note for Prop. 3 : The third equation solved by inspection and direct substitution is $a^3 - (-b - c + d)a^2 + (bc - cd - db)a - bcd = (a+b)(a+c)(a-d) = 0$. A form of factoring is used in the Lemmas, but not in the main argument.]

PROPOSITION 4.

The roots of the equation :

$$\begin{array}{l} aaa + baa - bca \\ \quad - caa - bda \\ \quad - daa + cda = - bcd \end{array}$$

are *c* or *d*, equal to the roots of *a* sought; *i. e.* $a = c$ or $a = d.$

For if *c* is put equal *a* in the equation

$$\begin{array}{l} aaa + baa - bca \\ \quad - caa - bda \\ \quad - daa + cda = - bcd \end{array}$$

for the root $a = c,$ by changing *a* into *c,* the equation becomes :

$$\begin{array}{l} ccc + bcc - bcc \\ \quad + ccc - bdc \\ \quad - dcc - cdc = + bcd \end{array}$$

. But this equality is itself apparent from the different redundant parts.
Therefore, $c = a$, satisfies the equation.
Likewise, if *d* is placed for the root *a,* the equation becomes :

$$\begin{array}{l} ddd + bdd - bcd \\ \quad - cdd - bdd \\ \quad - ddd + cdd = - bcd \end{array}$$

But the truth of this equality is similarly evident.

Therefore, by placing $d = a$, the equation is satisfied again.
Therefore the values of the roots sought are $a = c$ & $a = d$, as has been stated.
Moreover, another root equal to a cannot be given in addition to c or d, and this is shown in the following Lemma.

Lemma.

If it should be possible to give another root equal to a, not equal to the roots c or d, then let that be b, or some other.

Therefore by placing $b = a$:
$$\begin{array}{l} bbb + bbb - bcb \\ \quad - cbb - bdb \\ \quad - dbb + cdb = - bcd \end{array}$$

Therefore with the particular order : $2.bbb - 2.bbc = + 2.cbb - 2.cbd.$

Therefore . . . . $\dfrac{+bb - bd}{b} = \dfrac{+bb - bd}{c}$

Therefore $b = c$, which is contrary to the hypothesis.
or : . . . $.2.bbb - 2.bbc = +2.dbb - 2.dbc.$
That is . . . . $+ bbb - bbc = + dbb - dbc.$
Therefore . . . . $\dfrac{+bb - c}{b} = \dfrac{+bb - c}{d}$

Therefore $b = d$, which too is contrary to the hypothesis.
Therefore b is not equal to d, as has been put in place. In the same way, for other values except c & d , this result can be shown by the same deduction.
[Note for Prop. 4 : The fourth equation in modern terms is :
$a^3 - (-b + c + d)a^2 + (-bc + cd - db)a + bcd = (a + b)(a - c)(a - d) = 0$.]

Consequences

Two equations from two of the preceding theorems proposed are joined together and can be set out to be examined.
For if the equations are : . . . .
$$\begin{array}{lll} + aaa - baa - bca & & \\ + caa - bda & & \\ + daa + cda & = + bcd = & + bca - baa \\ & & + bda + caa \\ & & - cda + daa - aaa. \end{array}$$

But the form of the roots themselves are noted from the theorems. For the first root is $a = b$. For the second, truly $a = c$ or d. Which has been noted.
[We note that bcd is the product of *all* the roots of both equations. The left-hand equation is satisfied by setting $a = b$, while the right-hand equation is satisfied by setting $a = c$ or d; thus, the positive root b of the left-hand equation is the negative root of the right-hand equation, and vice-versa for the roots c and d for the right-hand equation. In this way, all the real roots of the cubic are covered, positive or negative.]

PROPOSITION 5.

The roots of the equation :
$$\begin{array}{l} aaa - baa + bca \\ \quad - caa + bda \\ \quad - daa + cda = + bcd \end{array}$$
are b or c or d, equal to the roots of a sought; *i. e.* $a = b$ or $a = c$ or $a = d$.

For if b is set equal to a in the equation :
$$\begin{array}{l} aaa - baa + bca \\ \quad - caa + bda \\ \quad - daa + cda = + bcd \end{array}$$
for the root $a = b$,
by changing a into b, the equation becomes :
$$\begin{array}{l} bbb - bbb + bcb \\ \quad - cbb + bdb \\ \quad - dbb + cdb = + bcd. \end{array}$$
But this equality is itself apparent from the different redundant parts.
Therefore, $b = a$, satisfies the equation.
In the same way, if c is placed for the root a, the equation becomes :
$$\begin{array}{l} ccc - bcc + bcc \\ \quad - ccc + bdc \\ \quad - dcc + cdc = + bcd \end{array}$$
But the truth of this equality similarly is shown by rejecting contrary parts.
Therefore, $c = a$, satisfies the equation. too.
In the same way, if d is placed for the root a, then
$$\begin{array}{l} ddd - bdd + bcd \\ \quad - cdd + bdd \\ \quad - ddd + cdd = + bcd \end{array}$$
But this equality is shown from the rejection of contradictory parts.
Therefore, $d = a$, satisfies the equation. too.
Therefore the roots are b, c, or d equal to a are the roots sought, as has been stated.
Moreover, that another root equal to a cannot be given in addition to b, c or d, is shown in the following Lemma.

Lemma.

If another root equal to a can be given, which is not equal to any of the roots b or c or d, then let f be that root, or any other.
Therefore by placing $f = a$, the equation becomes :
$$\begin{array}{l} fff - bff + bcf \\ \quad - cff + bdf \\ \quad - dff + cdf = + bcd \end{array}$$
Therefore with the particular order : $fff - cff + cdf - dff = + bff - bcf + bcd - bdf$

It follows . . . . $\left.\dfrac{+ff - cf + cd - df}{f}\right| = \left.\dfrac{+ff - cf + cd - df}{b}\right|$

Therefore $f = b$, which is contrary to the hypothesis. Or by changing the order it becomes . . . $fff - bff + bdf - dff = + cff - cbf + cbd - cdf$

Therefore . . . . $\dfrac{+ff - bf + bd - df}{f} = \dfrac{+ff - bf + bd - df}{c}$

Therefore $f = c$, which also is contrary to the hypothesis.
Or by changing the order thus, the equation
becomes . . . $fff - bff + bcf - cff = + dff - dbf + dbc - dcf$

Therefore . . . . $\dfrac{+ff - bf + bc - cf}{f} = \dfrac{+ff - bf + bc - cf}{d}$

Therefore $f = d$, which also is contrary to the hypothesis.

Therefore the root is not $f = d$, as has been placed. For by reasoning in the same manner for the others, it can be concluded that the root can be none other than one of b, c & d.
[Note for Prop. 5 : The fifth equation in modern terms is :
$a^3 - (b + c + d)a^2 + (bc + cd + db)a - bcd = (a - b)(a - c)(a - d) = 0$.]

Reduced equations.

PROPOSITION 6.

The roots of the equation :
$$\begin{aligned} aaa &- bba \\ &- bca \\ &- cca = - bbc \\ &\qquad\quad - bcc \end{aligned}$$
are b or c, equal to the roots of a sought; *i. e.* $a = b$ or $a = c$

For if $b = a$, and a is changed into b in the equation proposed, then
$$\begin{aligned} bbb &- bbb \\ &- bcb \\ &- ccb = - bbc \\ &\qquad\quad - bcc \end{aligned}$$

or if on setting $c = a$ & changing a into c, then the equation becomes
$$\begin{aligned} ccc &- bbc \\ &- bcc \\ &- ccc = - bbc \\ &\qquad\quad - bcc \end{aligned}$$
. But these equalities are apparent from the rejection of contradictory parts.
Therefore, the roots sought for the proposed equation are $a = b$ or c, as stated.
[Note for Prop. 6 : The 6th equation in modern terms is :
$a^3 - (0)a^2 + (-b^2 - bc - c^2)a + bc(b + c) = (a - b)(a - c)(a + b + c) = 0$.]

PROPOSITION 7.

The root of the equation :
$$\begin{aligned} aaa &- bba \\ &- bca \\ &- cca = + bbc \\ &\qquad\quad + bcc \end{aligned}$$
is $b + c$, equal to the root of a sought; *i. e.* $a = b + c$.

For if $b + c = a$, and a is changed into $b + c$ in the equation, then
$$\begin{aligned} bbb &- bbb \\ + 3.bbc &- bbc \\ + 3.bcc &- bbc \\ + ccc &- bcc \\ &- ccc = + bcc \\ &\qquad\quad + bcc \end{aligned}$$

But from the rejection of contradictory parts it is apparent,
to wit, that . . .
$$\begin{aligned} &+ bcc \\ &+ bcc = + bcc \\ &\qquad\quad + bcc \end{aligned}$$

Therefore, the root sought for the proposed equation is $a = b + c$, as stated.

Consequences.

Hence it is clear that this equation can be joined to the nearest preceding equation.
For they are . . . . .

$$\begin{aligned} aaa &- bba \\ &- bca \\ &- cca = + bbc \\ &\qquad\quad + bcc = + bba \\ &\qquad\qquad\qquad\quad + bca \\ &\qquad\qquad\qquad\quad + cca - aaa. \end{aligned}$$

And in the first $a = b + c$. In the second $a = b$ or c. Which it is sufficient to note.
[Note for Prop. 7 : The 7th equation in modern terms is :
$a^3 - (0)a^2 + (-b^2 - bc - c^2)a - bc(b + c) = (a + b)(a + c)(a - (b + c)) = 0$.]

PROPOSITION 8.

The roots of the equation :
$$\begin{aligned} aaa &- bbaa = -\frac{bbcc}{b+c} \\ &- bcaa \\ &- \frac{ccaa}{b+c} \end{aligned}$$
are b or c, equal to the root of a sought; *i. e.* $a = b + c$.

For if $b = a$, then :

$$\begin{array}{lll} +bbbb & -bbbb = & -\dfrac{bbcc}{b+c} \\ +\dfrac{cbbb}{b+c} & -cbbb & \\ & -\dfrac{ccbb}{b+c} & \end{array}$$

Or if $c = a$, then :

$$\begin{array}{lll} +bccc & -bbcc = & -\dfrac{bbcc}{b+c} \\ +\dfrac{cccc}{b+c} & -bccc & \\ & -\dfrac{cccc}{b+c} & \end{array}$$

But these equalities are shown.
Therefore, for the proposed equation, the roots are $a = b$ or c as stated.
[Note for Prop. 8 : The 8th equation in modern terms is :
$a^3 - (b^2 + bc + c^2)/(b + c).a^2 + (0)a + b^2c^2/(b + c) = (a - b)(a - c)(a + bc/(b + c)) = 0$.]

PROPOSITION 9.

The root of the equation :

$$\begin{array}{lll} aaa & +\dfrac{bbaa}{} & \\ & +\dfrac{bcaa}{} & \\ & +\dfrac{ccaa}{b+c} & = +\dfrac{bbcc}{b+c} \end{array}$$

is $a = \dfrac{bc}{b+c}$, equal to the root of a sought; i. e. $a = \dfrac{bc}{b+c}$.

For (by Prob. 5, Section 3) the binomial equation here proposed, has been reduced from the trinomial itself, by setting $\dfrac{bc}{b+c} = d$, and by changing the one into the other.

But (by Prop. 3 of this section) $a = d$ is the root of this trinomial.
But these equalities are shown.
Therefore, the root of this equation is $a = \dfrac{bc}{b+c}$, as was stated.
[Note for Prop. 9 : The 9th equation in modern terms is :
$a^3 + (b^2 + bc + c^2)/(b + c).a^2 + (0)a - b^2c^2/(b + c) = (a + b)(a + c)(a - bc/(b + c)) = 0$.]

Consequences.

Hence it is clear that this equation can be joined with the nearest preceding.

For they are . . . . .

$$\frac{aaa + bbaa + bcaa + ccaa}{b+c} = \frac{+bbcc}{b+c} = \frac{+bbaa + bcaa + ccaa - aaaa}{b+c}$$

And in the first $a = \frac{bc}{b+c}$; in the second $a = b$ or c, as stated.

[Again, all the real roots of the cubic can be shown in this way, without using a negative root.]

PROPOSITION 10.

The root of the equation : $aaa + 3.baa + 3.bba = + ccc - bbb$, is $c - b$, equal to the root of a sought; *i. e.* $a = c - b$.

For if $c - b = a$, and a is changed into $c - b$ in the equation, then

$$\left.\begin{array}{lr} & ccc - 3.bcc + 3.bbc - bbb = aaa, \\ \text{And . . .} & + 3.bcc - 6.bbc + 3.bbb = + 3.baa \\ \text{And . . .} & + 3.bbc - 3.bbb = + 3.bba \end{array}\right\} = + ccc - bbb$$

But from the rejection of contradictory parts the equality is apparent,

Therefore the root $a = c - b$. As stated.

[Note for Prop. 10 : The 10th equation in modern terms is :

$a^3 + 3ba^2 + 3b^2a + b^3 - c^3 = (a+b)^3 - c^3 = (a+b-c)((a+b)^2 - (a+b)c + c^2) = 0$.]

PROPOSITION 11.

The root of the equation : $aaa - 3.baa + 3.bba = + ccc + bbb$, is $c + b$, equal to the root of a sought; *i. e.* $a = c + b$.

For if $c + b = a$, and a changed into in $c + b$ in the equation, then

$$\left.\begin{array}{lr} & ccc + 3.bcc + 3.bbc + bbb = aaa, \\ \text{And . . .} & - 3.bcc - 6.bbc - 3.bbb = - 3.baa \\ \text{And . . .} & + 3.bbc + 3.bbb = + 3.bba \end{array}\right\} = + ccc + bbb$$

But the equality is apparent from the rejection of contradictory parts.

Therefore the root sought for the proposed equation is $a = c + b$. As stated.

[Note for Prop. 11 : The 11th equation in modern terms is :

$a^3 - 3ba^2 + 3b^2a - b^3 - c^3 = (a-b)^3 - c^3 = (a-b-c)((a-b)^2 - (a-b)c + c^2) = 0$.]

PROPOSITION 12.

The root of the equation : *aaa - 3.baa* + 3.*bba* = +*bbb* - *ccc,* is *b* - *c,* equal to the root of *a* sought; *i. e. a* = *b* - *c.*

For if *b* - *c* = *a,* and in the equation, *a* is changed into *b* - *c,* then

$$\left.\begin{array}{lr} & -ccc+3.bcc-3.bbc+bbb=aaa, \\ \text{And . . .} & -3.bcc+6.bbc-3.bbb=-3.baa \\ \text{And . . .} & -3.bbc+3.bbb=+3.bba \end{array}\right\}=+bbb-ccc$$

But the equality is apparent from the rejection of contradictory parts ,
Therefore the root sought for the proposed equation is *a* = *b* - *c*. As stated.
[Note for Prop. 12 : The 12th equation in modern terms is :
$a^3-3ba^2+3b^2a-b^3+c^3=(a-b)^3+c^3=((a-b)+c)((a-b)^2-(a-b)c+c^2)=0$.]

PROPOSITION 13.

The root of the equation : *aaa - 3.baa* + 3.*bba* = + *2.bbb,* is *2. b*, equal to the root of *a* sought; *i. e. a* = *2. b.*
For if *a* = *2.b,* by changing *a* into *2.b* in the equation, then

$$+8bbb-12.bbb+6.bbb=+2.bbb,$$

But the equality itself has become apparent.
Therefore the root is *a* = *2.b*. As stated.

[Note for Prop. 13 : The 13th equation in modern terms is :
$a^3-3ba^2+3b^2a-b^3-b^3=(a-b)^3-b^3=((a-b)-b)((a-b)^2+(a-b)b+b^2)=0$.]

Reduced equations.

PROPOSITION 14.

The root of the equation : *aaa* + *3.bca* = +*ccc* - *bbb,* is *c* - *b,* equal to the root of *a* sought; *i. e. a* = *c* - *b.*

For if *a* = c - *b* in the proposed equation *aaa* + *3.bca* = +*ccc* - *bbb* , by changing *a* into *c* - *b*,

$$\left.\begin{array}{lr} \text{then. . .} & ccc-3.bcc+3.bbc-bbb=+aaa \\ \text{And . . . . .} & +3.bcc-3.bbc=+3.bca \end{array}\right\}=+ccc-bbb$$

But this equality is apparent from the rejection of contradictory parts.
Therefore the root *a* = *c* - *b*. As stated.
[Note for Prop. 14 : The 14th equation in modern terms is :
$a^3-0.a^2+3bca+b^3-c^3=[a-(b-c)[a^2+(b-c)a+(b^2+bc+c^2)]=0.]$

PROPOSITION 15.

The root of the equation : *aaa - 3.bca = +ccc + bbb,* is *c + b,* equal to the root of *a* sought; *i. e. a = b - c.*

For if *a = b + c,* by changing *a* into *c + b* in the equation, then

$$aaa - 3.bca = +ccc + bbb\,,$$

becomes. . . $\left.\begin{array}{l} ccc + 3.bcc + 3.bbc + bbb = +aaa \\ \qquad\qquad - 3.bcc - 3.bbc = -3.bca \end{array}\right\} = +ccc + bbb$

And . . . . .

But the equality has become apparent, by the rejection of contradictory parts.
Therefore the root is *a = b + c*. As stated.
[Note for Prop. 15 : The 15th equation in modern terms is :
$a^3 - 0.a^2 - 3bca - b^3 - c^3 = [a - (b + c)[a^2 + (b + c)a + (b^2 - bc + c^2)] = 0.]$

PROPOSITION 16.

The root of the equation : *aaa + 3.bca = - ccc + bbb,* is *b - c,* equal to the root of *a* sought ; *i.e. a = b - c.*

For if *a = b - c,* by changing *a* into *b - c* in the equation, then

$$aaa + 3.bca = - ccc - bbb,$$

becomes. . . $bbb - 3.cbb + 3.ccb - ccc = +aaa \quad = - ccc + bbb$

And . . . . . $+ 3.cbb - 3.ccb = +3.bca$

But this equality has become apparent, by the rejection of contradictory parts.
Therefore the root is *a = b - c*. As stated.
[Note for Prop. 16 : The 16th equation in modern terms is :
$a^3 - 0.a^2 + 3bca - b^3 + c^3 = [a - (b - c)[a^2 + (b - c)a + (b^2 + bc + c^2)] = 0.]$

PROPOSITION 17.

The root of the equality : *aaa - 3.bba = +2.bbb,* is 2.*b,* equal to the root of *a* sought; *i.e. a* = 2.*b.*
For if *a* = 2.*b,* by changing *a* into 2.*b* in the equation, then

$$aaa - 3.bba = + 2.bbb,$$

becomes. . . $8.bbb - 6.bbb = + 2.bbb$
But this equality is apparent by itself.
Therefore the root is *a = 2.b*. As stated.
[Note for Prop. 17 : The 17th equation in modern terms is :
$a^3 - 0.a^2 - 3b^2a - b^3 - 2b^3 = (a - 2b)(a^2 + 2ba + b^2) = 0,$
following Prop.15., with $c = b$.]

Recurring [roots].

PROPOSITION 18.

The root of the equality : *aaa - bba + cda* = +*bcd,* is *b,* equal to the root of *a* sought; i.e. *a* = *b.*
For if *a* = *b*, by changing *a* into *b* in the equation, then

aaa - bba + cda = +*bcd,*

becomes. . . *bbb - bbb + cdb* = + *bcd*

But this equality is apparent by itself.
Therefore the root is *a* = *b*. As stated.
[Note for Prop. 18 : The 18th equation in modern terms is :
$a^3 - 0.a^2 - b^2a + cda - bcd = (a-b)(a^2 + ba + cd) = 0.$]

PROPOSITION 19.

The root of the equation : *aaa* + *baa* - *cca* = +*bcc,* is *c*, equal to the root of *a* sought; i.e. *a* = *c.*
For if *a* = *c,* by changing *a* into *c* in the equation, then

aaa + *baa* - *cca* = +*bcc,*

becomes . . . *ccc* + *bcc* + *ccc* = + *bcc.*

But this equality is apparent by itself.
Therefore the root is *a* = *c*. As stated.
[Note for Prop. 19 : The 19th equation in modern terms is :
$a^3 + b.a^2 - c^2a - bc^2 = (a-c)(a^2 + ba + bc) = 0.$]

PROPOSITION 20.

The roots if the equation : *aaa - baa - cca* = - *bcc,* are *b* or *c*; equal to the roots of *a* sought; *i.e. a* = *b* or *a* = *c.*

For if *a* = *b*, by changing *a* into *b* in the equation, then

aaa - baa - cca = - *bcc,*

becomes. . . *bbb - bbb - ccb* = -*bcc*

But this equality is apparent by itself.
Therefore a root is *a* = *b*.
The same is true if *a* = *c*, by changing *a* into *c* in the equation*,* then
it becomes. . . *ccc - bcc - ccc* = -*bcc*
This is also seen to be equal
Therefore the root is *a* = *c*.
Therefore *b* and *c* are the roots sought, equal to *a*. As stated.
[Note for Prop. 20 : The 20th equation in modern terms is :
$a^3 - b.a^2 - c^2a + bc^2 = (a-b)(a-c)(a+c) = 0.$]

PROPOSITION 21.

The root of the equation :
$$\begin{array}{llll} aaaa + baaa + & bcaa & & \\ + caaa + & bdaa & & \\ + daaa + & cdaa & + bcda & \\ - faaa - & bfaa & - bcfa & \\ - & cfaa & - bdfa & \\ - & dfaa & - cdfa & = + bcdf \end{array}$$
is f, equal to the root of a sought; *i. e.* $a = f$.

For if a is put equal f, for the root $a = f$ in the equation, by changing a into f, then :
$$\begin{array}{llll} ffff + bfff + & bcff & & \\ + cfff + & bdff & & \\ + dfff + & cdff & + bcdf & \\ - ffff - & bfff & - bcff & \\ - & cfff & - bdff & \\ - & dfff & - cdff & = + bcdf \end{array}$$

But this equality is itself shown from the different rejected redundant parts.
Therefore, $a = f$ satisfies the equation.
Moreover, another root equal to a cannot be given in addition to f, and this is shown in the following Lemma.

Lemma.

If it should be possible to give another root equal to a, not equal to the root f, then let that be b, or c or d or some other.
For if b is put equal a, then :
$$\begin{array}{llll} bbbb + bbbb & + bbbc & & \\ + cbbb & + bbbd & & \\ + dbbb & + bbcd & + bbcd & \\ - fbbb & - bbbf & - bbcf & \\ & - bbcf & - bbdf & \\ & - bbdf & - bcdf & = + bcdf. \end{array}$$
Hence, $2.bbbb + 2.bbbc + 2.bbbd + 2.bbcd = 2.bbbf + 2.bbcf + 2.bbdf + 2.bcdf$;
i. e. $bbbb + bbbc + bbbd + bbcd = bbbf + bbcf + bbdf + bcdf$;
Hence $\dfrac{bbb + bbc + bbd + bcd}{b} = \dfrac{bbb + bbc + bbd + bcd}{f}$
Hence, $b = f$, which is contrary to the hypothesis.
Therefore b is not equal to f, as has been put in place. In the same way, for the other values c & d, or any other value, this result can be shown by the same deduction.

[Note for Prop. 21 : The 21st equation in modern terms is :
$$a^4 + (b + c + d - f)a^3 + (bc + bd + cd - bf - cf - df)a^2 + (bcd - cdf - bdf - bcf)a - bcdf$$
$$= (a + b)(a + c)(a + d)(a - f) = 0.]$$

PROPOSITION 22.

The roots of the equation:

$$\begin{array}{llll} aaaa & -baaa & +bcaa & \\ & -caaa & +bdaa & \\ & -daaa & +cdaa & -bcda \\ & +faaa & -bfaa & +bcfa \\ & & -cfaa & +bdfa \\ & & -dfaa & +cdfa \quad = +\ bcdf \end{array}$$

are b, or c or d, equal to the roots of a sought; *i. e.* $a = b$, $a = c$, or $a = d$.

For if a is put equal to b, for the root $a = b$ in the equation, by changing a into b, then :

$$\begin{array}{llll} bbbb & -bbbb & +bbbc & \\ & -bbbc & +bbbd & \\ & -bbbd & +bbcd & -bbcd \\ & +bbbf & -bbbf & +bbcf \\ & & -bbcf & +bbdf \\ & & -bbdf & +bcdf \quad = +bcdf \end{array}$$

But this equality is itself shown from the different rejected redundant parts.
Therefore, $a = b$ satisfies the equation.

Likewise, if a is put equal to c, for the root $a = c$ in the equation, by changing a into c, then :

$$\begin{array}{llll} cccc & -bccc & +bccc & \\ & -cccc & +bdcc & \\ & -dccc & +cdcc & -bcdc \\ & +fccc & -bfcc & +bdfc \\ & & -cfcc & +bdfc \\ & & -dfcc & +cdfc \quad = +bcdf \end{array}$$

But this equality is itself returned from the different redundant parts.
Therefore, $a = c$ satisfies the equation.

Likewise, if a is put equal to d, for the root $a = d$ in the equation, by changing a into d, then :

$$\begin{array}{llll} dddd & -bddd & +bcdd & \\ & -cddd & +bddd & \\ & -dddd & +cddd & -bcdd \\ & +fddd & -bfdd & +bcfd \\ & & -cfdd & +bdfd \\ & & -dfdd & +cdfd \quad = +bcdf \end{array}$$

But this equality is itself returned from the different redundant parts.
Therefore, $a = d$ satisfies the equation.
Hence, the roots sought are $a = b$, $a = c$, and $a = d$, as stated.

Moreover, no other root equal to *a* can be given in addition to *b, c, d,* and this is shown in the following Lemma.

Lemma.

If it should be possible to give another root equal to *a*, which is not equal to the roots *b*, c, or *d*, then let that root be *f,* or some other.
For if *f* is put equal *a*, then :

$$\begin{array}{llll} ffff - bfff + bcff & & & \\ \quad - cfff + bdff & & & \\ \quad - dfff + cdff - bcdf & & & \\ \quad + ffff - bfff + bcff & & & \\ \qquad\quad - cfff + bdff & & & \\ \qquad\quad - dfff + cdff = + bcdf. & & & \end{array}$$

Hence, $2.ffff - 2.cfff + 2.cdff - 2.dfff = 2.bfff - 2.bcff + 2.bcdf - 2.bdff;$
i. e. $ffff - cfff + cdff - dfff = bfff - bcff + bcdf - bdff;$
Hence $\dfrac{fff - cff + dcf - dff}{f} = \dfrac{fff - cff + dcf - dff}{b}$
Hence, $f = b$, which is contrary to the hypothesis of the Lemma.
Therefore *b* is not equal to *f*, as has been put in place. In the same way, for any other value, this result can be shown by a similar deduction.

[Note for Prop. 22 : The 22nd equation in modern terms is :
$a^4 - (b + c + d - f)a^3 + (bc + bd + cd - bf - cf - df)a^2 - (bcd - cdf - bdf - bcf)a - bcdf$
$= (a - b)(a - c)(a - d)(a + f) = 0.$]

PROPOSITION 23.

The roots of the equation:

$$\begin{array}{l} aaaa - baaa + bcaa \\ \quad - caaa - bdaa \\ \quad + daaa - cdaa + bcda \\ \quad + faaa - bfaa + bcfa \\ \qquad\quad - cfaa - bdfa \\ \qquad\quad + dfaa - cdfa = - bcdf \end{array}$$

are *b*, or *c*, equal to the roots of *a* sought; *i. e. a = b, a = c.*

For if *a* is put equal to *b*, for the root *a = b* in the equation, by changing *a* into *b,* then :

$$\begin{array}{l} bbbb - bbbb + bcbb \\ \quad - cbbb - bdbb \\ \quad + dbbb - bfbb + bcdb \\ \quad + fbbb - cdbb + bcfb \\ \qquad\quad - cfbb - bdfb \\ \qquad\quad + dfbb - cdfb = - bcdf \end{array}$$

But this equality is itself shown from the different rejected redundant parts [note that the order has been changed in the equation].
Therefore, *a* = *b* satisfies the equation.

Likewise, if *a* is put equal to *c*, *a* = *c* satisfies the equation also.
Hence, the roots sought are *a* = *b, a* = *c*, as stated.

Moreover, no other root equal to *a* can be given besides *b, c,* and this is shown in the following Lemma.

Lemma.

If it should be possible to give another root equal to *a*, which is not equal to the roots *b*, c, then let that root be *d* or *f,* or some other.
For if *d* is put equal *a*, then :

$$\begin{array}{llll} dddd & - bddd & + bcdd & \\ & - cddd & - bddd & \\ & + dddd & - bfdd & + bcdd \\ & + fddd & - cddd & + bcfd \\ & & - cfdd & - bfdd \\ & & + fddd & - cfdd \quad = - bcdf. \end{array}$$

Hence, 2.*dddd* - 2.*cddd* + 2.*fddd* - 2.*cfdd* = 2.*bddd* - 2.*bcdd* + 2.*bfdd* - 2.*bcdf;*
i. e. *bddd* - *bcdd* + *bfdd* - *dcdf* = *dddd* - *cddd* + *fdddf* - *cfdd;*

Hence $\dfrac{ddd - cdd + fdd - cfd}{d} = \dfrac{ddd - cdd + fdd - cfd}{b}$

Hence, *d* = *b,* which is contrary to the hypothesis of the Lemma.
In a like manner, a contradiction can be established from the 16 terms of the equation [for the root *c*], in which *d* = *c* is similarly proposed. Hence, *a* is not equal to *d*, as was assumed.
Concerning *f* , or any other value besides *b* and *c*, the same pronouncement can be made by a similar deduction.

[Note for Prop. 23 : The 23rd equation in modern terms is :
$a^4 - (b + c - d - f)a^3 + (bc - bd - cd - bf - cf + df)a^2 + (bcd - cdf - bdf + bcf)a + bcdf$
$= (a - b)(a - c)(a + d)(a + f) = 0.$]

PROPOSITION 24.

The roots of the equation:

$$\begin{array}{llll} aaaa & - baaa & + bcaa & \\ & - caaa & + bdaa & \\ & - daaa & + cdaa & - bcda \\ & - faaa & + bfaa & - bcfa \\ & & + cfaa & - bdfa \\ & & + dfaa & - cdfa \quad = - bcdf \end{array}$$

are *b*, or *c*, or *d* or *f,* equal to the roots of *a* sought; *i. e. a* = *b, a* = *c, b* = *d, a* = *f.*

For if *a* is put equal to *b*, for the root *a* = *b* in the equation, by changing *a* into *b,* then :

$$\begin{array}{llll} bbbb & -\ bbbb & +\ bcbb & \\ & -\ cbbb & +\ bdbb & \\ & -\ dbbb & +\ bfbb & -\ bcdb \\ & -\ fbbb & +\ cdbb & -\ bcfb \\ & & +\ cfbb & -\ bdfb \\ & & +\ dfbb & -\ cdfb \quad = -\ bcdf \end{array}$$

But this equality is shown from the rejected redundant parts.
Therefore, $a = b$ satisfies the equation.

Likewise, if a is put equal to c, for the root $a = c$ in the equation, by changing a into c, then :

$$\begin{array}{llll} cccc & -\ bccc & +\ bccc & \\ & -\ cccc & +\ bdcc & \\ & -\ dccc & +\ cdcc & -\ bcdc \\ & -\ fccc & +\ bfcc & -\ bdfc \\ & & +\ cfcc & -\ bdfc \\ & & +\ dfcc & -\ cdfc \quad = -\ bcdf \end{array}$$

But this equality is itself shown to be returned from the different redundant parts.
Therefore, $a = c$ satisfies the equation.
Likewise, if a is put equal to d or f, for the root, similar equalities follow from the change.
This concludes the finding of the roots in a similar manner
Hence, the roots sought are $a = b$, $a = c$, $a = d$, $a = f$, as stated.

Moreover, no other root equal to a can be given besides b, c, d, or f and this is shown in the following Lemma.

Lemma.

If it should be possible to give another root equal to a, which is not equal to the roots b, c, d, or f, then let that root be g, or some other.
For if g is put equal a, then

$$\begin{array}{llll} gggg & -\ bggg & +\ bcgg & \\ & -\ cggg & +\ bdgg & \\ & -\ dggg & +\ cdgg & -\ bcdg \\ & -\ fggg & +\ bfgg & -\ bdfg \\ & & +\ cfgg & -\ bdfg \\ & & +\ dfgg & -\ cdfg \quad = -\ bcdf \end{array}$$

Hence, $gggg - cggg + cdgg - dggg + cfgg - fggg + dfgg - cdfg =$
$bggg - bcgg + bcdg - bdgg + bcfg - bfgg + bdfgf - bcdf;$

Hence $$\frac{ggg - cgg + cdg - dgg + \mathrm{cfg} - fgg + dfg - cdf}{g} = \frac{ggg - cgg + cdg - dgg + cfg - fgg + dfg - cdf}{b}$$

Hence, $g = b$, which is contrary to the hypothesis of the Lemma.
In a like manner, a contradiction can be established from the 16 terms of the equation, for the cases in which $g = c$, or $g = d$, or $g = f$ are similarly proposed in the correct order.
But that now set out concerning b is sufficient for an example. Hence, a is not equal to g, as was assumed. The truth lies in refuting the false nature of setting g equal to one of the remaining roots.
Hence, g is not equal to a, as was proposed, for any other g; this has been established by deduction from the equality.
[Note for Prop. 24 : The 24th equation in modern terms is :
$a^4 - (b+c+d+f)a^3 + (bc+bd+cd+bf+cf+df)a^2 - (bcd+cdf+bdf+bcf)a + bcdf$
$= (a-b)(a-c)(a-d)(a-f) = 0.$]

Reduced Equations.

PROPOSITION 25.

The roots of the equation:

$$\begin{array}{llll} aaaa & - bbaa & + bbca & \\ & - ccaa & + bbda & \\ & - ddaa & + bcca & \\ & - bcaa & + ccda & \\ & - bdaa & + bdda & \\ & - cdaa & + cdda & = + bcdf \\ & & +2.bcda & \quad + bccd \\ & & & \quad + bcdd. \end{array}$$

are b, or c, or d , equal to the roots of a sought; *i. e.* $a = b$, $a = c$, $b = d$.

For if a is put equal to b, for the root $a = b$ in the equation, by changing a into b, then :

$$\begin{array}{llll} bbbb & - bbbb & + bbcb & \\ & - ccbb & + bbdb & \\ & - ddbb & + bccb & \\ & - bcbb & + ccdb & \\ & - bdbb & + bddb & \\ & - cdbb & + cddb & = + bcdf \\ & & +2.bcdb & \quad + bccd \\ & & & \quad + bcdd. \end{array}$$

But this equality is shown from the rejected redundant parts.
Therefore, $a = b$ satisfies the equation.

Likewise, if a is put equal to c, for the root $a = c$ in the equation, by changing a into c, then :

$$\begin{array}{llll} cccc & -bbcc & +bbcc & \\ & -cccc & +bbdc & \\ & -ddcc & +bccc & \\ & -bccc & +ccdc & \\ & -bdcc & +bddc & \\ & -cdcc & +cddc & = +bcdf \\ & & +2.bcdc & \quad +bccd \\ & & & \quad +bcdd. \end{array}$$

But this equality is itself shown from the different rejected redundant parts.
Therefore, $a = c$ satisfies the equation.
Likewise, if a is put equal to d, for the root, similar equalities follow from the change.
For if a is put equal to d, for the root $a = d$ in the equation, by changing a into d, then :

$$\begin{array}{llll} dddd & -bbdd & +bbcd & \\ & -ccdd & +bbdd & \\ & -dddd & +bccd & \\ & -bcdd & +ccdd & \\ & -bddd & +bddd & \\ & -cddd & +cddd & = +bcdf \\ & & +2.bcdd & \quad +bccd \\ & & & \quad +bcdd. \end{array}$$

But this equality is itself shown from the different rejected redundant parts.
Therefore, $a = d$ satisfies the equation.
Hence, the roots sought are $a = b$, $a = c$, $a = d$, as stated.
[Note for Prop. 25 : The 25th equation in modern terms is :
$a^4 - (0)a^3 - (b^2 + bc + cd + bd + c^2 + d^2)a^2 + (c^2 d + bc^2 + b^2 d + b^2 c + bd^2 + cd^2 + 2.bcd)a$
$- bbcd - bccd - bcdd = (a-b)(a-c)(a-d)(a+b+c+d) = 0$.]

PROPOSITION 26.

The roots of the equation:

$$\begin{array}{llll} aaaa & -bbaaa & +bbcca & \\ & -ccaaa & +bbdda & \\ & -ddaaa & +bcdda & \\ & -bcaaa & +ccdda & \\ & -bdaaa & +bccda & \\ & \underline{-cdaaa} & +\underline{bbcda} & = +bbccd \\ & b+c+d & b+c+d & \quad +bbcdd \\ & & & \quad +\underline{bccdd.} \\ & & & \quad b+c+d \end{array}$$

are b, c, and d, equal to the roots of a sought; *i. e.* $a = b$, $a = c$, $b = d$.

For if a is put equal to b, for the root $a = b$ in the equation, by changing a into b, and the powers reduced to a common divisor, then :

$$\frac{bbbbb}{b+c+d}\;\frac{cbbbb}{}\;\frac{dbbbb}{} \;-\; \frac{bbbbb - ccbbb - ddbbb - bcbbb - bdbbb - cdbbb}{b+c+d} + \frac{bbccb + bbddb + ccddb + ccddb + bccdb + bbcdb}{b+c+d} = + \frac{bbccd + bbcdd + bccdd.}{b+c+d}$$

But this equality is shown from the separate contradictory parts.
Therefore, $a = b$ satisfies the equation.
Likewise, with a put equal to c or d for the roots by changing a, the equations follow.
From which it follows that these also are values of a equal to the root, as can be similarly concluded.
Hence, the roots sought are $a = b$, $a = c$, $a = d$, as stated.
[Note for Prop. 26 : The 26th equation in modern terms is :

$$a^4 - (b^2 + c^2 + d^2 + bc + cd + bd)/(b + c + d).a^3 - (0)a^2 +$$
$$(c^2d^2 + bcd^2 + b^2d^2 + b^2c^2 + bc^2d + b^2cd)/(b + c + d).a$$
$$-(b^2c^2d + bc^2d^2 + b^2cd^2)/(b + c + d) = (a - b)(a - c)(a - d)(a + (bc + cd + bd)/(b + c + d)) = 0.]$$

PROPOSITION 27.

The roots of the equation:

$$aaaa \frac{- bbcaaa - bbdaaa - bccaaa - bddaaa - ccdaaa - cddaaa - 2.bcdaaa}{bc + bd + cd} + \frac{bbccaa + bbddaa + ccddaa + bcddaa + bccdaa + bbcdaa}{bc + bd + cd} = + \frac{bbccdd}{bc + bd + cd}$$

are b, c, and d, equal to the roots of a sought; *i. e.* $a = b$, $a = c$, $b = d$.

For if a is put equal to b, for the root $a = b$ in the equation, by changing a into b, and the powers reduced to a common divisor, then :

$$\begin{array}{r}
bcbbb - bbcbb \\
bdbbb - bbdbb + bbccbb \\
\underline{cdbbb} - bccbbb + bbddbb \\
bc + bd + cd \ -bddbbb + ccddbb \\
- ccdbbb + bcddbb \\
- cddbbb + bccdbb \\
\dfrac{-2.bcdbbb}{bc+bd+cd} + \dfrac{bbcdbb}{bc+bd+cd} = + \dfrac{bbccdd}{bc+bd+cd}
\end{array}$$

But this equality is shown from the separate contradictory parts.
Therefore, $a = b$ satisfies the equation.
Likewise, with a put equal to c or d for the roots by changing a to c or d, equalities follow.
From which it follows that these also are values of a equal to the root, as can be similarly concluded.
Hence, the roots sought are $a = b$, $a = c$, $a = d$, as stated.

[Note for Prop. 27 : The 27th equation in modern terms is :
$$a^4 - (b^2c + b^2d + bd^2 + bc^2 + c^2d + cd^2 + 2.bcd)/(bc + cd + bd).a^3 - (b^2c^2 + b^2d^2 + c^2d^2 + bcd^2$$
$$+ bc^2d + b^2cd)/(cd + db + bc).a^2 + (0).a - b^2c^2d^2/(cd + db + bc)$$
$$= (a-b)(a-c)(a-d)(a + bcd/(cd + db + bc)) = 0.]$$

PROPOSITION 28.

The root of the equation:
$$\begin{array}{llll}
aaaa - bbaa & - bbca & & \\
- ccaa & - bbda & & \\
- ddaa & - bcca & & \\
- bcaa & - ccda & & \\
- bdaa & - bdda & & \\
- cdaa & - cdda & & \\
 & - 2.bcda & = & + bbcd \\
 & & & + bbcd \\
 & & & + bcdd.
\end{array}$$

is $b + c + d$, equal to the roots of a sought; *i. e.* $a = b + c + d$.
For (by Problem 12, Sect. 3), here the proposed trinomial equation is deduced from its own quadrinomial by putting $b + c + d = f$.
But, (by Problem 21, of this section), the root of this quadrinomial is $a = f$.
Hence the root of this trinomial is $a = b + c + d$, as stated.

[Note for Prop. 28 : The 28th equation in modern terms is :
$$a^4 - (0)a^3 - (b^2 + bc + cd + bd + c^2 + d^2)a^2 - (c^2d + bc^2 + b^2d + b^2c + bd^2 + cd^2 + 2.bcd)a$$
$$- b^2cd - bc^2d - bcd^2 = (a+b)(a+c)(a+d)(a-b-c-d) = 0.]$$

PROPOSITION 29.

The root of the equation:

$$
\begin{array}{llll}
aaaa & +\,bbaaa & -\,bbcca & \\
 & +\,ccaaa & -\,bbdda & \\
 & +\,ddaaa & -\,bcdda & \\
 & +\,bcaaa & -\,ccdda & \\
 & +\,bdaaa & -\,bccda & \\
 & \underline{+\,cdaaa} & \underline{+\,bbcda} & = +\,bbccd \\
 & b+c+d & b+c+d & +\,bbcdd \\
 & & & \underline{+\,bccdd.} \\
 & & & b+c+d
\end{array}
$$

is $\dfrac{bc + bc + cd}{b+c+d}$, equal to the roots of a sought; *i. e.* $a = \dfrac{bc + bc + cd}{b+c+d}$.

For (by Prop. 13, Sect. 3) here the trinomial equation is deduced from its quadrinomial by putting $f = \dfrac{bc + bc + cd}{b+c+d}$.

But (per Prop. 21 of this section), $a = f$ is the root of this quadrinomial.
Hence, the root of this trinomial is $a = \dfrac{bc + bc + cd}{b+c+d}$, as stated.

[Note for Prop. 29 : The 29th equation in modern terms is :

$$a^4 + (b^2 + c^2 + d^2 + bc + cd + bd)/(b + c + d).a^3 - (0)a^2 -$$

$$(c^2d^2 + bcd^2 + b^2d^2 + b^2c^2 + bc^2d + b^2cd)/(b + c + d).a$$

$$-(b^2c^2d + bc^2d^2 + b^2cd^2)/(b + c + d) = (a + b)(a + c)(a + d)(a - (bc + cd + bd)/(b + c + d)) = 0.]$$

PROPOSITION 30.

The root of the equation:

$$
\begin{array}{llll}
aaaa & +\,bbcaaa & & \\
 & +\,bbdaaa & +\,bbccaa & \\
 & +\,bccaaa & +\,bbddaa & \\
 & +\,bddaaa & +\,ccddaa & \\
 & +\,ccdaaa & +\,bcddaa & \\
 & +\,cddaaa & +\,bccdaa & \\
 & \underline{+2.bcdaaa} & +\,\underline{bbcdaa} & = +\,\underline{bbccdd} \\
 & bc+bd+cd & bc+bd+cd & bc+bd+cd
\end{array}
$$

is $\dfrac{bcd}{bc+bd+cd}$, equal to the roots of a sought;

i. e. $a = \dfrac{bcd}{bc+bd+cd}$.

For (by Prop. 14, Sect. 3) here the trinomial equation is deduced from its quadrinomial by putting $\dfrac{bcd}{bc + \mathrm{bc} + \mathrm{c}d} = f$.

But (per Prop. 22 of this section), $a = f$ is the root of this quadrinomial.
Hence, the root of this trinomial is $a = \dfrac{bcd}{bc + \mathrm{bc} + \mathrm{c}d}$, as stated.

[Note for Prop. 30 : The 30th equation in modern terms is :
$$a^4 + (b^2c + b^2d + bd^2 + bc^2 + c^2d + cd^2 + 2.bcd)/(bc + cd + bd).a^3 + (b^2c^2 + b^2d^2 + c^2d^2 + bcd^2$$
$$+ bc^2d + b^2cd)/(cd + db + bc).a^2 + (0).a - b^2c^2d^2/(cd + db + bc)$$
$$= (a + b)(a + c)(a + d)(a - bcd/(cd + db + bc)) = 0.]$$

PROPOSITION 31.

The roots of the equation:
$$\begin{array}{lll} aaaa + bdaa & + bbca & \\ + cdaa & + bcca & \\ - ddaa & + bdda & \\ - bbaa & + ccda & \\ - bcaa & - bbda & \\ - ddaa & - ccda & \\ & - 2.bcda & = - bbcd \\ & & - bccd \\ & & + bcdd. \end{array}$$

is b or c, equal to the roots of a sought; *i. e.* $a = b$ or $a = c$.
For if a is put equal to b, for the root $a = b$ in the equation, by changing a into b, then :
$$\begin{array}{lll} bbbb + bdbb & + bbcb & \\ + cdbb & + bccb & \\ - ddbb & + bddb & \\ - bbbb & + ccdb & \\ - bcbb & - bbdb & \\ - ddbb & - ccdb & \\ & - 2.bcdb & = - bbcd \\ & & - bccd \\ & & + bcdd. \end{array}$$

But this equality is shown from the cancellation of opposite parts.
Therefore, $a = b$ satisfies the equation.
Likewise, with a put equal to c for the root by changing a to c, the equality follows.
From which it follows that this also is a value of a equal to the root, as can be similarly concluded.
Hence, the roots sought are $a = b$, $a = c$, as stated.

[Note for Prop. 31 : The 31st equation in modern terms is :
$$a^4 - (0)a^3 - (b^2 + c^2 + d^2 + bc - cd - bd)a^2 + (-c^2d + bc^2 - b^2d + b^2c + bd^2 + cd^2 - 2.bcd)a$$
$$+ b^2cd + bc^2d - bcd^2 = (a-b)(a-c)(a+d)(a+b+c-d) = 0.]$$

PROPOSITION 32.

The roots of the equation:
$$aaaa\ \frac{\begin{array}{l}-bbaaa\\-bcaaa\\-ccaaa\\-ddaaa\\+bdaaa\\+cdaaa\end{array}}{b+c-d}\ \frac{\begin{array}{l}+bbcca\\+bbdda\\+bcdda\\+ccdda\\-bccda\\-bbcda\end{array}}{b+c-d} = \frac{\begin{array}{l}-bbccd\\+bbcdd\\+bccdd.\end{array}}{b+c-d}$$
are b and c, equal to the roots of a sought; *i. e.* $a = b, a = c.$

For if a is put equal to b, for the root $a = b$ in the equation, by changing a into b, and the powers reduced to a common divisor, then :

$$\frac{\begin{array}{l}+bbbbb\\+cbbbb\\-dbbbb\end{array}}{b+c+d}\ \frac{\begin{array}{l}-bbbbb\\-bcbbb\\-ccbbb\\-ddbbb\\+bdbbb\\+cdbbb\end{array}}{b+c-d}\ \frac{\begin{array}{l}+bbccb\\+bbddb\\+bcddb\\+ccddb\\-bbcdb\\-bccdb\end{array}}{b+c-d} = \frac{\begin{array}{l}-bbccd\\+bbcdd\\+bccdd.\end{array}}{b+c-d}$$

But this equality is shown from the separate contradictory parts.
Therefore, $a = b$ satisfies the equation.
Likewise, with a put equal to c for the root by changing a, the equality follows.
From which it follows that this also is a value of a equal to the root, as can be similarly concluded.
Hence, the roots sought are $a = b, a = c$, as stated.

[Note for Prop. 32 : The 32nd equation in modern terms is :
$$a^4 - (b^2 + c^2 + d^2 + bc - cd - bd)/(b + c - d).a^3 - (0)a^2 +$$
$$(c^2d^2 + bcd^2 + b^2d^2 + b^2c^2 - bc^2d - b^2cd)/(b + c - d).a$$
$$-(-b^2c^2d + bc^2d^2 + b^2cd^2)/(b + c - d) = (a-b)(a-c)(a+d)(a + (bc - cd - bd)/(b + c - d)) = 0.]$$

PROPOSITION 33.

The roots of the equation:

$$\begin{array}{llll}
aaaa & +bbaaa & -bbcca & \\
& +bcaaa & -bbdda & \\
& +ccaaa & -bcdda & \\
& +ddaaa & -ccdda & \\
& -bdaaa & +bbcda & \\
& \underline{-cdaaa} & +\underline{bccda} & = -bbcdd \\
& d-b-c & d-b-c & \quad -bccdd \\
& & & \quad +\underline{bbccd.} \\
& & & \quad d-b-c
\end{array}$$

are b and c , equal to the roots of a sought; *i. e.* $a = b$ or c.

For if a is put equal to b, for the root $a = b$ in the equation, by changing a into b, and the powers reduced to a common divisor, then :

$$\begin{array}{llll}
+dbbbb & +bbbbb & -bbccb & \\
-bbbbb & +bcbbb & -bbddb & \\
-\underline{cbbbb} & +ccbbb & -bcddb & \\
d-b+c & +ddbbb & -ccddb & \\
& -bdbbb & +bbcdb & \\
& \underline{-cdbbb} & +\underline{bccdb} & = -bbcdd \\
& d-b+c & d-b+c & \quad -bccdd \\
& & & \quad +\underline{bbccd.} \\
& & & \quad d-b+c
\end{array}$$

But this equality is shown from the separate contradictory parts.
Therefore, $a = b$ satisfies the equation.
Likewise, with a put equal to c for the root by changing a, the equality follows.
From which it follows that this also is a value of a equal to the root, as can be similarly concluded. Hence, the roots sought are $a = b$, $a = c$, as stated.

[Note for Prop. 33 : The 33rd equation in modern terms with denominator $d - b - c$ is :

$$a^4 - (b^2 + c^2 + d^2 + bc - cd - bd)/(b + c - d).a^3 - (0)a^2 -$$
$$(c^2d^2 + bcd^2 + b^2d^2 + b^2c^2 - bc^2d - b^2cd)/(b + c - d).a$$
$$+ (b^2c^2d - bc^2d^2 - b^2cd^2)/(b + c - d) = (a - b)(a - c)(a + d)(a + (bc - cd - bd)/(b + c - d)) = 0.]$$

PROPOSITION 34.

The roots of the equation:

$$\begin{array}{lll} aaaa + bbcaaa & & \\ + bccaaa & - bbccaa & \\ + bddaaa & - bbddaa & \\ + cddaaa & - bcddaa & \\ - bbdaaa & - ccddaa & \\ - ccdaaa & + bccdaa & \\ \underline{-2.bcdaaa} & + \underline{bccdaa} & = - \underline{bbccdd} \\ bd + cd - bc & bd + cd - bc & bd + cd - bc \end{array}$$

are b and c, equal to the roots of a sought; *i. e.* $a = b$, $a = c$.

For if a is put equal to b, for the root $a = b$ in the equation, by changing a into b, and the powers reduced to a common divisor, then :

$$\begin{array}{llll} + bdbbb & + bbcbbb & & \\ + cdbbb & + bccbbb & - bbccbb & \\ \underline{- bc\,bbb} & + bddbbb & - bbddbb & \\ bd + cd - bc & -cddbbb & - bcddbb & \\ & - bbdbbb & - ccddbb & \\ & - ccdbbb & + bbcdbb & \\ & \underline{-2.bcdbbb} & + \underline{bccdbb} & = - \underline{bbccdd} \\ & bd + cd - bc & bd + cd - bc & bd + cd - bc \end{array}$$

But this equality is shown from the separate contradictory parts.
Therefore, $a = b$ satisfies the equation.
Likewise, with a put equal to c for the root by changing a to c, the equality follow.
From which it follows that these also are values of a equal to the root, as can be similarly concluded.
Hence, the roots sought are $a = b$, $a = c$, as stated.

[Note for Prop. 34 : The 34th equation in modern terms is :

$$a^4 + (b^2c - b^2d + bd^2 + bc^2 - c^2d + cd^2 - 2.bcd)/(bc + cd + bd).a^3 - (b^2c^2 + b^2d^2 + c^2d^2 + bcd^2$$
$$- bc^2d - b^2cd)/(cd + db - bc).a^2 + (0).a - b^2c^2d^2/(cd + db - bc)$$
$$= (a - b)(a - c)(a + d)(a + bcd/(cd + db - bc)) = 0.]$$

PROPOSITION 35.

The roots of the equation:

$$\begin{array}{ll} aaaa & -bbba \\ & -bbca \\ & -bcca \\ & -ccca = -bbbc \\ & \qquad\quad -bbcc \\ & \qquad\quad -bccc. \end{array}$$

is said to be *b* or *c,* equal to the roots of *a* sought; *i. e.* $a = b$ or $a = c$.
For if *a* is put equal to *b,* then :

$$\begin{array}{ll} bbbb & -bbbb \\ & -cbbb \\ & -bbcc \\ & -bccc = -bbbc \\ & \qquad\quad -bbcc \\ & \qquad\quad -bccc. \end{array}$$

Or put $c = a$, then

$$\begin{array}{ll} cccc & -bbbc \\ & -bbcc \\ & -bccc \\ & -cccc = -bbbc \\ & \qquad\quad -bbcc \\ & \qquad\quad -bccc. \end{array}$$

The equalities can be seen.
Hence, the roots sought are $a = b$, $a = c$, as stated.

[Note for Prop. 35 : The 31st equation in modern terms is :
$$a^4 - (0)a^3 - (0)a^2 - (b^3 + b^2c + bc^2 + c^3)a + b^3c + b^2c^2 + bc^3$$
$$= (a-b)(a-c)(a^2 + (b+c)a + b^2 + bc + c^2) = 0.]$$

PROPOSITION 36.

The roots of the equation:

$$aaaa\ \frac{\begin{array}{l} -\,bbbaaa \\ -\,bbcaaa \\ -\,bccaaa \\ -\,cccaaa \end{array}}{bb+bc+cc} = \frac{-\ bbbccc}{bb+bc+cc}$$

are *b* or *c,* equal to the roots of *a* sought; *i. e.* $a = b$ or $a = c$.
For if *a* is put equal to *b,* then :

$$\frac{\begin{array}{l} bbbbbb \\ bbbbbc \\ bbbbcc \end{array}}{bb+bc+cc}\ \frac{\begin{array}{l} -\,bbbbbb \\ -\,bbbbbc \\ -\,bbbbcc \\ -\,bbbccc \end{array}}{bb+bc+cc} = \frac{-\ bbbccc}{bb+bc+cc}$$

Or put $c = a$, then

$$\frac{\begin{array}{l} bbcccc \\ bccccc \\ cccccc \end{array}}{bb+bc+cc}\ \frac{\begin{array}{l} -\,bbbccc \\ -\,bbcccc \\ -\,bccccc \\ -\,cccccc \end{array}}{bb+bc+cc} = \frac{-\ bbbccc}{bb+bc+cc}$$

The equalities can be seen.
Hence, the roots sought are $a = b,\ a = c,$ as stated.

[Note for Prop. 36 : The 36th equation in modern terms is :

$$a^4 - (b^3 + b^2c + bc^2 + c^3)/(b^2 + bc + c^2)a^3 - (0)a^2 - (0)a + b^3c^3/(b^2 + bc + c^2)$$
$$= (a-b)(a-c)(a^2 + (b+c)bc/(b^2 + bc + c^2).a + b^2c^2/(b^2 + bc + c^2)) = 0.]$$

PROPOSITION 37.

The roots of the equation:

$$aaaa\ \begin{array}{l} -\,bbaa \\ -\,ccaa \end{array} = -\ bbcc$$

are *b* or *c,* equal to the roots of *a* sought; *i. e.* $a = b$ or $a = c$.
For if $a = b,$ then :

$$bbbb\ \begin{array}{l} -\,bbbb \\ -\,bbcc \end{array} = -\ bbcc$$

Or put $c = a$, then

$$cccc\ \begin{array}{l} -\,bbcc \\ -\,bbcc \end{array} = -\ bbcc$$

The equalities can be seen. Hence, the roots sought are $a = b,\ a = c,$ as stated.

[Note for Prop. 37 : The 37th equation in modern terms is :

$$a^4 - (0)a^3 - (b^2 + c^2)a^2 - (0)a + b^2c^2 = (a-b)(a-c)(a+b)(a+c) = 0.]$$

PROPOSITION 38.

The root of the equation: *aaaa - baaa + cdfa = + bcdf.*
is *b,* equal to the root of *a* sought; *i. e. a = b.*
For if *a = b,* then :

bbbb - bbbb + cdfb = + cdfb.

The equalities can be seen. Hence, the root sought is *a = b,* as stated.

[Note for Prop. 38 : The 38th equation in modern terms is :
$a^4 - ba^3 - (0)a^2 + (cdf)a - bcdf = (a-b)(a^3 + cdf) = 0.$]

PROPOSITION 39.

The root of the equation: *aaaa + baaa - ccca = + bccc.*
is *c,* equal to the root of *a* sought; *i. e. a = c.*
For if *a = c,* then on changing *a* into *c* :

cccc + bccc - cccc = + bccc.

The equalities can be seen. Hence, the root sought is *a = c,* as stated.

[Note for Prop. 39 : The 39th equation in modern terms is :
$a^4 + ba^3 - (0)a^2 - c^3a - bc^3 = (a-c)(a^3 + (b+c)a^2 + c(b+c)a + bc^2) = 0.$]

PROPOSITION 40.

The roots of the equation: *aaaa - baaa - ccca = - bccc.*
are *b* and *c,* equal to the root of *a* sought; *i. e. a = b* or *a = c.*
For if *a = b,* then on changing *a* into *b* :

bbbb - bbbb - bccc= - bccc.

For which the truth of the equation is evident.
Hence, *a = c* satisfies the equation.
For if *a = c,* then on changing *a* into *c* :

cccc - bccc - cccc = - bccc.

For which the truth of the equation is evident.
Hence, the roots sought are *a = b* and *a = c ,* as stated.

[Note for Prop. 40 : The 40th equation in modern terms is :
$a^4 - ba^3 - (0)a^2 - c^3a + bc^3 = (a-b)(a-c)(a^2 + ca + c^2) = 0.$]