Common equations are to be reduced or parodised by the exclusion of some power, and by the change made of a substituted root.

The problem concerning the multiplication of the roots of reduced equations is set out in the present section, for equations that are suitable for this process. The root of a proposed equation, which is to be retained in the comparison, is to be multiplied by same given number.

Aequationum communium reductio per gradus alicuius parodici exclusionem & radicis supposititiae mutationem.
Problema de aequationum radicibus multiplicanda, aequationibus, quarum reductiones in praesenti Sectioni traduntur, reductioni praeparandis accomodum.
aequationis propositae, radicem servatã comparationis aequalitate, per quemcunque numerum datum multiplicare .

Let the quadratic equation be . . . $aa + ba = cc$

Therefore . . . $\begin{array}{r}1\\ aa\end{array} + \begin{array}{r}b\\ a\end{array} = \begin{array}{r}cc\\ 1\end{array}$

Because truly the root of the equation is to be doubled, in the first place, these three homogeneous [coefficients] are multiplied by the three proportional numbers in the duplicate ratio 1 : 2 : 4, in this order.

From which it follows . . . $\begin{array}{r}1\\ 1\\ aa\end{array} + \begin{array}{r}2\\ b\\ a\end{array} < \begin{array}{r}4\\ cc\\ 1\end{array}$,with the equality no longer true.

Therefore, for the equality to be again restored, the coefficients of the equation are multiplied by the same proportional numbers in reciprocal order, namely by 4: 2: 1.

Hence the equation becomes $\begin{array}{r}1\\ 1\\ aa\\ 4\end{array} + \begin{array}{r}2\\ b\\ a\\ 2\end{array} = \begin{array}{r}4\\ cc\\ 1\\ 1\end{array}$

For . . . $\begin{array}{r}1\\ 4\end{array} = \begin{array}{r}2\\ 2\end{array} = \begin{array}{r}4\\ 1\end{array}$ are equal.

Let . . . $2a = e$

Therefore . . . $\begin{array}{r}aa\\ 4\end{array}$ & $\begin{array}{r}a\\ 2\end{array}$ & $\begin{array}{r}1\\ 1\end{array} = ee$ & e & 1.

Therefore by substituting these in the place of the others, the equation become . . .

$$\begin{array}{r}1\\ 1\\ ee\end{array} + \begin{array}{r}2\\ b\\ e\end{array} = \begin{array}{r}4\\ cc\\ 1\end{array}$$

Therefore . . . $ee + 2.be = 4.cc$

Therefore the root a of the proposed equation has thus changed to twice the value $2a$ in e, while maintaining the equality; as was requested.

In which the root of a cubic equation is multiplied by three.

Let the cubic equation be . . . $aaa + baa + cca = ddd$

the root a of which is to be tripled.

Since . . . $aaa + baa + cca = ddd,$

this becomes . . .

$$\left.\begin{array}{r}1\\ \underline{aaa}\end{array}\right| + \left.\begin{array}{r}b\\ \underline{aa}\end{array}\right| + \left.\begin{array}{r}cc\\ \underline{a}\end{array}\right| = \left.\begin{array}{r}ddd.\\ \underline{1}\end{array}\right|$$

Since the root of the equation is to be tripled, in the first place these four coefficients are multiplied by 4 proportional numbers in the triplicate ratio 1: 3: 9: 27, in this order.

This becomes . . .

$$\left.\begin{array}{r}1\\ 1\\ \underline{aaa}\end{array}\right| + \left.\begin{array}{r}3\\ b\\ \underline{aa}\end{array}\right| + \left.\begin{array}{r}9\\ cc\\ \underline{a}\end{array}\right| < \left.\begin{array}{r}27\\ ddd\\ \underline{1}\end{array}\right|$$

with the equality no longer true.

Therefore for the equality to be again restored, the coefficients of the inequality are multiplied by the same proportional numbers in reciprocal order, to wit 27 : 9 : 3 : 1.

Hence the equation is : . .

$$\left.\begin{array}{r}1\\ 1\\ aaa\\ \underline{27}\end{array}\right| + \left.\begin{array}{r}3\\ b\\ aa\\ \underline{9}\end{array}\right| + \left.\begin{array}{r}9\\ cc\\ a\\ \underline{3}\end{array}\right| = \left.\begin{array}{r}27\\ ddd\\ 1\\ \underline{1}\end{array}\right|$$

For . . .

$$\left.\begin{array}{r}1\\ \underline{27}\end{array}\right| = \left.\begin{array}{r}3\\ \underline{9}\end{array}\right| = \left.\begin{array}{r}9\\ \underline{3}\end{array}\right| = \left.\begin{array}{r}27\\ \underline{1}\end{array}\right| \text{ are equal.}$$

Let . . . $3.a = e$

Therefore . . .

$$\left.\begin{array}{r}aaa\\ \underline{27}\end{array}\right| \;\&\; \left.\begin{array}{r}aa\\ \underline{9}\end{array}\right| \;\&\; \left.\begin{array}{r}a\\ \underline{3}\end{array}\right| \;\&\; \left.\begin{array}{r}1\\ \underline{1}\end{array}\right| = eee \;\&\; ee \;\&\; e \;\&\; 1.$$

Therefore by substituting these in place of the other terms, the equation becomes

$$\left.\begin{array}{r}1\\ 1\\ \underline{eee}\end{array}\right| + \left.\begin{array}{r}3\\ b\\ \underline{ee}\end{array}\right| + \left.\begin{array}{r}9\\ cc\\ \underline{e}\end{array}\right| = \left.\begin{array}{r}27\\ ddd\\ \underline{1}\end{array}\right|$$

Therefore the root a of the proposed equation has thus changed to the triple value $3a$ in e, while maintaining the equality; as was ordered.

Consequences.

For the multiplication of the root follows from the simple multiplication of the second coefficient, following in the same ratio, as in the above examples. Thus in the quadratic equation case, with the root a doubled, in like manner the coefficient b is twice the number, and in the cubic equation with the root a tripled, the coefficient likewise is three times the number. Indeed, if the problem concerning the multiplication of the coefficient can be understood and expressed, then conversely the notion of equipollence can be considered also. For by the multiplication of the coefficient by a number, the multiplication of the root is supposed. Moreover, the multiplication of the coefficient of the root is of consequence in the [final] resolution of equations, when the use of fractions is taken up; or in setting out how equations are to be solved, when the indivisibility of the coefficient of some number or power stands in the way: so beware of the implications of fractions [considered] of so great convenience, for by name only is the use of this particular art be seen.

[Note :Thoughts on the difficulty of finally solving cubic and perhaps quartic equations; however, the form of the equation can be changed after it has been reduced initially, for computational purposes.]

Consectarium.

Ex multiplicatione radicis sequitur multiplicatio coefficientis simplicis secundum eandem multiplicitatis rationem : ut in superioribus exemplis, in aequatione quadratica, radice a duplicata, coefficiens b per numerum item binarium, & in aequatione cubicae radice a triplacata, coefficiens b per numerum itidem ternarium multiplicatur. Adeo ut Problema si de coefficente multiplicando conceptum & enunctatum esset converso tantum sensu huic aequipollens foret. Nam coefficientis multiplicatio radicis multiplicationem praesupponit. Est autem coefficientis multiplicatio ista consequentialis in aequationibus resolvendis ad fractionis ubi opus fuerit tollendas, vel in aequationum reductionibus tractandis, ubi coefficientis numeri vel speciei indivisibilitas obstat, ad fractionum implicationes praecaueadas, tantae commoditatis, ut hoc solum nomine praecipuus huius artificii usus est videatur.

PROBLEM 1.

The equation $aaa - 3.baa = + ccc$, by placing $a = e + b$ is to be reduced to the equation $eee - 3.bbe = + ccc + 2.bbb$; or by placing $a = - e + b$ to the equation $eee - 3.bbe = - ccc - 2.bbb$ it is impossible to reduce the equation.

First by placing: . . . . . . $e + b = a.$

And . . . $ee + 2.eb + bb = aa.$

And . . . . $eee + 3.bee + 3.bbe + bbb = + aaa.$

Also . . . $- 3.bee - 6.bbe - 3.bbb = - 3.baa$ } $= + ccc.$

Thus by the rejection of contradictory [parts] and with the remaining terms in order then the equation is . . . . $eee - 3.bbe = + ccc + 2.bbb$;

the first equation required, of which the root $e = a - b.$

Secondly by placing: . . . . . . $- e + b = a.$

And . . . $ee - 2.eb + bb = aa.$

And . . . . $- eee + 3.bee - 3.bbe + bbb = + aaa.$

Also . . . $- 3.bbe + 6.bbe - 3.bbb = - 3.baa$ } $= + ccc.$

Thus by the rejection of contradictory [parts] the remaining order shall be . . . . $eee - 3.bbe = - ccc - 2.bbb$;

the second equation required which is shown to be impossible by the Lemma to follow.
And thus the reduction of the proposed equations to the required order is made.

Lemma.

The equation reduction $eee - 3.bbe = \begin{matrix} - ccc \\ -2.bbb \end{matrix}$ is impossible.

For by placing $e = b$, it becomes $bbb - 3.bbb = \begin{matrix} - ccc \\ -2.bbb. \end{matrix}$

Therefore . . . . ccc = 0, which is impossible.

Or by putting $c > b$; that is $c = b + d,$

then . . . . $\begin{matrix} + bbb \\ + 3.bbd \\ + 3.bdd - 3.bbb \\ + ddd - 3.bbd = - ccc - 2.bbb. \end{matrix}$

Therefore, with the contradictory [parts] separated; it is $\begin{matrix} + 3.bdd \\ ddd = - ccc. \end{matrix}$

Therefore . . . . . . $\begin{matrix} + 3.bdd \\ + ddd \\ + ccc = 0, \end{matrix}$ which is also impossible.

Or by putting $c < b$; that is $c = b - d,$

it will be . . . . $\begin{matrix} + bbb \\ - 3.bbd \\ + 3.bdd - 3.bbb \\ - ddd + 3.bbd = - ccc - 2.bbb. \end{matrix}$

Therefore, with the contradictory [parts] separated; it is $\begin{matrix} + ddd \\ - 3.bdd = + ccc. \end{matrix}$

But $+ ddd - 3.bdd = + d - 3.b \left| \underline{dd} \right|$

Therefore . . . ; $d > 3.b$. Therefore . . . . $d > b$.

But as $e = b - d$ is put in place, then $b > d,$ which is impossible too.

Therefore that equation is altogether impossible to be reduced.

[We are of course running up against the restriction set by Viete, and going back to the Greek masters, that only positive solutions of equations are to be considered.]

PROBLEM 2.

The equation $aaa + 3.baa = + ccc$, by putting $a = e - b,$ is to be reduced to the equation $eee - 3.bbe = \begin{matrix} + ccc \\ - 2.bbb. \end{matrix}$

Placing . . . . . . $e - b = a.$

Then . . . $eee - 3.bee + 3.bbe - bbb = + aaa.$

And . . . . . . $.ee - 2.be + bb = aa$

And hence . . . . . $+ 3.bbe - 6.bbe + 3.bbb = + 3.baa$ $\Big\}$ $= + ccc.$

Thus by the rejection of contradictory [parts] and with the remaining terms in order

there is . . . . $eee - 3.bbe = \begin{matrix} + ccc \\ - 2.bbb; \end{matrix}$

the equation required, of which the root $e = a - b.$

And thus the reduction of the proposed equation to the required order has been done.

PROBLEM 3.

The equation $aaa - 3.baa = - ccc$, by placing $a = b - e$, is to be reduced to the equation $eee - 3.bbe = + ccc - 2.bbb$; or by placing $a = e + b$ to the equation $eee - 3.bbe = - ccc + 2.bbb$ to be reduced.

First by placing: . . . . . . $b - e = a.$
Then we have. . . . . $- eee + 3.bee - 3.bbe + bbb = + aaa.$
And . . . $ee - 2.eb + bb = aa.$

And hence . . . . $- 3.bbe + 6.bbe - 3.bbb = - 3.baa = + ccc.$
Thus by the rejection of contradictory [parts] and with the remaining terms in order the equation is . . . . $eee - 3.bbe = + ccc - 2.bbb$;
the first equation required of which the root $e = b - a.$

Secondly by placing: . . . . . . $e + b = a.$
And it shall become . . . $eee + 3.bee + 3.bbe + bbb = + aaa.$
And . . . . $ee + 2.eb + bb = aa.$
And hence . . . . $- 3.bbe - 6.bbe - 3.bbb = - 3.baa$ $= + ccc.$
Thus by the rejection of contradictory [parts] the remaining order shall be . . . . $eee - 3.bbe = - ccc + 2.bbb$; the second equation required of which the root $e = a - b.$
And thus the required reduction to the other order has been done.

PROBLEM 4.

The equation $aaa + 3.baa + dda = + ccc$, by placing $a = e - b,$ is reduced to the equation

$$eee - 3.bbe + dde = + ccc - 2.bbb + bdd .$$

Placing . . . . . . $e - b = a.$
Then . . . $eee - 3.bee + 3.bbe - bbb = + aaa.$
And . . . . $ee - 2.be + bb = aa$
And hence . . . . . $+ 3.bbe - 6.bbe + 3.bbb = + 3.baa$
And . . . . $+ ddc - ddb = + dda$ $= + ccc.$
Hence it shall be . . . . $eee - 3.bbe + dde = + ccc - 2.bbb + bdd,$ the equation required, of which the root is $e = a - b.$
And thus the required reduction has been done.

PROBLEM 5.

The equation $aaa - 3.baa + dda = - ccc$, by placing $a = b - e$, is reduced to

the equation
$$\begin{array}{ll} eee - 3.bbe & \\ +dde & = + ccc \\ & - 2.bbb \\ & + bdd. \end{array}$$

Or by placing $a = e + b$, it is reduced to

the equation
$$\begin{array}{ll} eee - 3.bbe & \\ +dde & = - ccc \\ & + 2.bbb \\ & - bdd. \end{array}$$

First put . . . . . . $- e + b = a.$

And . . . . $ee - 2.be + bb = aa$

Then . . . $- eee + 3.bee - 3.bbe + bbb = + aaa.$

And . . . . $- 3.bbe + 6.bbe - 3.bbb = - 3.baa$

And . . . . $- ddc + bdd = + dda$ } $= + ccc.$

Hence by the rejection of contradictory parts and by transposition, the remaining order shall be

$$\begin{array}{ll} . \; . \; . \; . \; . \; . \; . \; eee - 3.bbe & \\ +dde & = + ccc \\ & - 2.bbb \\ & + bdd, \end{array}$$
the equation required, of which the root $e = b - a.$

In the second case . . . . . . $e + b = a.$

And . . . . $ee + 2.be + bb = aa$

Then . . . $eee + 3.bee + 3.bbe + bbb = + aaa.$

And . . . . $- 3.bbe - 6.bbe - 3.bbb = - 3.baa$

And . . . . $+ dde + bdd = + dda$ } $= - ccc.$

Hence by the rejection of contradictory [parts] the remaining order is,

$$\begin{array}{ll} . \; . \; . \; . \; . \; . \; . \; eee - 3.bbe & \\ +dde & = - ccc \\ & - 2.bbb \\ & - bdd, \end{array}$$
the second equation required, of which the root is $e = a - b.$

And thus the required reduction of the proposed equation to each has been done.

PROBLEM 6.

The equation $aaa + 3.baa - dda = + ccc$, by placing $a = e - b$ is reduced to

the equation
$$\begin{array}{ll} eee - 3.bbe & \\ - dde & = + ccc \\ & - 2.bbb \\ & - bdd . \end{array}$$

Placing . . . . . . $e - b = a.$

Then . . . . $ee - 2.be + bb = aa,$

and . . . $eee - 3.bee + 3.bbe - bbb = + aaa.$

And . . . . $3.bbe - 6.bbe + 3.bbb = + 3.baa$

and . . . . $- dde + bdd = - dda$ } $= + ccc.$

Hence by the rejection of contradictory [parts], and with the remaining in order, and by transposition

The equation becomes . . . .
$$\begin{array}{ll} eee - 3.bbe & \\ - dde & = + ccc \\ & - 2.bbb \\ & - bdd, \end{array}$$

the reduced equation of which the root is $e = a + b.$

And thus the ordered reduction has been done for the proposed equation.

PROBLEM 7.

The equation $aaa - 3.baa - dda = - ccc$, is to be reduced to the equation,

$$\begin{array}{l} eee - 3.bbe \\ \quad - dde = + ccc \\ \qquad - 2.bbb \\ \qquad - bdd, \end{array}$$

by placing $a = b - e$

or by placing $a = b + e$, to the equation

$$\begin{array}{l} eee - 3.bbe \\ \quad - dde = - ccc \\ \qquad + 2.bbb \\ \qquad + bdd . \end{array}$$

Placing . . . . $- e + b = a.$
Then . . . . $ee - 2.be + bb = aa,$
and . . . $- eee + 3.bee - 3.bbe + bbb = + aaa.$
And . . . . $- 3.bbe + 6.bbe - 3.bbb = - 3.baa,$
and . . . . $+ dde - bdd = - dda \quad = - ccc.$
Hence by the rejection of contradictory [parts] and with the remaining in order, and by transposition

It shall be . . . . $eee - 3.bbe$

$$\begin{array}{l} \quad - dde = + ccc \\ \qquad - 2.bbb \\ \qquad - bdd \end{array}$$

the first reduced equation of which the root $e = + b - a.$

Being placed . . . . $e + b = a.$
Then . . . . $ee + 2.be + bb = aa$
Then it will be . . . $eee + 3.bee + 3.bbe + bbb = + aaa.$
And . . . . $- 3.bbe - 6.bbe - 3.bbb = - 3.baa$
And . . . . $- dde - bdd = - dda \quad = - ccc.$
Hence by the rejection of contradictory [parts] and with the remaining in order,
It shall be . . . . $eee - 3.bbe$

$$\begin{array}{l} \quad - dde = - ccc \\ \qquad + 2.bbb \\ \qquad + bdd \end{array}$$

the second reduced equation of which the root $e = a - b.$

And thus the reduction ordered of the proposed equation has been done.

PROBLEM 8.

The equation $aaa - 3.baa - dda = + ccc$, is reduced to
the equation $eee - 3.bbe$

$$\begin{array}{l} \quad - dde = + ccc \\ \qquad + .bbb \\ \qquad + bdd, \end{array}$$

by placing $a = e + b,$

or to the equation $eee - 3.bbe$

$$\begin{array}{l} \quad - dde = + ccc \\ \qquad - 2.bbb \\ \qquad - bdd , \end{array}$$

by being placed $a = b - e.$

First put . . . . . . $e + b = a.$
Then . . . . $ee + 2.be + bb = aa$

$$\left.\begin{array}{ll} \text{hence} \ldots & eee + 3.bee + 3.bbe + bbb = + aaa. \\ \text{And} \ldots\ldots & - 3.bbe - 6.bbe - 3.bbb = - 3.baa \\ \text{And} \ldots\ldots & - dde - bdd = - dda \end{array}\right\} = + ccc.$$

Hence by the rejection of contradictory [parts] and with the remaining terms in order, The equation becomes . . . .

$$\begin{array}{lll} eee & -3.bbe & \\ & -dde & = +ccc \\ & & +2.bbb \\ & & +bdd, \end{array}$$

the first reduced equation of which the root $e = a - b$.

In the second place . . . . . . $-e + b = a$.

Then . . . . $ee - 2.be + bb = aa$

$$\left.\begin{array}{lr} \text{and} \ldots & -eee + 3.bee - 3.bbe + bbb = +aaa. \\ \text{And} \ldots & -3.bbe + 6.bbe - 3.bbb = -3.baa \\ \text{and} \ldots & +dde - bdd = -dda \end{array}\right\} = +ccc.$$

Hence by the rejection of contradictory [parts] and with the remaining terms in order, and by transposition the equation becomes . . . .

$$\begin{array}{lll} eee & -3.bbe & \\ & -dde & = -ccc \\ & & -2.bbb \\ & & -bdd, \end{array}$$

the second reduced equation of which the root is $e = b - a$.

And thus the reduction ordered of the proposed equation has been done .

PROBLEM 9.

The equation $aaa + 3.baa - dda = -ccc$, by placing $a = -e - b$, is reduced to the equation

$$\begin{array}{lll} eee & -3.bbe & \\ & -dde & = +ccc \\ & & +.bbb \\ & & +bdd, \end{array}$$

or by placing $a = e - b$ in the equation,

it is reduced to

$$\begin{array}{lll} eee & -3.bbe & \\ & -dde & = -ccc \\ & & -2.bbb \\ & & -bdd. \end{array}$$

First put . . . . . . $-e - b = a$.

Then . . . . $ee + 2.be + bb = aa$

$$\left.\begin{array}{lr} \text{Hence} \ldots & -eee - 3.bee - 3.bbe - bbb = +aaa. \\ \text{And} \ldots & +3.bee + 6.bbe + 3.bbb = +3.baa \\ \text{And} \ldots & +dde + bdd = -dda \end{array}\right\} = -ccc.$$

Hence by the rejection of contradictory [parts] and with the remaining terms ordered and transposed, the equation is . . . .

$$\begin{array}{lll} eee & -3.bbe & \\ & -dde & \left.\begin{array}{l} = +ccc \\ +2.bbb \\ +bdd, \end{array}\right\} \end{array}$$

the first reduced equation of which the root $e = -a - b$.

Secondly put . . . . . $e - b = a$.

Then . . . . $ee - 2.be + bb = aa$

$$\left.\begin{array}{lr} \text{Hence} \ldots & eee - 3.bee + 3.bbe - bbb = +aaa. \\ \text{And} \ldots & +3.bbe - 6.bbe + 3.bbb = +3.baa \\ \text{And} \ldots & -dde + bdd = -dda \end{array}\right\} = -ccc.$$

Hence by the rejection of contradictory [parts] and with the remaining terms in order, the equation is . . . .

$$\begin{array}{lll} eee & -3.bbe & \\ & -dde & = -ccc \\ & & -2.bbb \\ & & -bdd, \end{array}$$

the second reduced equation of which the root is $e = a + b$.

And thus the reduction ordered of the proposed equation has been done.

PROBLEM 10.

The equation $aaa - 3.baa + dda = + ccc$, is reduced the equation

$$\begin{aligned}&eee - 3.bbe\\ &\quad + dde = + ccc\\ &\qquad\qquad + .bbb\\ &\qquad\qquad - bdd,\end{aligned}$$ by placing $a = e + b$;

or is reduced to the equation

$$\begin{aligned}&eee - 3.bbe\\ &\quad + dde = - ccc\\ &\qquad\qquad - 2.bbb\\ &\qquad\qquad + bdd\end{aligned}$$,by setting $a = - e + b$.

In the first case. . . . . . $e + b = a.$
Then . . . . $ee + 2.be + bb = aa$

$$\left.\begin{aligned}\text{Hence . . . } & eee + 3.bee + 3.bbe + bbb = + aaa.\\ \text{And . . . . } & - 3.bee - 6.bbe - 3.bbb = - 3.baa\\ \text{And . . . . } & + dde + bdd = + dda\end{aligned}\right\} = + ccc.$$

Hence by the rejection of contradictory [parts] and with the remaining terms ordered,
the equation is . . . . $\begin{aligned}&eee - 3.bbe\\ &\quad + dde = + ccc\\ &\qquad\qquad + 2.bbb\\ &\qquad\qquad - ddb,\end{aligned}$

the first reduced equation of which the root $e = a - b$.
In the second case, put. . . . $- e + b = a.$
Then . . . . $ee - 2.be + bb = aa$

$$\left.\begin{aligned}\text{Hence . . . } & eee + 3.bee - 3.bbe + bbb = + aaa.\\ \text{And . . . . } & - 3.bbe + 6.bbe - 3.bbb = + 3.baa\\ \text{And . . . . } & - dde + bdd = + dda\end{aligned}\right\} = - ccc.$$

Hence by the rejection of contradictory [parts] and with the remaining terms in order and transposed,
the equation is . . . . $\begin{aligned}&eee - 3.bbe\\ &\quad + dde = - ccc\\ &\qquad\qquad - 2.bbb\\ &\qquad\qquad + bdd,\end{aligned}$

the second reduced equation of which the root is $a = b - e$.
And thus the reduction of the proposed equation ordered has been done.

PROBLEM 11.

The equation $aaa + 3.baa + dda = - ccc$, is reduced to the equation

$$\begin{aligned}&eee - 3.bbe\\ &\quad + dde = + ccc\\ &\qquad\qquad + .bbb\\ &\qquad\qquad - bdd,\end{aligned}$$ by placing $a =- e - b$, or to the equation

$$\begin{aligned}&eee - 3.bbe\\ &\quad + dde = - ccc\\ &\qquad\qquad - 2.bbb\\ &\qquad\qquad + bdd\end{aligned}$$, by placing $a = e - b$.

In the first place . . . . . . $- e - b = a.$
Then . . . . $ee + 2.be + bb = aa$

$$\left.\begin{aligned}\text{Hence . . . } & - eee - 3.bee - 3.bbe - bbb = + aaa.\\ \text{And . . . . } & 3.bee + 6.bbe + 3.bbb = + 3.baa\\ \text{And . . . . } & - dde - bdd = + dda\end{aligned}\right\} = - ccc.$$

Hence by the rejection of contradictory [parts] and with the remaining terms ordered,
the equation is . . . .

$$\begin{array}{lll} eee & -3.bbe & \\ & +dde & = +ccc \\ & & +2.bbb \\ & & -ddb, \end{array}$$

the first reduced equation of which the root $e = -a - b$.
In the second case. . . . . . $e - b = a$.
Then . . . . $ee - 2.be + bb = aa$
Hence . . . $eee - 3.bee + 3.bbe - bbb = +aaa$.
And . . . . $+3.bbe - 6.bbe + 3.bbb = +3.baa$
And . . . . $+dde - bdd = +dda$ $\Big\}$ $= -ccc$.
Hence by the rejection of contradictory [parts] and with the remaining terms in order,
the equation is . . . . $eee - 3.bbe$

$$\begin{array}{lll} & +dde & = -ccc \\ & & -2.bbb \\ & & +bdd, \end{array}$$

the second reduced equation of which the root is $a = e + b$.
And thus the ordered reduction of the proposed equation has been done.

PROBLEM 12.

The equation $aaa + 3.bba = 2.ccc$, by placing $a = \frac{ee - bb}{e}$ is reduced to the simple equation
$eee = +ccc + \sqrt{cccccc + bbbbbb}$.

Three continued proportionals are set down $e : b : \frac{b^2}{e}$.
And let the root of the proposed equation a be from the substitution of the extremes of the equality, namely

$$a = e - \frac{bb}{e} = \frac{ee - bb}{e}.$$

The equation is :

$$\left.\begin{array}{l} \dfrac{\begin{array}{cccc} +ee & -3.bb & +3.bb & bb \\ ee & ee & bb & bb \\ ee & ee & ee & bb \end{array}}{eee} = +aaa \\[2ex] \text{And} \;\; \dfrac{\begin{array}{cc} +3.bb & -3.bb \\ ee & bb \\ ee & ee \end{array}}{eee} = +3.bba \end{array}\right\} = 2.ccc$$

Therefore by the rejection of contradictory parts, and with the remaining terms ordered,
the equation is . . . . . $+eeeeee - bbbbbb = +2.ccceee$.
Therefore . . . . . $+eeeeee - 2.ccceee = +bbbbbb$.
Therefore . . . . . $eee = +ccc + \sqrt{cccccc + bbbbbb}$.
Which is the prescribed equation.

So therefore the reduction ordered of the equation to that prescribed has been done.

Consequences.

From that reduction of the proposed equation *aaa* + *3.baa* = *2.ccc*, the resolution of the equation itself can fortunately be deduced.

For $\overbrace{ccc + \sqrt{cccccc + bbbbbb}}^{I}$, $\overbrace{bbb}^{II}$, $\overbrace{-ccc + \sqrt{cccccc + bbbbbb}}^{III}$ are continued proportionals

hence also

$\overbrace{\sqrt{ccc + \sqrt{cccccc + bbbbbb}}}^{I}$, $\overbrace{b}^{II}$, $\overbrace{\sqrt{-ccc + \sqrt{cccccc + bbbbbb}}}^{III}$.

But . . . . . $\sqrt{ccc + \sqrt{cccccc + bbbbbb}} = e.$

Therefore . . . . . $\sqrt{-ccc + \sqrt{cccccc + bbbbbb}} = \frac{bb}{e}$.

Therefore . . . . . $\sqrt{ccc + \sqrt{cccccc + bbbbbb}} - \sqrt{-ccc + \sqrt{cccccc + bbbbbb}} = a.$

Therefore that binomial of the roots is involved with the binomials in the explanation of the proposed equation, which had to be shown.

[The second set of continued proportionalities follows from the first by taking the cube root: there is no distinction here between the cube root sign and the square root sign, you have to work out which one it is for consistency. In addition, as there are no brackets, you have to ascertain from the powers how far the root extends. It is hence worthwhile to consider the resolution in modern terms:

The reduced cubic equation is : $a^3 + 3b^2a = 2c^3$; this is solved by substituting $a = \frac{e^2 - b^2}{e} = e - b^2/e$.

The equation becomes: $(e - b^2/e)^3 + 3b^2(e - b^2/e) = 2c^3$; on expansion this becomes :

$e^3 - 3eb^2 + 3b^4/e - b^6/e^3 + 3b^2e - 3b^4/e = 2c^3$; or

$e^3 - b^6/e^3 = 2c^3$; or $(e^3)^2 - 2c^3e^3 - b^6 = 0.$

This gives :

$e^3 = c^3 \pm \sqrt{c^6 + b^6}$ and $e = (c^3 \pm \sqrt{c^6 + b^6})^{1/3}$

and $a = e - b^2/e = (c^3 \pm \sqrt{c^6 + b^6})^{1/3} - b^2/(c^3 \pm \sqrt{c^6 + b^6})^{1/3}$.]

Examples of the resolution with numbers.

20 = 6.*a* + *aaa* . . *a* = √ 3.)√108 + 10 - √ 3.)√ 108 - 10. = 2.

26 = 9.*a* + *aaa* . . *a* = √ 3.)√196 + 13 - √ 3.)√ 196 - 13. = 2.

7 = 6.*a* + *aaa* . . *a* = √ 3.) $\sqrt{\frac{81}{4}} + \frac{7}{2}$ - √ 3.)√ 3. $\frac{81}{4} - \frac{7}{2}$ = 1.

[Problem 12 amounts to showing that the reduced cubic (in the modern sense) $a^3 + 3b^2a = 2c^3$ can be resolved into the simple equation $e^3 = c^3 + \sqrt{(c^6 + b^6)}$, by making the substitution $a = e - b^2/e$: this is a form of Viete's substitution for solving reduced cubics, where the positive root only is considered.

The first example has $b^2 = 2$, and $a = e - 2/e$, giving $(e^3)^2 - 20e^3 - 8 = =$, giving $e^3 = 10 + \sqrt{108}$. In which case a = $\{10 + \sqrt{108}\}^{1/3} - 2/\{10 + \sqrt{108}\}^{1/3} = \{10 + \sqrt{108}\}^{1/3} - \{-10 + \sqrt{108}\}^{1/3} = 2$; in this case √ 3.) indicates the cube root of what follows. Likewise with the other two examples.]

PROBLEM 13.

The equation *aaa* - *3.baa* = *2.ccc*, by placing $a = \frac{ee + bb}{e}$ if *c* is greater than *b*, is reduced to the simple equation *eee* = + *ccc* + *ddd.* If *c* = *b*, in like manner to the simple equation *eee* = *ccc*. If however, *c* is less than *b*, then it is reduced to the equation *eee* = *ccc* + √-*dddddd,* which is impossible to be reduced.

Three continued proportionals are set down $e : b : \frac{b^2}{e}$.

And let the root a of the proposed equation be equal to the sum of the extremes, thus $a = e + \frac{b^2}{e}$, that is

$a = \frac{ee + bb}{e}$

Then the equation is :

$$\frac{+\,ee.ee.ee \;\; +3.bb.ee.ee \;\; +3.bb.bb.ee \;\; bb.bb.bb}{eee} = +aaa$$

And . . . .
$$\frac{-3.bb.ee.ee \;\; +3.bb.bb.ee}{eee} = +3.bba$$
$$\left.\begin{array}{l} +aaa \\ +3.bba \end{array}\right\} = 2.ccc$$

Therefore with the rejection of contradictory parts, and with the rest ordered,
the equation is . . . . . $+ eeeeee + bbbbbb = +2.ccceee.$
Therefore . . . . . $+ eeeeee - 2.ccceee + cccccc = + cccccc - bbbbbb.$
Therefore . . . . . $eee = + ccc + \sqrt{cccccc - bbbbbb}.$
If now in the first case by hypothesis c is greater than b, put
$cccccc - bbbbbb = dddddd$. Then $eee = ccc + \sqrt{dddddd}$, that is
$eee = ccc + ddd$. Hence, the equation in the first case has been prescribed.
But if in the second case by hypothesis b is equal to c, then
$cccccc - bbbbbb = 0.$
Therefore . . . $eee = ccc$, the prescribed equation of the second case.
If indeed, in the third case by hypothesis c is larger than b, placing
$cccccc - bbbbbb = -dddddd.$
Then . . . . $eee = ccc + \sqrt{-dddddd}$, the prescribed equation of the third impossible case (on account of $\sqrt{-dddddd}$).
So therefore the reductions ordered of the equations to those prescribed have been done.

Consequences of the first case.

From that reduction of the proposed equation $aaa - 3.baa = + 2.ccc$, the reduction itself is also deduced.

For $\overbrace{ccc + \sqrt{cccccc - bbbbbb}}^{\text{I}},\ \overbrace{bbb}^{\text{II}},\ \overbrace{ccc - \sqrt{cccccc - bbbbbb}}^{\text{III}}$ are continued proportionals

hence also

$$\overbrace{\sqrt{ccc + \sqrt{cccccc - bbbbbb}}}^{\text{I}},\quad \overbrace{b}^{\text{II}},\quad \overbrace{\sqrt{ccc - \sqrt{cccccc - bbbbbb}}}^{\text{III}}.$$

But . . . . . $\sqrt{ccc + \sqrt{cccccc - bbbbbb}} = e.$

Therefore . . . . . $\sqrt{ccc - \sqrt{cccccc - bbbbbb}} = \frac{bb}{e}$.

Therefore . . . . . $\sqrt{ccc + \sqrt{cccccc - bbbbbb}} + \sqrt{ccc - \sqrt{cccccc - bbbbbb}} = a.$

Therefore the binomial of these roots is involved with the roots of the required binomial, and the root of the proposed equation are set out, which was to be shown.

Examples of the resolution with numbers.

$40 = -6.a + aaa$ . . . $a = \sqrt{}\, 3.)20 + \sqrt{}.392$ ...+... $\sqrt{}\, 3.)20 - \sqrt{}.392 = 4.$
$72 = -24.a + aaa$ . . . $a = \sqrt{}\, 3.)36 + \sqrt{}.784$ ...+... $\sqrt{}\, 3.)36 - \sqrt{}.784 = 6.$
$9 = -6.a + aaa$ . . . . $a = \sqrt{3)}\,\frac{9}{2} + \sqrt{.\frac{49}{4}} \ldots + \ldots \sqrt{3)}\,\frac{9}{2} - \sqrt{.\frac{49}{4}} = 3.$

[The first example has $b^2 = 2$, and $a = e + 2/e$, giving $(e^3)^2 - 40e^3 + 8 = =$, giving $e^3 = 20 + \sqrt{392}$. In which case a = $\{20 + \sqrt{392}\}^{1/3} + 2/\{20 + \sqrt{392}\}^{1/3} = \{20 + \sqrt{392}\}^{1/3} + \{20 - \sqrt{392}\}^{1/3} = 4$; there are typographical errors in the second (27 instead of 72) and third examples (81 instead of 49), which have been corrected, and which then give the corresponding results].

Note 1.

The equation *aaa - 3.baa = + 2.ccc,* on account of the similarity between these three cases considered above, and the conic sections the hyperbola, the parabola, and the ellipse, for the three differences that occur, the excess, equality and deficiency, can be called by similar names, that is hyperbolic, parabolic, and elliptic. Hyperbolic is the case in which *c* is greater than *b*; parabolic in which *c* is equal to *b*; and elliptic the case in which *c* is less than *b*, and that on account of the incalculable kind of the irresolvable root.

Note 2.

In the two preceding equations, it sometimes occurred that the solution of the binomial cubic with involved roots could likewise be explained by binomial roots, which in turn can be explained finally in terms a simple root of an equation, constituting a sum or difference. The examples which follow are of this kind of solution. [This is accomplished by assuming the argument of the cube root to be of the form $(a + b\sqrt{n})^3$, and solving for integers *a* and *b*].

$52 = -3a + aaa$ . . . $a = 4.$

$a = \underbrace{\sqrt{}\, 3.)26 + \sqrt{}.675}_{2 + \sqrt{}.3}$ ...+... $\underbrace{\sqrt{}\, 3.)26 - \sqrt{}.675}_{2 - \sqrt{}.3} = 4.$

$\underbrace{2 + \sqrt{}.3 \quad + \quad 2 - \sqrt{}.3}_{4.}$

$276 = +3a + aaa$ . . . $a = 6.$

$a = \underbrace{\sqrt{}\, 3.)\sqrt{}.18252 + 135}_{\sqrt{}.12 + 3}$ ...-... $\underbrace{\sqrt{}\, 3.)\sqrt{}.18252 - 135}_{\sqrt{}.12 - 3} = 4.$

$\underbrace{\sqrt{}.12 + 3 \quad - \quad \sqrt{}.12 - 3}_{6.}$

$40 = -6a + aaa$ . . . $a = 4.$

$a = \underbrace{\sqrt{}\, 3.)20 + \sqrt{}.392}_{2 + \sqrt{}.2}$ ...+... $\underbrace{\sqrt{}\, 3.)20 - \sqrt{}.392}_{2 - \sqrt{}.2} = 4.$

$\underbrace{2 + \sqrt{}.2 \quad + \quad 2 - \sqrt{}.2}_{4.}$

$20 = +6a + aaa$ . . . $a = 2$.
$a = \sqrt{3.)}\sqrt{.108 + 10}$ ...-... $\sqrt{3.)}\sqrt{.108 - 10}$.

$\sqrt{.3} + 1$ - $\sqrt{.3} - 1$

2.

$\sqrt{.21632} = -6a + aaa$ . . . $a = \sqrt{.32}$.
$a = \sqrt{3.)}\sqrt{.5408} + \sqrt{.5400}$ ... + ...$\sqrt{3.)}\sqrt{.5408} - \sqrt{.5400}$

$\sqrt{.8} + \sqrt{.6}$ ... + ... $\sqrt{.8} - \sqrt{.6}$

$\sqrt{.32}$.

$\sqrt{.248832} = +24a + aaa$ . . . $a = \sqrt{.48}$.
$a = \sqrt{3.)}\sqrt{.62720} + \sqrt{.62208}$ - $\sqrt{3.)}\sqrt{62720} - \sqrt{.62208}$
$\sqrt{.20} + \sqrt{.12}$ ... - ... $\sqrt{.20} - \sqrt{.12}$

$\sqrt{.48}$.

PROBLEM 14.

The equation *aaaa + 4.baaa = + cccc*, by placing *a = e - b* can be reduced to
the equation *eeee - 6.bbee + 8.bbbe = + .cccc*
+ 3.bbbb.

Placing . . . . . . *e - b = a.*
Then . . . . *eee - 3.bee + 3.bbe - bbb = aaa.*
And the equation becomes . . *eeee - 4.beee + 6.bbee - 4.bbbe + bbbb = + aaaa*
And . . . . *+ 4.beee - 12.bbee + 12.bbbe - 4.bbbb = +4.baaa* } *= + cccc.*
Thus by the rejection of contradictory [parts] and with the remaining terms in order
the equation is . . . . *eeee - 6.bbee + 8.bbbe = + .cccc*
+ 3.bbbb.
The equation required of which the root *e = a + b.*
And thus the reduction to the required order has been done.

PROBLEM 15.

The equation *aaaa + 4.baaa = + cccc*, by placing *a = e + b* can be reduced to
the equation *eeee - 6.bbee - 8.bbbe = + cccc*
+ 3.bbbb; or by placing *a = - e + b*, can be reduced to the equation
eeee - 6.bbee + 8.bbbe = + cccc
+ 3.bbbb.

First by placing: . . . . . . *e + b = a.*
Then . . . . *eee + 3.bee + 3.bbe + bbb = + aaa.*

And it shall become . . . $eeee + 4.beee + 6.bbee + 4.bbbe + bbbb = + aaaa$
And . . . . $-4.beee - 12.bbee - 12.bbe - 4.bbbb = - 4.baaa$ $= + ccc.$
Thus by the rejection of contradictory [parts] and with the remaining terms in order
it shall be . . . . $eeee - 6.bbee - 8.bbbe = + cccc$
$+ 3.bbb$;
the first equation required of which the root $e = a - b.$

Secondly by placing: . . . . . . $- e + b = a.$
And . . . $-eee + 3.bee - 3.bbe + bbb = + aaa.$
And hence . . . $eeee - 4.beee + 6.bbee - 4.bbbe + bbbb = + aaaa.$
And . . . . $-4.beee - 12.bbee + 12.bbe - 4.bbbb = - 4.baaa$ $= + ccc.$
Thus by the rejection of contradictory [parts] and with the remaining terms in order
the equation is . . . . $eeee - 6.bbee + 8.bbbe = + cccc$
$+ 3.bbb$;
the first equation required of which the root $e = + b - a.$
And thus the reduction of the proposed equation ordered has been done as required.

PROBLEM 16.

The equation $aaaa - 4.baaa = - cccc$, by placing $a = e + b$ can be reduced to
the equation $eeee - 6.bbee - 8.bbbe = + cccc$
$+ 3.bbbb;$ or by placing $a = - e + b,$ can be reduced to the equation
$eeee - 6.bbee + 8.bbbe = - cccc$
$+ 3.bbbb.$

First by placing: . . . . . . $e + b = a.$
Then . . . . $eee + 3.bee + 3.bbe + bbb = + aaa.$
And it shall become . . . $eeee + 4.beee + 6.bbee + 4.bbbe + bbbb = + aaaa$
And . . . . $-4.beee - 12.bbee - 12.bbe - 4.bbbb = - 4.baaa$ $= - ccc.$
Thus by the rejection of contradictory [parts] and with the remaining terms in order :
The equation is . . . . $eeee - 6.bbee - 8.bbbe = - cccc$
$+ 3.bbb$;
for the first equation required with the root $e = a - b.$

Secondly by placing: . . . . . . $- e + b = a.$
Then . . . $-eee + 3.bee - 3.bbe + bbb = + aaa.$
And . . . $eeee - 4.beee + 6.bbee - 4.bbbe + bbbb = + aaaa.$
And . . . . $+4.beee - 12.bbee + 12.bbe - 4.bbbb = - 4.baaa$ $= -cccc.$
Thus by the rejection of contradictory [parts] and with the remaining terms in order
The equation is . . . . $eeee - 6.bbee + 8.bbbe = - cccc$
$+ 3.bbb$;
the second equation required with the root $e = b - a.$
And thus the reduction ordered of the proposed equation has been done as required.

PROBLEM 17.

The equation $aaaa + 4.baaa = - cccc$, by placing $a = e - b$ can be reduced to
the equation $eeee - 6.bbee + 8.bbbe = - cccc$
$+ 3.bbbb$
First by placing: . . . . . . $e - b = a.$
Then . . . . $eee - 3.bee + 3.bbe - bbb = + aaa.$
And it shall become . . . $eeee - 4.beee + 6.bbee - 4.bbbe + bbbb = + aaaa$
And . . . . $+ 4.beee - 12.bbee - 12.bbe - 4.bbbb = - 4.baaa$ $= - ccc.$
Thus by the rejection of contradictory [parts] and with the remaining terms in order
The equation is . . . . $eeee - 6.bbee + 8.bbbe = - cccc$
$+ 3.bbb$;

the equation required with the root $e = a + b$.
And thus the reduction ordered has been done.

PROBLEM 18.

The equation $aaaa + 4.baaa + ddda = + cccc$, by placing $a = e - b$, can be reduced to the equation

$$eeee - 6.bbee + 8.bbbe + ddde = \left.\begin{array}{l} - cccc \\ + 3.bbbb \\ + bddd. \end{array}\right\}$$

By placing . . . . . . $e - b = a$.
Then . . . . $eee - 3.bee + 3.bbe - bbb = + aaa$.

$$\left.\begin{array}{lr} \text{And it shall become . . .} & eeee - 4.beee + 6.bbee - 4.bbbe + bbbb = + aaaa \\ \text{And . . . .} & + 4.beee - 12.bbee + 12.bbe - 4.bbbb = + 4.baaa \\ \text{And . . . .} & + ddde - bddd = + ddda \end{array}\right\} = + cccc.$$

Thus by the rejection of contradictory [parts] and with the remaining terms in order
The equation is . . . .

$$eeee - 6.bbee + 8.bbbe + ddde = \left.\begin{array}{l} - cccc \\ + 3.bbbb \\ + bddd; \end{array}\right\}$$

the equation required with the root $e = a + b$.
And thus the reduction ordered has been done.

PROBLEM 19.

The equation $aaaa - 4.baaa - ddda = + cccc$, by placing $a = -e + b$, can be reduced to the equation

$$eeee - 6.bbee + 8.bbbe + ddde = \left.\begin{array}{l} + cccc \\ + 3.bbbb \\ + ddde; \end{array}\right\}$$

or by placing $a = e + b$, can be reduced to the equation

$$eeee - 6.bbee - 8.bbbe - ddde = \left.\begin{array}{l} - cccc \\ + 3.bbbb \\ + bddd. \end{array}\right\}$$

First by placing: . . . . . . $- e + b = a$.
Then . . . . $- eee + 3.bee - 3.bbe + bbb = + aaa$.

$$\left.\begin{array}{lr} \text{And it shall become . . .} & eeee - 4.beee + 6.bbee - 4.bbbe + bbbb = + aaaa \\ \text{And . . . .} & + 4.beee - 12.bbee + 12.bbe - 4.bbbb = - 4.baaa \\ \text{And . . . .} & + ddde - bddd = - ddda \end{array}\right\} = + cccc.$$

Thus by the rejection of contradictory [parts] and with the remaining terms in order
The equation is . . . .

$$eeee - 6.bbee + 8.bbbe + ddde = \left.\begin{array}{l} - cccc \\ + 3.bbb \\ + bddd; \end{array}\right\}$$

the first equation required with the root $e = - a - b$.
Secondly by placing: . . . . . . $e + b = a$.
Then . . . $eee + 3.bee + 3.bbe + bbb = + aaa$.

$$\left.\begin{array}{lr} \text{And it shall become . . .} & eeee + 4.beee + 6.bbee + 4.bbbe + bbbb = + aaaa. \\ \text{And . . . .} & - 4.beee - 12.bbee - 12.bbe - 4.bbbb = - 4.baaa \\ \text{And . . . .} & - ddde - bddd = - ddda \end{array}\right\} = + cccc.$$

Thus by the rejection of contradictory [parts] and with the remaining terms in order
The equation is . . . .

$$eeee - 6.bbee - 8.bbbe - ddde = \left.\begin{array}{l} - cccc \\ + 3.bbbb \\ + bddd; \end{array}\right\}$$

the second equation required with the root $e = a - b$.
And thus the reduction ordered of the proposed equation has been done as required.

PROBLEM 20.

The equation $aaaa + 4.baaa + ffaa = + cccc$, by placing $a = e - b$, can be reduced to the equation

$$\begin{array}{l} eeee - 6.bbee + 8.bbbe \\ \quad + ffee - 2.bffe = \left.\begin{array}{l} - cccc \\ + 3.bbbb \\ - bbff. \end{array}\right\} \end{array}$$

Placing . . . . . . $e - b = a$.
Then . . . . $eee - 3.bee + 3.bbe - bbb = + aaa$.

$$\left.\begin{array}{lr} \text{And it shall become . . .} & eeee - 4.beee + 6.bbee - 4.bbbe + bbbb = + aaaa \\ \text{And . . . .} & + 4.beee - 12.bbee + 12.bbe - 4.bbbb = + 4.baaa \\ \text{And . . . .} & + ffee + 2.bffe + bbff = + ffaa \end{array}\right\} = + cccc.$$

Thus by the rejection of contradictory [parts] and with the remaining terms in order
The equation is . . . .

$$\begin{array}{l} eeee - 6.bbee + 8.bbbe \\ \quad + ffee - 2.bffe = \left.\begin{array}{l} + cccc \\ + 3.bbbb \\ - bbff; \end{array}\right\} \end{array}$$

the equation required with the root $e = a + b$.
And thus the reduction ordered has been done.

PROBLEM 21.

The equation $aaaa - 4.baaa + ffaa = + cccc$, by placing $a = -e + b$, can be reduced to the equation

$$\begin{array}{l} eeee - 6.bbee + 8.bbbe \\ \quad + ffee - 2.bffe = \left.\begin{array}{l} + cccc \\ + 3.bbbb \\ - bbff; \end{array}\right\} \end{array}$$

or by placing $a = e + b$, can be reduced to the equation

$$\begin{array}{l} eeee - 6.bbee - 8.bbbe \\ \quad + ffee + 2.bbff = \left.\begin{array}{l} + cccc \\ + 3.bbbb \\ - bbff . \end{array}\right\} \end{array}$$

First by placing: . . . . . . $- e + b = a$.
Then . . . . $ee - 2.eb + bb = aa$
And . . . . $- eee + 3.bee - 3.bbe + bbb = + aaa$.

$$\left.\begin{array}{lr} \text{And it shall become . . .} & eeee - 4.beee + 6.bbee - 4.bbbe + bbbb = + aaaa \\ \text{And . . . .} & + 4.beee - 12.bbee + 12.bbe - 4.bbbb = - 4.baaa \\ \text{And . . . .} & + ffee - 2.bffe + bbff = + ffaa \end{array}\right\} = + cccc.$$

Thus by the rejection of contradictory [parts] and with the remaining terms in order
The equation is . . . . $eeee - 6.bbee + 8.bbbe$

$$+ ffee - 2.bffe = \left.\begin{array}{l} + cccc \\ + 3.bbbb \\ - bbff; \end{array}\right\}$$

the first equation required with the root $e = - a + b$.

Secondly by placing: . . . . . . $e + b = a$.
Then . . . $eee + 3.bee + 3.bbe + bbb = + aaa$.
And it shall become . . . $eeee + 4.beee + 6.bbee + 4.bbbe + bbbb = + aaaa$.
And . . . . $- 4.beee - 12.bbee - 12.bbe - 4.bbbb = - 4.baaa$
And . . . . $+ ffee + 2.bffe + bbff = + ffaa = + cccc$.

Thus by the rejection of contradictory [parts] and with the remaining in order
The equation is . . . . $eeee - 6.bbee - 8.bbbe - ddde = + cccc + 3.bbbb - bbff$;
the second equation required with the root $e = a - b$.
And thus the reduction ordered of the proposed equation has been done as required.

PROBLEM 22.

The equation $aaaa + 4.baaa + ffaa + ddda = + cccc$, by placing $a = e - b$, can be reduced to the equation

$$eeee - 6.bbee + 8.bbbe + ffee - 2.bffe + ddde = + cccc + 3.bbbb - bbff + bddd$$

Placing . . . . . . $e - b = a$.
Then . . . . $eee - 3.bee + 3.bbe - bbb = + aaa$.
And it shall become . . . $eeee - 4.beee + 6.bbee - 4.bbbe + bbbb = + aaaa$
And . . . . $+ 4.beee - 12.bbee + 12.bbe - 4.bbbb = + 4.baaa$
And . . . . $+ ffee - 2.bffe + bbff = + ffaa$
And . . . . $+ ddde - bddd = + ddda$ } $= + cccc$.

Thus by the rejection of contradictory [parts] and with the remaining terms in order
The equation is . . . .

$$eeee - 6.bbee + 8.bbbe + ffee - 2.bffe + ddde = + cccc + 3.bbbb - bbff + bddd;$$

the equation required with the root $e = a + b$.
And thus the reduction ordered has been done.

PROBLEM 23.

The equation $aaaa - 4.baaa + ffaa - dda = + cccc$, by placing $a = -e + b$, can be reduced to the equation

$$eeee - 6.bbee + 8.bbbe + ffee - 2.bffe + ddde = + cccc + 3.bbbb - bbff + bddd,$$

or by placing $a = e + b$ can be reduced to the equation

$$eeee - 6.bbee - 8.bbbe + ffee + 2.bbfe - ddde = + cccc + 3.bbbb - bbff + bddd$$

First by placing: . . . . $- e + b = a$.
Then . . . . $ee - 2.eb + bb = aa$
And . . . . $- eee + 3.bee - 3.bbe + bbb = + aaa$.

And it shall become . . . $eeee - 4.beee + 6.bbee - 4.bbbe + bbbb = + aaaa$
And . . . . $+4.beee - 12.bbee + 12.bbe - 4.bbbb = - 4.baaa$
And . . . . $+ ffee - 2.bffe + bbff = + ffaa$
And . . . . $+ ddde - bddd = - ddda$ $\Big\} = + cccc.$
Thus by the rejection of contradictory [parts] and with the remaining terms in order
The equation is . . . . $eeee - 6.bbee + 8.bbbe$
$+ ffee - 2.bffe$
$+ ddde = + cccc$
$+ 3.bbbb$
$- bbff$
$+ bddd,$

the first equation required with the root $e = - a + b$.

Secondly by placing: . . . . . . $e + b = a.$
Then . . . $eee + 3.bee + 3.bbe + bbb = + aaa.$
And it shall become . . . $eeee + 4.beee + 6.bbee + 4.bbbe + bbbb = + aaaa.$
And . . . . $- 4.beee - 12.bbee - 12.bbe - 4.bbbb = - 4.baaa$
And . . . . $- ddde - bddd = - ddda$ $\Big\} = + cccc.$
Thus by the rejection of contradictory [parts] and with the remaining terms in order
The equation is . . . . $eeee - 6.bbee - 8.bbbe$
$+ ffee + 2.bbfe$
$- ddde = + cccc$
$+ 3.bbbb$
$- bbff$
$+ bddd$
the second equation required with the root $e = a - b.$
And thus the reduction ordered of the proposed equation has been done as required.

PROBLEM 24.

The equation $aaaa - 4.baaa + ffaa - ddda = - cccc$, by placing $a = e + b$, can be reduced to the equation $eeee - 6.bbee - 8.bbbe$
$+ ffee + 2.ffbe$
$- ddde = - cccc$
$+ 3.bbbb$
$- ffbb$
$+ dddb,$
or by placing $a = - e + b$, can be reduced to the equation
$eeee - 6.bbee - 8.bbbe$
$+ ffee - 2.ffbe$
$+ ddde = - cccc$
$+ 3.bbbb$
$- bbff$
$+ bddd$.

Of these the reduction is similar to the preceding.

PROBLEM 25.

The equation $aaaa + 4.baaa - ffaa + ddda = + cccc$, by placing $a = e - b$, can be reduced to the equation

$$\begin{aligned} eeee - 6.bbee + 8.bbbe & \\ - ffee + 2.bffe & \\ + ddde = & \left.\begin{aligned} + cccc \\ + 3.bbbb \\ + ffbb \\ + bddd \end{aligned}\right\} \end{aligned}$$

Placing . . . . . . $+ e - b = a.$
Then . . . . $ee - 2.be + bb = aa$
And . . . . $eee - 3.bee + 3.bbe - bbb = + aaa.$

$$\left.\begin{aligned} \text{And it shall become} \quad eeee - 4.beee + 6.bbee - 4.bbbe + bbbb &= + aaaa \\ \text{And} \quad + 4.beee - 12.bbee + 12.bbe - 4.bbbb &= + 4.baaa \\ \text{And} \quad - ffee + 2.bffe - bbff &= - ffaa \\ \text{And} \quad + ddde - bddd &= + ddda \end{aligned}\right\} = + cccc.$$

Thus by the rejection of contradictory [parts] and with the remaining terms in order
The equation is . . . .

$$\begin{aligned} eeee - 6.bbee + 8.bbbe & \\ - ffee + 2.bffe & \\ + ddde = & \left.\begin{aligned} + cccc \\ + 3.bbbb \\ + ffbb \\ + dddb; \end{aligned}\right\} \end{aligned}$$

the equation prescribed with the root $e = a + b.$
And thus the reduction ordered of the proposed equation has been done.

PROBLEM 26.

The equation $aaaa + 4.baaa + ffaa - ddda = + cccc$, by placing $a = e - b$, can be reduced to the equation

$$\begin{aligned} eeee - 6.bbee + 8.bbbe & \\ + ffee - 2.bffe & \\ - ddde = & \left.\begin{aligned} + cccc \\ + 3.bbbb \\ - ffbb \\ - dddb \end{aligned}\right\} \end{aligned}$$

Placing . . . . . . $+ e - b = a.$
Then . . . . $ee - 2.be + bb = aa$
And . . . . $eee - 3.bee + 3.bbe - bbb = + aaa.$

$$\left.\begin{aligned} \text{And it shall become} \quad eeee - 4.beee + 6.bbee - 4.bbbe + bbbb &= + aaaa \\ \text{And} \quad + 4.beee - 12.bbee + 12.bbe - 4.bbbb &= + 4.baaa \\ \text{And} \quad + ffee - 2.bffe + bbff &= + ffaa \\ \text{And} \quad - ddde + bddd &= - ddda \end{aligned}\right\} = + cccc.$$

Thus truly by the rejection of contradictory [parts] and with the remaining terms in order
The equation is . . . .

$$\begin{aligned} eeee - 6.bbee + 8.bbbe & \\ + ffee - 2.bffe & \\ - ddde = & \left.\begin{aligned} + cccc \\ + 3.bbbb \\ - ffbb \\ - dddb; \end{aligned}\right\} \end{aligned}$$

the equation prescribed with the root $e = a + b.$
And thus the reduction ordered of the proposed equation has been done.

PROBLEM 27.

The equation $aaaa - 4.baaa + ffaa + dda = + cccc$, by placing $a = + e + b$, can be reduced to the equation

$$\begin{aligned} eeee - 6.bbee - 8.bbbe \\ + ffee + 2.bffe \\ + ddde = + cccc \\ + 3.bbbb \\ - ffbb \\ - dddb, \end{aligned}$$

or by placing $a = - e + b$, reduced to the equation

$$\begin{aligned} eeee - 6.bbee - 8.bbbe \\ + ffee - 2.bbfe \\ - ddde = + cccc \\ + 3.bbbb \\ - bbff \\ - dddb. \end{aligned}$$

First by placing: . . . . . . $e + b = a.$
Then . . . . $ee + 2.eb + bb = aa$
And . . . . $+ eee + 3.bee + 3.bbe + bbb = + aaa.$
And it shall become . . . $eeee + 4.beee + 6.bbee + 4.bbbe + bbbb = + aaaa$
And then . . . . $- 4.beee - 12.bbee - 12.bbe - 4.bbbb = - 4.baaa$
And . . . . $+ ffee + 2.bffe + bbff = + ffaa$
And . . . . $+ ddde + bddd = + ddda$ } $= + cccc.$
Thus by the rejection of contradictory [parts] and with the remaining terms in order
The equation is . . . .

$$\begin{aligned} eeee - 6.bbee - 8.bbbe \\ + ffee + 2.bffe \\ + ddde = + cccc \\ + 3.bbbb \\ - ffbb \\ - dddb, \end{aligned}$$

the first equation prescribed with the root $e = + a - b.$

Secondly by placing: . . . . . . $- e + b = a.$
Hence by a similar progression it shall be

$$\begin{aligned} eeee - 6.bbee - 8.bbbe \\ + ffee - 2.bbfe \\ - ddde = + cccc \\ + 3.bbbb \\ - bbff \\ - dddb, \end{aligned}$$

the second equation required of which the root $e = - a + b.$
And thus the reduction ordered of the proposed equation has been done as required.

PROBLEM 28.

The equation $aaaa + 4.baaa - ffaa - ddda = + cccc$, by placing $a = e - b$, can be reduced to the equation

$$\left.\begin{array}{l} eeee - 6.bbee + 8.bbbe \\ \quad - ffee + 2.ffbe \\ \qquad - ddde \end{array}\right. = \left\{\begin{array}{l} + cccc \\ + 3.bbbb \\ + ffbb \\ - dddb \end{array}\right.$$

Placing . . . . . . $+ e - b = a.$

Then . . . . $ee - 2.be + bb = aa$

And . . . . $eee - 3.bee + 3.bbe - bbb = + aaa.$

$$\left.\begin{array}{lrcl} \text{And it shall become . . .} & eeee - 4.beee + 6.bbee - 4.bbbe + bbbb & = & + aaaa \\ \text{And . . . .} & + 4.beee - 12.bbee + 12.bbe - 4.bbbb & = & + 4.baaa \\ \text{And . . . .} & - ffee - 2.bffe - bbff & = & - ffaa \\ \text{And . . . .} & - ddde - bddd & = & - ddda \end{array}\right\} = + cccc.$$

Thus truly by the rejection of contradictory [parts] and with the remaining terms in order it shall be . . . .

$$\left.\begin{array}{l} eeee - 6.bbee + 8.bbbe \\ \quad - ffee + 2.ffbe \\ \qquad - ddde \end{array}\right. = \left\{\begin{array}{l} + cccc \\ + 3.bbbb \\ + ffbb \\ - dddb \end{array}\right.$$

the equation prescribed with the root $e = + a + b.$

And thus the reduction ordered of the proposed equation has been done.

PROBLEM 29.

The equation *aaaa - 4.baaa - ffaa + dda=+ cccc*, by placing *a = + e + b,* can be reduced to the equation

$$\left.\begin{array}{l} eeee - 6.bbee - 8.bbbe \\ \quad - ffee - 2.bffe \\ \qquad + ddde \end{array}\right. = \left\{\begin{array}{l} + cccc \\ + 3.bbbb \\ + ffbb \\ - dddb, \end{array}\right.$$

or by placing *a = - e + b*, can be reduced to the equation

$$\left.\begin{array}{l} eeee - 6.bbee + 8.bbbe \\ \quad - ffee + 2.bbfe \\ \qquad - ddde \end{array}\right. = \left\{\begin{array}{l} + cccc \\ + 3.bbbb \\ + ffbb \\ - dddb . \end{array}\right.$$

First by placing: . . . . . . *e + b = a.*
Then . . . . *ee + 2.eb + bb = aa*
And . . . . + *eee + 3.bee + 3.bbe + bbb = + aaa.*

$$\left.\begin{array}{ll} \text{And it shall become . . .} & eeee + 4.beee + 6.bbee + 4.bbbe + bbbb = + aaaa \\ \text{And then . . . .} & - 4.beee - 12.bbee - 12.bbe - 4.bbbb = - 4.baaa \\ \text{And . . . .} & - ffee - 2.bffe - bbff = - ffaa \\ \text{And . . . .} & + ddde + bddd = + ddda \end{array}\right\} = + cccc.$$

Thus by the rejection of contradictory [parts] and with the remaining terms in order

The equation is . . . .

$$\left.\begin{array}{l} eeee - 6.bbee - 8.bbbe \\ \quad - ffee - 2.ffbe \\ \qquad + ddde \end{array}\right. = \left\{\begin{array}{l} + cccc \\ + 3.bbbb \\ + ffbb \\ - dddb \end{array}\right.$$

the first equation prescribed with the root *e = + a - b.*

Secondly by placing: . . . . . . *- e + b = a.*
Hence by a similar progression it shall be

$$\begin{array}{l} eeee - 6.bbee + 8.bbbe \\ \quad + ffee + 2.bbfe \\ \qquad - ddde = + cccc \\ \qquad\qquad + 3.bbbb \\ \qquad\qquad + bbff \\ \qquad\qquad - dddb, \end{array}$$

the second equation required with the root *e = - a + b.*
And thus the reduction ordered of the proposed equation has been done as required.

PROBLEM 30.

The equation *aaaa - 4.baaa - ffaa - dda=+ cccc*, by placing *a = + e + b,* can be reduced to the equation

$$\left.\begin{array}{l} eeee - 6.bbee - 8.bbbe \\ \quad - ffee - 2.bffe \\ \qquad - ddde \end{array}\right. = \left\{\begin{array}{l} + cccc \\ + 3.bbbb \\ + ffbb \\ + dddb, \end{array}\right.$$

or by placing *a = - e + b,* can be reduced to the equation

$$\left.\begin{array}{l} eeee - 6.bbee + 8.bbbe \\ \quad - ffee + 2.bbfe \\ \qquad + ddde \end{array}\right. = \left\{\begin{array}{l} + cccc \\ + 3.bbbb \\ + ffbb \\ + dddb. \end{array}\right.$$

First by placing: . . . . . . *e + b = a.*
Then . . . . *ee + 2.eb + bb = aa*
And . . . . + *eee + 3.bee + 3.bbe + bbb = + aaa.*

$$\left.\begin{array}{ll} \text{And it shall become . . .} & eeee + 4.beee + 6.bbee + 4.bbbe + bbbb = + aaaa \\ \text{And then . . . .} & - 4.beee - 12.bbee - 12.bbe - 4.bbbb = - 4.baaa \\ \text{And . . . .} & - ffee - 2.bffe - bbff = - ffaa \\ \text{And . . . .} & - ddde - bddd = - ddda \end{array}\right\} = + cccc.$$

Thus by the rejection of contradictory [parts] and with the remaining terms in order

The equation is . . . .

$$\left.\begin{array}{l} eeee - 6.bbee - 8.bbbe \\ \quad - ffee - 2.ffbe \\ \qquad - ddde \end{array}\right. = \left\{\begin{array}{l} + cccc \\ + 3.bbbb \\ + ffbb \\ + dddb \end{array}\right.$$

the first equation prescribed with the root *e = + a - b.*

Secondly by placing: . . . . . . *- e + b = a.*
Hence by a similar progression it shall be

$$\left.\begin{array}{l} eeee - 6.bbee + 8.bbbe \\ \quad - ffee + 2.bbfe \\ \qquad + ddde \end{array}\right. = \left\{\begin{array}{l} + cccc \\ + 3.bbbb \\ + bbff \\ + dddb, \end{array}\right.$$

the second equation required with the root *e = - a + b.*
And thus the reduction ordered of the proposed equation has been done as required.

PROBLEM 31.

The equation $aaaa + 4.baaa - ffaa = + cccc$, by placing $a = + e - b$, can be reduced to the equation

$$\begin{array}{l} eeee - 6.bbee + 8.bbbe \\ \qquad - 2.ffee + 2.ffbe = \left.\begin{array}{l} + cccc \\ +3.bbbb \\ + ffbb \end{array}\right\} \end{array}$$

Placing . . . . $+ e - b = a.$
Then . . . . $ee - 2.be + bb = aa$
And . . . . $eee - 3.bee + 3.bbe - bbb = + aaa.$

$$\left.\begin{array}{lrcl} \text{And it shall become . . .} & eeee - 4.beee + 6.bbee - 4.bbbe + bbbb & = & + aaaa \\ \text{And . . . .} & + 4.beee - 12.bbee + 12.bbe - 4.bbbb & = & + 4.baaa \\ \text{And . . . .} & - ffee + 2.bffe - ffbb & = & - ffaa \end{array}\right\} = + cccc.$$

Thus truly by the rejection of contradictory [parts] and with the remaining terms in order

The equation is . . . .
$$\begin{array}{l} eeee - 6.bbee + 8.bbbe \\ \qquad - ffee + 2.ffbe = \left.\begin{array}{l} + cccc \\ +3.bbbb \\ + ffbb, \end{array}\right\} \end{array}$$

the equation prescribed of which the root $e = + a + b.$
And thus the reduction ordered of the proposed equation has been done.

PROBLEM 32.

The equation $aaaa - 4.baaa - ffaa =+ cccc$, by placing $a = + e + b,$ can be reduced to the equation

$$\begin{array}{l} eeee - 6.bbee - 8.bbbe \\ \qquad - ffee - 2.bffe = \left.\begin{array}{l} + cccc \\ + 3.bbbb \\ + ffbb, \end{array}\right\} \end{array}$$

or by placing $a = - e + b$, can be reduced to the equation

$$\begin{array}{l} eeee - 6.bbee + 8.bbbe \\ \qquad - ffee + 2.bbfe = \left.\begin{array}{l} + cccc \\ + 3.bbbb \\ + ffbb. \end{array}\right\} \end{array}$$

First by placing: . . . . $e + b = a.$
Then . . . . $ee + 2.eb + bb = aa$
And . . . . $+ eee + 3.bee + 3.bbe + bbb = + aaa.$

$$\left.\begin{array}{lrcl} \text{And it shall become . . .} & eeee + 4.beee + 6.bbee + 4.bbbe + bbbb & = & + aaaa \\ \text{And then . . . .} & - 4.beee - 12.bbee - 12.bbe - 4.bbbb & = & - 4.baaa \\ \text{And . . . .} & - ffee - 2.bffe - bbff & = & - ffaa \end{array}\right\} = + cccc.$$

Thus by the rejection of contradictory [parts] and with the remaining terms in order

The equation is . . . . .
$$\begin{array}{l} eeee - 6.bbee - 8.bbbe \\ \qquad - ffee - 2.ffbe = \left.\begin{array}{l} + cccc \\ + 3.bbbb \\ + ffbb \end{array}\right\} \end{array}$$

the equation prescribed with the root $e = + a - b.$

Secondly by placing: . . . . $-e + b = a.$
Hence by a similar progression The equation is

$$\left.\begin{array}{l} eeee - 6.bbee + 8.bbbe \\ \quad - ffee + 2.bbfe \end{array}\right. = \left.\begin{array}{l} + cccc \\ + 3.bbbb \\ + ffbb, \end{array}\right\}$$

the second equation required with the root $e = -a + b.$
And thus the reduction ordered of the proposed equation has been done as required.

PROBLEM 33.

The equation $aaaa + 4.baaa - ddda = + cccc$, by placing $a = - e + b,$ can be reduced to the equation

$$\left.\begin{array}{l} eeee - 6.bbee + 8.bbbe \\ \quad - ddde \end{array}\right. = \left.\begin{array}{l} + cccc \\ + 3.bbbb \\ - dddb, \end{array}\right\}$$

Placing: . . . . $+ e - b = a.$
Then . . . . $ee - 2.eb + bb = aa$
And . . . . $+ eee - 3.bee + 3.bbe - bbb = + aaa.$

$$\left.\begin{array}{lll} \text{And it shall become . . .} & eeee - 4.beee + 6.bbee - 4.bbbe + bbbb & = + aaaa \\ \text{And then . . . .} & + 4.beee - 12.bbee + 12.bbe - 4.bbbb & = + 4.baaa \\ \text{And . . . .} & - ddde + dddb & = - ddda \end{array}\right\} = + cccc.$$

Thus by the rejection of contradictory [parts] and with the remaining terms in order

The equation is . . . .

$$\left.\begin{array}{l} eeee - 6.bbee + 8.bbbe \\ \quad - ddde \end{array}\right. = \left.\begin{array}{l} + cccc \\ + 3.bbbb \\ - dddb \end{array}\right\}$$

the equation prescribed with the root $e = + a + b.$
And thus the reduction ordered of the proposed equation has been done as required.

PROBLEM 33.

The equation $aaaa - 4.baaa + ddda = + cccc$, by placing $a = e + b,$ can be reduced to the equation

$$\left.\begin{array}{l} eeee - 6.bbee - 8.bbbe \\ \quad + ddde \end{array}\right. = \left.\begin{array}{l} + cccc \\ + 3.bbbb \\ - dddb, \end{array}\right\}$$

Placing: . . . . $+ e + b = a.$
Then . . . . $ee + 2.eb + bb = aa$
And . . . . $+ eee + 3.bee + 3.bbe + bbb = + aaa.$

$$\left.\begin{array}{lll} \text{And it shall become . . .} & eeee + 4.beee + 6.bbee + 4.bbbe + bbbb & = + aaaa \\ \text{And then . . . .} & - 4.beee - 12.bbee - 12.bbe - 4.bbbb & = - 4.baaa \\ \text{And . . . .} & + ddde + dddb & = - ddda \end{array}\right\} = + cccc.$$

Thus by the rejection of contradictory [parts] and with the remaining in order

The equation is . . . .

$$\begin{array}{l} eeee - 6.bbee - 8.bbbe \\ \quad + ddde \end{array} = \begin{array}{l} + cccc \\ + 3.bbbb \\ - dddb, \end{array}$$

the equation prescribed of which the root $e = + a - b.$
And thus the reduction ordered of the proposed equation has been done as required.

And thus the tract of the first part has been explained, preparatory to the numerical Exegetic [Definition 9], *the second part which now follows in the main comprising the practise of the numerical Exegetic itself.*