# HYDRODYNAMICS SECTION TEN.

*Concerning the properties and motions of elastic fluids, but especially those of air.*

§. 1. Now with elastic fluids, for these it will be allowed for us to go on to assign such a constitution, which may agree with all the known properties at this stage, so that thus an approach may be provided for the remaining [properties] not yet investigated well enough. Moreover the particular properties of elastic fluids placed in accordance with that are : 1st that they shall be heavy, 2nd that amongst themselves they may spread out in all directions, unless they may be contained, and 3rd so that they may permit themselves continually to be compressed more and more by increasing forces of compression : Thus air is compared, to which chiefly our present thoughts relate.

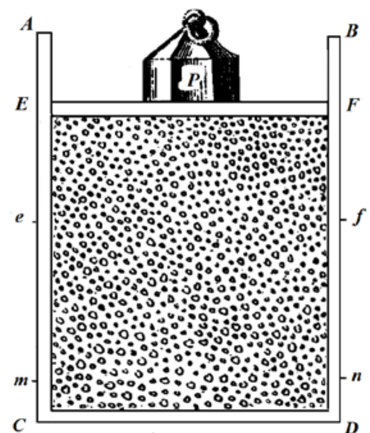


Fig. LVI

§. 2. And thus imagine a cylindrical vessel placed vertically *ACDB* (Fig. 56) and into that the moveable opening *EF*, to which the weight *P* may be resting above : hence the cavity *ECDF* may contain the smallest particles [corpuscles] thence moving with the most rapid motion : thus the corpuscles, while they impinge on the opening *EF* and support the same by their continually repeated impacts, compose an elastic fluid which expands with the weight *P* moved back or diminished: which with the same increased may be compressed and which can hardly do otherwise than to exert a weight on the horizontal base *CD*, even if they were not endowed with an elastic force: for whither the corpuscles may remain at rest or be moving around, they do not change weight, thus so that the base sustains both the weight as well as the elasticity [*i.e.* pressure] of the fluid. Therefore we will substitute such a fluid in place for air, which agrees with the primary properties of elastic fluids, and thus we will explain other properties, which have been found in air now, and we will explain and make clear other and further properties not yet considered fully enough.

§. 3. We will consider the corpuscles enclosed by the cavity of the cylinder as infinite in number, and when they occupy the space *ECDF*, then we may say those to form natural air, to the measures of which all are to be referred : and thus the weight *P* holding the lid *EF* in place does not differ from the pressure of the atmosphere above pressing down, as hence we will designate by *P* in the following.

However it may be observed that this pressing force is not at all equal to the absolute weight of a vertical cylinder of the air in the atmosphere resting on the lid *EF*, which authors have inconsiderately agreed on at this stage: but this pressing force is to the magnitude [area] of the lid *EF*, as the weight of the whole atmosphere over the whole surface of the earth is to the surface of the earth, being the fourth proportion.

$$[\text{i.e. Pressure} = \frac{\text{wt. of whole atmosphere}}{\text{Surface of the earth}} \times EF\ ]$$

§. 4. Now the weight $\pi$ may be sought, that it may act to condense the air in the volume *ECDF* into the volume *eCDf*, with the same velocities of the particles in each air, surely for the natural and the condensed air : moreover, there shall be $EC = 1$ and $eC = s$: Truly since the lid *EF* is moved to *ef*, a greater pressure is endured by the fluid in two ways : *firstly*, because the number of particles in relation to the volume, to which they are enclosed, is now greater, and *secondly* because any particle makes repeated impulses more often: so that we may consider the increments correctly, which depend on the first cause, we will consider the particles as if at rest, and the number of these, which are near the lid in the position *EF*, we will make $= n$, and the like number for the position of the lid at $ef = n : \left(\frac{eC}{EC}\right)^{\frac{2}{3}}$, or $= n : s^{\frac{2}{3}}$.

[ Following K.F. we have :

(a) Bernoulli puts the number of particles spread out immediately under the surface of the lid capable of reaching *EF* equal to *n*. If you want to relate them to the volume *ECDF*, these must be raised to the power $n^{\frac{3}{2}}$, the number of particles per unit volume. At present no indication is given as to why this must be the case.

(b) Initially we de not denote *EC* by 1, but by *a*, and the surface area of the lid by $F = a^2 f$, where *f* is a fraction relating the original lid area to the height *a* squared . ]

(c) Now $\rho^3$ is the total number of particles contained in the volume *ECDF* then it is related to the size of this volume, note the subscripts *a* and *s* by :

$$\rho_a^3 = \frac{\rho^3}{a \cdot F} = \frac{\rho^3}{a^3 \cdot f} = \left(\sqrt{n}\right)^3 .$$

(d) The whole compressed space *eCDf* appears to contain the same number of particles $\rho^3$. Now let $eC = a_s = a \cdot s = EC \cdot s$, wherein *s* again denotes a known ratio, then, as this volume will be designated the same number of particles, the condensed density:

$$\rho_s^3 = \frac{\rho^3}{a_s \cdot F} = \frac{\rho^3}{a^3 s \cdot f} = \left(\sqrt{n_s}\right)^3 .$$

Here, the particle number $n_s$ is again immediately below the lower lid *F*, and $\rho_s^3$ is greater than $\rho_a^3$.

(e) The ratio $n_s : n$ itself then gives

$$\rho_s^3 : \rho_a^3 = \frac{\rho^3}{a^3 s \cdot f} : \frac{\rho^3}{a^3 f} = \frac{1}{s} : 1 = \left(\sqrt{n_s}\right)^3 : \left(\sqrt{n}\right)^3 = n_s^{\frac{3}{2}} : n^{\frac{3}{2}} .$$

Therefore it follows

$$\frac{1}{s}\cdot n^{\frac{3}{2}}=1\cdot n_s^{\frac{3}{2}} \text{ or } n_s=n:s^{\frac{2}{3}}=(\text{after } d)n:\left(\frac{eC}{EC}\right)^{\frac{2}{3}}. \quad Q.E.D.]$$

However it should be observed by us that the lower part is not to be compressed more than the upper part, such is the case, just as with the weight $P$ being infinitely greater than with its own weight of fluid: Hence it is evident, the force of the fluid is to be given by this name, so that the numbers are as $n$ and $n:s^{\frac{2}{3}}$, that is, as $s^{\frac{2}{3}}$ to 1. Truly which pertains to the other increment arising from the *second cause*, that may be found by looking into the motion of the particles; and thus it will be apparent the impulses there happen more often, when the particles are placed closer to each other in turn : Evidently the number of impulses will be inversely as the average distance between the surfaces of the particles: Thus these mean distances will be determined.

We may consider the particles to be spheres, and the mean distance between the centres of the globules for the position *EF* of the lid we will call *D*, and we will designate the diameter of a globule by *d*: thus the mean distance between the surfaces of the globules will be $=D-d$ : truly it is apparent in the position *ef* of the lid, the mean distance between the centres of the globules $=D\sqrt[3]{s}$, and therefore the mean distance between the surfaces of the globules $=D\sqrt[3]{s}-d$. Therefore with respect of the second cause the force of the natural air *ECDF* to the force of the compressed air *eCDf* will be as $\frac{1}{D-d}$ to $\frac{1}{D\sqrt[3]{s}-d}$, or as $D\sqrt[3]{s}-d$ to $D-d$ : Indeed with both causes taken together the predicted forces will be as, $s^{\frac{2}{3}}\times\left(D\sqrt[3]{s}-d\right)$ to $D-d$.

For the ratio *D* to *d* , we can substitute another more understandable ratio : namely if we may consider the lid *EF* to descent with an immense weight as far as into the position *mn*, in which all the particles touch each other, and the line *mC* we may call *m*, *D* will be to *d* as 1 to $\sqrt[3]{m}$, with which ratio substituted, finally the forces of the natural air *ECDF* and of the compressed air *eCDf* will be as $s^{\frac{2}{3}}\times\left(\sqrt[3]{s}-\sqrt[3]{m}\right)$ to $1-\sqrt[3]{m}$, or as $s-\sqrt[3]{mss}$ to $1-\sqrt[3]{m}$. Therefore there is

$$\pi=\frac{1-\sqrt[3]{m}}{s-\sqrt[3]{mss}}\times P.$$

§. 5. From all these phenomena we are able to conclude natural air can be condensed greatly, and to be compressed into an almost infinitely small space ; therefore making $m=0$, there becomes $\pi=\frac{P}{s}$, thus so that the pressing weights shall be nearly in the inverse ratio of the volumes, which the air in different states of compression occupies; which has been confirmed by experiment in many ways. And certainly I have accepted this rule safely in air more rarefied than normal ; but truly also it shall be the case that I have not examined the rule well enough in air of greater density : nor whether indeed experiments were performed with that accuracy which is required here: there is a need for defining the value of the letter *m* uniquely, but with that requiring to be put in place most accurately and certainly with the air compressed strongly; moreover the degree of heat in the air, while it is compressed, shall be kept constant with great care.

§. 6. Meanwhile the elasticity of the air may be increased not only by condensation, but also by the increase in the heat, and because the heat has extended everywhere with the increased internal motion of the particles, it follows, the elasticity of the air increased without changing the volume, to increase more intensely the motion of the particles in the air, which agrees correctly with our hypothesis : for it is evident, a greater weight *P* to be required there towards containing the air in the place *ECDF*, when the air particles are moving around with a greater velocity: Indeed it is not difficult to see the weight *P* is going to follow the square ratio of this velocity, thus because from the increase of it velocity, both the number of impacts as well as the strength of the same may increase equally, each truly separately shall be proportional to the weight *P*.

Therefore if the velocity of the particles of air may be called *v*, the weight will be, which acts to support the lid *EF* in place, = *vvP* and in the position $ef, = \frac{1-\sqrt[3]{m}}{s-\sqrt[3]{mss}} \times vvP$, or approximately $= \frac{vvP}{s}$, because as we have seen *m* is an exceedingly small number in relation to unity and the number *s*.

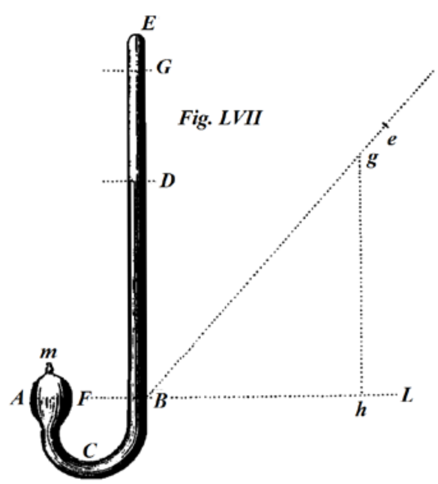
[We may note that if $P_1, T_1$ are the initial pressure and absolute temperature of an ideal gas, and $P_2, T_2$ the final pressure and temperature, then $P_2 = \frac{T_2 \times P_1}{T_1} = \frac{\bar{v}_2^2}{\bar{v}_1^2} \times P_1$, where the mean square speeds are used for particles of the same mass; and the general gas equation can be written in a number of ways. Thus, in this sense, the above assertion is correct.]

§. 7. It may be shown by that same theorem, which I have put in place in the preceding paragraph, evidently, *in any air of whatever density but with the same degree of heat endowed, the elasticities to be as the densities, and therefore also the increments of the elasticities, which arise with the heat increased equally, to be proportional to the densities*, that same theorem, I say, Amontons had established from experiments and considered in the *Memoires de l'Acad. R. des Se. de Paris for the year* 1702. The understanding of this theorem is, if for example a weight of 100 lbs resting on a surface may act to sustain the air of a medium heat, and then while the heat itself may be increased the air shall be able to carry 120 lb. on the same surface and with the same volume, to be as the same air condensed into half the volume, and endowed with the same steps in the heat, shall be able to carry respectively 200 lb. and 240 lb., thus so that the increments 20 lb. & 40lb. generated in both places from the increased heat shall be proportional to the densities. Again it may be confirmed of air, which we may call temperate, the spring to be to the spring of the air with the heat from boiling water, approximately as 3 to 4 or more accurately as 55 to 73. [This works out from the gas laws to be around $8^0$C.] But I have known from experiments performed the air of the warmest climates, and in such countries the summer to be especially seething hot, at no time to be with such a spring [*i.e.* pressure], of the amount Amontons attributed to the temperate air ; indeed I can persuade myself nor under the equator itself at no time is the air to be of this heat. But I consider my own experiments to be more trustworthy than those of Amontons, because thus in these the air has not kept its volume and nothing of that variation was accounted for in the calculation of the author. Here for the day 25th December 1731, old calendar, which was the coldest of the air of Petersburg, the pressure to be taken to the pressure of similar air, in common with the heat endowed by boiling water, was as 523 to l000. [This works out from the gas laws to be around -80$^0$C, which seems doubtful.]

But for the year 1733 the day 21st Jan. the cold was much more intense and for that I have observed the corresponding elasticity [*i.e.* pressure] of the air to be less than half that which similar air heated to boiling water [around -87$^0$C]. But when the maximum heat of the air in a place of shade in the year 1731 had an elasticity around $\frac{3}{4}$ and more accurately $\frac{100}{76}$ of that which the coldest air had, and $\frac{2}{3}$ of that which the air of the same had with the heat of boiling water: therefore the greatest variations of the heat in the air of places here may be contained between the limits 3 and 4, which I have read in England do not go beyond the limits 7 and 8. But I consider the heat of the air to be intolerable to the bodies of animals, the elasticity of which may equal three quarters of the elasticity of the air equal to the heat of boiling water.

§. 8. From the known ratio between the different elasticities of air enclosed in the same volume, it is easy to deduce a measure of the heat, which may pertain to that air, but only if we may agree on defining air with twice as much, three times as much heat etc., which definition is arbitrary, and not considered in the nature of things ; to me indeed the heat of the air does not seem to be incongruous, if generally a proportional of its density be set up to be a proportional of its elasticity. But first the degree of the heat, by which the measure of the rest may be taken, may be selected from the boiling of rainwater, because for this with doubts removed every country the degree of heat is approximately the same.

Thus with these accepted the heats of boiling water, of the air in the hottest summertime, and of the air in the coldest winter times in these countries will be approximately as 6, 4 and 3. I may say now in whatever manner I have come upon these numbers, so that from the most accurate of experiments, of which the success is certainly different from that of Amonton, it may be able to bring a judgment.




§. 9. Indeed I have used an ordinary barometer ACBE (Fig. 57), and that I have sealed carefully hermetically at *m;* in this way I have changed the instrument into a thermometer of the air not liable to barometric changes [*i.e.* the forerunner of the gas thermometer]. Indeed with the increased heat the spring of the air *AmF* becomes higher and the column of mercury *BD*, which the captive air sustains, and if the volume *AmF* can be estimated to be just as small, the heat [*i.e*. temperature] shall be in the ratio of the height *BD* (by §§. 7 and 8) and the measure of the heat with the aid of this thermometer to be specifically defined everywhere. For if the instrument may be immersed in boiling rainwater and the point *G* may be observed in a vertical situation to which the surface of the mercury rises ; and then some other degree of heat would have been required to be defined, which the mercury sustained at the point *D* while it was observed, everywhere this heat will be to the heat of boiling water as *BD* to *BG.* And since the ratio *BD* to *BG* shall be constant, whatever the height *BG* were, the same degree of heat, that we are discussing, can be replicated easily everywhere in all places. Moreover I have divided *BG* into one hundred or one thousand small parts and I have defined the height *BD* by small parts of this kind.

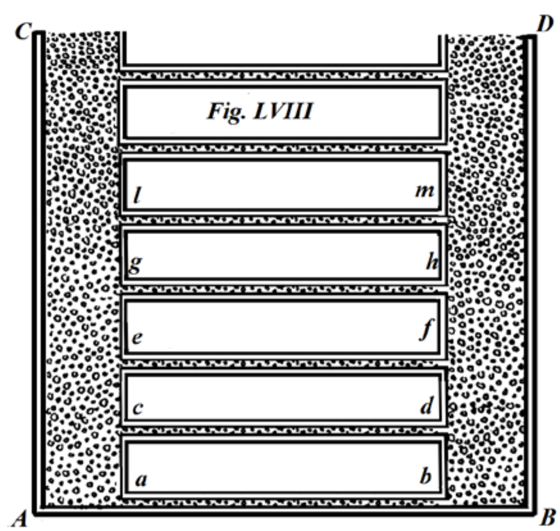
I say nothing about the ways of producing more sensitive thermometers of this kind ; and of these whoever wishes easily will be able to think out more. But care is required, so that the height *BE* shall not be less than 4 feet, indeed so that it may be greater, if also one may have in mind to find the degree of heat of other boiling fluids, which often is greater than that of water. If smaller thermometers of this kind may be desired, these are able to be made thus, so that at the time of sealing the small glass cavity *AF* at *m* with the fire of a lamp brought near towards rarefying the air contained in that, and then at once the seal may be made, and lest there may be a delay in injecting the seal, first the glass ampoule can be drawn into a capillary tube, which may be melted at once by the heat of a flame drawn near. In this manner I have obtained thermometers not more than four or six inches long, but of small strength. Besides it is of great concern, that the volume *ED* shall be free of all air, so much as it can become a vacuum, nor shall we be sure enough about that vacuum when we have seen the end of the mercury reach to *E*, because it can happen that air, which before was in the volume *ED*, itself may be recovered from the pores of the mercury, and again may occupy the former space in the descent of the mercury [Recall that at the time, substances such as gunpowder were believed to be porous, and the pores filled with air.]: it will be safer to examine the part *DE* by bringing up a flame: for if from the heat of the flame the surface *D* shall not change position, it will be a sure indication the volume *ED* to be free of air.

§. 10. In the preceding paragraph we have considered the volume *AmF* occupied by the air to be as if infinite in the ratio to the volumes *DG* or *DE:* But if truly it were only eight times or ten times greater, in no way would that be allowed to be considered as infinite without a noticeable error: and hence I guess some error to have arisen in defining the spring of the air with Amonton's experiments of moderate heat.

Therefore so that an experiment may be made the most accurate, it will be required to proceed thus : Should the lower surface of the mercury have been at *AF* and the horizontal drawn at *AL;* then for the degree of any amount of heat requiring to be defined the instrument may be inclined, then the surface of the mercury shall be at the point *g* (which is the same place at which the mercury stopped with the degree of heat of boiling water in the situation of the vertical thermometer), and then the measure of the vertical height *gh* may be taken, which will be to the height *GB* truly as the spring of the air, of which the heat is required to be found, to the spring of the air equal to the heat of boiling water. Therefore truly the heats will be properly in the ratio of the height *gh.* Before I may stop this argument, it will be convenient to note (since by some people perhaps it will be observed, which was put by us, the *first degree of heat* chosen from boiling water may not always nor everywhere itself to be agreed on entirely) because in place of the heat of boiling water, the thermometer also shall be able to make sure and fixed measures, if the density of air or its specific gravity may be investigated by experiment and likewise the height of the barometer may be noted. Indeed if the thermometer were to be inclined, then the surface of the mercury would be in the position *g*, and at that time the height of the barometer 28 *Paris* inches, and a cubic foot of the air, in which it was situated, had a weight of 600 *Nuremberg grains*, the height of the vertical *gh* or the *first* degree of heat to be considered. But if in another place and at the time the height of the barometer were 29 *Paris inches*, and the weight of a cubic foot of air, which another thermometer embraces (in which one has in mind to define the *first degree* of heat) shall be 500 *Nuremberg grains* and then the surface of the mercury in the thermometer again shall be at *g*, the height of the vertical of the first degree agreeing will be $\frac{29\cdot 600}{28\cdot 500}\times gh$. In the use of the thermometer the instrument shall be inclined always, then the surface of the mercury shall be at *g*: I have wanted this method to be placed nearby so that it would be evident how easy it shall be in the established theory to give the measure of the heat: Truly in practice I shall present another much easier and accurate enough to this

§. 11. We may come now to considering the atmosphere of air, which is not forced together by some alien superimposed weight, but by its proper mass. But in the *first place* we will examine the pressures down both of vertical columns of air and the equilibriums of these between themselves as well as with columns of mercury in barometers. In the *second place* we will probe the elasticities of the air in various atmospheres with the heights above sea level and the corresponding barometer heights. And from these premises, we will be satisfied with most other phenomena relating to changes of the atmosphere.

§. 12. *AC* and *BD* (Fig. 58) shall be two vertical tubes of equal cross-sections each of indefinite height: Then imagine narrower horizontal tubes *ab, cd, ef, gh, lm* &c. with a number as if infinite, each open and adjoining onto the vertical tubes. Therefore consider everywhere air particles occupying these tubes to be moving around with the same velocity, and thus to have the same degree of heat : Thus there is no doubt, why the bases *A* and *B* may not by themselves be pressed likewise with an equal weight resting above (because without doubt it is the weight of the indefinite column of air itself *AC* or *BD*).




Also you understand, if at equal heights such as at *g* and *h* you can imagine diaphragms [*i.e.* membranes] and you can think the air to be absent from *gA* and *hB* below these, even now these diaphragms to be pressed on equally at both ends and the weights of the columns of air *gC* and *hD* to be superimposed on the diaphragms. Therefore if the weight of the whole column of air

*AC* or *BD* may be called *A*, and the weight of the column of air *gC* or *hD* may be considered as *B*, the weight of the air intercepted between *A* and *g* or *B* and $h = A - B$, the overlying weight from the base *A* or $B = A$, and the weight pressing on the diaphragm at *g* or $h = B$.

§. 13. But if the particles in the tubes *AC* and *BD* may be moving around with an unequal velocity, the matter will be otherwise : yet whatever the difference of the speeds and heats at the individual places, it is apparent nevertheless the pressing on the parts of the tube is going to be the same on each end placed at the same height, just as at *g* and *h*, and hence the diaphragms, if they may be imagined put at the same height at both ends, to be going to sustain an equal pressure. For if you may say the pressure at *g* to be smaller than the pressure at *h*, because there will be nothing that may impede the flow of air from *BD* into *AC* along the transverse tube *hg*, and thus that put in place will dispute the state of permanence that we suppose.

And thus since places put at the same altitude may be pressed equally by the superimposed air, the densities will be (by §. 6) at whatever homologous places, such as at *g* & *h*, approximately in the inverse square ratio of the velocities, with which they are moving around in these places.

[Here Daniel Bernoulli seems to be indicating a form of the *V* vs. *T* ideal gas law with *P* constant; the long narrow tubes enables the pressure to be the same, while essentially insulating the two air columns, so that the temperatures in each can be different; however, we are still far from an absolute temperature scale …..]

§. 14. The consequence from the preceding paragraph is, the air pressure of places everywhere is to be the same at equal heights above the surface of the sea, if the atmosphere may be considered in a state of permanent equilibrium, and with no disturbances from the wind, whatever the difference of the heat in diverse parts of the atmosphere should be : Therefore it will be required wheresoever of the locations under the equator or under the pole the height of mercury in barometers shall be the same, which shall be place on the surface of the sea or at equal heights above that, if the atmosphere shall be liable to no changes. Moreover I place the waters from the surface of the sea to be defined according to a common equilibrium put in place, not because that generally shall be necessary, but because at this time nothing different will have been observed : indeed the courses of the water (*the currents*) in many places of the ocean, which always are directed towards the same region, show that this hypothesis is not to be accepted with all rigor.

§. 15. Now I have noted the density of the air in any of the vertical tubes in place depends on the corresponding heat : And since the degrees of heat may be different with the equilibrium remaining, the densities also can be different: and thus the densities may be put at $g = D$, and at $h = \delta$; and at each end there may be imagined two layers of equal heights and infinitely small *dx*, with the height *Ag* or *Bh* being put = *x* : Thus the weight of the column of air $Ag = \int Ddx$ and of the column $Bh = \int \delta dx$ : and in this manner both the integrated columns as well as the weight of any part can be defined: Meanwhile it is apparent, the nature of the matter minimally to require, that the weights of the columns *AC* and *BD* or of *Ag* and *Bh* or finally *gC* and *hD* shall be equal to each other, whenever (by §. 13) the pressures both at the bases *A* and *B* as well as at the diaphragms *g* and *h* shall be equal to each other; that will be regarded at first perhaps with a certain wonder, to be able to arise that the base *A* may sustain another pressure than the weight of the indefinite column of air *AC* resting above that, since with everything in its permanent state, as it may be generally seen, the openings *a*, *c*, *e*, *g* etc. shall be considered to be separately closed, so that reasonably there is no case to doubt, why the pressure of the base *A* shall not itself be the weight of the column of air superposed : truly this small concern itself will be removed in the following way: we may imagine each column of a finite height (although indeed they rise without end as long as the particles maintain some motion, yet they are ended, if the same particles in the upper part of the column shall be deprived of motion, and thus produce a simple heavy fluid without any elasticity); with this in place it is evident: 1st each column rises to the common height of the opening of the transverse tubes, which are present everywhere ; 2nd the upper layer to be equally dense on both sides, because they are at the equilibrium position and have a common height. From this it is obvious now, whereby it is not required to consider the transverse tubes to be stopped up, which it was arranged to show. It is evident also from these, the pressures everywhere to be proportional to the upper layer [at this level], from which it follows, which now was shown in §.13, the pressures from each side are to be equal to each other at equal heights. If now the columns nowhere shall be ended, it will be agreed to consider the final level at equal heights or to imagine diaphragms on both sides loaded with equal weights, thus so that nothing may depart from the strength of the demonstration.

§. 16. Therefore when the mercury falls in a barometer on being carried from a lower place such as *A* to a higher place *g*, it does not follow that the weight of the column of mercury, by which the mercury falls, to be equal to the weight of the column of air of the same diameter and height *Ag*, which thus is asserted by some people [*i.e.* only being the case approximately for an isothermal atmosphere]. And certainly with all else being equal the column of mercury descending will be the same both in winter time as in summer time, since opinion has it that it [*i.e.* the pressure] must be less in a warm time, then in a cold time: Also the same will be the case in southern and northern countries.

Thence it is apparent what shall be required to be assessed by that method, which has been used in England for some time and Du Hamel reviewed in *Hist. Acad. Sc. Paris.*[1701, where de la Hire's experiments are discussed], towards investigating the ratio between the specific gravities of air and mercury : certainly with the height of mercury both in a lower place as well as in a higher one, the specific gravities of air and mercury were established, so that the difference of the heights of mercury in a barometer between the places of observation could be intercepted: Even if air of the same density may be put in place from the deepest place of observation as far as to the other, thence it will still not be permitted to judge the specific of this in the ratio to mercury. This alone is what it is permitted to deduce from the experiment:

Clearly we may consider the whole covering of the air around the earth and the intercept between both places of observation, and the weight of this shell to the surface of the earth, shall be as the weight of a column of mercury, of such an amount in the barometer descending to its base; evidently these are from that because the sum of the bases *A* and *B* may support a certain sum of weights, which the columns of air *AC* and *BD* have, nor yet may any base be pressed separately by the weight of its column, and because the same must be understood, with the columns *Ag* and *Bh* removed, by the columns *gC* and *hD,* with the diaphragms incumbent at the positions *g* and *h*. Therefore the experiment not only indicates the specific gravity of the air, in which it has been made, but may determine as well the *average* specific gravity of all the air close to the earth ; certainly the former is variable, but the other without any doubt to remain almost the same constantly.

We may make a computation of this *average specific gravity* of all the air, which enfolds the earth : Truly by many experiments, which were taken in different places with a little elevation above the sea, that was agreed to be raised approximately 66 feet to correspond to a descent of one line [*i.e.* $\frac{1}{12}^{\text{th}}$ of an inch] in the barometer. Thence it follows, because the mean specific gravity of the air to mercury by the reckoning, shall be at the height of one line to a height of 66 *feet,* that is, as 1 to 9504; therefore with the specific gravity of mercury placed $=1$, the mean specific gravity of the air $=0,000105$. Surely it is remarkable that the mean weight of the air is to be so great: for I am indeed certain with the raging cold here of all places, the specific gravity of the air scarcely yet to be so great as the amount we have now shown for the average state of all the air around the earth: but at the equator it will be much less and with everything correctly considered I will not believe the mean weight of the air, which will be contained between each latitude of 60 degrees, to run beyond 0, 000090; with which put in place the mean weight of the surrounding air from each pole to 30 degrees (which volume makes up a little more than an eighth part of the whole surface of the earth) $=0,000210$, which is the double of the densest places here: but at the pole itself, especially at the Antarctic, certainly the air will be heavier and scarcely ten times lighter than water, when it is the coldest and densest.

§. 17. Now we come to changes both of the atmosphere as well as of the barometer: Therefore we will consider two barometers each in the place of the deepest air, the one at *A*, the other at *B*, and in each we may place the mercury to be suspended at the same height: Later at *A* we may consider the air suddenly to be made warmer: Thus we see it may be, so that the same air may be rarefied: nor yet thence any change of the barometer is going to be produced, if the air may not be at rest, even if all the air may be expelled from *AC* into *BD* : but with that inertia put in place a certain pressure arises in all directions and most noticeably in the region *A.* Therefore the height of the mercury in each barometer will increase according to the time, and it will increase more in *A* than in *B.* It will be the opposite, if immediately a certain great mass of air may be condensed by the cold around the barometer *A* or *B.*

§.18. This may be seen the single cause, which largely may be able to effect a change with barometers placed at *A* or *B*, because with this removed the bases *A* and *B* always are pressed equally, clearly with some one weight, which shall be of half of the columns of air AC and *BD* taken together, which indeed is a constant sum of the weights. If we wish to apply this to the atmosphere, it is required to be observed *A* and *B* represent places lowest in the atmosphere, which indeed were placed on the surface of the earth, if the air was not required to penetrate into the bowels of the earth: because truly the matter would have to be considered otherwise, were the analogous places of the bases *A* and *B* agreed to be within the earth.

§. 19. Now the barometers may be considered to be placed at *g* and *h* ; and in both the mercury suspended at the same height: with these in place a cause is imagined to arrive, by which the column *Ag* either alone or in conjunction with the associate *Bh* may become warm and itself may expand. From these it is evident, if either none of the air shall be at rest, to be, so that the pressures of the air at *g* and *h* may increase, because with these in place now a greater amount of air may accumulate then before ; doubtless the weight of all the air approaching was driven upwards from *Ag* and *Bh* by the heat. And so that we may indicate these with symbols, we may make the weight of the column *Ag*, before the new degree of heat has arrived, $= A$, the other $Bh = \alpha$, the weight of the column $gC = B$, of the column $hD = \beta$: the weight of the rarefied column $Ag = C$, the weight of the column *Bh* likewise rarefied $= \gamma$: the height of the mercury at *g* before the expansion of the air *Ag* and $Bh = l$, the height of the same after its expansion $= x$, and we will have this analogous ratio :

$$B + \beta : l \ :: B + A - C + \beta + \alpha - \gamma : x;$$

from which there is :

$$x = \frac{B + A - C + \beta + \alpha - \gamma}{B + \beta} l.$$

Therefore the mercury rises less from the rarefied air by the height

$$x - l = \frac{A - C + \alpha - \gamma}{B + \beta} l =$$

(with everything equal in each tube)

$$\frac{A - C}{B} l.$$

But again with the air cooling in *Ag* & *Bh* the mercury in each barometer descends again.

It is required to be observed here, in this manner with a little change of the heat in *Ag* and *Bh* a notable variation in the barometer arises on account of the sign of the density in the lower parts, when it can happen, if that much more or the air may be contained in the part *Ag* (indeed infinitely more, if the air may be able to be condensed into an infinitely small space by an infinite pressure) then in the remaining *gC*, even if infinitely long. From which if the weight *A* certainly shall be greater than the weight *B*, and likewise with the cause rarefying the air remaining, the given weight *C* maintains the ratio to *A;* because thus this generally shall be, the apparent rise of the mercury by a minimum degree of heat arising at *Ag* is able to be any size.

Indeed if it may be imagined, the sides *Ag* and *Bh* certainly to be narrower besides the cross-sections in *gC* and *hD*, it is understood the variations of the barometer from the increase or decrease of the degree of heat in *Ag* and *Bh* thus becomes less remarkable, because the weights $A$ and $\alpha$ and these $C$ and $\gamma$ from the first proportionals decrease in this manner; but yet the barometric variations , which arise from this cause, even now will be able to be considered large in some manner.

§. 20. Thus while these are being considered, truly similar shall be the barometric variations in the main part from the rapid changes of heat in hidden underground cavities. There are many and those very large cavities of this kind have been known about now for a very long time : because also in the solid earth pores are able to make hidden chambers: if you gather together all the cavities ( both which are formed from caves, as well as by pores containing air) to a depth below the surface of the earth 20 000 or 30 000 feet and you compare the capacity of these with the solid crust of the earth with the same depth, and even thousands or hundreds of thousands other small weights, it will be surely sufficient even now to explain that cause for the maximum changes of the barometer. These will be evident as I think uniquely from the previous paragraph.

Moreover the places which are better suited for underground cavities, these are liable to more winds and barometric changes, on account of the slow motion of the air, which perhaps is the reason, because towards the equator, where almost everything is sea, minor variations may be observed in the barometer as in these northern regions.

§. 21. It may be deduced from the same source, to some extent the exhalations of water vapour from the pores of the earth are able to bring together barometric variations : but certainly that will be small: for if so much water vapour were supplied, so great a maximum amount of rain can be decided, thence it ascends scarcely a single line of mercury in the barometer, besides which because this cause shall not thus be fast, why its effect in the whole atmosphere likewise may not be distributed generally, and thus for sure at a certain place all may vanish. For if we may consider the whole atmosphere, which will surround the earth, certainly it cannot be noticed that it can be oppressed by vapors now less now more. Indeed I will have brought forwards with all the remaining exposition in the account of §. 20, for great and swift changes are able to happen in the bowels that indicate the motion of the earth, which often can be sensed for as far a handed miles at the same time, and other phenomena of this kind. A certain cause is required to be submitted towards explaining the barometric changes expressed ; for now I have advised the slow ones distributed in the whole mass of air to be of no effect, and I have shown that in §.14. And for that reason few changes are being made, which occur at once in the atmosphere above the surface of the earth.

§. 22. And this equally may be considered the cause because the moon, which has so much effect towards disturbing the oceans, anyone would have noted from careful observations, would extend no effect on the barometer: and if also the remaining causes, which prevail to produce some change in the atmosphere, may act a little, without doubt in all places to be equally distant from the surface of the sea, the same height of mercury would be presented to the senses. This height can be called the *mean,* and may be determined approximately by that method which was used be Johan Jacob Scheuchzer, by observing daily the barometric height for a long period of time and by taking a mean between all the heights.

And by using this careful manner of consideration from many observations which were sent to him from many places, the most celebrated author put in place the mean height

| | | |
|---|---|---|
| Padua .......... | 27 *inches.* | $11\frac{1}{2}$ *Paris lines* |
| Paris .......... | 27 *in.* | $9\frac{1}{2}$ *l.* |
| Turin. .......... | 27 *in.* | $1\frac{1}{4}$ *l.* |
| Basil ......... | 26 *in.* | $10\frac{1}{8}$ *l.* |
| Zurich ........... | *26 in.* | $6\frac{1}{2}$ *l.* |
| Mount St. Gotthard | 21 *in.* | $27\frac{1}{2}$ *l.* |

§. 23. It has been noted that the differences of these mean heights arise from unequal places above sea level. For now in Pascal's time experiments were taken for the descent of the mercury in the barometer carried from the deepest to the highest place. Thence philosophers to enquire into the mutual proportion of cause and effect: Diverse rules regarding this matter were produced by various authors: The particular rule, to which even now many adhere, is this, that the heights of the places may follow the proportion of the logarithms, which correspond to the heights of the barometer. [This had been established originally be Edmund Halley in the *Phil. Trans.* for May 1686] This particular rule has been based on that, because the density of the air everywhere shall be proportional to the weight of the air resting above: but here this principle is applied badly, because it prevails only for air of the same heat, nor is this certain at every height of the air, however much may be present in the same column of air ; if truly thus it shall be that the heat shall be equal, it is required to be admitted, thus the rule may considered to be correct enough.

But experiments plainly are contrary to the rule; therefore there is not the same degree of heat everywhere through the whole height of the vertical air column, now so that I may make it plain, I have set up certain experiments accurately, so that I myself may be persuaded, but yet, which gives me pain, at different times and places ; certainly the experiments set up by our institution at the same time and on the same mountain may agree more, with only the difference in the heights taken; but such is the case, except for small distances of locations, that no one as far as I know at this time has published everything for the circumstances required to be known.

(I) At the height 1070 *Paris ft.* from the surface of the sea the barometer fell $16\frac{1}{3}$ *lin.* when at the surface of the sea it maintained a height of 28 *in.* $4\frac{2}{3}$ *lin.* (others put simply 28 *in.;* but in the leaves which De Lisle [Joseph-Nicolas Delisle was the Petersburg astronomer and Siberian explorer at this time] communicated to me had 28 *in.* $4\frac{2}{3}$ *lin*). Therefore on putting the elasticity of the air at the surface of the sea, as henceforth I will put always $=1$, the elasticity was found at the upper place that I will indicate by $E=0,9520$.

(II) At a height from the surface of the sea of 1542 *Paris ft.* the mercury in the barometer fell $21\frac{1}{2}$ *lin.* which held a height of 28 *in. 2 lin.* at sea level: this therefore became $E=0,9364$.

(III) At the height of the mountain peak on the island of Tenerife 13158 *Paris ft.* the mercury stood at a height of 17 *poll.* 5 *lin.* from the surface of the sea, while at the surface of the sea it held a height of 27 *in.*10 *lin.* from which there was in place $E=0,6257$.

(IV) If the descents of the mercury may be observed accurately at smaller altitudes, it is found that a descent of one line corresponds to a height of 65 or 66 *ft.* Therefore at a height of 65 *ft.* there is $E=0,9970$.

These observations are on record here and there : but the third I have from De Lisle and it was prepared and read by R. P. Feuillée in person before the *Societate Reg. Scient. Paris.*: and it is that rock against which all theories press, which have seen the light to this point.

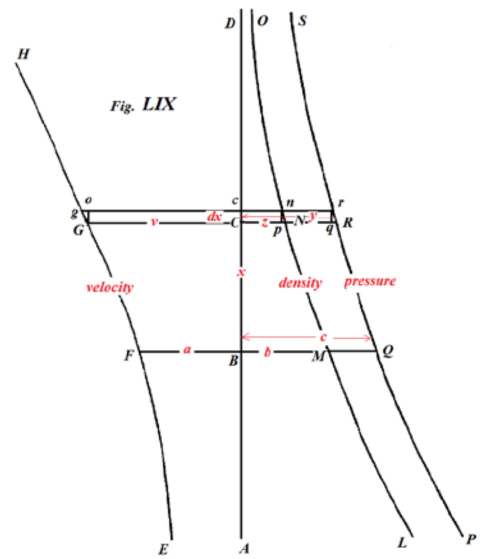
§. 24. As may now be apparent, to what extent these may agree with the position of the logarithms or the scale of the height of the corresponding elasticities, we will put the height of the place from the surface of the sea requiring to be defined by a certain number of feet $=x$; the spring of the air at the surface of the sea we will designate by 1, and the spring of the air at the height $x$ we will put $=E$. Moreover it may be observed the atmosphere now to be considered by us invariable or rather constantly similar to itself, thus so that the springs of the air at the surface of the sea and at some height $x$ maintain a constant ratio. For if certainly it were unequal at different heights of the atmosphere, with no proportion maintained, the springs would be changed with time in a variable manner, unable to be worked out by a sensible rule. From these premises we may now consider the equation $\alpha \log E = x$, where the coefficient $\alpha$ will be determined from a single observation : and we may use from the observation first observation and there will be $\alpha \log 0,9520 = 1070$, and hence $\alpha$ (following Vlacqian logarithms) $=-50194$. Therefore for this business, if the logarithms must be satisfied, putting $-50194 \log E = x$, or

$\log \frac{1}{E} = \frac{x}{50194}$. But according to the standard form of this equation, if for the second observation there may be put $x=1542$, there is found $E=0,9317$, but that observation itself indicates $\mathrm{E}=0,9364$: the difference between the hypothesis and the observation is more than one and a half lines, which however noticeable is with respect to the small difference had of the vertical heights.

If now again for the third observation there may be put $x=13158$, there becomes from the hypothesis $E=0,5469$, while the experiment has indicated $E=0,6257$: which difference is excessive, as that shall not be maintained by any logarithmic manner : for this difference prevails to be more than two inches with two lines.

§. 25. With logarithms rejected the consequence is the elasticities at different heights of the atmosphere are by no means to be proportional to the densities, or what amounts to the same, the average degree of heat to be different at different heights. Others therefore by whom this defect had been noted correctly, were thinking out rules from these other experiments : yet none of these rules it can be said fitted experiment III (§. 23) well enough. Truly, I think it can scarcely be hoped to find a law that may follow nature: from which indeed otherwise as from the most trivial of conjectures an account of the mean velocities of the particles in the air may follow. Yet by chance I have come upon some hypothesis, which corresponds not badly with the phenomena: but first I will give the curve with the law for any of the velocities, then I will come down to this special hypothesis.

§. 26. *AD* shall be a vertical line (Fig. 59); *QF* a horizontal line touching the surface of the sea ; *BF* shall denote the mean velocity of the particles of air at the surface of the sea, *BM* the mean density and *BQ* the elasticity, which in every place is equal to the same height [of mercury]. Then it may be considered to draw the curves *EFH*, *LMO*, *PQS* through the points *F*, *M*, *Q* or the scales, which at all heights, such as *BC*, with the applied lines *CG*, *CN*, *CR* will denote the mean velocities of all the particles of the air, the mean densities and the mean elasticities. Now from two given curves the third may be allowed to be found from that, because the elasticities (as it has been taught from experiment and was in §§. 3, 4, 5 and 6) shall be approximately in a ratio composed from the square of the velocities in the aforesaid manner and of the simple density.




Indeed I have advised in the aforementioned place this proportion itself is not to be exactly true, because the air can have an infinite spring or to be compressed by an infinite force, but it cannot be condensed into an infinitely small space: because still in air which be natural or squared times denser shall have this , *because namely the elasticities shall be in a ratio composed from the square of the velocities of the particles and from the simple density*, even now from experiments was seen generally to correspond to the senses, that without any sensible error as we were able to use for the natural air of the atmosphere resting above the sea, if indeed for that truly to become more accurate the rarer the air shall be.

With that preparation we may put the calculation in place

$$BF = a,\ BM = b,\ BQ = c,\ BC = x,\ Cc = dx,$$

$$CG = v,\ CN = z,\ CR = y, \text{and there will be } y : c = vvz : aab \text{ or } y = \frac{cvvz}{aab}$$

Again because the elasticity measured is the weight of the overlying air, there will be $qR(i.e. -dy) =$ to the weight of the layer of air intercepted between *C* & *c*, which is proportional to the density of the air *z* and to the height of the layer *dx* : therefore there is

$-dy = \frac{zdx}{n}$ or $z = \frac{-ndy}{dx}$ [*n* is the constant of proportionality; there is no reason to suspect it is the same as previously, *i.e*. the particle density], with which value substituted into the equation ( $y = \frac{cvvz}{aab}$) there is found :

$$y = \frac{cvv}{aab} \times \frac{-ndy}{dx}$$

or
$$-\frac{dy}{y} = \frac{aabdx}{ncvv}.$$

§. 27. If the velocity of the particles of air may be put the same at any height, evidently $= a$, there becomes $\frac{-dy}{y} = \frac{bdx}{nc}$, with the integration due makes, $\log\frac{c}{y} = \frac{bx}{nc}$; truly we have seen in §. 24 that hypothesis not to be confirmed enough by experiment. Truly by another test, if I have put $v = \sqrt{(aa + mx)}$ or $vv = aa + mx$, which is the law in the motions of freely falling bodies: nor that without success; thus indeed there becomes

$$\frac{-dy}{y} = \frac{aabdx}{naac + mncx},$$

or

$$\log\frac{c}{y} = \frac{aab}{mnc}\log\frac{aa + mx}{aa}.$$

In this equation with a little more generality in which *m* and *n* even now are arbitrary, again I have taken a risk, whether it is not possible to put $\frac{aab}{mnc} = 1$, and that also I have considered to be appropriate : thus truly I have obtained :

$$\log\frac{c}{y} = \log\frac{aa + mx}{aa} \quad \text{or} \quad \frac{c}{y} = \frac{aa + mx}{aa} \quad \text{or} \quad \frac{y}{c} = \frac{aa}{aa + mx}.$$

That hypothesis indicates the elasticity of the air everywhere to be in the inverse square ratio of the velocities, by which the particles of the air may be moving around, or *CR* to *BQ* to be as $BF^2$ to $CG^2$; and since *EFH* is a parabola by hypothesis upon the vertical axis *AD*, having the point *B* at the distance $\frac{aa}{m}$, it follows the curve *PQS* is a hyperbola ; truly I have taken the said distance $\frac{aa}{m}$ to be $= 22000$ ft., so that from the observations in

§. 23 it may be satisfied approximately. Thence from such now the equation specified will be produced $\frac{y}{c}=\frac{22000}{22000+x}$. Truly for the curve there is found $LMO$ (by §. 26) , or $\left(\text{because } \frac{aa}{vv}=\frac{22000}{22000+x}=\frac{y}{c}\right)$ after this substitution there will be produced

$$\frac{z}{b}=\left(\frac{22000}{22000+x}\right)^2.$$

§. 28. In order that it may be apparent, to what extent our hypothesis may agree with the experiments in §. 23, we may put into the equation for the elasticities successively for $x$, 1070, 1542, 13158, & 65; thus respectively there is come upon $\frac{y}{c}=0,9536$; $\frac{y}{c}=0,9345$; $\frac{y}{c}=0,6257$, and $\frac{y}{c}=0,99705$: but the observations indicate $\frac{y}{c}=0,9520$; $\frac{y}{c}=0,9364$; $\frac{y}{c}=0,6257$, and $\frac{y}{c}=0,9970$. The third observation disagrees strongly from the other hypothesis since our one plainly agrees, nor do the remaining differ by more than 0,0019 small parts, which prevail in the height of the barometer as the three fifth part of a line. But no one who was experimenting, as the observations of the barometer were vague and barely agreeing between themselves, certainly will not care about such a small difference. Meanwhile this matter itself I consider no more than a doubtful hypothesis, nor otherwise on account of the calculated cause presented in §§. 26 and 27, so that as the account I may give, by which it can happen that the vertical heights may not correspond to the logarithms of the barometric heights, such as must come about, if the heat is to be uniform through the whole atmosphere: for with the calculation in place and with a comparison made of this with the experiments I have seen, to seem to me that I am unable to explain sufficiently this matter from the different weight of the particles of air at different distances from the centre of the earth, just as Newton attempted by setting the weights of these to decrease in the ratio of the squares of the distances from the centre of the earth, which hypothesis at heights of 13000 *Paris ft.* by not departing a sensible difference, do not produce a difference from the hypothesis of uniform gravity. Similarly I fell into the opinion finally the increase of the centrifugal force of the particles of air in the greater altitudes could contribute somewhat here; but equally with a calculation put in place I no longer hold this opinion. Meanwhile I do not think it to be absurd, if we may say the average heat of the air to be greater there, where the surface of the sea is more distant. But I wish that it may be observed properly, this discussion to be about the *average* heat in the free atmosphere: thus indeed it can happen, so that the actual heat indeed in mountains may not increase from these other causes, nor yet may the hypothesis be overturned thence, when indeed §§.15 and 16 had been demonstrated, the weight of the column of mercury in the barometer is not required to be agreed to be precisely equal to the weight of a column of air taken in that region, but the average of all the columns pressing on the earth: and therefore I think thus, from the different densities.

§. 29. If everywhere there shall be an equal amount of heat, everywhere the densities with the pressures shall be perceived proportional, and the vertical heights will correspond to the logarithms of the barometric heights: But truly that I put to disagree with experiment: nor still will I believe that a difference of heat can intercede at two placed situated in turn a little distance from each other, because the heat in a rarer body, such as air is, soon is distributed uniformly, unless a cause shall be present always, which may heat the air in the vicinity.

But it is another matter in more distant places; nor indeed do I think it absurd the air in place at the poles may be even ten times denser, than at the equator, but only if the air at both places may be taken near the surface of the earth ; but at greater heights everywhere the difference will be less between the density of the air which corresponds at the poles and of that which corresponds at the equator, with all else being equal, and therefore certainly the densities of the air decrease unequally from the surface of the earth, and much more at the poles than at the equator: therefore in this way it can happen, that under the poles the actual densities of the air at small heights may decrease for example in the ratio as $(22000+x)^4$ to $22000^4$ on account of the heat increase, and at the equator they may not decrease noticeably, on account of the diminution of heat, which diminution of heat may be confirmed near the equator from this because the peak of mount Pico through a space of nearly ten months shall be covered with snow, while on the island of Tenerife at no time do they get snow. Therefore not absurdly the mean densities can be agreed to be diminished in the ratio as $(22000+x)^2$ to $22000^2$, as was assumed in §. 27, while the elasticities everywhere decrease in the ratio as $22000+x$ to $22000$; and indeed nor can these differ at the same height from the surface of the earth, unless from some fortuitous causes arising and with a small duration.

§. 30. In lands, which are contained between the fortieth and sixtieth degrees of latitude, it is probable the densities decrease in the same ratio as the elasticities approximately; and I have wished on account of that reasoning to set out the risks which thence may arise from the theory of refraction, about which matter I may add now something.
…………………………………………………………………………………………….

*A digression concerning the refraction of rays passing through the atmosphere.*

$(\alpha)$ A most noteworthy property of rays incident from one medium into another and that confirmed by innumerable experiments, so that the angle of incidence to the angle of refraction maintains a constant ratio: in addition also it is apparent, if the refraction becomes infinitely small, that is, if the differences of each sine may have an infinitely small ratio to the other sine, to be as the sine of the angle, which is intercepted between the incident ray extended and the refracted ray, may have the same ratio to the total sine , as the difference of the sines of the angles of incidence and refraction to the cosine of the angle of incidence. Truly that angle intercepted, which I have just proposed, between the incident ray prolonged and the refracted ray, henceforth I will call the *differential angle of refraction.* Therefore it follows, because there shall be with all else equal, the sine of the differential angle of refraction to be proportional to the sine of the angle of incidence divided by the cosine of the same angle.

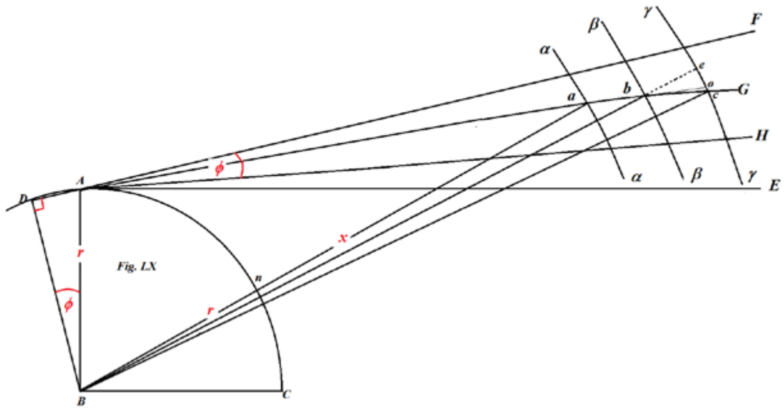
[In a usual notation, the refraction law between two media is expressed by $\frac{\sin i}{\sin r} = \mu$; if however, if the angle of refraction is almost the same as the angle of incidence, we can put $\frac{\sin i}{\sin(i-di)} = \mu \approx 1$; where $\sin(i-di) \approx \sin i - \sin di \cdot \cos i$ and $\cos di = 1$. Hence,

$\frac{\sin i}{\mu} = \sin i - \sin di \cdot \cos i$ and $\sin di = \frac{\left(\sin i - \frac{\sin i}{\mu}\right)}{\cos i} = (\mu - 1)\frac{\sin i}{\cos i}$; Here $\mu - 1$ is the constant of proportionality, which in this case is taken to be related to the density of the air.]

$(\beta)$ Again experiments instruct us, if the ray from air may be incident on another with a different density, the *differential of the angle of refraction* with all else equal to be proportional to the difference of the density.

Moreover experiments into this matter, as far as can be done with the greatest accuracy, were undertaken by Hauksbee certainly both with condensed air, as well as with the most rare air, which yet were to have had no effect: the way in which they were carried out is described in the *English Transactions* [*See F. Hauksbee, Physico-Mechanical Experiments*…..London 1719, p.225-230.]: but the success of all the experiments returned this, that they proved the sine of the angle of the differential of the refraction to the whole sine to be as $5\frac{1}{8}$ inches to 2588 feet, when the ray was incident from natural air into a space with the air evacuated at an angle of thirty two degrees, that is, as 1 to 6060, and with the same in place, with the angle changed from thirty two degrees into a semi-right angle, as 1 to 3787 (by §. $\alpha$ ). Thence it is deduced, if the ray is incident from natural air into a vacuum at any angle whatever, the sine of the angle of incidence to the sine of the angle of refraction to be as 3787 to 3786.

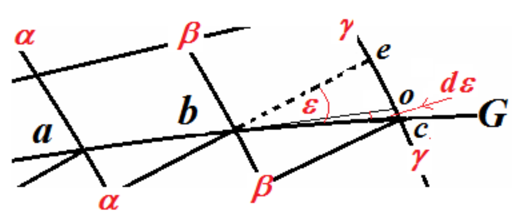
Newton assumed in his *Treatise on Optics* in place of this ratio, that it lies between 3201 & 3200, and he deduced that amount of refraction from the observations of astronomers : but he put in place the amount of refraction to be the same, if the layers refracting the rays shall be parallel, in whatever ratio of the mean densities they may decrease, but only if the difference may remain the same in the first and final layers ( see Newt. *Treatise opt.*, p. 321 French edition). Concerning the rest under diverse circumstances, certainly the refraction can be variable, because the air, which we call natural, shall be liable to many more changes, both from warmth and cold, while from the pressure of the atmosphere, which both agree according to the density of the air being formed, to which density the refractions of the incident rays are proportional with all else being equal. Also Hauksbee advised the same in reviewing the experiments, just as we have alleged, and that on account of the ratio of the state of the air was defined properly , which it was with the experiments he took up.



[There are errors in the following section; the point *e* is not marked on the original diagram; see note 23 in G. K. Mikhailov's treatment in Vol. V of the works ; Mikhailov disagrees with K. F., who thinks there is a flaw in the argument ; clearly the tangent of the angle of incidence is used incorrectly in the following, presumably a continued typographical error, for eventually the ratio used in the integration seems ok. I have put in the correct and indicated the incorrect form in brackets in this section.]

$(\gamma)$ Now the arc of a circle *AC* (Fig. 60) will have been drawn from the centre of the earth *B*, in the plane of which there is the ray of light *AG* [from a star; note that the scale of the diagram is very misleading: evidently the earth's radius is much greater than indicated, while the atmosphere is a thin layer in comparison]: moreover this ray *AG* by its nature will be curving inwards [thus, it bends progressively away from the normal as it goes from more to less dense air], so that it converges to an asymptote, and *AH* may be put parallel to this asymptote*;* the horizontal *AE* may be drawn, and the right line *AF* which is a tangent to the curve *AG* at *A.* Thus we see the angle *HAE* to be the measure of the angle of the true height of the star, and the angle *FAE* a measure of the apparent height, and the angle *FAH* to be the angle of refraction: but the angle *FAH* likewise is the sum of all the *angles of the differential refraction,* or the sum of all the contact angles such as the angle *cbo.*

Two elements of the curve *ab, bo* may be considered, and through the points *a, b, o* the arcs $\alpha\alpha, \beta\beta, \gamma\gamma$ may be considered to be drawn with the common centre *B*: and the density of the air in $\alpha\alpha\beta\beta$ shall be $= D$*;* the density of the air $\beta\beta\gamma\gamma$ will be $D - dD$, equal (by §§ $\alpha, \beta$) to the sine of the contact angle at *b* divided by the whole sine, or the contact angle itself shall be proportional to the difference of the densities *dD* multiplied by the ratio of the sines of the angles of incidence and refraction, that is, multiplied by $\frac{eo}{be}$, $\left[\frac{be}{eo} \text{ in the original.}\right]$.



[ Above we have shown that $\sin d\varepsilon = dD \cdot \tan\varepsilon = dD \cdot \frac{eo}{be}$. The last ratio is inverted above.]

Truly if *BD* may be drawn perpendicular to *FA* produced, it is evident, $\frac{eo}{be}$ and $\frac{BD}{Do}$ scarcely to differ, and thus because the ray shall be almost a right line and thus the triangle *BDo* may be had for rectilinear, and similar to the triangle *beo.*

[This is tantamount to saying that *BE* and *bo* are parallel; more plausibly, we may assert that for the infinitesimal triangle *ebo* and the triangle *BDb* are both right angled, and have equal and opposite angles at *b*. In which case, for the infinitesimal ratio *eo* to *be*, we can use instead the ratio *BD* to *Db*, which amounts to the author's ratio at last.]

Therefore the angle sought *FAH* shall be proportional to $\int \frac{BD}{Do} \times dD$.

[Thus in modern terms, considering the refractive index of the air $n-1$ to be proportional to the density of the air, we may consider the path traced out by the ray to be expressed in the form $n_1 \sin\theta_1 = n_2 \sin\theta_2 = \text{etc.}$; here the normal is the arc $\beta\beta$ and *ab* is the incident ray, while *bo* is the refracted ray; hence we can write $(1+D)\sin ebc = (1+D-dD)\sin ebo.$ This gives approximately

$$\frac{1+D-dD}{1+D} \approx 1-dD \approx \frac{\sin ebc}{\sin ebo}; dD = 1-\frac{\sin ebc}{\sin ebo}$$

$$= \frac{\sin\varepsilon - \sin(\varepsilon - d\varepsilon)}{\sin\varepsilon} = \frac{\sin\varepsilon - \sin\varepsilon\cos d\varepsilon + \cos\varepsilon\sin d\varepsilon}{\sin\varepsilon} = \frac{\cos\varepsilon\sin d\varepsilon}{\sin\varepsilon};$$

$$\therefore dD = \frac{\cos\varepsilon\sin d\varepsilon}{\sin\varepsilon} \text{ or } \sin d\varepsilon = dD \cdot \tan\varepsilon.]$$

$(\delta)$ And by putting these parts into place and by considering everywhere the density $D = \frac{22000}{22000+x}G$, where $x$ expresses the line *na* [*n* is not marked on the original diagram; see note 24 *loc.sit*.] by the number of Paris ft. and *G* denotes the density of the air at the place of observation, I have found what follows. The sine of the apparent height of the star shall be $= F$, the cosine $= f$, the radius of the earth $= r$ being expressed in the number of Paris ft. : the number 22000 may be indicated by *a*: again the whole sine may be put $= 1$, the *angle of the differential refraction* for a ray incident on natural air from a vacuum $= g$, with the angle of incidence less than half a right angle: Finally for brevity there may be put $2r - 2a = \alpha$; $-FFrr + 2ar - aa = \beta$ *:* and $\beta$ a positive or negative number; it will be positive, if the apparent height of the star were very small and indeed less than $2^{o}$, 44'; otherwise it will be negative: In the former case the angle sought *FAH* hence will be obtained in this way:

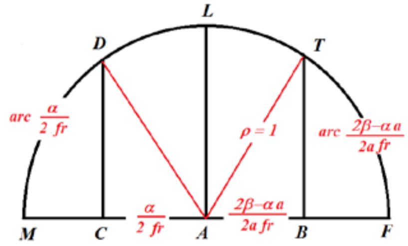


*Fig. LXI*

Truly the semicircle *MLF* (Fig. 61) may be made, of which the radius $AM = 1$; taking $AC = \frac{\alpha}{2fr}$, $AB = \frac{2\beta - \alpha a}{2afr}$, and the perpendiculars *CD, BT* are drawn to *MC*, and the angle *FAH*

$$= \frac{-fFrr}{\beta} g + \frac{far}{\beta} g + \frac{far\alpha \times DT}{2\beta\sqrt{\beta}} g .$$

[Note that the arc $DT = \pi - \sin^{-1}\frac{\alpha}{2fr} - \sin^{-1}\frac{2\beta - a\alpha}{2afr}$; This is Bernoulli's solution; that shown by K. F. replaced $f$ by $F$ in these arcs.]

In the case, when $\beta$ is negative, likewise the angle *FAH*

$$= \frac{-far}{\beta} g + \frac{fFrr}{\beta} g + \frac{far\alpha \times}{2\beta\sqrt{\beta}} g \times \log \frac{\left(\alpha - 2\sqrt{\beta}\right) \times \left(Fr - \alpha + \sqrt{\beta}\right)}{\left(\alpha + 2\sqrt{\beta}\right) \times \left(Fr - \alpha - \sqrt{\beta}\right)}.$$

[Evaluation of the arc DT, according to K. Flierl :

a) Although K. F. has come upon the above errors, he has left the original text in place in his German translation, so that the reader can make a comparison with his discussion following. However, due to the incredible mess someone made typing out the manuscript, his own version is almost useless, and it has taken a lot of patience to resolve matters to some extent. G.K.M comments on this negative aspect of his work also. Bernoulli puts in place the equation for air density he had developed based on §§ 26 and 27 :

$$D = \frac{22000}{22000 + x} G .$$

According to this, at the observation site, that is, at the surface $x = 0$; the density is equal G. Substituting $x = \infty$, we obtain $D = 0$, *i.e.* the vacuum in space. However, be aware that this angle *g* is very difficult to determine.

Further he put in place the differential refraction angle, the sum of which acting on the initial angle *g* gives the total refraction (*FAH*) from vacuum to the surface of the earth; according to the final sentence of the second paragraph of ($\beta$), for all layers of elements *g* is given a constant value : the value of the angle which accomplished the same refraction for normal air . However, be aware that this angle g is very difficult to

determine. The difference between the densities at the beam exit (*G*) and at the entrance (0) $G-0=G$
Further he puts in place the differential refraction angles, the sum of which at the end of part $\gamma$ shall be proportional to $\int \frac{BD}{Do}\times dD$; thus, $\left(FAH\right)=\int_{\infty}^{0}\frac{BD}{Do}\cdot g\cdot dD$ gives the total refraction (*FAH*) from vacuum to the surface, according to the final sentence of the second paragraph of ( $\beta$ ) for all the layers of elements (see Figure 60), where *g* is a constant, taken as the initial angle.
The basic difference *dD* of two consecutive layers *dx* is found from
$D=\frac{22000}{22000+x}\cdot G=\frac{a}{a+x}\cdot G$ to be $dD=-\frac{adx}{\left(a+x\right)^2}\cdot G$. From that the equation of the angle *FAH* becomes

$$\left(FAH\right)=-\int_{\infty}^{0}\frac{BD}{Do}\cdot g\cdot\frac{adx}{\left(a+x\right)^2}$$

The angle of the apparent height of the star shall be $\varphi$. Then, with
$F=\sin\varphi$ and $f=\cos\varphi$, the path

$$BD=r\cdot\cos\varphi=rF \text{ and } Do=\sqrt{Bo^2-BD^2}=\sqrt{\left(r+x\right)^2-r^2F^2}.$$

Accordingly, from the first equation :

$$FAH=-\int_{\infty}^{0}\frac{raF\cdot g\cdot dx}{\left(a+x\right)^2\sqrt{\left(r+x\right)^2-r^2F^2}}\ .$$

Transformation of the second equation:
With the substitution $a+x=u, dx=du$, the argument of the root in the denominator becomes $\left(r+x\right)^2-r^2F^2=\left(r+u-a\right)^2-r^2F^2=u^2+2\left(r-a\right)u+r^2-r^2F^2-2ar+a^2$,

or with $r^2-r^2F^2=r^2f^2$, the equation becomes:

$$u^2+2\left(r-a\right)u+r^2f^2-2ar+a^2=u^2+2\left(r-a\right)u-\left(-r^2f^2+2ar-a^2\right).$$

Bernoulli introduces the following abbreviations :

$$2r-2a=\alpha \text{ and } -r^2f^2+2ar-a^2=\beta$$

and notices for the latter, that $\beta$ will be positive for values of $\varphi<2°44'$ and negative for values of $\varphi>2°44'$
(The angle of 2 ° 44' must arise out of the equation $-r^2f_0^2+2ar-a^2=0$.
The calculation results in $\varphi=2°42'54''$ when $a=22000\,ft$. and $r=19600000$ ). Now

there is almost $F_0^2 = \cos^2 2^0 43' = 0.97 \approx 1$, and hence $-r^2 f^2 + 2ar - a^2 = \beta$ becomes almost $\beta_0 = 2ar - r^2 - a^2 = -(a-r)^2 < 0.$ Flierl notes that Bernoulli remarks that the small values will be positive and not negative, contrary to what he has just found, which Mikhailov disputes in turn without providing any evidence, although it seems that Flierl is correct as we find out by reading on. However, let us continue: The following development of the integral of second equation can now be made :

$$\int_{\infty}^{0} \frac{dx}{(a+x)^2 \sqrt{(r+x)^2 - r^2F^2}} = \int_{\infty}^{a} \frac{du}{u^2\sqrt{u^2 + \alpha u - \beta}} = \int_{\infty}^{a} \frac{du}{u^2\sqrt{U}} \quad (1),$$

where $U = uu + \alpha u - \beta,\ \alpha = 2r - 2a$ and $\pm\beta = -rrff + 2ar - aa.$
Instead the limits $0$ and $\infty$ now the limits are set as $a$ and $\infty$; then $(a+x)_{x=0} = u = a$ .
We obtained first for the general solution (1) :

$$\int \frac{du}{u^2\sqrt{U}} = +\frac{\sqrt{U}}{\beta u} + \frac{\alpha}{2\beta}\int \frac{du}{u\sqrt{U}} \qquad (2),$$

as you can convince yourself by returning the differentiation :

For if we consider the function

$\frac{\sqrt{U}}{\beta u} = \frac{\sqrt{u^2+\alpha u-\beta}}{\beta u}$, and differentiate, we obtain :

$$\frac{d}{du}\cdot\frac{\sqrt{u^2+\alpha u-\beta}}{\beta u} = -\frac{\sqrt{u^2+\alpha u-\beta}}{\beta u^2} + \frac{2u+\alpha}{2\beta u\sqrt{u^2+\alpha u-\beta}}$$

$$= -\frac{\sqrt{u^2+\alpha u-\beta}}{\beta u^2} + \frac{1}{\beta\sqrt{u^2+\alpha u-\beta}} + \frac{\alpha}{2\beta u\sqrt{u^2+\alpha u-\beta}}$$

$$= \frac{\beta-\alpha u}{\beta u^2\sqrt{u^2+\alpha u-\beta}} + \frac{\alpha}{2\beta u\sqrt{u^2+\alpha u-\beta}}.$$

$$\therefore \frac{\sqrt{u^2+\alpha u-\beta}}{\beta u} = \int\frac{(\beta-\alpha u)du}{\beta u^2\sqrt{u^2+\alpha u-\beta}} + \int\frac{\alpha du}{2\beta u\sqrt{u^2+\alpha u-\beta}}$$

$$= \int\frac{du}{u^2\sqrt{u^2+\alpha u-\beta}} - \alpha\int\frac{du}{2\beta u\sqrt{u^2+\alpha u-\beta}}, \text{ or :}$$

$$\frac{\sqrt{U}}{\beta u} = \int\frac{du}{u^2\sqrt{U}} - \frac{\alpha}{2\beta}\int\frac{du}{u\sqrt{U}}.$$

or, as above (2): $\int \frac{du}{u^2\sqrt{U}} = +\frac{\sqrt{U}}{\beta u} + \frac{\alpha}{2\beta}\int \frac{du}{u\sqrt{U}}$. Thus, the above integration is correct.

To resolve further, the integral on the right side will need to be considered, if $\beta$ shall be positive or negative.
The integration for $\beta > 0$ or $-\beta < 0$:
We complete the square for the function $U$ on the r.h.s. :

$$U = uu + \alpha u - \beta = \left(u + \frac{\alpha}{2}\right)^2 - \frac{\alpha^2}{4} - \beta$$

$$\frac{\alpha}{2\beta}\int \frac{du}{u\sqrt{U}} = \frac{\alpha}{2\beta}\int \frac{du}{u\sqrt{\left(u+\frac{\alpha}{2}\right)^2 - \left(\frac{\alpha^2}{4}+\beta\right)}} \qquad (3)$$

Following *Praktische Funcktionenlehre* von Friedr. Tölke, Vol. 1, 2nd Ed., p. 127, for which the solution applies, for $b < a^2$

$$\frac{\alpha}{2\beta}\int \frac{du}{u\sqrt{U}} = -\frac{\alpha}{2\beta}\cdot\frac{1}{\sqrt{-\beta}}\cdot \operatorname{arc}\cosh\left(\frac{-\beta+\frac{1}{2}\alpha u}{u\sqrt{\frac{\alpha^2}{4}-(-\beta)}}\right) = +\frac{\alpha}{2\beta}\cdot\frac{1}{\sqrt{\beta}}\cdot i\cdot \operatorname{arc}\cosh\left(\frac{\alpha u - 2\beta}{u\sqrt{\alpha^2+4\beta}}\right)$$

$$= +\frac{\alpha}{2\beta}\cdot\frac{1}{\sqrt{\beta}}\cdot \arccos\left(\frac{\alpha u - 2\beta}{u\sqrt{\alpha^2+4\beta}}\right) \qquad (4);$$

Now we have the whole solution of the integral of the above equation, for the case $\frac{\alpha^2}{4}+\beta > 0$, on putting the limits $(a,\infty)$ in place for $u$ in $(2)$:

$$\int_\infty^a \frac{du}{u^2\sqrt{U}} = +\left[\frac{\sqrt{U}}{\beta u}\right]_\infty^a + \frac{\alpha}{2\beta}\int_\infty^a \frac{du}{u\sqrt{U}};$$

For the lower limit :

$$\left[\frac{\sqrt{U}}{\beta u}\right]_\infty = \left(\frac{u\sqrt{1+\frac{\alpha}{u}-\frac{\beta}{u^2}}}{\beta u}\right)_\infty = \frac{1}{\beta};\ \left(\frac{\alpha u - 2\beta}{u\sqrt{\alpha^2+4\beta}}\right)_\infty = \left(\frac{\alpha}{\sqrt{\alpha^2+4\beta}}\right)_\infty = \frac{\alpha}{\sqrt{\alpha^2+4\beta}} = \frac{\alpha}{2rF}.$$

And for the upper limit for relevant terms :

$$\left[\frac{\sqrt{U}}{\beta u}\right]^a = \left[\frac{\sqrt{uu+\alpha u-\beta}}{\beta u}\right]^a = \frac{\sqrt{aa+(2r-2a)a-rrff-2ar+aa}}{\beta u} = \frac{rf}{\beta a} \text{ and}$$

$$\frac{\alpha u-2\beta}{u\sqrt{\alpha^2+4\beta}}=\frac{\alpha a-2\beta}{a\sqrt{4rr-4rrff}}=\frac{\alpha a-2\beta}{2arF}$$

Now we have the whole solution of the integral of the above equation, on putting the limits $(a,\infty)$ in place on the integral :

$$\int_\infty^a \frac{du}{u^2\sqrt{U}}=\frac{rf}{\beta a}-\frac{1}{\beta}-\frac{\alpha}{2\beta\sqrt{\beta}}\left(\arccos\frac{\alpha a-2\beta}{2arF}-\arccos\frac{\alpha}{2rF}\right),$$ and finally we find the refraction angle for $\beta>0$ :

$$FAH=-\frac{rrfF}{\beta}\cdot g+\frac{raF}{\beta}\cdot g+\frac{raF}{2\beta\sqrt{\beta}}\cdot g\left(\pi-\arccos\frac{2\beta-\alpha a}{2arF}-\arccos\frac{\alpha}{2rF}\right).$$

We have thus elucidated the last term in brackets of Bernoulli's first equation, except that in the second and third terms instead of the sine *f* of the apparent star height, thus instead the cosine F. Thus it appears that only the first term is unchanged, due to the interchange of f and F in the other terms. Thus at this point there is an actual error, but the numbers of the table generated agree well with Bessel's calculations, which would have resulted by calculation from his equations. This leads to the conclusion that he possessed the correct solutions in his manuscript and the confusion between f and F somehow appeared at the time of printing.

Integration for the case $\beta<0$.

If $f=\sin\varphi>\sin 2^0 43'$, then $\beta=-rrff+2ar-aa$ becomes negative and the root argument in $\sqrt{U}=\sqrt{uu+\alpha u-\beta}$ becomes positive in the third equation. In order to avoid errors in signs, we put rather $c=-\beta=rrff-2ar+aa$ and $\sqrt{U}=\sqrt{uu+\alpha u+c}$. According to the same reference as before, we have

$$\int\frac{du}{uu\sqrt{U}}=-\frac{\sqrt{U}}{cu}-\frac{\alpha}{2c}\int\frac{du}{u\sqrt{U}}=-\frac{\sqrt{U}}{cu}+\frac{\alpha}{2c\sqrt{c}}\cdot\log\left(\frac{2\sqrt{c}\sqrt{U}}{u}+\frac{2c}{u}+\alpha\right).$$

Putting in place the limits $(a,\infty)$ on the integral, we obtain :

$$\int_\infty^a\frac{du}{uu\sqrt{U}}=-\frac{rf}{ca}+\frac{1}{c}+\frac{\alpha}{2c\sqrt{c}}\cdot\left[\log\left(\frac{2\sqrt{c}\cdot rf}{a}+\frac{2c}{a}+\alpha\right)-\log\left(2\sqrt{c}+\alpha\right)\right].$$

and hereby the refraction angle is found for $\beta<0$ :

$$FAH = \frac{Far}{c}\cdot g + \frac{Ffrr}{c}\cdot g + \frac{Far\alpha}{2c\sqrt{c}}\cdot g\log\left(\frac{a\left(2\sqrt{c}+\alpha\right)}{2\sqrt{c}\cdot rf + 2c + \alpha a}\right)$$

Finally, K.F. has verified Bernoulli's amended equations by a calculation of the observed known value of FAH = 5 '28 " from the apparent height of 10°, and similarly the apparent height of 20°, were found to be close matches.]

$(\varepsilon)$ Following this hypothesis by putting 19600000 for the radius of the earth, for any apparent height of the star it will be possible to determine its astronomical refraction, if the value of the angle *g* were found best from experiment; because truly it is with great difficulty this value can be defined with sufficient accuracy, it will be more prudent in any particular case to define it by astronomical refraction, and from this the remaining can be deduced by calculation. We may assume for example at a height of ten degrees the refraction to be 5 *min.* 28 *sec.*,[the modern value is taken as 5.3'] and most Parisian astronomers adhere to this hypothesis. We will find this table of refraction :

| App. ht. of star. | refract. | App. ht. of star. | refract. |
|---|---|---|---|
| *0 grad.* | 34 *min.* 55 *sec.* | *50 grad.* | *0 min.* $52\frac{1}{2}$ *sec.* |
| 5 | 9 .... 45… | 55 | …44 |
| 10 | 5 .... 28… | 60 | …$36\frac{1}{5}$ |
| 15 | 3 .... 44… | 65 | …$29\frac{1}{4}$ |
| 20 | 2 .... 48… | 70 | … 23 |
| 25 | 2 .... 12… | 75 | …17 |
| 30 | 1 .... 47… | 80 | …11 |
| 35 | 1 .... 29… | 85 | …$5\frac{1}{2}$ |
| 40 | 1 .... 14… | 90 | …0 |
| 45 | 1 .... $2\frac{1}{2}$… | | |

Truly because the refractions follow the ratio of the letter *g*, that is, the angles of the differential refraction of a ray less than half a right angle from ordinary air incident in and because this angle is proportional to the density of natural air, or of the air which the observer breathes, it is apparent even if the air shall be constantly heavy similarly with vapors (which at this point we have removed from consideration), it still happens that astronomical refractions shall be exceedingly variable. Clearly they are greater at sea level than in mountains, and that difference will be notable even in the medium heights of mountains: besides the greater owns will be in a freezing rather than in a warm time and by this cause alone in these lands the minimum refraction can increase by a fourth part: finally also the refractions will be greater with a barometer high rather than a low. Moreover, the refractions can be defined correctly all the time, if vapors shall not be a concern, if the instrument, which was described in §. 9 and which may represented in Fig. 57, likewise may be used with a barometer; for if you divide the height of mercury in the barometer by the height of mercury in the other instrument, [which is a primitive kind of

thermometer] you will have the density of the air, to which with all else equal the refraction is being made proportional. Nor do I doubt, why the refraction of the sun shall be less than the refraction of the remaining stars, because the heat of the sun not only expands the air and diminishes the density of the air a little.
……………………………………………………………………………………….

§. 31. From these matters which are concerned with the motions of particles of the air, on which certainly the heat of the air depends, to be especially true from what was advised in §.10, it is apparent the same degree of heat of the air is present, whenever the same ratio exists between its elasticity and density ; the barometer indicates the elasticity ; the density we gather from the specific gravity of the air; and thence as we have seen in §.10, it will be possible to obtain a fixed degree of heat [*i.e.* by using a fixed or standard pressure], if the heat of boiling water may seem to be uncertain, just as Fahrenheit had observed that it depended on the weight of the overlying atmosphere. The instruments which indicate the density of the air at individual moments can be understood easily and many have been described.

Here it is required to observe that ratio, in the manner said between the elasticity of the air and its density, likewise to shows the height of homogeneous air, and because henceforth our discussion will be about that height, it will be convenient to define that first correctly, as we may go on to other matters.

§. 32. If we may imagine a vertical column of air of uniform density and with the mercury of barometer put at equilibrium, the *height* of this column I will call the height of the *homogeneous air* for the given density.

And because the specific gravity of moderately dense air is to the specific gravity of mercury as 1 to 11000 and that average height of the mercury in the barometer for places little above sea level shall be $2\frac{1}{3}$ *Paris ft.*, the height of the homogeneous moderately dense air will be 25666 ft.

From this definition it is apparent these heights, we are discussing now, to be less there, when the air to which the height must correspond is denser, and when the height of mercury is smaller. Therefore if the degree of heat shall be the same in the mountains and at sea level, also the height of the air will be the same in both places, because for the same degree of heat, the density of the air follows the ratio of the elasticity of the air or of the height of the mercury in the barometer. Again it is apparent the height of the homogeneous air at sea level certainly decreases from the equator towards the poles, because the cold and density extend out and the density of the air is increased with the elasticity remaining, and in the same regions [the height of the mercury] to be less in wintertime than in summertime.

§. 33. There are many things which are related towards defining the motion of the air, whose solution depends on the height of the homogeneous air : Among these also is the propagation of sound and its speed: For although the speed of sound may be defined in many different ways from each other, which we can consider by its propagation, thus from the different manners, so that now the speed may be seen to be that which must be due to the air of a homogeneous atmosphere, again which may correspond to half the height, or even to half the height multiplied by multiplied by the ratio of a circumscribed square to the area of a circle, yet all the opinions are in agreement with that, so that the speed of sound shall be proportional to the square root of the height of the homogeneous air with that, in which it may be propagated. Thus if this matter may itself be considered, sound will be propagated faster in warmer air than in colder air, with the height of the barometer as low (I say nothing about following or against the wind); many experiments have been undertaken both in Italy and in England regarding this matter, and these latter ones set out the average speed of sound to correspond to 1140 *English ft.* being completed in one second. But because at one and the same place named here the height of the homogeneous atmosphere is variable and departs from that with many barometric changes jointly with changes of the heat from 3 as far as 4, everywhere the speed of sound will be variable, even if the wind may change nothing, and the speed in these lands may be contained within the limits $\sqrt{3}$ and $\sqrt{4}$, or 173 & 200.

§. 34. I come now to various questions requiring to be solved which can be composed concerning the motion of air similar to these, which we have considered in the preceding concerning the motion of non-elastic fluids.

Problem.

The motion shall be required to be defined of air escaping from a vessel through a small opening into an infinite space with the air evacuated.

Solution.

It is apparent from the nature of the question the local internal motion of the air to be unaffected when it itself expands, while a certain amount escapes through the opening, t: Therefore here this customary ascent potential, which a particle may acquire, while it is being expelled, is required to be considered, and comparing with the *actual descent*, or rather with the diminution of the elasticity, which the internal air has got. [The science of thermodynamics had not yet progressed to considering adiabatic processes ; in this case the cooling of the ejected air on expansion.]Truly so that we may reduce the whole matter to our method using non-elastic fluids [*i.e.* ordinary hydrodynamics ; here an equivalent cylinder of a height that generates an equal velocity by a particle falling from rest, according to the *vis viva* principle, to the velocity of the particles ejected by the actual vessel], we will consider a vertical cylinder of a size common with the proposed vessel and of such a height, as great as the height of homogeneous air with the air inside ; truly this cylinder, if it may be agreed to be filled with air, but not elastic, the lower air will be expelled with the same velocity through an opening, by which the air in the proposed vessel by its elasticity will itself be expelled. Moreover, in the first case it must be ejected with the velocity corresponding to the height of the cylinder itself, and therefore in the latter case. But it is required to be observed, the height which we have devised for the cylinder, must be the same always, because the elasticity and density of the air are diminished in the same ratio, but we do not consider the heat to be changed. Therefore if the height of the homogeneous air (which depends on the heat of the internal air) may be called $A$, the air constantly flows out with the velocity $\sqrt{A}$. Nor yet, which the calculation has shown, shall the vessel itself ever be emptied, because the air flowing out shall continue to become rarer, so that as we may understand from the equation, we may put the density or the initial amount of the flow of the air $=1$; the density or amount of air left after a definite time shall be $=x$, and the time itself will be $=t$ , because the constant speed is, $-dx = axdt$ , [*i.e.* the rate of change of the density is decreasing in proportion to the density] where the size of the constant *a* is understood to be defined from the magnitude of the vessel, from the cross-section of the opening and from the height *A:* hence $\frac{-dx}{x} = adt$ and $\log\frac{1}{x} = at$; moreover the value of the coefficient *a* is found in this way. Because there was put by us $-dx = axdt$ , from the beginning the outflow will be $-dx = adt$ . Now the first element [increment] of the density ( $-dx$ ) may be changed into a cylinder based on the opening as constructed ; but the height of this cylinder itself $= -Ldx$ , if *L* shall be the height of the cylinder constructed above the same opening, and having a common capacity with the proposed vessel [*i.e.* $L =$ volume of vessel/area of opening ]: again this length $-Ldx$ is that [fraction of the height that may be traversed in the time *dt*], which may be transversed in the time increment *dt*, and because it is accustomed to have put the time increment equal to the interval traversed divided by the velocity, this will be $dt = \frac{-Ldx}{\sqrt{A}}$ ; this value may be substituted into the equation $-dx = axdt$ and there will be found $-dx = \frac{-aLdx}{\sqrt{A}}$, or $a = \frac{\sqrt{A}}{L}$. Hence this is the final equation :

$$\log\frac{1}{x} = \frac{t\sqrt{A}}{L}.$$

If it may be wished to express the time by a certain number of seconds, which we will call *n*, and by the interval *s* is understood the distance which a moving body has resolved by falling freely from rest within a single second, [i.e. $s = \frac{1}{2}g$ in modern terms; in Bernoulli's time, the acc. of grav. was taken as $= \frac{1}{2}$, so that our equation $s = \frac{1}{2}gt^2 \to \frac{1}{4}t^2$, and $t = 2\sqrt{s}$ and thus a time n times longer becomes $t = 2n\sqrt{s}$ ] there will be required to put $t = 2n\sqrt{s}$ ,and thus there arises

$$\log\frac{1}{x} = \frac{2nt\sqrt{As}}{L}.$$

Problem.

§. 35. The motion is sought of denser air escaping from a vessel through a very small opening into infinitely rarer external air, with the same degree of heat considered in each air.

Solution.

Let the density of the internal air be $= D$, the density of the external air $= \delta$, the density of the internal air remaining after a given time *t* shall be $= x$, the height of the homogeneous air ( either in the ratio of the internal air or the external air; nor indeed can it be different, if each air shall be endowed with the same degree of heat, and thus the densities and elasticities shall decrease in equal proportion) $= A$. The height of the homogeneous air is sought everywhere, which may have the same pressure or spring as the external air, and the density of which shall be the same as the internal air: this height at the beginning will be $\frac{\delta \cdot A}{D}$, and after the time *t* it will become $\frac{\delta \cdot A}{x}$. But it is apparent the velocity of the air escaping to be such everywhere, which shall correspond to the difference of the heights defined $A$ and $\delta A$; and thus after the time *t* the velocity of the air escaping $= \sqrt{A - \frac{\delta \cdot A}{x}}$.

[Thus, a measure of the residual pressure in the vessel for some intermediate density is the height of equivalent air column of normal air.]

Again the decrements of the densities ($-dx$) are proportional to the quantities of air escaped, which have a ratio composed from the velocity $\left(\sqrt{A - \frac{\delta \cdot A}{x}}\right)$, from the density ($x$), and from the increment of the time ($dt$): thus so that there is

$$-dx = a\sqrt{A - \frac{\delta \cdot A}{x}}xdt ,$$

where *a* is a constant number, which by the method of the preceding paragraph shall become $= \frac{1}{L}$, with the meaning of this letter used retained in that place ; and with this value substituted there arises

$$-dx = \frac{dt}{L} \times \sqrt{(Axx - \delta \cdot Ax)}$$

or

$$\frac{-dx}{\sqrt{(xx - \delta \cdot x)}} = \frac{dt\sqrt{A}}{L} :$$

And with the integration due being accomplished, there becomes :

$$\log \frac{\left(\sqrt{x} - \sqrt{x - \delta}\right) \times \left(\sqrt{D} + \sqrt{D - \delta}\right)}{\left(\sqrt{x} + \sqrt{x - \delta}\right) \times \left(\sqrt{D} - \sqrt{D - \delta}\right)} = \frac{t\sqrt{A}}{L},$$

[*i.e.* to show the correctness of the procedure, we can differentiate to restore the original :

$$\log\frac{\left(\sqrt{x}-\sqrt{x-\delta}\right)\times\left(\sqrt{D}+\sqrt{D-\delta}\right)}{\left(\sqrt{x}+\sqrt{x-\delta}\right)\times\left(\sqrt{D}-\sqrt{D-\delta}\right)}=\frac{t\sqrt{A}}{L},$$

$$\log\left(\sqrt{x}-\sqrt{x-\delta}\right)+\log\left(\sqrt{D}+\sqrt{D-\delta}\right)-\log\left(\sqrt{x}+\sqrt{x-\delta}\right)-\log\left(\sqrt{D}-\sqrt{D-\delta}\right)=\frac{t\sqrt{A}}{L},$$

$$\frac{1}{\sqrt{x}\times\sqrt{x-\delta}}=-\frac{dt\sqrt{A}}{dxL};\frac{1}{\sqrt{x}\times\sqrt{x-\delta}}=-\frac{dt\sqrt{A}}{dxL},$$

which agrees with the original equation.]
or again on putting, as in the preceding paragraph, $t=2n\sqrt{s}$, there will be

$$\log\frac{\left(\sqrt{x}-\sqrt{x-\delta}\right)\times\left(\sqrt{D}+\sqrt{D-\delta}\right)}{\left(\sqrt{x}+\sqrt{x-\delta}\right)\times\left(\sqrt{D}-\sqrt{D-\delta}\right)}=\frac{2n\sqrt{As}}{L}.$$

Corollary 1.

§. 36. All the efflux will happen in a finite time where in this matter the question differs from the other preceding: Moreover the air stops to flow out, when $x=\delta$, and then there shall become

$$n=\frac{L}{2\sqrt{As}}\times\log\frac{\left(\sqrt{D}+\sqrt{D-\delta}\right)}{\left(\sqrt{D}-\sqrt{D-\delta}\right)}.$$

For example let $A=26000$ *Paris ft.*; the proposed vessel may contain one cubic foot of air, moreover the opening may have a cross-section of 1 square line, and there will be $L=20736$ [ $=12^4$]; above the internal air may be put from the start to be of twice the density of the outer air ; moreover it is agreed that $s=15\frac{1}{12}$ *Paris ft.* [Huygens had measured this as early as 1659; see Prop. XXVI of the *Horologia*] and therefore

$$n=\frac{20736\sqrt{3}}{\sqrt{181\cdot 26000}}\log\frac{\left(\sqrt{2}+1\right)}{\left(\sqrt{2}-1\right)}=29,2$$

which indicates the air everywhere is going to composed in equilibrium in a time a little more than twenty nine seconds, and after that all the efflux is going to cease. But a contraction of the fluid can happen, which the fluid may endure in front of the opening (see Sect. IV) and to which we have paid no attention in the computation, so that the time may be increased nearly in the ratio as 1 to $\sqrt{2}$. [This of course does not happen in the case of air, as opposed to water.]

Corollary 2.

§. 37. If it may be imagined the air does not at once flow out via the opening, but through a long tube, therefore the velocity will not be changed, but only if the capacity of the long tube shall be as if infinitely small in relation to the capacity, which is in the vessel itself ; but it may be considered the density of the air, as long as it is in the tube, to be the same as the density of the air enclosed in the vessel, nor yet, which I shall show below, the elasticity of the air in a greater tube is with the elasticity of the external air, which surrounds the tube. Thus it follows, the air of wind is to be denser than still air, but not to have more elasticity: but yet the difference of the density will be very small too ; for a wind, which makes a speed of 30 ft per second, scarcely will become a 1700$^{\text{th}}$ part denser than the neighbouring air equally warm and at rest.

Problem.

§. 38. To define the influx of air through a very small opening into a vessel filled with rarefied air, again with the same degree of heat in both places.

Solution.

From the beginning the vessel was completely empty, and after a time *t* the density of the internal air $= x$; thus it may be found by following almost the same procedure, as we have used in paragraph thirty five, and with the same denominations kept either

$$\frac{dx}{\sqrt{(D-x)}} = \frac{dt\sqrt{AD}}{L}$$

or

$$t = 2n\sqrt{s} = \frac{2L}{\sqrt{A}} = \frac{2L\sqrt{(D-x)}}{\sqrt{AD}}.$$

Therefore the number of seconds, in which the whole vessel will be filled, while there shall be equilibrium between each air, is expressed by $\frac{L}{\sqrt{AS}}$: and the time of filling is twice that when it may be filled if the air flows in constantly with the initial velocity. In the case where the capacity of the vessel holds a cubic foot and the opening equals a square line [*i.e.* $\frac{1}{144}^{\text{th}}$ part of a square inch], it will be filled in a time of approximately thirty three seconds, unless the filling may be retarded due to the influence by the contraction of the jet of air.

§. 39. We have put in place the properties of various elastic fluids, whether they be of moving fluids, or fluids at rest: One remains not to be omitted, by which elastic fluids differ from non-elastic ones, clearly this, because with an elastic fluid, even at rest the *vis* *viva* shall be in place, not because in the manner of other moving bodies it may be able to raise itself to a certain height, nor even do we consider a local motion here in that, but because by its spring such an ascent may be able to be generated in other heavy bodies. Moreover it will be allowed, as I hope, in the following to use the *vis viva* for a body by designating the innate *elastic compression*, where nothing else may be understood there than the *ascent potential*, which the elastic body is able to communicate to other bodies before all its elastic force will have been lost.

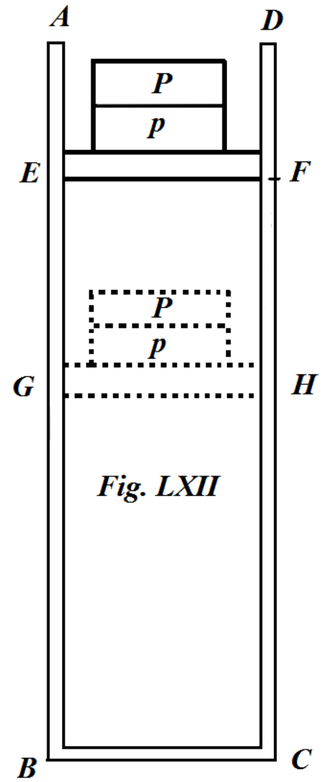


Fig. LXII

This deserves to be noted in the preceding, because just as the descent of a give body through a given height, however it happens, produces the same living force [*vim vive*] in the body, thus also the elasticity of a body or fluid, after it has been reduced to some other given degree from a given tension or condensation, that itself always receives the same living force and again shall be able by an opposite change to communicate it to other bodies.

Regarding the living forces of this kind I may now perform a few measurements on the innate elastic compression for a fluid: the argument is worthy of attention, because there the measurements of the forces may lead back to machines required to be moved by air, either by fire or by other moving forces of this kind, of which several new kinds perhaps shall be able to be devised but not be without an addition of practical mechanical insight, and will be able to be understood perfectly.

§. 40. So that we may begin from air into a vacuum, we will consider the vertical cylinder *ABCD* put in place (Fig. 62) with the support *EF*, which considered weightless may be able to be moved up or down freely. The space *EBCF* encloses air, but the whole cylinder is imagined to be put in a vacuum : The pressure of the air *EBCF* shall be able to sustain a weight as great as *p*, which will be equal to the pressure of the atmosphere, if this air shall be natural. Now another weight *P* may be placed on top*:* thus it happens that the support falls to *GH* and may be moved about between the points *H* and *F* by a reciprocal motion. So that we may define the motion, we will use the common hypothesis, so that with all the rest remaining the same, the pressures of the air shall be proportional to the densities.

And thus if there were $FC = a,\ FH = x$ [*i.e.* not the equilibrium position: this occurs when $p \cdot a = (P + p) \cdot (a - x_{eq})$; thus, at any other distance *x*, the imbalance between $P + p$ and $\frac{pa}{a - x}$ contributes an unbalanced pressure $(P + p) - \frac{pa}{a - x}$ ]; the velocity of the support at the position $GH = v$, the pressure will be, by which the support *GH* may be pushed in order to descend further, $= P + p - \frac{a}{a - x} p,$ and the force may be considered to be equal to this pressure, to which the weight resting on the support may give rise ; [Note again that here the acceleration of gravity is taken as 1, which has the convenience that mass and weight are described by the same number, taken synonymous with the pressure here, and in this case the *vis viva* equation has the fraction of a half present.] ; therefore if you divide this force by the mass you will have the accelerating force, which multiplied by the element of the time $\frac{dx}{v}$ will give the increment of the velocity *dv;* that is

$$dv = \left(P + p - \frac{ap}{a-x}\right) \times \frac{dx}{v} : (P+p),$$

or

$$\tfrac{1}{2}(P+p)vv = (P+p)x - ap\log\frac{a}{a-x}.$$

But from the descent of the weight $(P+p)$ through the height $x$ the potential *vis viva* $(P+p)x$ arises, and when the support is at the situation *GH*, the actual *vis viva* of the body $(P+p)$ is $\tfrac{1}{2}(P+p)vv$, that is, $(P+p)x - ap\log\frac{a}{a-x}$, which is deficient from the first quantity by $ap\log\frac{a}{a-x}$, and this has passed into the compression of the air.

And thus I say it is not possible for the air occupying a volume $a$ to be condensed into the volume $a-x$, *that may not depend on the vis viva, which may be generated from the descent of a weight p through the height* $ap\log\frac{a}{a-x}$, *in whatever manner the compression were made; moreover it can happen in an infinite number of ways.* Truly I will illustrate this rule by one or another example.

[The similarity to the result obtained from Boyle's law for the work done in the isothermal compression of a gas is at once apparent.]

Let the base of the cylinder be of one square foot, the initial height *FC* of two feet: and that may be held in a volume *BF* of such air as shall be accustomed to be the average on the surface of the earth, which surface *EF* shall be able to carry 2240 pounds [*i.e.* the normal atmospheric pressure]: putting $x = 1$, thus so that the *vis viva* may be had, by which two cubic feet of natural air can be forced into a volume of one cubic foot in vacuo: and this *vis viva* will be $= 2 \times 2240 \times \log 2 = 3105$, that is, by such which may be generated by the fall of a body with a weight of 3105 pounds through a height of one foot. Therefore and in turn, if a cubic foot of air may be had with double the density of natural air, with its aid a weight of 3105 pounds may be raised to a height of one foot in vacuo, while it has assumed the density of natural air.

Again under the same remaining circumstances the same air shall be expanded into double the volume as before, now occupying a height of four feet in the cylinder, and again this may be condensed into a volume of one cubic foot; for this a compression *vis viva* shall be required, which is expressed by $4 \times 1120 \log 4$, which is greater by two with the former. Therefore if in vacuo there may be had a cubic foot of air denser by twofold of natural air, with the aid of that a weight of 6210 pounds will be able to be raised to a height of one foot, while it assumes the density half the density of natural air, or a weight of 9315 *lb.* while with the natural air it shall become four times rarer.

Thence it follows, if the air in a volume shall be able to expand itself indefinitely and everywhere it may keep the elasticity proportional to the density, the *vis viva* for a finite quantity of air becomes infinite.

§. 41. But this is relevant for the valuation of the *vis viva*, which shall be put in place for the air in vacuo : for the computation for denser air will be a little otherwise, which has been put into the atmosphere: for here the maximum degree of the expansion cannot be extended beyond the equilibrium with the air of the atmosphere: hence it is foreseen easily in the preceding, if for example a cubic foot of denser air may be had with the double of natural air, the *vis viva* which shall be able to be elicited into the atmosphere from this compressed air, to be far from infinite. But the *vis viva* of this kind are to be determined in this way.
[By now it is apparent that the *vis viva*, or *living force* introduced by Leibniz, as considered by Daniel Bernoulli, is to be identified with the kinetic energy of the body; whether the factor of a half is included depends on the value adopted for the acceleration of gravity at the time. Although it had been measured quite accurately by Huygens many years before, it was taken usually either as a half or as one, mainly for computational convenience, it would seem.]

§. 42. *EBCF* shall be natural air and in equilibrium with the external air; moreover it may be understood by *p* the pressure of the atmosphere on the support *EF*, which certainly is in equilibrium with the pressure of the internal compressed air. The weight *P* may be imposed on the same support *P;* for the air had been compressed into the volume *GBCH*, and the support with the weight *P* pressing at the position *GH* may have the velocity *v;* and with the remaining denominations retained there will be :

$$dv = \left( P + p - \frac{ap}{a-x} \right) \times \frac{dx}{v} : P,$$

or

$$Pvdv = \left( P - \frac{xp}{a-x} \right) dx,$$

which integrated gives

$$\tfrac{1}{2} Pvv = Px + px - ap \log \frac{a}{a-x}.$$

Now truly from the descent of the weight *P* through the height *x* the *vis viva Px* will have been generated, from which as for the same weight with the velocity *v* from the motion the part present is $\frac{1}{2} Pvv$ or $Px + px - ap \log \frac{a}{a-x}$; therefore the part of the *vis viva* which has been transferred to the air is $=-px+ap\log\frac{a}{a-x}$, which is less than defined by the other in §. 40.

For example a cubic foot of denser air may be had for twice normal; the vis viva may be found, which that air permits, while it takes the density of the natural air around, that which is produced by the free fall of a body with the weight 865 *lb.* through a height of one foot.

By like thinking, a cubic foot of air denser by three times natural air is considered to have a *vis viva* such as may correspond to the fall of a body of weight of 2898 *lb*. through an altitude of one foot, which number surely is produced when there is put $p=2240$, as in §. 40; $a=3$, and $x=2$.

§. 43. It is evident from this agreement between the conservation of the innate *vis viva* for compressed air and for a body falling freely from a given height, there is nothing hoped for according to the preferred use of machines succeeding from the principle of compressed air, and generally the rules I have shown in the preceding section prevail. Because truly it can happen in many ways, that the air may acquire a compression or greater spring not by force but naturally, certainly it is the hope, to be able to devise a great saving from the nature of things of this kind for machines being moved, just as Amontons has demonstrated already a way of moving machines by the force of fire. I have convinced myself if all the *vis viva* latent in a cubic foot of coal, is elicited from the same by combustion, expended usefully towards moving a machine, because more thence may be able to be used, than by the daily labour of eighteen men. And indeed charcoal while being burned not only augments the elasticity of the air significantly, but also generates a huge amount of a new air [carbon monoxide and dioxide had yet to be discovered].

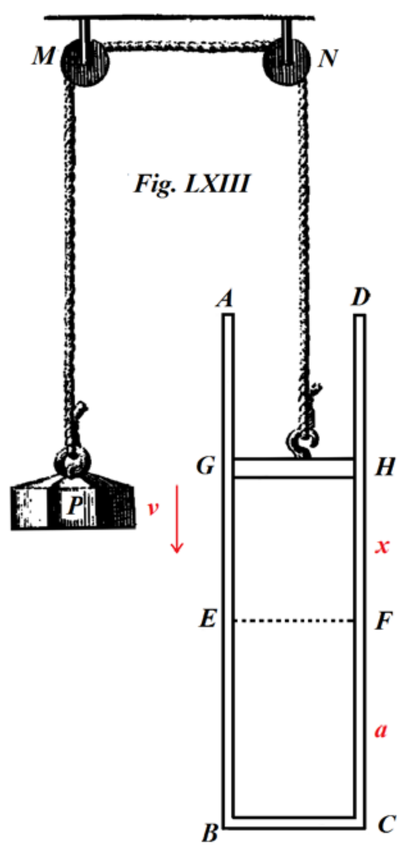
Thus Hales in *Vegetable Statiks* obtained from half a cubic inch of charcoal 180 cubic inches of air that was generated with the same elasticity of natural air ; therefore a cubic foot of charcoal will give 360 cubic feet of air. But if in §. 42, the *vis viva* may be sought which shall be able to generate from a cubic foot of natural air in turn a cubic foot 360 denser, that will be found to agree with a weight of 3938000 pounds dropped from a height of one foot: and if besides the elasticity of that air may become greater by four from the heat of the burning charcoal, this *vis viva* will agree with a weight of 15752000 *lb.* dropped from the same height. But is with difficulty to devise a suitable machine to this end. Besides there are many other natural things, which not only heat the air by compression, but also by heating the surrounding air act to return the same with more elasticity: such is quicklime mixed with soft water, and all fermentations; an incredible force is present in water rendered into vapour by the force of fire; a most ingenious machine which provides water moved to the whole town of London according to this principle and that described by the most celebrated Weidler [*De machina hydraulica Londinensi,* ….; the machine in question was invented by Thomas Savery, and described in *Phil. Trans*. no. 253 (1699)]. Especially true it deserves to be wondered at, the effect which can be expected from gunpowder: Indeed the calculation of certain experiments I conducted, which I have added below, I have shown the elasticity of gunpowder to be reckoned to be more than ten thousand times that of natural air, thus with everything considered carefully it will come about probably, its elasticity to be incredibly greater :

but we may consider the elasticity of the expanded gas from the gunpowder to decrease in the ratio with the density: and with these in place the *vis viva* may be found for a cubic foot of gunpowder put in place, if in §. 42 there may be put $a = 10000, x = 9999, p = 2240$ and assuming $-px + ap\log\frac{a}{a-x}$, which amount shall become equal to 183913864. Therefore the machine is giving in theory, which with the aid of a single cubic foot of gunpowder may be able to raise 183 913 864 pounds to a height of one foot, which the labor even of a hundred robust men within the space of a day I would not believe able to perform, whatever machine they may be using. But it is probable, as I have said, the effect of the gunpowder to be far greater; but certainly it is not less, for the calculation depends on the height, to which a iron ball ejected from a cannon shall be able to ascend in vacuo, in which kind of experiments a greater part of the gunpowder is lost.

This truth may be understood more, if the same calculation may be noted (which before we have made for the effect, which by being shown, itself arose from the condensed air restituted) to proceed also for the air which by natural circulation shall become indeed not more dense but still more elastic from the increase in the heat : thus for example as often as a cubic foot of ordinary air with the increase of the heat acquired twice the spring, by its aid a weigh of 865 pounds can be raised to a height of one foot, but only if the machine being used may be the most perfect.

But the effect of all the matters here established depends on the increase both on the density as well as on the heat of the air.

§. 44. Meanwhile the *vis viva* required to be expended for moving a machine is able to be obtained not only from heated compressed air, but also from air made rarer or cooler. For wherever the equilibrium has been raised, a *vis viva* is present, which can be expended, if a machine ought to be devised, for raising loads and turning armaments around. But the method determining the *vis viva*, which is able to elicit from the air occupying a given volume of a given density and of a given heat, with all changes being made, is the same as that which we have used in §. 42.




§. 45. Truly again there was a vertical cylinder *ABCD* (Fig. 63) with a moveable diaphragm *EF:* consider the air *EBCF*, as in §. 42, to be natural and in equilibrium with the external air : moreover the pressure force of this air on *EF* may be called *p*: Imagine then the weight *P*, so that by means of the rope drawn across two pulleys *M* and *N* it may be attached to the diaphragm, and the same may be drawn towards *AD*, and thus the diaphragm will arrive at *GH* from the position *EF*: And then again there may be put $FC = a,\ FH = x$, the velocity of the diagram at the position *GH* or of the weight at the position $P$ will be $= v$ ; with these in place if §§. 40 and 42 may be brought together, it will be apparent now to be :

$$dv = \left( P + \frac{ap}{a-x} - p \right) \times \frac{dx}{v} : P$$

or

$$Pvdv = \left( P - \frac{px}{a+x} \right) dx,$$

which integrated gives

$$\tfrac{1}{2} Pvv = Px - px + ap \log \frac{a+x}{a}.$$

But again from the descent of the weight *P* through the height *x* the *vis viva Px* would be produced, while meanwhile with the velocity of the weight itself *v* by the motion only the *vis viva* $\tfrac{1}{2} Pvv$ or $Px - px + ap\log \frac{a+x}{a}$ is present; therefore the *vis viva* which is left, evidently $-px + ap\log \frac{a+x}{a}$ passes to the air with the restoration of the equilibrium between the internal air, and again with the restitution of the equilibrium between the internal and external air that *vis viva* will be able to be poured across as it pleases: Therefore if you may have a volume of air *GBCH* filled with air of which the density shall be to the density of the external air as *CF* to *CH*, in the *vis viva* there will be the power $-ap\log \frac{a+x}{a}$.

But truly this *vis viva* properly may adhere either to the internal or external air, it is a play on words; because it suffices from the equilibrium brought between each air such a *vis viva* can be obtained, while restitution is allowed. For example there may be had a cubic foot of air with twice the rareness of ordinary air, for which hypothesis we will agree on putting in place [Recall that normal atmospheric pressure is approx. 2240 lb. per sq.ft.] $p = 2240 lb.$, $a = \tfrac{1}{2} ft.$ and $x = \tfrac{1}{2} ft.$ and the *vis viva*, which is being discussed, $= 1120 - 1120\log 2 = 344$, that is, that which may be generated from the fall by a weight of 344 *lb.* from a height of one foot. [Note that the cylinder has a cross-section of 1 sq.ft., and that initially we have natural air in the volume *a*.]

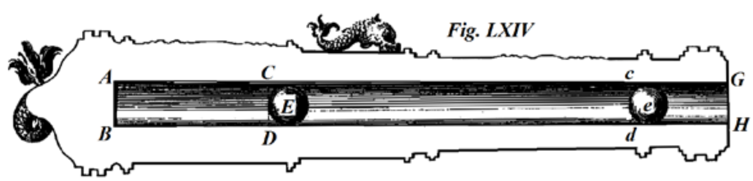
If the cubic foot shall be full of air four times rarer than natural air, now the *vis viva* will be sought (clearly on putting $p = 2240,$ and $a = \tfrac{1}{4},\ x = \tfrac{3}{4}) = 1680 - 560\log 4 = 904$, or of such as which arises from the free fall of a weight 904 *lb.* through a height of one foot.

If finally a cubic foot of air may be had from entirely empty air, the weight is required to be $p = 2240, a = 0, \&\ x = 1$: and thus the *vis viva* sought will be $= 2240 \times \left(1 - 0\log \tfrac{1}{0}\right)$; but it may be agreed $0\log \tfrac{1}{0}$ to be infinitely small in comparison with unity ; therefore

this number is $=2240$, which shows 2240 lb. can be raised by this *vis viva* to a height of one foot.
[If we consider an ideal gas at NTP, the volume is 22.4 litres, since a litre is 0.0353147 cu.ft., the standard volume is approx. 0.8 cu.ft., and hence the energy stored in 1 cu.ft. of air is roughly $RT\times\frac{5}{4}=8\times300\times\frac{5}{4}=3000J.$ at $27^0$C. We contrast this with the Bernoulli number of approx. $1000\text{Kg}\times10m/\sec^2\times0.3m.=3000J.$ Thus the error is appreciable but not beyond the bounds of possibility, we have assumed an ideal gas.]

§. 46. The amazing force of exceedingly compressed air is related to the present argument, but particularly of the blast of air of ignited gunpowder in the use of cannons and of the air used in air rifles. Concerning these which I have discussed separately I have added the following section.
[Note : The Latin *aura* used here can define anything from a breeze to a blast of air, amongst other things, such as modified air, as used above, where it refers to another kind of gas.]

……………………………………………………………………………………………..
*Concerning the force of the compressed air and of the blasts of ignited gunpowder for projecting balls in the use of air rifles and cannon.*




(I) *AG* (Fig. 64) shall be the length of the barrel for a cannon or gun placed horizontally, and it may be called $=a$ *: AC* shall denote the length of the volume, which the compressed air or the air from the ignited gunpowder may occupy from the start of the explosion, and let $AC=b$ : the weight of the ball being ejected $E=1$; moreover we may put the ball to fill the cavity of the barrel exactly and that to be moving freely within : the density of the compressed air in the volume *AD* may itself be had to the density of natural air as *n* to 1: Finally the weight of a column of mercury *P* (the base of which is *CD* and the height of which shall be the same as in a barometer). Moreover we will make use of the hypothesis, either the ball is propelled from the compressed air or from the blast of the gunpowder, the force of that propelling fluid is to be proportional to the density.

With these put in place for the calculation, we will consider the ball at the position *e*, on putting $Ac=x$, and the velocity of the ball at this place will be $=v$ *;* thus the force at the position *e* [*i.e.* the excess over atmospheric pressure] propelling the ball will be $=\left(\frac{nb}{x}-1\right)\times P$, which divided by the mass 1 and multiplied into the element of volume *dx* gives the increment of half the square of the velocity; so that there becomes :

$$vdv = \left(\frac{nb}{x} - 1\right) \times Pdx,$$

or

$$\tfrac{1}{2}vv = (b - x + nb \log \frac{x}{b})P.$$

There may be put $x = a$: the height due to the velocity , by which the ball is sent off; this height may be called $\alpha$ and there will be

$$\alpha = (b - a + nb \log \frac{a}{b}) \times P.$$

(II) For example in the air rifle the length of the piece or $a$ shall be $= 3$ *Paris ft.*, the length $AC = 4$ *in.*, and the air held in *AD* shall be ten times denser or $n = 10$, the diameter of the barrel or of the ball ejected shall be of three lines and its specific gravity in the ratio with mercury shall be as 10 to 17. *P* will be about $= 286$; and thence $\alpha$ will be found $= 2788$, with the indication the ball is going to be ejected with a velocity which would be able to rise in a vacuum to a height of 2788 *ft*. From the preceding formula it is gathered the strongest shot of the ball for the same amount of elastic pressure shall be, if the length of the piece shall be made $= nb$. Truly if the mind may be turned to the other impediments, which the globe suffers besides its inertia and the resistance of the external air in its passage through the barrel of the gun, it is apparent the length of the barrel will be least long for the strongest shot required to be produced. If the length *nb* shall be far greater than the length *a*, thus as in stronger shots, without sensible error there will be $\alpha = nbP \log \frac{a}{b}$.

If the cannon shall be erected vertically, the calculation becomes a little different but for the strongest shots the difference is negligible. Therefore because the shots we will consider henceforth only shall be the strongest, for brevity therefore we may put :

$$\alpha = nbP \log \frac{a}{b}.$$

(III) Just as used in the preceding paragraph we will determine the height owed [*i.e.* corresponding] to the velocity by which the ball is driven out, by the given elastic force of the air ejecting the ball, thus in turn it is apparent, the elastic force of the air can be deduced from that height observed, for there is $n = \alpha : \left(bP \log \frac{a}{b}\right)$.

Thence the elastic force of the gunpowder if not defined accurately, perhaps can be reduced to terms which certainly will it will surpass. But you will ask, who shall be able to determine the height $\alpha$ by experiment ; to which I may respond, that to be able to be deduced accurately enough from the time, which the ball ejected from the explosion rises vertically to the top point, while it is falling to the earth with an account had of the air resistance in the calculation. I will transcribe here the experiments reviewed in the *Comm. Acad. Petrop. book.* 2, *pp.* 338 and 339, the calculation of which I have set up with an account made from the hypothesis of air resistance, and the specific gravity of iron and of air to be as 7650 to 1 and the air, in which the ball rose, to be of uniform density : the ratio of the specific gravities was seen to be a little greater than it ought, but the error would be compensated in the highest shots from the diminution of the air towards higher places. " The position of the cannon was favorably positioned with all accuracy to the perpendicular : the individual experiments were repeated, and with the individual cannon in turn repositioned and made firm in place. Moreover, the length of the barrel was 7,7 *Eng. ft.* ; the diameter of the ball was 0,2375 *ft.* ; the diameter of barrel was not measured, neither was the size of the touch hole : however much of the gunpowder to be used was weighed in turn and the time from the point of the explosion to the point when the ball struck the earth was defined by a pendulum : the following table shows, both what was observed, as well as what was thence elicited by calculation "

| Amount of gunpowder expressed in Holland ounces. I | Observed time of ascent and descent in min. sec. II | Trajectory height with air resistance calc. in Eng. ft. III | Ascent time calc. in resisting air in min. sec. IV | Descent time calc. in resisting air in min. sec. V | Trajectory height in vacuo as calc. in Eng. ft. VI | Ascent and descent in vacuo in min. sec. VII |
|---|---|---|---|---|---|---|
| $\frac{1}{2}$ | 11 | 486 | 5,42 | 5,58 | 541 | 11,6 |
| 2 | 34 | 4550 | 14,37 | 19,63 | 13694 | 58 |
| 4 | 45 | 7819 | 16,84 | 28,16 | 58750 | 121 |

" For the same cannon and with the same ball, but with the former diminished by one foot and seven tenths parts, thus so that the length of the barrel left shall be exactly 6 *Eng.ft.,* the following table has been inserted with the same rules."

| I | II | III | IV | V | VI | VII |
|---|---|---|---|---|---|---|
| $\frac{1}{2}$ | 8 | 257 | 3,95 | 4,05 | 274 | 8,2 |
| 2 | 20,5 | 1665 | 9,74 | 10,76 | 2404 | 24,5 |
| 4 | 28 | 3187 | 12,5 | 15,5 | 6604 | 40,5 |
| 6 | 32,5 | 4304 | 13,9 | 18,6 | 11810 | 54,3 |
| 8 | 38 | 5643 | 15,54 | 22,46 | 22394 | 74 |

There are many things, which the succession of these experiments thus return doubtful, so that there shall be nothing, which may validate the same elasticity of the blast. I could believe the maximum inequality to arise from that, because the smallest part of the gunpowder may be ignited at once from the start of the explosion, as the greater part while finally it may be set alight, when the ball is close to the opening of the cannon, and because finally a maximum part may be ejected not ignited: perhaps it happens on this account alone, that the elastic force of the propelling blast may be hundreds of times greater, than what was produced by the force of the experiment, with no account taken of this matter: that for me very probably could happen, because there with the with 4 ounces powder used in the 7,7 *ft.* long cannon the ball shot was able to rise in vacuo to a height of 58750 *ft.,* whereby with the same amount of powder but with the cannon shortened by

1,7 ft., the shot corresponded to a height in vacuo of 6604 feet, which height scarcely extended to a ninth part of the earlier one : I guess from the comparison of each experiment, the maximum amount of powder in the longer cannon was ignited while the ball was already near the opening, nor to be more distant from that by more than 1,7 *ft*.

Also the throwing of the ball is diminished by the size of the touch hole, as well as by the space which is left between the ball and the inner surface of the barrel, because through each a notable part of the blast may escape unused : but just as so much diminished it does not arise thence, how much of that I had presumed to put into the calculation : finally I may add in the following calculation, so that a method may be had of putting in place the greatest limits on the force of the gunpowder, which even now certainly may be exceeded.

(IV) Because it is the third of the experiments which shows the maximum elasticity of the blast with the cannon assumed not yet shortened, which indicates the ball to be able to rise by its accepted impetus to a height $\alpha = 58750$ *Eng.ft.* Moreover the length of the barrel *AG* or $a$, was $= 7,7$ : the length *AC* (as much as I can estimate from the cross-section of the barrel and from the weight of the gunpowder) was $= 0,08$. Finally the value of *P* itself (or of the weight of a column of mercury, of which the base shall be a great circle of the ball and of which the height shall be 30 *Eng. in.* on account of the weight of the iron ball to be designated by one) is found by putting the specific gravity between mercury and iron so that 17 to 10 becomes $= 26,8$ :

[For $P = \frac{\text{density of } Hg}{\text{density of } Fe} \times \frac{\pi D^2/4 \times \text{Ht. mercury column}}{4\pi D^3/24 \times \text{Diameter of bore}} = \frac{3}{2} \times \frac{\rho_{Hg}}{\rho_{Fe}} \times \frac{Ht.}{D}$ ]

And since by §. III there shall be approximately $n = \alpha : \left( bP \log \frac{a}{b} \right)$, there becomes $n = 6004$. From which it follows, if the blast of ignited gunpowder may have its elasticity proportional to the density, its maximum elasticity will be six thousand times greater than that of ordinary air.

(V) But truly if we shall consider the useless part of the blast, which flies out through the light-hole and the gap left by the ball, we will find a greater part of the elasticity: The calculation which is required for solving this question, since it shall not be very long or involved, I have not hesitated to use the hypotheses a little more freely, by which it may be greatly facilitated: although these hypotheses shall not all be with true rigor, yet they shall not produce a notable error. In the first place I may put each opening, through which the exhaust gas shall be able to fly out, to be as if infinitely small on account of the cross-section of the barrel; with this assumed the velocity at individual instants will be able to be estimated at once with which the exhaust may fly out, to be judged at once from the pressure alone :but a hypothesis of this kind can happen for any fluid without any sensible error, then also since the openings are not excessively small at different places so that we can deduce the corollary from our theory, and much easier to be able to be assumed in a very elastic fluid and which will be seen from that, how an increment of the *ascent potential* in the account of the internal motion is less long in the account of the *ascent potential* of the particles escaping from the fluid through the opening, because it is

expelled by the property of elasticity, rather than being ejected by the force of gravity : for in the *first place* the local internal motion is by far less than in the other. *Secondly* the elastic force of the blast of the ignited gunpowder is so great, that the contrary pressure of the atmosphere shall not deserve to be attended to; *thirdly* the velocity of the ball in the cannon however great, yet can be agreed as minimal on account of the velocities, by which the exhaust gas may escaped through one gap or the other, because evidently the inertia of this exhaust can not only be exceedingly small with regard to the inertia which is present in the ball: on the strength of these hypotheses the exhaust will fly out through each opening with the same velocity, since with the velocity put otherwise, at the light-hole $=\sqrt{A}$,and with the velocity of the ball $=v$, the velocity of the exhaust at the gap left between the ball and the surface of the barrel may be said to be $=\sqrt{A}-v$. I come now to the solution.
[Note that Bernoulli has returned to the original definition of the vis viva, *i.e*. without the factor of one half.]

(VI) It is to be noted initially, if the elasticities of the exhaust gas may be considered to be proportional to the densities, to become so that the exhaust shall fly out with the same velocity constantly through each opening, as we have seen in problem §. 34, and that velocity will be named, which may be generated from a height of such homogeneous gas, the weight of which gas taken shall be held together, lest it may expand. Therefore the said velocity will be determined in this way: let the weight of the ball $=1$, the elasticity or weight which may be able to hold together the exhaust gas of the powder just ignited, in that state of compression *ACDB* , $=P$, the weight of the powder to be used $=p$;the weight of the exhaust gas just after the ignition also $=p$; and if the length *AC* is puts $=b$, it is evident the height of the homogeneous exhaust gas, which the weight *P* may have, to become $=\frac{P}{p}b$;therefore the velocity with which the exhaust gas freshly produced may fly out through the light-hole $=\sqrt{\left(\frac{P}{p}b\right)}$, and it will be ejected with the same velocity during the whole explosion, and that not only through the light-hole but also approximately by the gap left between the ball and the barrel.

(VII) Now again the cross-section of the barrel shall be $=F$; the gap intercepted between the ball and the barrel shall be $=f$, with the cross-section of the light-hole $=\varphi$; the length of the barrel $=a$; the amount of exhaust gas from the start of the explosion $=g$. Then the ball may be understood to arrive at *e* from *E* , and calling $Ac=x$; the amount of exhaust gas remaining there in the cannon at that point of time $=z$; the velocity of the ball in this position $=v$;the remaining denominations now were set out as before.

Because the elasticity by the hypothesis is directly as the quantity and inversely as the volume, the elasticity of the exhaust gas remaining in *AcdB* $=\frac{zb}{gx}P$ : which certainly is not wholly expended in propelling the ball, but only a part of this, which itself may be had to the whole as $F-f$ to *F*. And thus there is, on putting *dt* for the element of time,

$$dv = \frac{F-f}{F} \times \frac{zb}{gx} P \times dt.$$

[Again an expression of Newton's second law as we now know it.]

But by the method I have shown in §. 34, where the amount of air flowing out in a given minute time was defined specifically, there is found

$$-dz = \frac{f+\varphi}{F} \times \frac{z}{x} \times \sqrt{\left(\frac{P}{p} b\right)} \times dt\ ;$$

from a comparison of these two equations there arises

$$-dz = \frac{f+\varphi}{F-f} \times \frac{g}{b} \times \frac{\sqrt{b}}{\sqrt{Pp}} \times dv$$

which integrated, with the constant of integration added, gives

$$z = g - \frac{f+\varphi}{F-f} \times \frac{g}{b} \times \frac{\sqrt{b}}{\sqrt{Pp}} \times v\,.$$

If now the value may be substituted in the first equation this value found for $z$, and likewise there may be put $\frac{dx}{v}$ for *dt*, the equation becomes

$$vdv = \frac{F-f}{f} \times \frac{b}{x} \times P \times dx - \frac{f+\varphi}{F} \times \frac{\sqrt{bP}}{x\sqrt{p}} \times vdx$$

or

$$\frac{Fvdv\sqrt{p}}{(F-f) \times bP\sqrt{p} - (f+\varphi) \times v\sqrt{bP}} = \frac{dx}{x}\,,$$

which equation after its due integration, on making $x = a$, will go into this :

$$\log \frac{a}{b} =$$

$$\left(-F(f+\varphi)v\sqrt{p} - F(F-f) \times p\sqrt{bP} \times \log\left(1 - \frac{(f+\varphi)v}{(F-f) \times \sqrt{bPp}}\right)\right) : \left((f+\varphi)^2 \times \sqrt{Pb}\right).$$

[As G.K.Mikhailov has indicated in vol. V, note 67, of the Collected Works, Bernoulli is in error here as part of his work in this section uses the old vis viva formula, and part the new; thus the dependent results in the following section are also in error numerically; this sort of mistake was not unusual at the time, and Euler fell into the same trap occasionally.]

(VIII) Now if by experiment the value of *v* has become known, thence it will be possible to deduce the value of *P*, which denotes the elasticity of the exhaust gas of the gunpowder not yet expanded: Which we will illustrate as an example, we will make use of the same experiment, as we have established now in article IV, so that it may become apparent thence, what increase of elasticity may be proven from the flying off of the exhaust gas. Thus the calculation therefore may be put in place.

Because the weight of the ball, which was three pounds, we have assigned by unity, the four ounces of powder used being expressed by $\frac{1}{12}$, therefore $p=\frac{1}{12}$. The measures of the openings which we consider I have not accepted: but it is customary to put in place the gap left by the ball in a similar cannon around a fifteenth part of the cross-section of the barrel; the cross-section of the light-hole here I consider to be almost negligible ; and thus I may put in place $F=15; f=1; \varphi=0$ : Then again there may be had $a=7,7; b=0,08$; the height to which the ball may be able to rise in vacuo shall be given by $\frac{1}{2}vv=58750$, or $v=343$: Therefore this will be the final equation of the above article :

$$\log 96=\frac{-5251}{\sqrt{P}}+17,5\log\frac{\sqrt{P}}{\sqrt{P}-300},$$

for which it will be satisfied approximately when there is put $\sqrt{P}=534$ and therefore $P=285156$, which a column of mercury of the same cross-section with the cross-section of the cannon, of which the height in turn shall be more than 10000 greater than of a common barometer; moreover we have found the number *n* above in *art.* IV (which signifies the same) $=6004$. Therefore now we may confirm completely (even everywhere in which we have ignored proving the greater force of the powder) the minimum elastic force of the gunpowder to be at least ten thousand times greater than the elastic force of ordinary air. Moreover likewise it is apparent from the comparison of the numbers 10 000 and 6004, round about how mach of the strength of the powder departs from the gaps often mentioned. Indeed I might have thought that the decrement were greater: But I have confirmed this by a calculation into the matter about which some time ago a certain man knowledgeable in these things wished me to be more certain, evidently to have observed no notable decrease in cannons themselves, when the light-hole should be above the measured cross-section in lasting use in a siege.

(IX) Truly so that certain corollaries may be able to be deduced more easily from our equation although only approximately true, we will change the logarithmical quantities into a series. Moreover there is :

$$-\log\left(1-\frac{(f+\varphi)v}{(F-f)\times\sqrt{bPp}}\right)=$$

$$=\frac{(f+\varphi)v}{(F-f)\times\sqrt{bPp}}+\frac{(f+\varphi)^2 vv}{2(F-f)^2\times bPp}+\frac{(f+\varphi)^3 v^3}{3(F-f)^3\times bPp\sqrt{bPp}}+\text{etc.}$$

And thence with the value substituted into the final equation of *art.* VII it becomes :

$$\log\frac{a}{b}=\frac{Fvv}{2(F-f)\cdot bP}+\frac{F\cdot(f+\varphi)v^3}{3(F-f)^2\times bP\sqrt{bPp}}+\text{etc.}$$

We will note here this same equation agrees perfectly with the final equation of *art.* II if the openings $f$ and $\varphi$ may be put $=0$: which indeed here is indicated by $\frac{1}{2}vv$ and $P$ there is *a* and *nP,* with the remaining denominations in agreement.

(X) So that it may become apparent, this equation may attend to showing how much approximately the height of the shot may be diminished by the openings, if these openings may be minimized. By $\alpha$ the height may be understood to which a ball shall be able to reach in vacuo, if no quantity of exhaust gas is considered to escape through the openings, and the decrease of that height from the eruption of exhaust gas through the same openings will be approximately this

$$\left((2\alpha)^{\frac{3}{2}}\times(f+\varphi)\right):\left(3F\times\sqrt{bPp}\right).$$

So that in the same cannon and with the same amount of powder used and with the weight of the ball remaining, the decreases of the shots will be proportional to the cross-sections of the openings. The decreases follow almost in the same square root ratio of the of the amounts of powder used, with the rest equal ; because indeed the logarithms of the magnitudes of the numbers increase in a much smaller ratio and the numbers themselves because above there is $\alpha=bP\log\frac{a}{b}$, with everything else constant it will be possible to have set *a* proportional to *b* itself, because *P* is not affected by *b*. But the decreases, about which the discussion is about, with everything else equal, follow the ratio of the quantities $\left(\alpha^{\frac{3}{2}}\right):\left(\sqrt{bp}\right)$ or in the ratio of the quantity $\frac{b}{\sqrt{p}}$*;* truly *p* itself, which denotes the weight of the powder used, is as *b;* therefore the aforesaid decrease follows approximately in the ratio $\sqrt{b}$ , which square root itself is the amount of powder used. Therefore in the account had of the shots, the decreases are much greater with the weaker shots than with the stronger, and these also are seen to confirm the experiments reviewed in *art.* III: for I can see no other reason, why in the first table of the experiments of the balls trajected in vacuo, with two ounces of powder taken, ought to have been more than twenty six times higher than taken with half an ounce, and why with the amount of powder soon doubled to 4 ounces a throw only four times higher may be produced after the calculation, than with a quantity of two ounces.

(XI) The amounts remaining in each table compare the inequalities of the experiments, these as I have said above, I have derived in the main part from that effect, because the powder may not be all ignited, nor all this which may be ignited may be considered from the flame at the start of the explosion. Nor surely will we marvel that, when we consider carefully the whole time of the explosion in *exper.* 4 *tab.*1 to be effected in not even a hundredth part of a second. Therefore since a certain maximum part of the powder to be ejected not ignited, nor a small part left to be ignited later, than was put into the calculation; and since besides a notable part of the powder shall be falsified with vapors and materials from the ground, which cannot be set alight, it follows a greater part is present with the elastic parts kindled, than what was determined in the calculation of *art.* X; perhaps it is greater than ten or twenty parts.

But truly the elasticity evidently shall be ten thousand times greater than that of ordinary air, of such an amount as experiment has shown ; thence it follows that elasticity of exhaust gas, which is elicited from the ignition of gunpowder, either is not to be composed of ordinary air or the elasticities to increase in a greater ratio with the increase than the densities: for the density of air is not possible, which has just arisen from the ignition of the powder, to be more than one thousand times the density of ordinary air, even if the whole powder were composed of compressed air, which I conclude from the specific gravity of powder with the ratio to air.

But meanwhile immediately it is disturbed, or the elastic exhaust gas made, which is deduced from the bodies, the air may or may not be ordinary, which question I cannot decide.

Yet if the gunpowder may be considered to be a thousand time denser than natural air and with ten thousand times the elasticity, then it follows from §.4, the air compressed by an infinite force cannot be condensed more than 1331 times, and following the same rule the elasticity of natural air four times denser shall be to the elasticity of natural air as $4\frac{1}{4}$ to 1. [*i.e.* substituting the values $\frac{\pi}{P}=10000$ and $s=\frac{1}{1000}$ into the formula $\pi=\frac{1-\sqrt[3]{m}}{s-\sqrt[3]{mss}}\times P$ gives the maximum compression $m=1331$; again $m=1331$ and $s=\frac{1}{4}$ gives $\frac{\pi}{P}=4\frac{1}{4}$.]

But truly whether experiments set up by others, which make a ratio of these elasticities accurately as 4 to 1, were made with sufficient accuracy and whether the heat of the air while it was being compressed remained the same? I do not know. But it is plausible, the same exhaust gas which lies hidden in the pores of the gunpowder, to be the cause of the elasticity of the elastic bodies, or from the contraction of fibers: for while [the particles] swarm in the smallest cavities, if the body may be rendered into an unusual form by a certain force, the elastic exhaust gas is compressed, and while together with the cavities it returns to the most capacious shape, the body is restored to it original shape and length.

## HYDRODYNAMICAE SECTIO DECIMA.

*De affectionibus atque motibus fluidorum elasticorum, praecipue autem aëris.*

§. 1. Fluida nunc elastica consideraturis licebit nobis talem iis affingere constitutionem, quae cum omnibus adhuc cognitis conveniat affectionibus, ut sic ad reliquas etiam nondum satis exploratas detur aditus. Fluidorum autem elasticorum praecipuae affectiones in eo positae sunt: 1°. ut sint gravia, 2°. ut se in omnes plagas explicent, nisi contineantur, & 3°. ut se continue magis magisque comprimi patiantur crescentibus potentiis compressionis: Ita comparatus est aër, ad quem potissimum praesentes nostrae pertinent cogitationes.

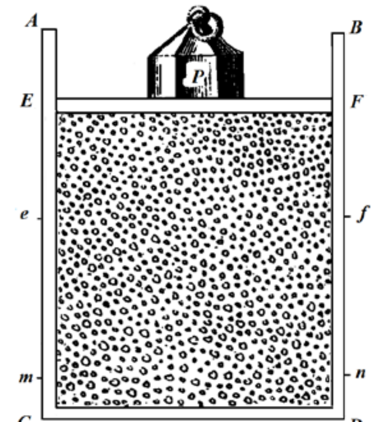


Fig. LVI

§. 2. Finge itaque vas cylindricum verticaliter positum *ACDB* (Fig. 56) atque in illo operculum mobile *EF*, cui pondus *P* super incumbat: contineat cavitas *ECDF* corpuscula minima motu rapidissimo hinc inde agitata: sic corpuscula, dum impingunt in operculum *EF* idemque suis sustinent impetibus continue repetitis, fluidum componunt elasticum quod remoto aut diminuto pondere *P* sese expandit: quod eodem aucto condensatur & quod in fundum horizontalem *CD* haud aliter gravitat, ac si nulla virtute elastica esset praeditum: sive enim quiescant corpuscula sive agitentur, non mutant gravitatem, ita ut fundum tum pondus tum elasticitatem fluidi sustineat. Tale igitur fluidum quod cum primariis convenit fluidorum elasticorum affectionibus substituemus aëri, atque sic alias, quae jam in aëre detectae fuerunt, explicabimus aliasque nondum satis perpensas ulterius illustrabimus proprietates.

§. 3. Corpuscula cavitati cylindri inclusa considerabimus tanquam numero infinita, & cum spatium *ECDF* occupant, tunc aërem illa dicemus formare naturalem, ad cujus mensuras omnia sunt referenda: atque sic pondus *P* operculum detinens in situ *EF* non differt a pressione Atmosphaerae superincumbentis, quam proinde per *P* in sequentibus designabimus.

Notetur autem hanc pressionem minime aequalem esse ponderi absoluto cylindri verticalis aërei operculo *EF* in atmosphaera superincumbentis, quod hactenus inconsiderate affirmarunt auctores: sed est pressio ista aequalis quartae proportionali ad superficiem terrae, magnitudinem operculi *EF* & pondus totius atmosphaerae in superficiem terrae

§. 4. Quaeratur jam pondus $\pi$, quod aërem *ECDF* in spatium *eCDf* condensare valeat, positis velocitatibus particularum in utroque aëre, naturali scilicet & condensato, iisdem: sit autem $EC = 1$ & $eC = s$ : Cum vero operculum *EF* transponitur in *ef*, majorem a fluido patitur nisum duplici modo: *primo* quod numerus particularum ratione spatii, cui includuntur, major nunc est, & *secundo* quod quaevis particula saepius impulsum repetit:

ut recte calculum ponamus incrementi, quod a *prima* pendet causa, particulas considerabimus ceu quiescentes, atque numerum earum, quae operculo in situ *EF* sunt contiguae, faciemus $= n$, & erit numerus similis pro situ operculi in $ef = n:\left(\frac{eC}{EC}\right)^{\frac{2}{3}}$, seu $= n : s^{\frac{2}{3}}$.

Notetur autem fluidum a nobis considerari non magis condensatum in parte inferiori, quam in superiori, quale est, cum pondus *P* veluti infinite majus est pondere proprio fluidi: Perspicuum hinc est, hoc nomine vim fluidi esse, ut sunt numeri *n* & $n : s^{\frac{2}{3}}$, id est, ut $s^{\frac{2}{3}}$ ad 1. Quod vero attinet ad alterum incrementum a *secunda* proveniens *causa*, invenitur id respiciendo motum particularum; atque sic apparet impulsus eo saepius fieri, quo propius ad se invicem sitae sunt particulae: Erunt scilicet impulsuum numeri reciproce ut distantiae mediae inter superficies particularum: Istaeque distantiae mediae ita determinabuntur.

Particulas ponemus esse sphaericas, distantiamque mediam inter centra globulorum pro situ operculi *EF* vocabimus *D*, diametrumque globuli designabimus per *d*: ita erit distantia media inter superficies globulorum $= D - d$ : patet vero in situ operculi *ef* fore distantiam mediam inter centra globulorum $= D\sqrt[3]{s}$, atque proinde distantiam mediam inter superficies globulorum $= D\sqrt[3]{s} - d$. Igitur respectu secundae causae erit vis aëris naturalis *ECDF* ad vim aëris compressi *eCDf* ut $\frac{1}{D-d}$ ad $\frac{1}{D\sqrt[3]{s}-d}$, seu ut $D\sqrt[3]{s} - d$ ad $D - d$ : Conjunctis vero ambabus causis erunt praedictae vires, $s^{\frac{2}{3}} \times \left(D\sqrt[3]{s} - d\right)$ ad $D - d$.

Rationi *D* ad *d* aliam substituere possumus magis intelligibilem: nempe si putemus operculum *EF* pondere infinito depressum descendere usque in situm *mn*, in quo particulae omnes se tangunt, atque lineam *mC* vocemus *m*, erit *D* add ut 1 ad $\sqrt[3]{m}$, qua ratione substituta, erunt tandem vires aëris naturalis *ECDF* & compressi *eCDf* ut $s^{\frac{2}{3}} \times \left(\sqrt[3]{s} - \sqrt[3]{m}\right)$ ad $1 - \sqrt[3]{m}$, seu ut $s - \sqrt[3]{mss}$ ad $1 - \sqrt[3]{m}$. Est igitur

$$\pi = \frac{1-\sqrt[3]{m}}{s-\sqrt[3]{mss}} \times P.$$

§. 5. Ex omnibus phaenomenis judicare possumus aërem naturalem admodum condensari posse, & fere in spatiolum infinite parvum comprimi; facta igitur $m = 0$, fit $\pi = \frac{P}{s}$, ita ut pondera comprimentia sint fere in ratione inversa spatiorum, quae aër diversimode compressus occupat; quod multiplex experientia confirmavit. Et potest certe haec regula tuto accipi in aëre rariore quam est naturalis; an vero etiam possit in aëre admodum densiori, non satis exploratum habeo: nec dum enim fuerunt experimenta ea accuratione, quae hic requiritur, instituta: unico opus est ad definiendum valorem litterae *m*, sed eo accuratissime instituendo & quidem cum aëre vehementer compresso; gradus autem caloris in aëre, dum comprimitur, sollicite invariatus conservetur.

§. 6. Elasticitas interim aëris nonsolum a condensatione augetur, sed & ab aucto calore, & quia constat calorem intendi ubique crescente motu particularum intestino, sequitur, elasticitatem aëris spatium non mutantis auctam, intensiorem arguere motum in particulis aëris, quod cum hypothesi nostra recte convenit: perspicuum enim est, eo majus requiri pondus *P* ad continendum aërem in situ *ECDF*, quo majori velocitate particulae aëreae agitantur: Imo non difficile est videre pondus *P* secuturum rationem duplicatam istius velocitatis, ideo quod ab aucta velocitate tum numerus impetuum tum intensitas eorundem aequaliter crescat, utrumque vero seorsim proportionale sit ponderi *P*.

Igitur si velocitas particularum aërearum dicatur *v*, erit pondus, quod in situ operculi *EF* sustinere valet, = *vvP* & in situ $ef = \frac{1-\sqrt[3]{m}}{s-\sqrt[3]{mss}} \times vvP$, vel proxime $= \frac{vvP}{s}$, quia ut vidimus *m* numerus admodum exiguus est ratione unitatis & numeri *s*.

§. 7. Istud theorema, quod in praecedente paragrapho apposui, quo nempe indicatur, *in omni aëre cujuscunque densitatis sed eodem caloris gradu praedito elasticitates esse ut densitates, atque proinde etiam incrementa elasticitatum, quae fiunt a calore aequaliter aucto, proportionalia esse densitatibus*, istud, inquam, theorema experientia edoctus fuit D. Amontons idemque recensuit *dans les Memoires de l'Acad. R. des Se. de Paris pour l'annec* 1702. Sensus istius theorematis est, si v. gr. aër naturalis mediocris caloris pondus 100 *lb.* datae superficiei impositum sustinere valeat, atque deinde calor ipsius augeatur donec 120 *lb.* eadem superficie eodemque volumine ferre possit, fore ut idem aër in dimidium spatium condensatus & iisdem caloris gradibus praeditus respective ferre possit 200 *lb.* & 240 *lb.*, ita ut incrementa 20 *lb.* & 40 *lb.* utrobique ab aucto calore genita sint densitatibus proportionalia. Affirmat porro aëris, quem vocat temperatum, elaterem esse ad elaterem aëris ejusdem cum aqua bulliente caloris, proxime ut 3 ad 4 vel accuratius ut 55 ad 73. At ego institutis experimentis cognovi aërem calidissimum, qualis maxime fervente in hisce terris est aestate, tanti nondum esse elateris, quantum D**.** Amontons aëri tribuit temperato; imo nec sub ipso aequatore aërem unquam ejus esse caloris mihi persuadeo. Meis autem magis fidendum esse puto experimentis quam Amontonianis, ideo quod in his aër non conservarit suum volumen ejusque variationis nulla ab Auctore habita fuerit ratio in calculo. Aëris qui hic Petropoli frigidissimus fuit die 25. Decembr. 1731 st. vet. elaterem deprehendi esse ad elaterem similis aëris, communi cum aqua bulliente calore praediti, ut 523 ad l000.

Sed anno 1733 d. 21. Jan. multo intensius fuit frigus eique respondere observavi aëris elasticitatem infra dimidiam ejus quam habet similis aër ad aquam bullientem calefactus. Sed cum esset maximus aëris calor in loco umbroso ann. 1731, elasticitatem habuit proxime $\frac{3}{4}$ & accuratius $\frac{100}{76}$ ejus quam habuit aër frigidissimus & $\frac{2}{3}$ ejus quam habet aër ejusdem cum aqua bulliente caloris: maximae igitur caloris variationes in aëre hic locorum continentur intra terminos 3 & 4, quos in Anglia non ultra terminos 7 & 8 excurrere legi. Calorem autem aëris, cujus elasticitas tres quartas exaequet partes elasticitatis aëris instar aquae bullientis calidi, corpori animali fere intolerabilem esse puto.

§. 8. Ex cognita ratione inter diversas ejusdem aëris eodemque spatio inclusi elasticitates, facile est deducere mensuram caloris, qui ad aërem pertineat, si modo conveniamus in definiendo calore duplo, triplo &c., quae definitio arbitraria est, neque in rerum natura posita; mihi quidem videtur non incongrue aëris calorem si communis sit densitatis proportionalem statui ejus elasticitati. Primus autem caloris gradus, a quo reliqui mensuram accipiant, sumetur ab aqua pluviali bulliente, quia huic procul dubio ubique terrarum idem proxime caloris gradus est.

His ita acceptis erunt calores aquae bullientis, aëris tempore aestivo calidissimi & aëris tempore hyemali frigidissimi in hisce terris proxime ut 6, 4 & 3. Dicam nunc quemadmodum hosce invenerim numeros, ut de accuratione experimentorum, quorum successus ab Amontonianis diversus admodum est, judicium ferri possit.

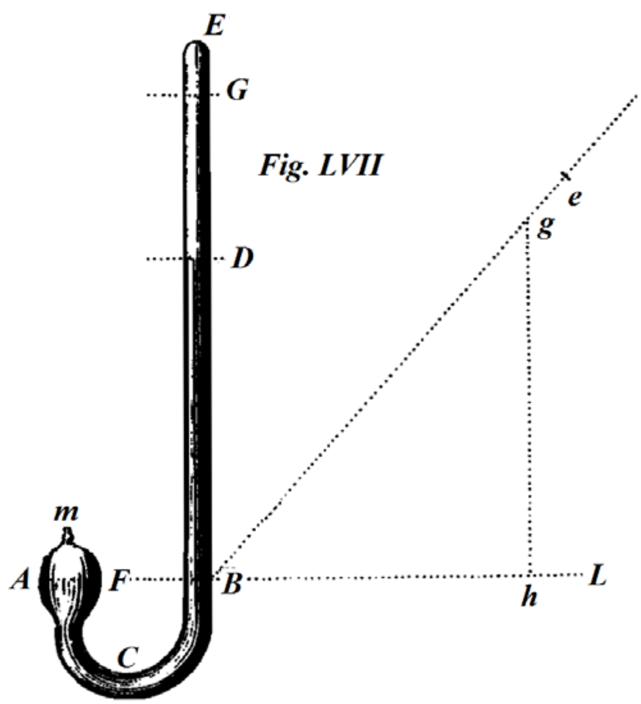


*Fig. LVII*

§. 9. Barometro nempe usus sum ordinario ACBE (Fig. 57), idque hermetice sigillari curavi in *m;* hoc modo instrumentum mutavi in thermometrum aëreum mutationibus barometricis non obnoxium: Crescente enim calore intenditur elaterium aëris *AmF* altiorque fit columna mercurii *BD*, quam aër captus sustinet, & si spatiam *AmF* veluti infinitum censeri posset, esset calor in ratione altitudinis *BD* (per §§. 7 & 8) atque hujus thermometri ope poterit mensura caloris ubique specifice definiri. Si enim immergatur instrumentum aquae bullienti pluviali in situ verticali observeturque punctum *G* ad quod superficies mercurii ascendit; fueritque deinde alius caloris gradus qualiscunque definiendus, qui mercurium sustinuisse ad punctum *D* usque observatus fuerit, erit utique calor iste ad calorem aquae ferventis ut *BD* ad *BG*. Et cum ratio *BD* ad *BG* constans sit, quaecunque fuerit altitudo *BG*, erit idem caloris gradus, de quo sermo est, ubique locorum facile imitabilis. Poterit autem *BG* in centum aut mille dividi particulas atque hujusmodi particulis altitudo *BD* definiri.

Nihil dico de modis hujusmodi thermometra sensibiliora reddendi; eorum quisque facile excogitabit plures, qui volet. Curetur autem, ut altitudo *BE* non sit infra 4 pedes, imo ut major sit, si etiam aliorum fluidorum bullientium gradus caloris, qui saepe major est quam in aqua, experiri animus sit. Si minora hujusmodi thermometra desiderentur, poterunt ea ita fieri, ut tempore sigillationis in *m* ampulla vitrea *AF* igni lampadis apponatur ad rarefaciendum aërem in illa contentum, tuncque protinus sigillatio fiat, & ne sigillationi mora injiciatur, poterit prius ampulla vitrea in tubulum capillarem duci, qui vel leviter flammae admotus illico colliquescat. Hoc modo thermometra obtinui non ultra quatuor aut sex pollices longa, sed parvae virtutis. Caeterum multum refert, ut spatium *ED* sit ab omni aëre, quantum fieri potest. vacuum, neque de isto vacuo satis certi erimus cum viderimus in situ instrumenti horizontali mercurium extremitatem *E* attingere, quia fieri potest, ut aër, qui antea in spatio *ED* fuit, sese in poros mercurii recipiat, rursusque pristinum spatium occupet descendente mercurio: tutius erit examen admovendo partem *DE* flammae: si enim a calore flammae superficies *D* locum non mutet, indicium erit certum vacuum esse ab aëre spatium *ED.*

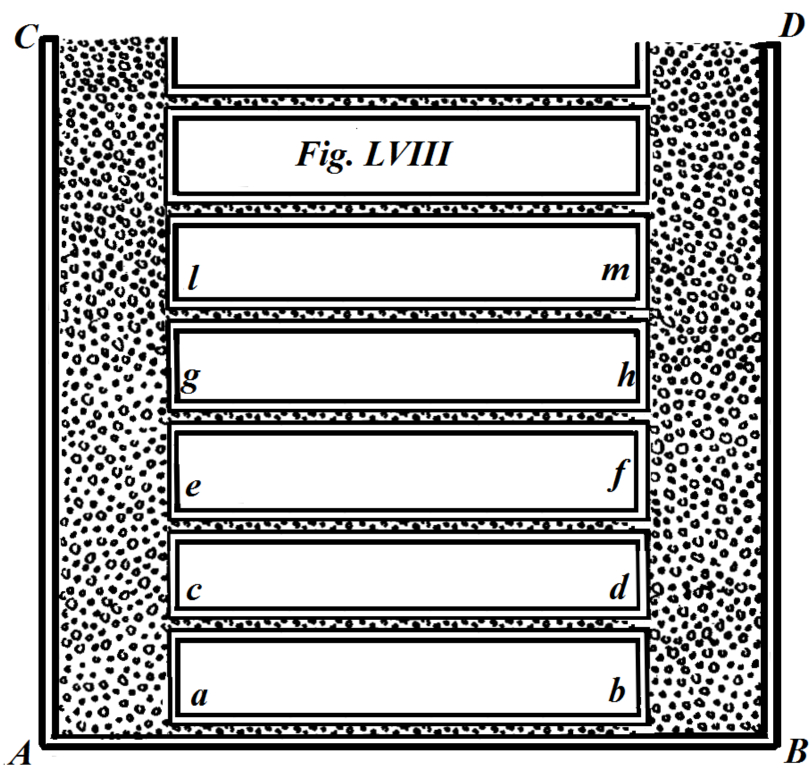
§. 10. In praecedente paragrapho consideravimus spatium *AmF* ab aëre occupatum veluti infinitum ratione spatii *DG* aut *DE:* Quod si vero fuerit tantum octuplo vel decuplo majus, nondum licebit illud sine notabili errore tanquam infinitum considerare: atque hinc conjicio ortum esse errorem aliquem in definiendo elatere aëris mediocriter calidi in experimentis Amontonianis.

Ut igitur accuratissime fiat experimentum, ita procedendum erit: Fuerit superficies mercurii inferior in *AF* ducaturque horizontalis in *AL;* deinde pro caloris gradu qualicunque definiendo inclinetur instrumentum, donec superficies mercurii sit in puncto *g* (quod idem est in quo mercurius subsistebat a gradu caloris aquae ferventis in situ thermometri verticali), tuncque capiatur mensura altitudinis verticalis *gh,* quae erit ad altitudinem *GB* vere ut elater aëris, cujus calor definiendus est, ad elaterem aëris instar aquae ferventis calidi. Sic igitur calores erunt proprie in ratione altitudinum *gh.* Priusquam hoc argumentum abrumpam, notasse conveniet (quandoquidem aliquibus fortasse videbitur *primum,* qui a nobis positus fuit, *caloris gradum* ab aqua bulliente desumtum non semper nec ubique sibi omnino constare quod loco caloris aquae bullientis thermometrum etiam possit certis & fixis mensuris fieri, si experimento densitas aëris exploretur seu ejus gravitas specifica simulque altitudo barometri notetur. Si enim thermometrum inclinetur, donec superficies mercurii fuerit in *g,* & eo tempore altitudo barometri fuerit 28 *poll. Paris.* atque pes cubicus aëris, in quo thermometrum positum est, pondus habuerit 600 *gran. Norimb.*, poterit altitudo verticalis *gh* ceu *primus* caloris gradus considerari. Si autem alio loco & tempore altitudo barometri fuerit 29 *poll. Paris.* & pondus pedis cub. aëris, qui ambit aliud thermometrum (in quo *primum* caloris *gradum* definire animus est) sit 500 *gran. Norimb.* ac denique superficies mercurii in thermometro rursus sit in *g,* erit altitudo verticalis primo caloris gradui conveniens $\frac{29\cdot 600}{28\cdot 500}\times gh$. In usu thermometri inclinetur semper instrumentum, donec superficies mercurii sit in g: Volui methodum hanc apponere ut appareret quam facile sit in theoria fixam dare caloris mensuram: In praxi vero alteram multo faciliorem satisque accuratam huic praetulerim.

§. 11. Veniamus nunc ad aëris considerandam atmosphaeram, quae non a superincumbente pondere alieno, sed propria coercetur mole. *Primo* autem examinabimus pressiones columnarum aërearum verticalium atque aequilibria earum tum inter se tum cum columna mercuriali in barometris. *Secunda* elasticitates aëris in variis atmosphaerae altitudinibus supra mare atque altitudines respondentes barometricas rimabimur. Atque his praemissis, plurimis satisfaciemus phaenomenis aliis ad mutationes atmosphaerae pertinentibus.

§. 12. Sint duo tubi aequalis amplitudinis verticales *AC* & *BD* (Fig. 58) uterque indefinitae altitudinis: Deinde finge tubulos strictiores horizontales *ab, cd, ef, gh, lm* &c. numero veluti infinitos, utrinque apertos & hiantes in tubos verticales. Puta praeterea ubique aëreas particulas hos tubos occupantes eadem velocitate agitari, eundemque adeo caloris gradum habere: Ita dubium nullum est, quin funda *A* & *B* aequaliter premantur simulque ipsis aequale pondus (quod scilicet ipsum est pondus columnae aëreae indefinitae *AC* vel *BD)* superincumbat.

Intelligis etiam, si in aequalibus altitudinibus veluti in *g* & *h* diaphragmata fingas atque abesse putes aërem inferiorem *gA* & *hB*, etiamnum ista diaphragmata utrinque




aequaliter premi & aequalia esse pondera columnarum aërearum *gC* atque *hD* diaphragmatibus superjacentium. Si igitur pondus totius columnae aëreae *AC* vel *BD* dicatur *A,* & pondus columnae aëreae *gC* vel *hD* ponatur *B*, erit pondus aëris inter *A* & *g* sive *B* & *h* intercepti $= AB$, pondus fundo *A* vel *B* superjacens $= A$, & pondus diaphragmati in *g* vel *h* incumbens *B.*

§. 13. At si inaequali velocitate in tubis *AC* & *BD* particulae agitentur, res alia erit: tamen quaecunque fingatur velocitatum & calorum in singulis locis diversitas, patet nihilominus utrobique aequaliter pressum iri partes tubi in eadem altitudine positas, veluti in *g* & *h*, atque proinde diaphragmata, si fingantur utrobique in eadem altitudine posita, aequalem pressionem sustentura esse. Si enim dicas minorem esse pressionem in *g* quam in *h*, nihil erit quod fluxum aëris ex *BD* in *AC* per tubulum transversum *hg* impediat, sicque ista positio contra statum *permanentiae*, quem supponimus, pugnabit.

Cum itaque loca in eadem altitudine posita aequaliter a superincumbente aëre premantur, erunt (per §. 6) densitates in locis homologis quibuscunque, veluti in *g* & *h*, proxime in reciproca ratione quadrata velocitatum, quibus in illis locis particulae agitantur.

§. 14. Consequens est ex praecedente paragrapho, ubique locorum eandem esse aëris pressionem in aequalibus a superficie maris altitudinibus, si atmosphaera in statu permanente aequilibrii posita nullisque agitata ventis putetur, quaecunque fuerit caloris differentia in diversis atmosphaerae partibus: Igitur ubique terrarum sub aequatore & sub polo eadem sit oportet altitudo mercurii in barometris, quae in superficie maris aut in aequalibus super illam altitudinibus posita sunt, si atmosphaera nullis obnoxia sit mutationibus. Pono autem aquas a superficie maris terminatas ad commune aequilibrium esse positas, non quod id omnino necesse sit, sed quod nulla adhuc observata fuerit differentia: imo cursus *(les courans)* aquarum in multis oceani locis, qui ad eandem perpetuo diriguntur plagam, hanc hypothesin non omni rigore accipiendam esse ostendunt.

§. 15. Jam notavi densitatem aëris in quovis tuborum verticalium loco pendere a calore respondente: Et cum diversi esse possint caloris gradus manente aequilibrio, diversae quoque esse poterunt densitates: ponantur itaque densitates in $g = D$, in $h = \delta$; finganturque utrobique duo strata altitudinis aequalis & infinite parvae *dx*, posita altitudine *Ag* vel $Bh = x$: Ita erit pondus columnae aëreae $Ag = \int Ddx$ & columnae $Bh = \int \delta dx$: atque hoc modo poterit tum integrae columnae tum cujusvis partis pondus definiri: Interim apparet, minime requirere rei naturam, ut sint pondera columnarum *AC* & *BD* vel *Ag* & *Bh* vel denique *gC* & *hD* inter se aequalia, quamvis (per§. 13) pressiones tam in funda *A* & *B* quam in diaphragmata *g* & *h* sint inter se aequales; mirum id prima intuitu quibusdam fortasse erit, fieri posse ut fundum *A* aliam sustineat pressionem quam est pondus columnae aëreae indefinitae *AC* ei superincumbentis, quandoquidem omnibus in statu suo permanentibus, ut fere videtur, concipi possint orificia *a, c, e, g* &c. singula obturata, quo sane in casu dubium nullum est, quin pressio fundi *A* sit ipsum columnae aëreae superjacentis pondus: hunc vero scrupulum sibi quisque eximet hunc in modum: fingamus utramque columnam terminatae altitudinis ( quamvis enim sine fine assurgant quamdiu particulae motum aliquem servant, attamen terminatae erunt, si eaedem particulae in suprema columnarum parte motu destitutae sint, sicque simplex fluidum grave omni elasticitate destitutum efficient); hoc posito apparet 1°. columnam utramque ad communem assurgere altitudinem apertis tubulis transversalibus, qui ubique adsunt;

2°. suprema strata utrobique esse aeque densa, quia sunt ad aequilibrium posita & communem habent altitudinem. Ex hoc jam obvium est, quare non liceat tubulos transversales considerare ceu obturatos, quod ostendere constitui. Perspicuum quoque est ex se, pressiones ubique proportionales esse ponderi supremi strati, ex quo consequens est, quod jam §.13 indicatum fuit, pressiones ab utraque parte aequales inter se esse sub aequalibus altitudinibus. Si jam columnae nusquam terminatae sint, licebit mente ultima concipere strata aut sub aequalibus altitudinibus diaphragmata fingere utrobique aequali pondere onerata, sic ut nihil vi demonstrationis inde decedat.

§. 16. Igitur quum in barometro ex loco humiliori veluti *A* in altiorem *g* transportato mercurius descendit, non sequitur pondus columnae mercurialis, quae in barometro descendit, aequale esse ponderi columnae aëreae ejusdem diametri & altitudinis *Ag*, quod ab aliquibus ita asseritur. Et profecto caeteris paribus columna mercurii descendens eadem erit tam tempore hyemali quam aestivo cum ex sententia illa deberet tempore calido esse minor, quam tempore frigido: Eadem quoque erit in locis meridionalibus & septentrionalibus.

Patet exinde quid censendum sit de illa methodo, qua in Anglia aliquando usos esse recenset D. Du Hamel in *Hist. Acad. Sc. Paris.*, ad indagandam rationem inter gravitates specificas aëris & mercurii: Observata nimirum altitudine mercurii in loco humiliori, tum etiam in altiori, gravitates specificas in aëre & mercurio statuerunt, ut erat differentia altitudinum mercurii in barometro ad altitudinem inter locos observationum interceptam: Etiamsi aër ejusdem densitatis ponatur ab imo observationis loco ad alterum usque, non licet tamen inde judicare de ejus gravitate specifica ratione mercurii. Quicquid ab experimento colligere licet, hoc solum est:

Consideremus scilicet integram crustam aëream terram ambientem atque inter ambo observationis loca interceptam, & erit pondus istius crustae ad superficiem terrae, ut pondus columnae mercurialis, qualis in barometro descendit ad basin ejus; manifesta haec sunt ex eo quod summa basium *A* & *B* sustinet quidem summam ponderum, quae habent columnae aëreae *AC* & *BD*, neque tamen quaevis basis premitur suae columnae pondere seorsim, & quod idem resectis columnis *Ag* & *Bh* intelligi debet de columnis *gC* & *hD*, diaphragmatis in *g* & *h* positis incumbentibus. Igitur experimentum non tam gravitatem specificam aëris, in quo factum est, indicat quam omnis aëris terrae proximi gravitatem specificam *mediam* determinat; prior admodum variabilis est, altera procul dubio constanter eadem fere permanet.

Faciamus computum *gravitatis specificae* istius *mediae* aëris omnis, qui terram ambit: Multis vero experimentis, quae in diversis locis parum supra mare elevatis sumta fuerunt, id constat, elevationi 66 pedum proxime descensum respondere unius lineae in barometro. Sequitur inde, quod aëris gravitas specifica media ratione mercurii sit, ut altitudo unius lineae ad altitudinem 66 *ped.*, id est, ut 1 ad 9504; ergo posita gravitate specifica mercurii $=1$, erit gravitas specifica *media* aëris $=0,000105$. Notabile est profecto tantam esse hanc gravitatem mediam aëris: certus enim sum vel maxime saeviente hic locorum frigore, aëris gravitatem specificam vixdum tantam esse, quantam nunc exhibuimus pro statu medio omnis aëris terram ambientis: at sub aequatore multo erit minor & omnibus recte perpensis non crediderim *gravitatem mediam* aëris, qui inter utramque latitudinem 60 *gr.* continetur, ultra 0, 000090 excurrere; quo posito erit *gravitas media* aëris ab utroque polo ad 30 gradus terram cingentis (quod spatium paullo plusquam octavam totius terrae superficiei efficit partem) = 0,000210, quae dupla est aëris hic locorum densissimi: sub ipso autem polo, praesertim antarctico, admodum gravior erit aër & fortasse aqua vix decies levior, cum est frigidissimus atque densissimus.

§. 17. Veniamus nunc ad mutationes tum atmosphaerae tum barometri: Considerabimus ergo duo barometra utrobique in imo aëris loco posita, alterum in *A*, alterum in *B*, & in utroque mercurium ad eandem altitudinem suspensum ponemus: Postea in *A* subito aërem admodum calefieri fingamus: Ita videmus fore, ut idem aër rarefiat: neque tamen inde ulla barometri mutatio proditura esset, si nullam aër haberet inertiam ad motum, etiamsi omnis aër ex *AC* in *BD* transpellatur: posita autem ista inertia supervenit quaedam pressio in omnes plagas eaque maxime sensibilis in regione *A.* Crescet igitur ad tempus altitudo mercurii in utroque barometro, magisque crescet in *A* quam in *B.* Contrarium erit, si extemplo magna quaedam aëris massa barometro *A* vel *B* vicina a frigore condensetur.

§.18. Haec unica videtur causa, quae aliquam in barometris in *A* vel *B* positis efficere possit mutationem, quia hac remota funda *A* & *B* semper aequaliter premuntur, nempe unusquisque pondere, quod sit dimidium columnarum aërearum AC & *BD* simul sumtarum, quae quidem ponderum summa constans est. Si haec ad atmosphaeram applicare velimus, notandum est funda *A* & *B* repraesentare loca ima atmosphaerae, quae quidem in superficie terrae posita forent, si aër terrae viscera penetrare nequiret: quia vero res secus se habet, erunt loca fundis *A* & *B* analoga intra superficiem terrae censenda.

§. 19. Putentur nunc barometra in *g* & *h* posita; sitque in ambobus mercurius ad eandem altitudinem suspensus: his positis causa fingatur supervenire, qua columna *Ag* sive sola sive conjunctim cum socia *Bh* calefiat atque sese expandat. His perspicuum est, si vel nulla aëris sit inertia, fore, ut pressiones aëris in *g* & *h* crescant, quia his locis major nunc aëris quantitas supereminet quam antea; accessit nimirum pondus omnis aëris, qui ex *Ag* & *Bh* a calore fuit sursum propulsus. Atque ut haec symbolis indicemus, faciemus pondus columnae Ag, antequam novus caloris gradus superveniret, $=A$, alterius $Bh=\alpha$, pondus columnae $gC=B$, columnae $hD=\beta$: pondus columnae *Ag* rarefactae $=C$, pondus columnae *Bh* itidem rarefactae $=\gamma$: altitudo mercurii in *g* ante expansionem aëris *Ag* & $Bh=l$, altitudo similis post istam expansionem $=x$, & habebimus hanc analogiam

$$B+\beta:1::B+A-C+\beta+\alpha-\gamma:x;$$

unde est
$$x=\frac{B+A-C+\beta+\alpha-\gamma}{B+\beta}l.$$

Igitur ascendet mercurius ab rarefacto aëre inferiore per altitudinem

$$x-l=\frac{A-C+\alpha-\gamma}{B+\beta}l=$$

(positis omnibus in utroque tubo paribus)

$$\frac{A-C}{B}l.$$

Refrigescente autem rursus aëre in *Ag* & *Bh* iterum descendet mercurius in utroque barometro.

Notandum hic est, posse hoc modo a parvula caloris mutatione in *Ag* atque *Bh* notabilem oriri in barometro variationem ob insignem aëris densitatem in partibus inferioribus, qua fieri potest, ut in parte *Ag* multo plus aëris contineatur (imo infinities, si aër vi infinita pressus in infinite parvum spatium condensari ponatur) quam in reliqua *gC*, etiamsi longitudine infinita. Unde si pondus *A* admodum majus sit pondere *B,* simulque manente causa aërem rarefaciente, pondus *C* datam servet rationem ad *A;* quod ita fere sit, apparet ascensum mercurii a minimo caloris gradu superveniente in *Ag* posse utcunque magnum esse.

Equidem si fingatur, partes *Ag* & *Bh* strictiores admodum esse prae amplitudinibus in *gC* & *hD,* intelligitur variationes barometri ab aucto diminutove caloris gradu in *Ag* & *Bh* ita fieri minus notabiles, quia pondera *A* & $\alpha$ ipsaque *C* & $\gamma$ prioribus proportionalia hoc modo decrescunt; attamen variationes barometricae, quae ab hac causa proveniant, etiamnum utcunque magnae concipi poterunt.

§. 20. Haec dum ita perpenduntur, verisimile fit variationes barometricas maxima parte petendas esse a celeribus caloris mutationibus in cryptis subterraneis. Multas esse easque permagnas hujusmodi cryptas jam diu notum est: in terra etiam solida pori facere possunt quod cryptae: si omnes cavitates ( tum quae a cavernis, tum quae a poris aërem continentibus formantur) ad altitudinem infra superficiem terrae 20 000 aut 30 000 pedum colligas earumque capacitatem compares cum soliditate crustae terrestris ejusdem altitudinis, hancque vel millies aut centies millies altera minorem ponas, erit profecto etiamnum sufficiens causa ista ad maximas barometri mutationes explicandas. Haec ut puto ex praecedente paragrapho unicuique perspicua erunt.

Caeterum loca quae sunt cryptis propiora, ea magis & ventis & barometri mutationibus erunt obnoxia, ob aëris ad motum inertiam, quae fortasse ratio est, quod versus aequatorem, ubi omnia fere pontus, minores variationes in barometro observentur quam in locis hisce septentrionalibus.

§. 21. Ex eodem fonte deducitur, aliquid etiam ad variationes barometricas conferre posse exhalationes aqueas ex terrae poris: sed certe parum id erit: si enim tantum aquae vapores suppeditarint, quantum maxima pluvia decidere potest, vix inde unica linea mercurius ascendet in barometro, praeterquam quod haec causa non sit ita celeris, quin illius effectus in totam atmosphaeram simul fere distribuatur, atque sic pro certo quodam loco totus evanescat. Si enim totam consideramus Atmosphaeram, quae terram ambit, animadverti certe non poterit esse eam vaporibus nunc minus nunc magis oneratam. Equidem rationem §. 20 expositam omnibus reliquis praetulerim, magnas enim & celeres in terrae visceribus fieri posse mutationes indicant terrae motus, qui saepe ad centum usque milliaria eodem tempore sentiuntur, & alia hujuscemodi phaenomena. Ad mutationes barometricas explicandas imprimis requiritur causa quaedam subita; jam enim monui lentas in integram distribui aëris massam nulliusque esse effectus, idque demonstravi §.14. Atque hanc ob causam parvi faciendas esse mutationes, quae immediate fiant in atmosphaera supra terrae superficiem.

§. 22. Et haec videtur pariter causa quod luna, quae tantae est efficaciae ad oceani aquas agitandas, nullum, qui observationibus diligentissimis observari potuerit, effectum exerat in barometrum: sique causae etiam reliquae, quae mutationem aliquam alicubi in Atmosphaera producere valent, paullatim agerent, foret procul dubio in omnibus locis a superficie maris aeque distantibus eadem constanter mercurii altitudo ad sensus. Haec altitudo *media* vocari potest & proxime determinabitur eo modo quo usus est Joh. Jacobus Scheuchzer, observando quotidie altitudinem barometricam per longum temporis tractum sumendoque inter omnes mediam.

Atque hac circumspectione usus celeberrimus Auctor ex multis observationibus, quae ad ipsum ex pluribus transmissae fuerunt locis, posuit altitudinem mediam

| | | |
|---|---|---|
| Patavii .......... | 27 *poll.* | $11\frac{1}{2}$ *fin. Paris.* |
| Parisiis .......... | 27 *poll.* | $9\frac{1}{2}$ *l.* |
| Tur ini. .......... | 27 *poll.* | $1\frac{1}{4}$ *l.* |
| Basileae ......... | 26 *poll.* | $10\frac{1}{8}$ *l.* |
| Tiguri ........... | *26poll.* | $6\frac{1}{2}$ *l.* |
| In monte Gothardi | 21 *poll.* | $27\frac{1}{2}$ *l.* |

§. 23. Diversitates istarum altitudinum *mediarum* ab inaequalibus locorum supra mare elevationibus provenire notum est. Jam enim Pascalii tempore experimenta sumta fuere de descensu mercurii in barometro ex loco profundiori in altiorem lato. Inde Philosophi in mutuam causae & effectus proportionem inquirere: Diversae in hanc rem variis auctoribus prodiere regulae: Praecipua, cui etiamnum plurimi adhaerent, haec est, quod altitudines locorum proportionem sequantur logarithmorum, qui altitudinibus barometri respondent. Fundata est haec regula praecipue super eo, quod densitas aëris ubique proportionalis sit ponderi aëris superincumbentis: male autem hic applicatur istud principium, quod pro aëre ejusdem caloris tantum valet, neque res certa est in omni altitudine aëris, quamvis in eadem columna verticali existentis; si vero ita sit, calorem aequalem esse, fatendum est, sic satis recte regulam se habere.

At experimenta regulae plane sunt contraria; igitur non est ubique idem caloris gradus per totam columnae aëreae verticalis altitudinem, quod ut nunc planum faciam, apponam experimenta quaedam accurate, ut mihi persuadeo, instituta, sed tamen, quod doleo, diversis temporibus locisque; magis utique instituto nostro convenirent experimenta eodem tempore in eodemque monte, diversis tantum altitudinibus, sumta; talia autem, nisi pro mediocribus locorum altitudinibus, nulla adhuc quantum scio extant cum omnibus quae scire oportet circumstantiis.

(I) In altitudine 1070 *ped. Paris.* a superficie maris barometrum descendit $16\frac{1}{3}$ *lin.* cum in superficie maris altitudinem teneret 28 *poll.* $4\frac{2}{3}$ *lin.* (alii ponunt simpliciter 28 *poll.;* in schedis autem quas D. De Lisle mecum communicavit habetur 28 *poll.* $4\frac{2}{3}$ *lin*). Igitur posita elasticitate aëris in superficie maris, uti deinceps semper ponam, $=1$, inventa fuit elasticitas in loco superiori quam designabo per $E = 0,9520$.

(II) In altitudine a superficie maris 1542 *ped. Paris.* descendit mercurius in barometro $21\frac{1}{2}$ *lin.* qui in mari ad altitudinem 28 *poll. 2lin.* suspensus haesit: hic igitur fuit $E = 0,9364$.

(III) In altitudine montis Pici super Insula Teneriffa 13 158 *ped. Paris.* a superficie maris stetit mercurius ad altitudinem 17 *poll.* 5 *lin.* dum in superficie maris teneret altit. 27 *poll.* 10 *lin.* unde eo in loco fuit $E = 0,6257$.

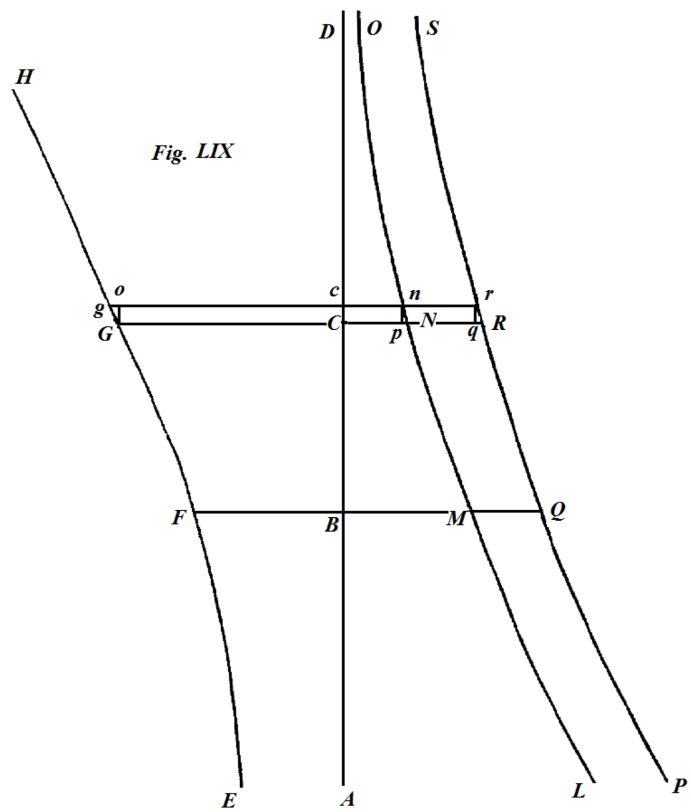
(IV) Si in minoribus altitudinibus accurate descensus mercurii observentur, reperitur descensum unius lineae respondere altitudini 65 aut 66 *ped.* Igitur in altitudine 65 *ped.* est $E = 0,9970$.

Extant passim hae observationes: tertiam autem habeo a D.[no] De Lisle fuitque a R. P. Feuillée instituta atque coram *Societate Reg. Scient. Paris.* praelecta: estque illa scopulus, ad quem omnes, quae adhuc lucem aspexerunt, theoriae illidunt.

§. 24. Ut jam pateat, quousque haec cum positione logarithmicae ceu scalae altitudinum elasticitatibus respondentium conveniant, ponemus altitudinem loci a superficie maris certo numero pedum Parisinorum definiendam $= x$; elaterem aëris in superficie maris designabimus per 1, & elaterem aëris in altitudine $x$ ponemus $= E$. Notetur autem atmosphaeram nunc nobis considerari invariatam aut saltem sibi constanter similem, ita ut elateres aëris in superficie maris & in altitudine quacunque $x$ constantem servent rationem. Si enim admodum inaequaliter in diversis atmosphaerae altitudinibus, nulla servata proportione, elateres inconstantia temporis mutentur, sane nulla excogitari poterit regula. His praemissis ponamus nunc aequationem $\alpha \log E = x$ ubi coefficiens $\alpha$ unica determinabitur observatione: utamur observatione prima & erit $\alpha \log 0,9520 = 1070$, hincque $\alpha$ (secundum logarithmos Vlacquianos) $-50194$. Igitur pro hoc negotio, si logarithmica satisfacere debeat, ponendum esset $-50194 \log E = x$, sive $\log \frac{1}{E} = \frac{x}{50194}$.
Ad hujus autem aequationis normam, si ponatur pro secunda observatione $x = 1542$, invenitur $E = 0,9317$, ipsa autem observatio indicat $\mathrm{E} = 0,9364$: differentia inter hypothesin & observationem est plus quam sesquilineae, quae sane notabilis est respectu habito ad differentiam parvam altitudinum verticalium.

Si jam porro pro tertia observatione ponatur $x = 13158$, fit ex hypothesi $E = 0,5469$, dumexperimentumindicavit $E = 0,6257$: quae differentianimiaest, quam ut ullo modo logarithmica servari possit: valet enim haec differentia plus quam duos pollices cum duabus lineis.

§. 25. Rejecta logarithmica consequens est elasticitates in diversis atmosphaerae altitudinibus nequaquam esse densitatibus proportionales, aut quod eodem recidit, diversum esse in diversis altitudinibus medium caloris gradum. Aliae igitur ab aliis, quibus defectus iste probe fuit notatus, fuerunt excogitatae regulae: earum tamen nulla ad experimentum III (§. 23) satis accommodata dici potest. Veram, quam natura sequatur, legem invenire, rem esse puto vix sperandam: quis enim aliter quam levibus conjecturis




assequetur rationem velocitatum mediarum in particulis aëreis. Incidi tamen forte in aliquam hypothesin, quae phaenomenis non male respondet: prius autem pro quacunque velocitatum lege curvam dabo, quam ad specialem istam hypothesin descendam.

§. 26. Sit linea verticalis *AD* (Fig. 59); *QF* horizontalis radat superficiem maris; denotet *BF* velocitatem mediam particularum aërearum in superficie maris, *BM* densitatem mediam & *BQ* elasticitatem, quae in omni loco aeque alto eadem est. Deinde per puncta *F, M, Q* ductae concipiantur curvae *EFH, LMO, PQS* ceu scalae, quae in omnibus altitudinibus, veluti *BC,* applicatis *CG, CN, CR* denotent velocitates medias particularum aërearum, densitates medias & elasticitates medias. Datis nunc duabus curvis tertiam licet determinare ex eo, quod elasticitates (ceu experientia docuit & §§. 3, 4, 5 & 6 explicatum fuit) sint proxime in ratione composita ex quadrato velocitatum modo dictarum & simplici densitatum.

Ipse quidem monui praedicto loco hanc proportionem non posse exacte esse veram, quia aër quidem elaterem potest habere infinitum seu vi infinita comprimi, non potest autem in spatium plane infinite parvum condensari: quia tamen in aëre qui sit naturali vel quadruplo densior, haec proprietas, *quod nempe elasticitates sint in ratione composita ex quadrato velocitatum particularum & simplici densitatum,* experimentis etiamnum ad sensus omnino respondere visa fuit, illa sine ullo sensibili errore uti poterimus pro aëre naturali atmosphaerae mari incumbentis, siquidem eo accuratius vera sit quo rarior est aër.

His ad calculum praeparatis ponemus

$BF = a,\ BM = b,\ BQ = c,\ BC = x,\ Cc = dx,$

$CG = v,\ CN = z,\ CR = y,$ & erit $y : c = vvz : aab$ seu $y = \frac{cvvz}{aab}$

Quia porro elasticitatis mensura est pondus superincumbentis aëris, erit $qR(-dy) =$ ponderi strati aërei intercepti inter *C & c*, quod proportionale est aëris densitati *z* & altitudini strati $dx$ : est igitur $-dy = \frac{zdx}{n}$ seu $z = \frac{-ndy}{dx}$, quo valore substitute in aequatione ( $y = \frac{cvvz}{aab}$) habetur

$$y = \frac{cvv}{aab} \times \frac{-ndy}{dx}$$

vel

$$-\frac{dy}{y} = \frac{aabdx}{ncvv}.$$

§. 27. Si ponatur velocitas particularum aërearum in omni altitudine eadem, nempe $= a,$ fiet $\frac{-dy}{y} = \frac{bdx}{nc}$, facta debita integratione, $\log\frac{c}{y} = \frac{bx}{nc};$ istam vero hypothesin non satis experimentis confirmari vidimus §. 24. Igitur alia tentata, posui $v = \sqrt{(aa + mx)}$ vel $vv = aa + mx$ , quae lex est in motibus corporum libere cadentium: neque id sine successu; ita vero fit

$$\frac{-dy}{y} = \frac{aabdx}{naac + mncx},$$

vel

$$\log\frac{c}{y} = \frac{aab}{mnc}\log\frac{aa + mx}{aa}.$$

In hac aequatione paullo generaliori in qua *m* & *n* etiamnum arbitrariae sunt, porro periculum feci, num non posset poni $\frac{aab}{mnc} = 1$, atque id etiam apte fieri vidi: sic vero obtinui $\log\frac{c}{y} = \log\frac{aa + mx}{aa}$ vel $\frac{c}{y} = \frac{aa + mx}{aa}$ aut $\frac{y}{c} = \frac{aa}{aa + mx}$. Indicat ista hypothesis esse elasticitates aëris ubique in ratione reciproca quadrata velocitatum, quibus particulae aëreae agitantur, sive esse *CR* ad *BQ* ut $BF^2$ ad $CG^2$; atque cum *EFH* ex hypothesi parabola est super axe *AD* verticem habens infra punctum *B* ad distantiam $\frac{aa}{m}$, sequitur esse curvam *PQS* hyperbolam; dictam vero distantiam $\frac{aa}{m}$ sumendam esse $=22000$ pedum animadverti, ut observationibus §. 23 proxime satisfiat. Inde talis jam prodit aequatio specifica $\frac{y}{c}=\frac{22000}{22000+x}$. Pro curva vero *LMO* invenitur (per§. 26) , seu $\left(\text{quia } \frac{aa}{vv}=\frac{22000}{22000+x}=\frac{y}{c}\right)$ prodit post hanc substitutionem $\frac{z}{b}=\left(\frac{22000}{22000+x}\right)^2$.

§. 28. Ut appareat, quousque hypothesis nostra conveniat cum experimentis §. 23, ponemus in aequatione pro elasticitatibus successive pro *x*, 1070, 1542, 13158, & 65; ita invenitur respective $\frac{y}{c}=0,9536$; $\frac{y}{c}=\ 0,9345$; $\frac{y}{c}=0,6257$, atque $\frac{y}{c}=0,99705$: observationes autem indicant $\frac{y}{c}=0,9520$; $\frac{y}{c}=0,9364$; $\frac{y}{c}=0,6257$, atque $\frac{y}{c}=0,9970$. Observatio tertia aliis hypothesibus inimicissima cum nostra plane conspirat, nec reliquae plusquam 0,0019 particulis dissentiunt, quae in altitudine barometri tres quintas unius lineae partes valent. Nemo autem qui expertus fuerit, quam vagae & parum inter se consentientes fuerint observationes barometricae, tantillam differentiam admodum curabit. Ipse interim hanc rem non aliter quam hypothesin precariam considero, neque aliam ob causam calculum §§. 26 & 27 praemisi, quam ut rationem darem, qua fieri possit ut altitudines verticales non respondeant logarithmis altitudinum barometricarum, prouti deberet fieri, si per totam atmosphaeram uniformis esset calor: instituto enim calculo factaque comparatione ejus cum experimentis mihi videre visus sum, non posse rem hanc a diversa particularum aërearum gravitatione in diversis a centro terrae distantiis sufficienter explicari, prouti Newtonus tentavit statuendo gravitationes harum particularum decrescere in ratione quadrata distantiarum a centro terrae, quae hypothesis in altitudinibus 13000 *pedes Paris.* non excurrentibus sensibilem differentiam non efficit ab hypothesi uniformis gravitationis. Similiter ego aliquando incidi in opinionem auctam vim centrifugam particularum aërearum in majoribus altitudinibus aliquid hic contribuere posse; at pariter instituto calculo opinioni huic non amplius adhaesi. Interim non puto, absurdum esse, si dicamus calorem aëris medium eo majorem esse, quo magis a superficie maris distet. Velim autem ut probe notetur, hic sermonem esse de calore *medio* in libera atmosphaera: sic enim fieri potest, ut calor realis quidem in montibus non crescat ex causis aliis, nec tamen inde hypothesis evertatur, quandoquidem §§.15 & 16 jam demonstratum fuerit, pondus columnae mercurii in barometro non praecise censendum esse aequale ponderi columnae aëreae in illa regione sumtae, sed ponderi medio omnium columnarum terrae insistentium: De diversis densitatibus itaque sic sentio.

§. 29. Si aequalis esset ubique calor, forent utique densitates elasticitatibus ad sensus proportionales, responderentque altitudines verticales logarithmis altitudinum barometricarum: At vero id experimentis repugnare pono: neque tamen crediderim in duobus locis parum a se invicem dissitis notabilem intercedere posse caloris differentiam, quia calor in corpore rariore, ut est aër, mox uniformiter distribuitur, nisi perpetua adsit causa, quae aërem vicinum calefaciat.

Alia autem res est in locis remotioribus; nec enim absurdum puto aërem vel decies densiorem statuere sub polis, quam sub aequatore, si modo aër utrobique accipiatur superficiei terrae proximus ; at in magnis altitudinibus minor utique erit differentia inter densitatem aëris qui polis & ejus qui aequatori respondet caeteris paribus, & propterea inaequaliter admodum decrescent a superficie terrae densitates aëris & multo magis decrescent sub polis quam sub aequatore: hoc igitur modo fieri posset, ut sub polis densitates aëris reales in parvis altitudinibus v. gr. decrescant in ratione ut $(22000+x)^4$ ad $22000^4$ ob auctum calorem, & sub aequatore vix sensibiliter decrescant, ob diminutum calorem, quae caloris diminutio prope aequatorem confirmatur ex eo quod culmen montis Pici per decem fere mensium spatium sit nive obtectum, dum in ipsa Teneriffae insula nunquam ut ferunt ningit. Igitur non absurde densitates mediae censeri possunt diminui in ratione ut $(22000+x)^2$ ad $22000^2$, ut §. 27 assumtum fuit, dum elasticitates ubique decrescant in ratione ut $22000+x$ ad $22000$; neque enim hae in iisdem a superficie terrae altitudinibus differre possunt, nisi a causis fortuito supervenientibus & parum durantibus.

§. 30. In terris, quae intra quadragesimum & sexagesimum latitudinis gradum continentur, probabile est densitates in eadem proxime ratione decrescere qua elasticitates; hancque ob rationem volui periculum facere, quaenam inde refractionum theoria oriatur, qua de re nunc quaedam adjiciam.

…………………………………………………………………………………………….

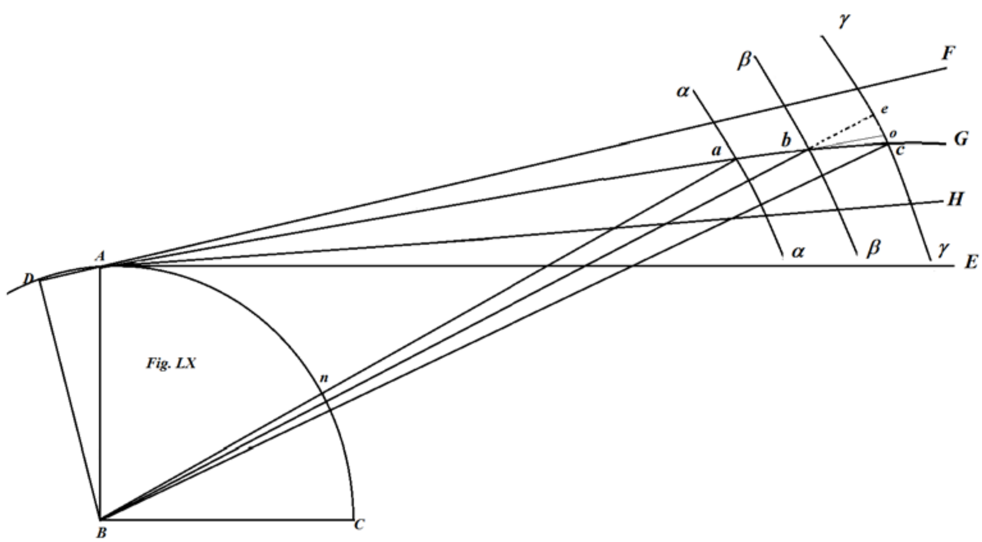
*Digressio de refractione radiorum per atmosphaeram transeuntium.*

$(\alpha)$ Proprietas est notissima radiorum ex uno medio in aliud incidentium eaque innumeris experimentis confirmata, quod angulus incidentiae ad angulum refractionis constantem servat rationem: praeterea etiam patet, si refractio fiat infinite parva, id est, si differentia utriusque sinus rationem habeat infinite parvam ad alterutrum sinum, fore ut sinus anguli, qui intercipitur inter radium incidentiae prolongatum & radium refractum, eandem habeat rationem ad sinum totum, quam habet differentia sinuum angulorum incidentiae & refractionis ad cosinum anguli incidentiae. Illum vero, quem modo allegavi, angulum interceptum inter radium incidentiae prolongatum & radium refractum, deinceps vocabo *angulum refractionis differentialem.* Exinde sequitur, quod sit caeteris paribus sinus *anguli refractionis differentialis* proportionalis sinui anguli incidentiae diviso per cosinum ejusdem anguli.

$(\beta)$ Experimenta porro docent, si radius ex aëre in aërem diversae ab altero densitatis incidat, esse *angulum refractionis differentialem* caeteris paribus differentiae densitatum proportionalem.

Experimenta autem hanc in rem, quantum fieri potest, sumta fuerunt a D. Hauksbee, accuratissime, tum de aëre admodum condensato, tum etiam de aëre rarissimo, qui tandem pro nullo haberi poterat: modus quo instituta fuerunt describitur in *Transactionibus Anglicanis :* Successus autem omnium experimentorum huc redit, ut arguant fuisse sinum *anguli refractionis differentialis* ad sinum totum ut $5\frac{1}{8}$ pollices ad 2588 pedes, cum radius incideret ex aëre naturali in spatium ab aëre vacuum sub angulo triginta duorum graduum, id est, ut 1 ad 6060, & iisdem positis, mutato angulo triginta duorum graduum in semirectum, ut 1 ad 3787 (per $\S.\alpha$ ). Inde deducitur, si radius ex *aëre naturali* in vacuum sub angulo quocunque incidat, esse sinum anguli incidentiae ad sinum anguli refractionis ut 3787 ad 3786.

Newtonus loco hujus rationis assumit in *Tract.* suo *optico* illam, quae est inter 3201 & 3200, eamque deducit ex refractionum quantitate ab Astronomis observata: statuit autem quantitatem refractionis eandem esse, si strata radium refringentia sint parallela, in quacunque caeterum ratione densitates medii decrescant, si modo in primo & ultimo strato densitatum differentia eadem maneat ( vid. Newt. *Tract. opt., pag.* 321 edit. gall.). De reliquo sub diversis circumstantiis non potest non admodum esse variabilis refractio, quod aër, quem vocamus naturalem, multis mutationibus sit obnoxius, tum a calore & frigore, tum a pressione atmosphaerae, quae ambo concurrunt ad densitatem aëris formandam, cui densitati refractiones radiorum in vacuum incidentium sunt proportionales caeteris paribus. Eadem etiam monuit D. Hauksbee in recensione experimentorum, quae modo allegavimus, eamque ob rationem statum aëris, qui erat, cum experimenta sumeret, probe definivit.




$(\gamma)$ Fuerit nunc *AC* (Fig. 60) arcus circuli terrestris centro *B* ductus, in cujus plano radius luminis *AG* est: erit autem iste radius incurvatus *AG* eius indolis, ut convergat ad asymptoton, huicque asymptotae parallela putetur *AH;* ducatur horizontalis *AE,* rectaque *AF* quae tangat in *A* curvam *AG.* Ita videmus fore angulum *HAE* mensuram altitudinis astri verae, & angulum *FAE* mensuram altitudinis apparentis, angulumque *FAH* fore angulum refractionis: est autem angulus *FAH* idem quod summa omnium *angulorum refractionis differentialium,* seu angulorum contactus qualis est angulus *cbo.*

Considerentur duo elementa curvae *ab, bo,* & per puncta *a, b, o* ducti intelligantur centra communi *B* arcus $\alpha\alpha, \beta\beta, \gamma\gamma$ : sitque densitas aëris $\alpha\alpha\beta\beta = D$ *;* densitas aëris $\beta\beta\gamma\gamma = D - dD$ , erit (per $\S\S.\alpha, \beta$ ) sinus anguli contactus in *b* divisus per sinum totum, seu ipse angulus contactus, proportionalis differentiae densitatum $dD$ multiplicatae per rationem sinuum angulorum incidentiae & refractionis, id est, multiplicatae per $\frac{be}{eo}$. Si vero ducatur $BD$ perpendicularis ad $FA$ productam, perspicuum est, vix differre $\frac{be}{eo}$ & $\frac{BD}{Do}$, ideo quod radius fere sit rectus sicque possit triangulum $BDo$ pro rectilineo haberi & simili cum triangulo $beo$. Igitur erit angulus quaesitus $FAH$ proportionalis $\int \frac{BD}{Do} \times dD$.

$(\delta)$ Hisce vestigiis insistendo ponendoque esse ubique densitatem $D = \frac{22000}{22000 + x} G$, ubi $x$ exprimit lineam $na$ numero pedum Parisinorum & $G$ denotat densitatern aëris in loco observationis, inveni quod sequitur. Sit sinus altitudinis astri apparentis $= F$, cosinus $= f$, radius terrae $= r$ numero pedum Parisinorum exprimendus: indicetur numerus 22000 per a: ponatur porro sinus totus $= 1$, *angulus refractionis differentialis* pro radio ex aëre naturali in vacuum sub angulo semirecto incidentis $= g$ : Denique brevitatis ergo fiat $2r - 2a = \alpha$; $-FFrr + 2ar - aa = \beta$ : & erit $\beta$ aut numerus affirmativus aut negativus; affirmativus erit, si altitudo apparens sideris parva fuerit & quidem infra $2^{o}$, 44'; secus erit negativus: In priori casu obtinebitur angulus quaesitus $FAH$ hunc in modum:

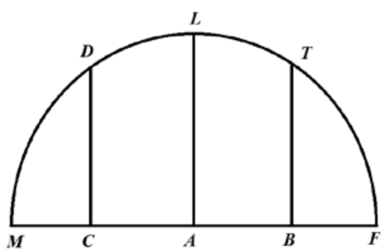


Fig. LXI

Fiat nempe semicirculus $MLF$ (Fig. 61) cujus radius $AM = 1$; sumatur $AC = \frac{\alpha}{2f}$, $AB = \frac{2\beta - \alpha a}{2afr}$, ducanturque $CD$, $BT$ ad $MC$ perpendiculares & erit

$$\text{angulus } FAH = \frac{-fFrr}{\beta} g + \frac{far}{\beta} g + \frac{far\alpha \times DT}{2\beta\sqrt{\beta}} g .$$

In casu, quo $\beta$ est negativus, erit idem

$$\text{angulus } FAH = \frac{-far}{\beta} g + \frac{fFrr}{\beta} g + \frac{far\alpha \times}{2\beta\sqrt{\beta}} g \times \log \frac{\left(\alpha - 2\sqrt{\beta}\right) \times \left(Fr - \alpha + \sqrt{\beta}\right)}{\left(\alpha + 2\sqrt{\beta}\right) \times \left(Fr - \alpha - \sqrt{\beta}\right)}.$$

$(\varepsilon)$ Secundum istas hypotheses ponendo pro radio terrae 19600000, poterit pro omni altitudine sideris apparentis ejus determinari refractio astronomica, si bene experimento inventus fuerit valor anguli *g;* quia vero difficile admodum est hunc valorem cum sufficiente accuratione definire, consultius erit in casu aliquo particulari astronomice refractionem definire, & ex hoc reliquos calculo subducere. Assumamus v. gr. in altitudine decem graduum refractionem esse 5 *min.* 28 *sec.*, cui hypothesi plerique Astronomi Parisiis adhaerent. Inveniemus hance refractionis tabulam :

| altit. sid. appar. | refract. | altit. sid. appar. | refract. |
|---|---|---|---|
| *0 grad.* | 34 *min.* 55 *sec.* | *50 grad.* | *0 min.* $52\frac{1}{2}$ *sec.* |
| 5 | 9 .... 45… | 55 | ……44 |
| 10 | 5 .... 28… | 60 | ……$36\frac{1}{5}$ |
| 15 | 3 .... 44… | 65 | ……$29\frac{1}{4}$ |
| 20 | 2 .... 48… | 70 | …… 23 |
| 25 | 2 .... 12… | 75 | ……17 |
| 30 | 1 .... 47… | 80 | ……11 |
| 35 | 1 .... 29… | 85 | ……$5\frac{1}{2}$ |
| 40 | 1 .... 14… | 90 | ……0 |
| 45 | 1 .... $2\frac{1}{2}$… | | |

Quia vero refractiones sequuntur rationem litterae *g*, id est, *anguli refractionis differentialis* radii sub angulo semirecto ex aëre naturali in vacuum incidentis & quia iste angulus proportionalis est densitati aëris naturalis, seu aëris, quem observator respirat, patet si vel aër constanter similiter vaporibus esset oneratus (a quibus animum adhuc abstraximus), non posse tamen fieri, quin refractiones astronomicae sint admodum variabiles. Majores nempe erunt in superficie maris quam in montibus, eritque notabilis differentia vel in mediocribus montium altitudinibus: majores praeterea erunt tempore frigido quam calido, haecque sola causa in hisce terris refractiones minimum quarta parte augere potest: denique majores etiam erunt refractiones barometro alto quam humili. Poterunt autem, si vapores nullo sint obstaculo, refractiones omni tempore recte definiri, si instrumentum, quod §. 9 descriptum fuit quodque Fig. 57 repraesentat, simul adhibeatur cum barometro; si enim altitudinem mercurii in barometro dividas per altitudinem mercurii in altero instrumento, habebis densitatem aëris, cui caeteris paribus refractio proportionalis est facienda. Neque dubito, quin refractio solis minor sit refractionibus reliquorum siderum, quod calor solis aërem non mediocriter expandit aërisque densitatem diminuit.
……………………………………………………………………………………….

§. 31. Ex iis quae de agitatione particularum aërearum, aqua utique calor aëris pendet, praesertim vero, quae §. 10 monita fuerunt, apparet gradum eundem caloris aëri inesse, quoties eadem ratio intercedit inter ejus elasticitatem atque densitatem; elasticitatem indicat barometrum; densitatem concludimus ex gravitate aëris specifica; atque inde ut vidimus§. 10, gradus obtineri poterit caloris fixus, si aquae bullientis calor incertus videatur, prouti D. Fahrenheit observatus fuit pendere a pondere atmosphaerae incumbentis. Instrumenta quae singulis momentis densitatem aëris indicant facile excogitari possunt atque a multis descripta fuerunt.

Notandum hic est rationem illam modo dictam inter aëris elasticitatem ejusque densitatem simul exhibere altitudinem aëris homogenei, & quia nobis deinceps sermo erit de ista altitudine, convenit illam recte prius definire, quam ad alia pergamus.

§. 32. Si fingamus columnam aëream verticalem uniformis densitatis & cum mercurio barometri ad aequilibrium compositam, erit altitudo illius columnae *altitudo* quam voco *aëris homogenei* pro data densitate.

Et quia aëris mediocriter densi gravitas specifica est ad gravitatem specificam mercurii ut 1 ad 11000 ipsaque altitudo media mercurii in barometro pro locis parum a superficie maris elevatis sit $2\frac{1}{3}$ *ped. Paris.,* erit altitudo aëris homogenei mediocriter densi 25666 pedum.

Patet ex ista definitione altitudines illas, de quibus nunc dicimus, eo minores esse, quo densior est aër, cui altitudo respondere debet, & quo minor est altitudo mercurii in barometro. Igitur si idem sit caloris gradus in montibus & in superficie maris, eadem quoque erit utrobique altitudo aëris homogenei, quia pro eodem caloris gradu aëris densitas rationem sequitur aëris elasticitatis seu altitudinis mercurii in barometro. Apparet porro altitudinem aëris homogenei in superficie maris admodum decrescere ab aequatore versus polos, quia frigus intenditur densitasque aëris augetur manente elasticitate, & in iisdem regionibus minorem esse tempore hyemali quam aestivo.

§. 33. Multa sunt quae ad motum aëris definiendum pertinent, quorum solutio pendet ab altitudine aëris homogenei: Inter haec etiam est propagatio soni ejusque celeritas: Quamvis enim celeritas soni diversimode definiatur a diversis, quos concipere possumus de ejus propagatione, modis ita, ut nunc videatur celeritatem eam esse quae debeatur altitudini aëris homogenei, nunc quae dimidiae altitudini respondeat, aut etiam dimidiae altitudini multiplicatae per rationem quadrati circulo circumscripti ad aream circuli, omnes tamen opiniones in eo conveniunt, quod celeritas soni proportionalis sit radici altitudinis aëris homogenei cum eo, in quo propagatur. Si ita se res habeat, celerius propagatur sonus in aëre calido quam frigido, barometro alto quam humili (nihil dicam de ventis secundis aut contrariis); multa in hanc rem partim in Italia partim in Anglia sumta fuerunt experimenta, haecque posteriora docuerunt celeritatem soni mediam respondere 1140 *ped. Angl.* intra minutum secundum perficiendis. At quia in uno eodemque loco variabilis est altitudo atmosphaerae homogeneae nominatimque hic locorum excurrit a mutationibus barometricis junctis cum mutationibus caloris a 3 usque ad 4, variabilis erit ubique celeritas soni, si vel nihil mutent venti, eaque celeritas in hisce terris continebitur intra terminos $\sqrt{3}$ & $\sqrt{4}$, seu 173 & 200.

§. 34. Venio jam ad varias quae fingi possunt de motu aëris quaestiones solvendas similes illis, quas de motu fluidorum non elasticorum in praecedentibus habuimus.

## Problema.

Sit motus definiendus aëris ex vase per foramen exiguum erumpentis in spatium infinitum ab aëre vacuum.

Solutio.

Apparet ex natura quaestionis insensibilem esse motum localem aëris interni quo sese expandit, dum certa sui quantitas per foramen erumpit: Igitur hic solus *ascensus potentialis*, quem particula aërea, dum expellitur, acquirit, considerandus est, atque comparandus cum *descensu actuali* vel potius cum diminutione elasticitatis, quam aër internus habet. Ut vero totam rem ad methodum nostram pro fluidis non elasticis adhibitam reducamus, considerabimus cylindrum verticalem communis cum vase proposito amplitudinis atque tantae altitudinis, quanta est altitudo aëris homogenei cum aëre interno; is vero cylindrus, si simili aëre plenus censeatur, sed non elastico, eadem velocitate suo pondere aërem infimum expellet per foramen, qua aër in vase proposito sua elasticitate se ipsum expellit. In priori autem casu ejicitur velocitate quae debetur ipsi altitudini cylindri, ergo & in posteriori. Notandum autem est, altitudinem quam pro cylindro finximus, perpetuo eandem esse, quia aëris elasticitas & densitas in eadem ratione diminuuntur, calorem autem non mutari ponimus. Igitur si altitudo aëris homogenei (quae a calore aëris interni pendet) dicatur *A*, effluet aër constanter velocitate $\sqrt{A}$. Nec tamen, quod calculus ostendit, vas ipsum unquam evacuatur, quia aër effluens fit continue rarior, quod ut aequatione comprehendamus, ponemus densitatem seu quantitatem aëris a fluxus initio $=1$; densitatem seu quantitatem aëris post definitum tempus residui $=x$, tempusque ipsum $=t$, erit, quia velocitas constans est, $-dx = axdt$, ubi per *a* intelligitur quantitas constans definienda ex magnitudine vasis, amplitudine foraminis & altitudine *A:* hinc $\frac{-dx}{x} = adt$ & $\log\frac{1}{x} = at$*;* reperitur autem valor coefficientis *a* hoc modo. Quia positum a nobis fuit $-dx = axdt$, erit ab initio effluxus $-dx = adt$. Jam mutetur elementum primum $(-dx)$ in cylindrum foramini ceu basi superinstructum; erit autem altitudo istius cylindruli $=-Ldx$, si *L* sit altitudo cylindri super eodem foramine extructi & communem cum vase proposito capacitatem habentis: haec porro longitudo $-Ldx$ illa est, quae tempusculo *dt* percurritur, & quia poni solet tempusculum aequale spatio percurso diviso per velocitatem, erit hic $dt = \frac{-Ldx}{\sqrt{A}}$; substituatur iste valorum in aequatione $-dx = axdt$ & habebttur $-dx = \frac{-aLdx}{\sqrt{A}}$, sive $a = \frac{\sqrt{A}}{L}$. Est proinde aequatio finalis haec:

$$\log\frac{1}{x} = \frac{t\sqrt{A}}{L}.$$

Si tempus exprimere lubeat per certum minutorum secundorum numerum, quem vocabimus *n*, & intelligatur per *s* spatium quod mobile absolvit cadendo libere a quiete intra unum minutum secundum, erit ponendum $t = 2n\sqrt{s}$, sicque fiet

$$\log\frac{1}{x} = \frac{2nt\sqrt{As}}{L}.$$

Problema.

§. 35. Quaeritur motus aëris densioris in aërem externum rariorem infinitum ex vase per foramen valde parvum erumpentis, posito in utroque aëre eodem caloris gradu.

Solutio.

Sit densitas aëris interni initialis $= D$, densitas aëris externi $= \delta$, densitas aëris interni post datum tempus $t$ residui $= x$, altitudo aëris homogenei ( sive ratione aëris interni sive externi; nec enim diversa esse potest, si uterque aër eodem calore praeditus sit, sicque densitates & elasticitates in pari ratione decrescant) $= A$. Quaeratur ubique altitudo aëris homogenei, qui habeat eandem pressionem seu elaterem cum aëre externo & cujus densitas eadem sit cum aëre interno: haec altitudeo ab initio erit $\frac{\delta A}{D}$, & post tempus $t$ erit $\frac{\delta A}{x}$. Patet autem veloctitatem aëris erumpentis talem ubique fore, quae respondeat differentiae definitarum altitudinum $A \,\&\, \delta A$; est itaque post tempus $t$ velocitas aëris erumpentis $= \sqrt{A - \frac{\delta A}{x}}$.

Sunt porro decrementa densitatum ($-dx$) proportionalia quantitatibus aëris erumpentis, quae rationem habent compositam ex velocitate $\left(\sqrt{A - \frac{\delta A}{x}}\right)$, ex densitate ($x$) & ex tempusculo ($dt$): sic igitur est

$$-dx = a\sqrt{A - \frac{\delta A}{x}} xdt ,$$

ubi $a$ est numerus constans qui per methodum praecedentis paragraphi fit $= \frac{1}{L}$, retenta significatione hujus litterae ibidem adhibita; hocque valore substituto oritur

$$-dx = \frac{dt}{L} \times \sqrt{(Axx - \delta Ax)}$$

seu

$$\frac{-dx}{\sqrt{(xx - \delta x)}} = \frac{dt\sqrt{A}}{L} :$$

Factaque debita integratione fit:

$$\log \frac{\left(\sqrt{x} - \sqrt{x - \delta}\right) \times \left(\sqrt{D} + \sqrt{D - \delta}\right)}{\left(\sqrt{x} + \sqrt{x - \delta}\right) \times \left(\sqrt{D} - \sqrt{D - \delta}\right)} = \frac{t\sqrt{A}}{L},$$

aut posito rursus, ut in praecedente paragrapho, $t = 2n\sqrt{s}$, erit

$$\log\frac{\left(\sqrt{x}-\sqrt{x-\delta}\right)\times\left(\sqrt{D}+\sqrt{D-\delta}\right)}{\left(\sqrt{x}+\sqrt{x-\delta}\right)\times\left(\sqrt{D}-\sqrt{D-\delta}\right)}=\frac{2n\sqrt{As}}{L}.$$

Corollarium 1.

§. 36. Omnis effluxus fit tempore finito qua in re ista quaestio ab altera praecedente differt: Cessat autem aër effluere, cum est $x=\delta$, & tunc fit

$$n=\frac{L}{2\sqrt{As}}\times\log\frac{\left(\sqrt{D}+\sqrt{D-\delta}\right)}{\left(\sqrt{D}-\sqrt{D-\delta}\right)}.$$

Sit v. gr. $A = 26000$ *ped. Paris.;* contineat vas propositum unum pedem cubicum, foramen autem habeat amplitudinem unius lineae quadratae, erit $L = 20736$; ponatur insuper aërem internum ab initio duplo fuisse densiorem externo; est autem ut constat $s = 15\frac{1}{12}$ *ped. Paris.* Fiet igitur

$$n=\frac{20736\sqrt{3}}{\sqrt{181\cdot 26000}}\log\frac{\left(\sqrt{2}+1\right)}{\left(\sqrt{2}-1\right)}. = 29,2$$

quod significat aërem utrumque ad aequilibrium compositum iri tempore paullo majori quam viginti novem minutorum secundorum, post idque omnem effluxum cessaturum. Fieri autem potest a contractione, quam fluida prae foramine patiuntur (vid. Sect. IV) & ad quam nullam fecimus in computo attentionem, ut tempus istud augeatur fere in ratione ut 1 ad $\sqrt{2}$.

Corollarium 2.

§. 37. Si fingatur aërem non immediate per foramen effluere, sed per longum tubum, non mutabitur propterea velocitas, si modo totius tubi capacitas sit veluti infinite parva ratione capacitatis, quae in vase ipso est; videtur autem densitatem aëris, quamdiu in tubo est, eandem esse cum densitate aëris vasi inclusi, nec tamen, quod demonstrabo inferius, elasticitas aëris in tubo major est elasticitate aëris externi, qui tubum circumdat. Consequens inde est, ventum aërem esse densiorem aëre quiescente, sed non magis elasticum: attamen densitatum differentia parvula quoque erit; ventus enim, qui vel 30 pedes singulis minutis secundis conficit, aërem vicinum, aeque calidum & quietum, vix una millesima septingentesima parte densitate superabit .

Problema.

§. 38. Definire influxum aëris per foramen valde parvum in vas aëre rariore plenum, posito rursus utrobique eodem caloris gradu.

Solutio.

Fuerit vas ab initio omnino vacuum, & post tempus *t* ponatur densitas aëris interni $= x$; sic reperietur iisdem fere vestigiis insistendo, quibus in trigesimo quinto paragrapho usi sumus, retentisque iisdem denominationibus sive

$$\frac{dx}{\sqrt{(D-x)}}=\frac{dt\sqrt{AD}}{L}$$

sive

$$t=2n\sqrt{s}=\frac{2L}{\sqrt{A}}=\frac{2L\sqrt{(D-x)}}{\sqrt{AD}}.$$

Numerus igitur minutorum secundorum, quo totum vas impletur, donec inter utrumque aërem aequilibrium sit, exprimitur per $\frac{L}{\sqrt{AS}}$: & est tempus repletionis duplum illius quo repleretur si velocitate initiali constanter influeret aër. In casu quo capacitas vasis pedem cubicum continet & foramen lineam quadratam aequat, fit repletio tempore propemodum triginta trium minutorum secundorum, nisi contractione venae aëreae influentis repletio retardetur.

§. 39. Exposuimus varias fluidorum elasticorum sive motorum sive quiescentium proprietates: Unum superest non omittendum, quo fluida elastica differunt a non-elasticis, hoc scilicet, quod fluido elastico vel quiescenti *vis viva* insita sit, non quod instar aliorum corporum motorum se ad certam altitudinem elevare possit, neque enim motum localem in illo hic consideramus, sed quod elatere suo talem ascensum in aliis corporibus gravibus generare possit. Licebit autem, quod spero, in sequentibus uti vocabulo *vis vivae corpori elastico compresso insitae,* quando nihil aliud eo intelligitur quam *ascensus potentialis,* quem corpus elasticum aliis corporibus communicare potest priusquam totam suam vim elasticam perdiderit.

Meretur hic in antecessum notari, quod sicut descensus corporis dati per datam altitudinem, utcunque fiat, eandem constanter vim vivam in corpore producit, ita quoque elastrum sive fluidum elasticum postquam a dato tensionis seu condensationis gradu ad datum alium gradum fuit reductum utcunque, id semper eandem vim vivam in se recipiat rursusque contraria mutatione alii corpori communicare possit.

De hujusmodi viribus vivis fluido elastico compresso insitis earundemque mensuris paucis nunc agam: dignum attentione argumentum est, quod eo reducantur mensurae virium pro machinis aëre, aut igne aut aliis hujusmodi viribus motricibus, quarum fortasse plures novae non sine insigni mechanicae practicae incremento & perfectione excogitari poterunt, movendis

§. 40. Ut incipiamus ab aëre in vacuo, considerabimus cylindrum verticaliter positum *ABCD* (Fig. 62) cum sustentacula *EF*, quod omni pondere destitutum liberrime sursum deorsumque moveri possit. Sit spatio *EBCF* aër inclusus, totus autem cylindrus in vacuo positus fingatur: Sit pressio aëris *EBCF* tanta qua sustinere possit pondus *p*, quod aequale erit pressioni columnae atmosphaerae, si aër iste sit naturalis. Superveniat jam aliud pondus *P:* ita fiet ut operculum descendat in *GH* motibusque reciprocis ad puncta *H* & *F* agitetur. Ut motum definiamus, utemur hypothesi ordinaria, quod pressiones aëris caeteris paribus sint densitatibus proportionales.

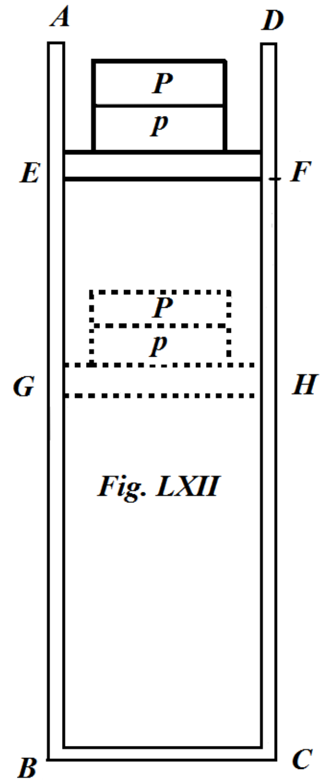


*Fig. LXII*

Fueritiaque $FC = a,\ FH = x$;velocitas sustentaculi in situ $GH = v$,erit pressio, qua sustentaculum *GH* ad ulteriorem descensum urgetur,

$= P + p - \frac{a}{a-x}p,$ huicque pressioni aequalis censenda est vis, quae pondus sustentaculo incumbens animat; igitur si hanc vim dividas per massam habebis vim accelerantem, quae multiplicata per tempusculum seu per $\frac{dx}{v}$ dabit incrementum velocitatis *dv;* est itaque

$$dv = \left(P + p - \frac{ap}{a-x}\right) \times \frac{dx}{v} : (P+p),$$

vel

$$\tfrac{1}{2}(P+p)vv = (P+p)x - ap\log\frac{a}{a-x}.$$

Sed ex descensu ponderis $(P+p)$ per altitudinem *x* generatur *vis viva potentialis* $(P+p)x$, & cum sustentaculum est in situ *GH*, in est corpori $(P+p)$ *vis viva actualis* $\tfrac{1}{2}(P+p)vv$, id est, $(P+p)x - ap\log\frac{a}{a-x}$, quae a priori deficitquantitate $ap\log\frac{a}{a-x}$, haecque in compressionem aëris transiit.

Dico itaque *non posse aërem occupantem spatium a condensari in spatium* $a - x$, *quin vis viva impendatur, quae generatur ex descensu ponderis p per altitudinem* $ap\log\frac{a}{a-x}$, *quocunque modo illa compressio facta fuerit; potest autem modis fieri infinitis.* lstam vero regulam uno nunc alterove exemplo illustrabo.

Sit basis cylindri unius pedis quadrati, altitudo initialis *FC* duorum pedum: contineaturque in spatio *BF* aër qualis in superficie terrae medius esse solet, qui ferre possit superficie *EF* 2240 libras: ponatur $x = 1$, ut sic habeatur vis *viva*, qua duo pedes cubici aëris naturalis in spatium unius pedis cubici coerceri possunt in vacuo: eritque ista *vis viva* $= 2 \times 2240 \times \log 2 = 3105$, id est, talis quae generatur lapsu libero corporis 3105 librarum per altitudinem unius pedis. Ergo & vicissim, si habeatur pes cubicus aëris naturali duplo densioris, poterit illius ope pondus elevari 3105 librarum ad altitudinem unius pedis in vacuo, dum aëris naturalis densitatem assumit.

Sit porro sub iisdem reliquis circumstantiis idem aër in spatium duplum, quam antea fuit, expansus, occupans nunc in cylindro altitudinem quatuor pedum, isque rursus condensetur in spatium unius pedis cubici; requiretur ad hanc compressionem *vis viva*, quae exprimitur per $4\times 1120\log 4$, quae priore duplo major est. Igitur in vacuo si habeatur pes cubicus aëris naturali duplo densioris, poterit illius ope pondus elevari 6210 librarum ad altitud. unius pedis, dum aëris naturalis dimidiam densitatem assumit, aut pondus 9315 *lib.* dum aëre naturali fit quadruplo rarior.

Consequens inde est, si aër in spatium expandere se possit infinitum & ubique elasticitatem servet densitati proportionalem, quantitati aëris finitae vim vivam inesse infinitam.

§. 41. Haec autem pertinent ad aestimationem vis vivae, quae aëri in vacuo posito insita sit: paullo alius fit computus pro aëre densiore, qui in atmosphaera positus est: hic enim maximus expansionis gradus non ultra aequilibrium cum aëre atmosphaerae extendi potest: facile hinc est in antecessum praevidere, si v. gr. habeatur pes cubicus aëris naturali duplo densioris, vim vivam quae in atmosphaera ab hoc aëre compresso elici possit, minime esse infinitam. Poterunt autem hujusmodi vires vivae hunc in modum determinari.

§. 42. Sit aër *EBCF* naturalis & in aequilibrio cum aëre externo; intelligatur autem per *p* pressio atmosphaerae in sustentaculum *EF*, quae quidem cum pressione aëris interni nondum condensati in aequilibrio est. Imponatur eidem sustentacula pondus *P;* fueritjam aërcondensatus in spatium *GBCH*, habeatque sustentaculum pondere *P* oneratum in situ *GH* velocitatem *v;* erit retentis reliquis denominationibus

$$dv=\left(P+p-\frac{ap}{a-x}\right)\times\frac{dx}{v}:P,$$

vel

$$Pvdv=\left(P-\frac{xp}{a-x}\right)dx,$$

quae integrata dat

$$\tfrac{1}{2}Pvv=Px+px-ap\log\frac{a}{a-x}.$$

Jam vero descensu ponderis *P* per altitudinem *x* genita fuit *vis viva Px*, de qua eidem ponderi ceu velocitate *v* moto inest pars $\tfrac{1}{2}Pvv$ seu $Px+px-ap\log\frac{a}{a-x}$; pars igitur vis vivae quae ad aërem transiit est $=-px+ap\log\frac{a}{a-x}$, , quae minor est altera §. 40 definita.

Habeatur v. gr. pes cubicus aëris naturali duplo densioris; invenietur vis viva, quam iste aër amittit, dum aëris naturalis circumfusi densitatem assumit, ea quae lapsu libero corporis 865 *lib.* per altitudinem unius pedis generatur.

Pari sensu pes cubicus aëris naturali triplo densioris vim vivam habere intelligitur talem quae respondeat lapsui libero corporis 2898 *lib*. per altitud. unius pedis, qui numerus nempe prodit cum ponitur $p = 2240$, ut§. 40; $a = 3$, & $x = 2$.

§. 43. Perspicuum est ex hoc consensu inter conservationem virium vivarum aëri compresso & corpori a data altitudine delapso insitarum, nullam esse ad usum machinarum perficiendum praerogativam sperandam ex principio aëris comprimendi, & ubique valere regulas in praecedente sectione exhibitas. Quia vera multis modis fit, ut aër non vi sed natura sit compressus aut elaterem naturali majorem acquirat, spes certe est, posse hujusmodi rebus naturalibus magna ad machinas movendas compendia excogitari, prouti D. Amontons jamjam docuit modum movendarum machinarum vi ignis. Mihi persuadeo si omnis vis viva, quae in carbonum pede cubico latet, ex eodemque combustione elicitur, utiliter ad machinam movendam impendatur, quod plus inde profici possit, quam labore diurno octo aut decem hominum. Etenim carbones dum comburuntur aëris elasticitatem nonsolum insigniter augent, sed & ingentem aëris novi quantitatem generant.

Ita Halesius in *Veget. statiks* deprehendit ex semipollice cubico carbonis 180 pollices cubicos aëris ejusdem cum aëre naturali elasticitatis fuisse generatos; ergo pes cubicus carbonum aërem dabit ad 360 *ped. cub.* Sed si§. 42 Quaeratur vis viva quae generari possit a pede cubico aëris naturali 360 vicibus densioris, invenietur illam convenire cum pondere 3938000 librarum ab altitudine unius pedis delapso: atque si praeterea aëris illius elasticitas a calore carbonum incensorum quadruplo fieri major ponatur, conveniet ista vis viva cum pondere 15752000 *lib.* ab eadem altitudine delapso. Difficile autem est machinam ad hunc finem aptam excogitare. Multae praeterea aliae sunt res naturales, quae nonsolum aërem fovent compressum, sed & aërem circumfusum calefaciendo eundem magis elasticum reddere valent: tales sunt calx viva cum aqua dulci mista, omniaque fermentantia; aquae in vapores vi ignis redactae incredibilis vis inest; machina ad hoc est Londini ingeniosissima quae hoc principio motus aquas toti urbi erogat eamque descripsit Cl. Weidlerus. Praesertim vera considerari meretur stupendus, qui a pulvere pyrio expectari possit, effectus: Calculo enim quorundam sumtorum experimentorum subducto, quem infra adjiciam, edoctus fui elasticitatem pulveris pyrii accensi plus decies millies superare elasticitatem aëris naturalis, imo omnibus bene perpensis probabile fit, elasticitatem ejus esse incredibiliter majorem: ponamus autem aurae pulveris pyrii accensi expansae elasticitatem decrescere in simili ratione cum densitate: hisce positis invenietur vis viva pedi cubico pulveris pyrii insita, si in §. 42 ponatur $a = 10000, x = 9999, p = 2240$ & sumatur $-px + ap\log\frac{a}{a-x}$, quae quantitas sic fit aequalis 183913864. Igitur machina datur in theoria, quae ope unius pedis cubici pulveris pyrii possit elevare 183 913 864 libras ad altitudinem unius pedis, quem laborem vel centum homines robustissimi intra unius diei spatium perficere posse non crediderim, quacunque machina utantur. Probabile autem est, ut dixi, effectum pulveris pyrii longe majorem esse; certe autem non minor est, calculus enim innititur altitudini, ad quam globus ferreus ex tormento bellico ejectus in vacuo ascendere possit, in quo experimentorum genere maxima pulveris pyrii pars perit.

Ista vera magis percipientur, si notetur eundem calculum (quem antea fecimus pro effectu, qui ex aëre condensato sese restituente oritur, demonstrando) procedere etiam pro aëre qui naturali circumfuso non quidem magis densus sed tamen ab aucto calore magis elasticus fit: ita v. gr. quoties pes cubicus aëris ordinarii augmento caloris duplum elaterem acquisivit, potest ejus ope pondus 865 librarum ad altitudinem unius pedis elevari, si modo machina adhibeatur perfectissima.

Ab auctis autem aëris tum densitate tum calore pendent omnium rerum hic expositarum effectus.

§. 44. Interim non solum ab aëre condensato calefactove vis viva pro machinis movendis impendenda obtineri potest, sed & ab aëre rariore aut frigidiore. Ubicunque enim aequilibrium sublatum est, *vis viva* adest, quae impendi potest, si debita machina excogitetur, ad onera elevanda machinamentaque circumagenda. Methodus autem determinans *vim vivam,* quae ab aëre datae densitatis datique caloris spatium datum occupante elici potest, mutatis mutandis eadem est cum illa quam§. 42 adhibuimus.

§. 45. Fuerit nempe rursus cylindrus verticalis *ABCD* (Fig. 63) cum diaphragmate mobili *EF:* puta aërem *EBCF*, ut §. 42, naturalem & in aequilibrio cum aëre externo: pressio autem aëris cujus vis in *EF* dicatur p: Finge dein pondus *P*, quod mediante fune trans duas trochleas *M* & *N* ducto cum diaphragmate cohaëreat, idemque versus *AD* trahat, perveneritque sic diaphragma ex situ *EF* in *GH:* Denique ponatur rursus $FC = a$, $FH = x$, velocitas diaphragmatis in situ *GH* seu ponderis in situ $P = v$; his positis si conferantur §§. 40 & 42, patebit fore nunc

$$dv = \left( P + \frac{ap}{a-x} - p \right) \times \frac{dx}{v} : P$$

vel

$$Pvdv = \left( P - \frac{px}{a+x} \right) dx,$$

quae integrata dat

$$\tfrac{1}{2} Pvv = Px - px + ap \log \frac{a+x}{a}.$$

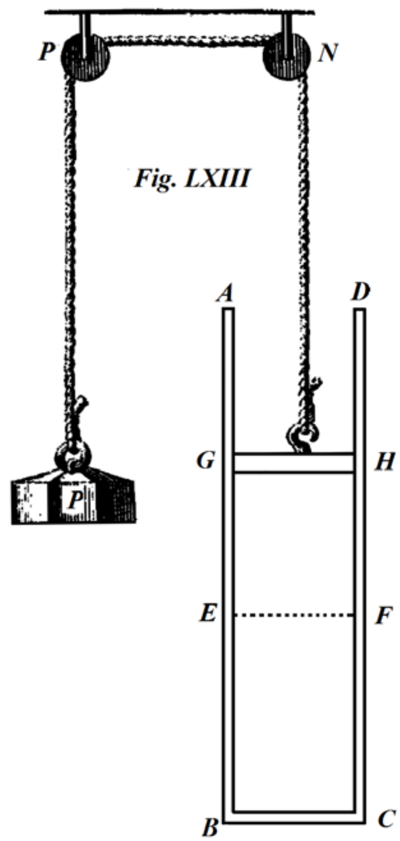


Fig. LXIII

At rursus descensu ponderis *P* per altitudinem *x* producta fuit *vis viva Px,* dum ipsi interim ponderi velocitate *v* moto inest tantum *vis viva* $\tfrac{1}{2} Pvv$ seu $Px - px + ap\log \frac{a+x}{a}$: igitur *vis viva,* quae residua est, nempe $-px + ap\log \frac{a+x}{a}$ ad aërem transiit rursusque restitutione aequilibrii inter aërem internum & externum illa *vis viva* ad alia corpora pro lubitu transfundi poterit: Igitur si habeas spatium *GBCH* aëre plenum cujus densitas sit ad densitatem aëris externi ut *CF* ad *CH*, in potestate erit *vis viva* $- ap\log\frac{a+x}{a}$.

An vero ista vis viva aëri inhaereat proprie externa an interno, logomachia est; sufficit quod a sublato aequilibrio inter utrumque aërem talis vis viva obtineri potest, dum restitutio permittitur. Habeatur v. gr. pes cubicus aëris naturali duplo rarioris, cui hypothesi quadrabunt positiones $p = 2240 lib.$, $a = \frac{1}{2}\, ped. \& \; x = \frac{1}{2}\, ped$ . & erit vis viva, de qua sermo est, $= 1120 - 1120\log 2 = 344$, id est, ea quae generatur lapsu libero 344 *lib.* ab altitudine unius pedis.

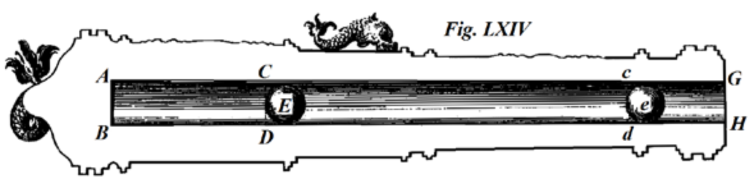
Si pes cubicus sit aëre repletus, qui naturali sit quadruplo rarior, erit jam vis viva quaesita (posito nempe $p = 2240, \& \; a = \frac{1}{4}, \; x = \frac{3}{4}) = 1680 - 560\log 4 = 904$, seu talis quae oritur lapsu libero ponderis 904 *lib.* per altit. unius pedis.

Si denique habeatur pes cubicus ab aëre omnino vacuus, ponendum est $p = 2240, a = 0, \& \; x = 1$: atque sic erit vis viva quaesita $= 2240 \times \left(1 - 0\log\frac{1}{0}\right)$; constat autem esse $0\log\frac{1}{0}$ infinite parvum prae unitate; est igitur numerus iste $= 2240$, qui indicat posse hac vi viva 2240 libras ad altitudinem unius pedis elevari.

§. 46. Pertinet ad praesens argumentum stupenda vis aëris admodum condensati, sed praesertim aurae pulveris pyrii accensi in usu sclopetorum pneumaticorum & tormentorum bellicorum. De his quae seorsim commentatus sum huic sectioni adjiciam.

…………………………………………………………………………………………..

*De vi aëris condensati & aurae pulveris pyrii accensi ad globos projiciendos in usu sclopetorum pneumaticorum & tormentorum bellicorum.*




(I) Sit *AG* (Fig. 64) longitudo animae in tormento sclopetove horizontaliter posito, voceturque $= a$: denotet *AC* longitudinem spatii, quod aër condensatus seu aura pulveris pyrii accensi occupat ab initio explosionis, sitque $AC = b$ : pondus globi ejiciendi $E = 1$; ponimus autem, globum cavitatem animae exacte replere & liberrime in illa moveri: densitas aëris condensati in spatio *AD* se habeat ad densitatem aëris naturalis ut *n* ad 1: Denique ponatur pondus columnae mercurii (cujus basis est *CD* & cujus altitudo eadem sit quae in barometro) *P*. Utemur autem hypothesi, sive globus propellatur ab aëre condensato sive a pulveris pyrii aura, potentiam illius fluidi propellentis proportionalem esse densitati.

His ad calculum praeparatis, globum considerabimus in situ *e*, ponendo $Ac = x$, velocitatemque globi in hoc situ $= v$; sic erit potentia globum in situ *e* propellens

$=\left(\frac{nb}{x}-1\right)\times P$, quae divisa per massam 1 ductaque in elementum spatii $dx$ dat incrementum dimidium quadrati velocitatis; unde fit

$$vdv=\left(\frac{nb}{x}-1\right)\times Pdx,$$

sive

$$\tfrac{1}{2}vv=(b-x+nb\log\frac{x}{b})P.$$

Ponatur $x=a$: habetur altitudo debita velocitati, quacum globus exploditur; vocetur ista altitude $\alpha$ & erit

$$\alpha=(b-a+nb\log\frac{a}{b})\times P.$$

(II) Sit v. gr. in sclopeto pneumatico longitudo animae seu $a=3$ *ped. Paris.*, longitudo $AC=4$ *poll.*, fueritque aër captus in *AD* naturali decies densior seu $n=10$, diameter animae seu globuli ejiciendi trium linearum ejusque gravitas specifica ratione mercurii ut 10 ad 17. Erit *P* praeterpropter $=286$; indeque invenitur $\alpha=2788$, indicio globum ejectum iri velocitate qua in vacuo ad altitudinem 2788 *ped.* ascendere possit. Ex praecedente formula colligitur jactum globi vehementissimum fore pro eadem aurae elasticae quantitate, si longitudo animae fiat $=nb$. Si vero animus ad impedimenta alia, quae globus praeter inertiam suam & resistentiam aëris externi in transitu suo per sclopeti animam patitur, advertatur, apparet longitudinem animae ad jactum vehementissimum producendum requiri longe minorem. Si longitudo *nb* admodum major sit longitudine *a,* quod ita est in jactibus fortioribus, erit sine sensibili errore $\alpha=nbP\log\frac{a}{b}$.

Si tormentum sit verticaliter erectum, fit aliquantum diversus calculus sed pro vehementioribus jactibus differentia nequit esse sensibilis. Igitur quia jactus deinceps considerabimus tantum vehementissimos, brevitatis ergo ponemus; $\alpha=nbP\log\frac{a}{b}$.

(III) Prouti in praecedentibus altitudinem determinavimus debitam velocitati qua globus exploditur, ex data vi elastica aurae globum ejicientis, ita vicissim patet, ex observata illa altitudine vim aurae elasticam deduci posse, est enim $n=\alpha:\left(bP\log\frac{a}{b}\right)$.

Exinde poterit vis elastica pulveris pyrii si non accurate definiri, saltem ad terminos reduci, quos certe superabit. At quaeres, qui altitudo $\alpha$ experimento determinari possit; ad quod respondeo, posse eam sat accurate colligi ex tempore, quod globus verticaliter sursum ejectus ab explosionis puncta insumit, dum in terram delabitur habita in calculo aëris resistentiae ratione. Transscribam huc experimenta in *Comm. Acad. Petrop. tom.* 2, *pp.* 338 & 339 recensita, quorum calculum institui factis ratione aëris resistentiae hypothesibus, gravitates specificas ferri & aëris esse ut 7650 ad 1 & aërem, in quo globus ascendit, uniformis esse densitatis: gravitatum specificarum ratio paullo major assumta fuisse videtur quam debebat, sed compensabitur in altissimis jactibus error a diminutione aëris densitatum versus superiora. "Tormenti situs omni accuratione ad perpendiculum erat accommodatus & singulis vicibus in hunc situm reponebatur atque firmabatur: singula experimenta fuerunt repetita. Erat autem longitudo animae 7,7 *ped. Angl.;* diameter globi erat 0,2375 *ped. ;* diameter animae mensurata non fuit neque magnitudo luminis accensorii: qualibet vice ponderabatur quantitas pulveris pyrii adhibiti & pendulo definiebatur tempus a puncto explosionis ad punctum, quo globus in terram cecidit: tabula sequens exhibet, tum quae observata, tum quae calculo inde eruta fuerunt "

| quant. pulv. pyr. numero unciar. holl. express. I | tempus asc. & descens in min. sec. observ. II | altit. jactus in aëre resist. per calculum in ped.Angl. III | temp. asc. in aëre resist. per calculum in min. sec. IV | temp. desc. in aëre resist. per calculum in min. sec. V | altit. jactus in vacuo per calculum in ped. Angl. VI | temp. ascens. & desc. in vacuo per calc. in min. sec. VII |
|---|---|---|---|---|---|---|
| $\frac{1}{2}$ | 11 | 486 | 5,42 | 5,58 | 541 | 11,6 |
| 2 | 34 | 4550 | 14,37 | 19,63 | 13694 | 58 |
| 4 | 45 | 7819 | 16,84 | 28,16 | 58750 | 121 |

" Pro eodem tormento eodemque globo, sed priori diminuto pede uno cum septem decimis partibus, sic ut longitudo animae residua esset praecise 6 *ped. Angl.,* inservit sequens tabula eadem lege constructa."

| I | II | III | IV | V | VI | VII |
|---|---|---|---|---|---|---|
| $\frac{1}{2}$ | 8 | 257 | 3,95 | 4,05 | 274 | 8,2 |
| 2 | 20,5 | 1665 | 9,74 | 10,76 | 2404 | 24,5 |
| 4 | 28 | 3187 | 12,5 | 15,5 | 6604 | 40,5 |
| 6 | 32,5 | 4304 | 13,9 | 18,6 | 11810 | 54,3 |
| 8 | 38 | 5643 | 15,54 | 22,46 | 22394 | 74 |

Multa sunt, quae successum horum experimentorum ita reddunt dubium, ut nullum sit, quod eandem aurae elasticitatem arguat. Maximam ego inaequalitatem ex eo oriri crediderim, quod minima pars pulveris inflammetur statim ab explosionis initio, quod magna pars tum demum accendatur, cum globus orificio tormenti jam proximus est, & quod maxima denique pars non inflammata ejiciatur: facit fortasse haec sola ratio, ut vis elastica aurae globum propellentis sit centies major, quam quae vi experimenti, nulla habita istius rei ratione, prodit: id mihi valde probabile fit, ex eo quod adhibito in tormento 7,7 *ped.* longo pulvere ad 4 uncias globus in vacuo jactu suo ascendere potuerit ad altitudinem 58750 *ped.,* cum eadem pulveris quantitate eodemque tormento sed 1,7 pede decurtato jactus respondent altitudini in vacuo 6604 pedum, quae altitudo vix ultra nonam partem prioris excurrit: Ex comparatione utriusque experimenti conjicio, maximam pulveris quantitatem in tormento longiore inflammatam fuisse dum globus jamjam esset orificio proximus neque ab ipso ultra 1,7 *ped.* amplius distaret.

Diminuitur quoque jactus globi a magnitudine luminis accensorii, ut & ab hiatu qui inter globum & internam animae superficiem relinquitur, per quod utrumque notabilis aurae pars inutilis avolat: tanta autem inde diminutio non oritur, quantam illam nondum posito calculo praesumseram: adjiciam tamen in sequentibus calculum, ut methodus habeatur vi pulveris pyrii longissimos statuendi limites, quos etiamnum certe transgrediatur.

(IV) Quod maximam ostendit aurae elasticitatem est experimentum tertium cum tormento nondum decurtato sumtum, quod indicat ascendere potuisse globum accepto impetu ad altitudinem $\alpha = 58750$ *ped. Angl.* Erat autem longitudo animae *AG* seu $a = 7,7$: longitudo *AC* (quantum ex amplitudine animae & gravitate pulveris pyrii conjicio) erat $= 0,08$. Denique valor ipsius *P* (seu ponderis columnae mercurialis, cujus basis sit circulus maximus globi & cujus altitudo sit 30 *poll. Angl.* ratione ponderis globi ferri designati per unitatem) invenitur posita gravitate specifica inter mercurium & ferrum ut 17 ad 10 $= 26,8$: Et cum per§. III sit proxime $n = \alpha : \left( bP \log \frac{a}{b} \right)$, erit $n = 6004$. Unde sequitur, si aura pulveris pyrii inflammati elasticitatem habeat suae densitati proportionalem, esse illius maximam elasticitatem minimum sexies millies majorern elasticitate aëris ordinarii.

(V) At vero si jam consideremus partem aurae inutilem, quae avolat per lumen accensorium & hiatum a globo relictum, majorem elasticitatem inveniemus: Calculus qui ad hanc quaestionem solvendam requiritur, cum non parum prolixus atque intricatus sit, non haesitavi hypotheses adhibere paullo liberiores, quibus admodum facilitatur: quamvis ipsae hypotheses non sint omni rigore verae, errorem tamen notabilem producere non possunt. *Primo* ponam utramque aperturam, per quam aura evolare possit, esse veluti infinite parvam ratione animae amplitudinis; hoc posito poterit singulis momentis velocitas, cum qua aura avolat, aestimari immediate ex pressione sola: hujusmodi autem hypothesin sine ullo sensibili errore fieri posse pro omni fluido, tunc etiam cum foramina non sunt admodum exigua, passim ut corollarium ex theoria nostra deduximus, & multo facilius assumi posse in fluido valde elastico facile quisque videbit ex eo, quod incrementum *ascensus potentialis* ratione motus interni longe minus est ratione *ascensus potentialis* particulae per foramen exilientis in fluido, quod a propria elasticitate expellitur, quam quod gravitatis vi ejicitur: in priori enim minor est motus localis internus quam in altero. *Secundo* aurae pulveris pyrii inflammati vim elasticam tantam esse, ut nisus atmosphaerae contrarius attendi non mereatur; *tertio* velocitatem globi in tormento utut permagnam, tamen minimam censeri posse ratione velocitatis, qua aura per hiatum utrumque avolat, quia nempe inertia istius aurae non potest non admodum esse exigua ratione inertiae quae globo inest: vi istius hypotheseos avolabit aura per utramque aperturam eadem velocitate, cum alias posita velocitate in lumine accensorio $= \sqrt{A}$, & velocitate globi $= v$, velocitas aurae in hiatu a globo ad superficiem animae relicto dicenda esset $= \sqrt{A} - v$. Venio nunc ad solutionem.

(VI) Primo notandum est, si elasticitates aurae censeantur densitatibus proportionales, fore ut aura constanter eadem velocitate per utramque aperturam avolet, uti vidimus in problemate §. 34, istaque velocitas nominatim talis erit, quae generetur ab altitudine aurae homogeneae, cujus pondus auram captam coercere possit, ne se expandat. Igitur determinabitur dicta velocitas hoc modo: sit gravitas globi $=1$, elasticitas seu pondus quod auram pulveris modo inflammati *ACDB* in illo compressionis statu coercere possit, $=P$, pondus pulveris adhibiti $=p$; erit pondus aurae pulveris modo inflammati etiam $=p$; sique longitudo *AC* ponitur $=b$, patet altitudinem aurae homogeneae, quae pondus *P* habeat, fore $=\frac{P}{p}b$; igitur velocitas quacum aura recens nata per lumen accensorium avolat est $=\sqrt{\left(\frac{P}{p}b\right)}$, eademque velocitate durante tota explosione ejicietur, idque non solum per lumen accensorium, sed & proxime per hiatum inter globum & animam relictum.

(VII) Sit nunc porro amplitudo animae $=F$; hiatus interceptus inter globum & animam = *f*, amplitudo luminis accensorii $=\varphi$; longitudo animae $=a$; quantitas aurae ab initio explosionis $=g$. Intelligatur deinde globus pervenisse ex *E* in *e*, dicaturque $Ac=x$; quantitas aurae eo temporis puncto in tormento residua $=z$; velocitas globi in isto situ $=v$; reliquae denominationes fuerunt jam antea explicatae.

Quoniam elasticitas per hypothesin est directe ut quantitas & reciproce ut spatium, erit elasticitas aurae in *AcdB* residuae $=\frac{zb}{gx}P$: quae quidem non tota in propellendum globum impenditur, sed tantum pars ejus, quae se habeat ad totam ut $F-f$ ad *F*. Est itaque posito *dt* pro elemento temporis

$$dv=\frac{F-f}{F}\times\frac{zb}{gx}P\times dt.$$

Per methodum autem §. 34 exhibitam, ubi quantitas aëris dato tempusculo effluens specifice definita fuit, invenitur

$$-dz=\frac{f+\varphi}{F}\times\frac{z}{x}\times\sqrt{\left(\frac{P}{p}b\right)}\times dt\ ;$$

ex comparatione harum duarum aequationum oritur

$$-dz=\frac{f+\varphi}{F-f}\times\frac{g}{b}\times\frac{\sqrt{b}}{\sqrt{Pp}}\times dv$$

quae cum debitae constantis additione integrata dat

$$z = g - \frac{f+\varphi}{F-f} \times \frac{g}{b} \times \frac{\sqrt{b}}{\sqrt{Pp}} \times v.$$

Si jam in aequatione prima substituatur valor iste inventus pro *z*, simulque ponatur $\frac{dx}{v}$ pro *dt*, fiet

$$vdv = \frac{F-f}{f} \times \frac{b}{x} \times P \times dx - \frac{f+\varphi}{F} \times \frac{\sqrt{bP}}{x\sqrt{p}} \times vdx$$

sive

$$\frac{Fvdv\sqrt{p}}{(F-f) \times bP\sqrt{p} - (f+\varphi) \times v\sqrt{bP}} = \frac{dx}{x},$$

quae aequatio post debitam sui integrationem, facta $x = a$, abit in hanc

$$\log\frac{a}{b} =$$
$$\left(-F(f+\varphi)v\sqrt{p} - F(F-f) \times p\sqrt{bP} \times \log\left(1 - \frac{(f+\varphi)v}{(F-f) \times \sqrt{bPp}}\right)\right) : \left((f+\varphi)^2 \times \sqrt{Pb}\right).$$

(VIII) Si jam per experimentum innotuerit valor ipsius *v*, poterit inde deduci valor ipsius *P*, qui denotat elasticitatem aurae pulveris pyrii nondum expansae: Quod ut exemplo illustremus, eodem utemur experimento, quod jam articulo IV exposuimus, ut appareat inde, quodnam ab avolatione aurae elasticitatis augmentum arguat. Sic igitur ponetur calculus.

Quia pondus globi, quod erat trium librarum, indicavimus per unitatem, erunt quatuor unciae pulveris adhibitae exprimendae per $\frac{1}{12}$ igitur $p = \frac{1}{12}$. Mensuras aperturarum, quas consideramus, non accepi: solet autem hiatus a globo relictus constituere in simili tormento praeterpropter partem decimam quintam amplitudinis animae; amplitudinem luminis accensorii hic fere negligi posse puto; itaque statuam $F = 15; f = 1; \varphi = 0$: Deinde habetur rursus $a = 7,7; b = 0,08$; altitude ad quam globus in vacuo ascendere possit seu $\frac{1}{2}vv = 58750$, seu $v = 343$: Igitur aequatio ultima superioris articuli haec erit

$$\log 96 = \frac{-5251}{\sqrt{P}} + 17,5\log\frac{\sqrt{P}}{\sqrt{P} - 300},$$

cui proxime satisfit cum sumitur $\sqrt{P} = 534$ & proinde $P = 285156$, quod efficit pondus columnae mercurialis ejusdem cum anima tormenti amplitudinis, cujus altitudo sit plusquam 10000 vicibus major altitudine communi barometri; invenimus autem supra *art.* IV numerum *n* (qui idem significabat) $=6004$. Ergo jam toto affirmabimus (ubique enim quae neglеximus majorem vim pulveri arguunt) inesse pulveri pyrio vim elasticam minimum decies millies majorem vi elastica aëris ordinarii. Apparet autem simul ex comparatione numerorum 10 000 & 6004, quantum circiter vi pulveris decedat ab hiatibus saepe dictis. Equidem istud decrementum majus putassem: Confirmatus autem sum hoc calculo in re de qua aliquando me certiorem voluit vir harum rerum gnarus, nullum nempe se in tormentis notabile observasse decrementum, cum lumen accensorium diuturno usu supra modum amplificatum esset in obsidio.

(IX) Verum ut ex aequatione nostra quaedam corollaria deduci possint faciliora quamvis proxime tantum vera, mutabimus quantitatem logarithmicalem in seriem. Est autem

$$-\log\left(1-\frac{(f+\varphi)v}{(F-f)\times\sqrt{bPp}}\right)=$$

$$=\frac{(f+\varphi)v}{(F-f)\times\sqrt{bPp}}+\frac{(f+\varphi)^2 vv}{2(F-f)^2\times bPp}+\frac{(f+\varphi)^3 v^3}{3(F-f)^3\times bPp\sqrt{bPp}}+\text{etc.}$$

Istoque valore substituto in aequatione ultima *art.* VII fit

$$\log\frac{a}{b}=\frac{Fvv}{2(F-f)\cdot bP}+\frac{F\cdot(f+\varphi)v^3}{3(F-f)^2\times bP\sqrt{bPp}}+\text{etc.}$$

Notabimus hic istam aequationem perfecte convenire cum aequatione ultima *art.*II si aperturae *f* & $\varphi$ ponantur $=0$: quod enim hic indicatur per $\frac{1}{2}vv$ & *P* ibi est *a* & *nP*, convenientibus denominationibus reliquis.

(X) Ut appareat, quantum proxime altitudo jactus ab aperturis diminuatur, si istae aperturae sint minimae, inserviet haec aequatio. Intelligatur per $\alpha$ altitudo ad quam globus pervenire possit in vacuo, si nulla aurae quantitas per aperturas avolare ponatur, & erit decrementum istius altitudinis ab eruptione aurae per easdem aperturas oriundum proxime hoc

$$\left((2\alpha)^{\frac{3}{2}}\times(f+\varphi)\right):\left(3F\times\sqrt{bPp}\right).$$

Unde in eodem tormento adhibitaque eadem pulveris quantitate & manente globi pondere, erunt decrementa jactuum proportionalia amplitudinibus aperturarum. Decrementa eadem fere sequuntur rationem subduplicatam quantitatum pulveris adhibitarum caeteris paribus; quia enim logarithmi magnorum numerorum in multo minori crescunt ratione ac numeri ipsi & quoniam insuper est $\alpha=bP\log\frac{a}{b}$, poterit caeteris paribus statui *a* proportionale ipsi *b*, quia *P* non afficitur a *b*. Sed decrementum, de quo sermo est, ceteris paribus rationem sequitur quantitatis $\left(\alpha^{\frac{3}{2}}\right):\left(\sqrt{bp}\right)$ seu ratione quantitatis $\frac{b}{\sqrt{p}}$ ; ipsum vero *p*, quod pondus denotat pulveris adhibiti, est ut *b;* igitur decrementum praedictum sequitur proxime rationem $\sqrt{b}$ , quae subduplicata est quantitatis pulveris adhibiti. Igitur ratione habita jactuum, decrementa multo majora sunt in jactibus debilibus, quam vehementioribus, idque etiam experimenta *art.* III recensita confirmare videntur: non video enim aliam rationem, cur in prima tabula experimentorum globi jactus in vacuo, sumtis duabus pulveris unciis, plus quam vigesies sexies altior esse debuerit, quam cum uncia dimidia sumeretur, & cur mox duplicata pulveris quantitate ad 4 uncias jactus tantum quadruplo altior post calculum prodeat, quam quantitate duarum unciarum.

(XI) Quae reliquae in utraque tabula comparent experimentorum inaequalitates, eas ut supra dixi, maximam partem derivo ab eo, quod pulvis non omnis inflammatur, nec is qui inflammetur omnis statim ab initio explosionis flammam concipiat. Neque certe id mirabimur, cum perpendimus totum explosionis tempus in *exper.* 4 *tab.*1 nequidem centesimam unius minuti secundi partem efficere. Igitur cum certum sit maximam pulveris partem non inflammatam ejici, nec exiguam partem reliqui tardius inflammari, quam in calculo positum fuit; cumque praeterea notabilis pulveris pars fucata sit vaporibus materiaque terrestri, quae non accenditur, sequitur longe majorem inesse elasticitatem partibus accensis, quam quae experimenti calculo *art.* X determinata fuit; fortasse decies aut centies major est.

At vero sit tantum talis, quam experimentum ostendit, elasticitate nempe aëris ordinarii decies millies major; sequitur inde auram illam elasticam, quae ex pulvere pyrio accenso elicitur, aut non aërem esse communem aut elasticitates in majori ratione crescere quam densitates: non potest enim densitas aëris, qui a pulvere modo inflammato oritur, esse plus quam millies densitate aëris ordinarii major, si pulvis vel totus ex aëre compresso compositus sit, quod ex gravitate pulveris specifica ratione aëris concludo.

Quaestio interim jamdudum est agitata, an aura elastica factitia, quae ex corporibus deducitur, aër sit ordinarius nec ne, quam ego quaestionem non decidam. Si tamen ponatur, pulverem pyrium aërem esse naturali millies densiorem & decies millies magis elasticum, tum ex§. 4 sequetur, aërem vi infinita compressum non posse pluribus quam 1331 vicibus condensari, & secundum eandem regulam foret aëris naturali quadruplo densioris elasticitas ad elasticitatem aëris naturalis ut $4\frac{1}{4}$ ad 1.

An vero experimenta ab aliis instituta, quae harum elasticitatum rationem faciunt accurate ut 4 ad l, sufficiente accuratione facta fuerint & an calor aëris dum comprimebatur idem permanserit? nescio. Verosimile autem est, eandem auram quae in poris pulveris pyrii latet, causam esse elasticitatis corporum elasticorum aut villorum contractilium: dum enim in cavernulis scatet, si corpora in figuram insolitam vi quadam redigantur, comprimitur aura elastica, cavernulisque dum reddit figuram capacissimam corpus restituit in pristinam figuram & longitudinem.