CHAPTER II

THE PLEASURES AND PRINCIPLES OF HARMONY

1. In this chapter I have decided to investigate how from the sensations arising that affect our senses, some may be pleasing to us and others displeasing. I certainly do not consider the necessity to show that there should always be a reason for this, concerning what we may find pleasing or displeasing, nor what without thought may be a delight to us. Indeed since now it may be admitted by most as if an axiom, nothing can happen in the world without sufficient reason, nor from that will it be required to doubt, that whichever of these may please, some reason be given. Therefore with this point conceded also the opinion of those must vanish, who consider music to depend only on the will of men and music only to please according to our customs and to displease the barbarian, as he may be unaccustomed to our ways.

2. Certainly I do not deny, and below I will show my approval for this, that it can happen in practice by hearing frequently, that some melody may begin to be pleasing for us, which at first had displeased, and vice versa. Yet this cannot be contrary to the principle of the sufficient reason, as it may be called ; for not only is an account of the object itself required to be sought, as to why it may please or displease someone, but also according to the sense, what the mind itself may form from the image offered, by which the image of the object may be represented mainly in the mind, this also is required to be respected and to be considered as well. Since which things may be able to be formed in different ways in different people and also in the same way at different times, then it is no wonder the same thing may please some and truly may displease others.

3. But I see now, how such an argument may be drawn against us and our tradition ; certainly the principles of harmony and the related rules cannot be put in place and for this reason we ourselves and all of those, who have tried to submit music to these rules, have labored on an empty and futile cause. If indeed some find delight in other ways and these things which please, are at once different and opposite to the views of others, how are the precepts to be presented to unite these sounds in such a manner that they may represent the pleasure of the melody to be heard ? And the rules, if which may be found, either will be so general, that they may not have any use, or to be neither stable nor constant, but will have to be adapted to the taste of the listeners ; which not only may require an infinite amount of labour, but at once will destroy all the certainty from the music.

4. But the musician is required to bear himself in a manner similar to the architect, who erects buildings without caring about following certain fundamental laws established in nature itself, with the edifices frequently devoid of judgement ; which even if they may not be pleasing to the ignorant, yet he is content while they may be approved by the intelligentsia. For in music just as in architecture too, while diversity is the taste of different people, so that, what may be pleasing with some, others may reject the same. In

this regard so that in all other matters just as also in music those people are required to be followed mainly, the taste of whom is perfect and the judgement from the sense of things perceived be freed from all fault. Of this kind are those, who not only have been accepted as being acute and pure as far as hearing the nature of the sound is concerned, but also everything which may be presented to the ear is perceived exactly, and will be brought together whole to those who pass judgement.

5. Since all sounds, as has been shown in the preceding chapter, shall be nothing other than of a series of vibrations produced in the air in a certain order, we may perceive a sound distinctly, if we may sense everything arising from the small blows on the organ of hearing and we acknowledge the order of these ; and besides, when all the strokes are not equally strong, if indeed we may have be able to account for the relative strength of the individual blows. Therefore listeners of this kind are required for the judgement concerning the kind of music being delivered, who shall be both endowed both with an acute sense of hearing and also may possess such a degree of understanding, so that the order, by which the vibrations of the air striking the organs of hearing, may be perceived and from that they may be able to judge. Indeed this, as will be shown in the following, is of necessity for being recognised, or what actual pleasure may be present in the proposed musical work and to what degree it may be held.

6. On account of which before all we will use a work, so that in which we may define the cause, which shall be, why it may be pleasing or displeasing to us, and why there may be any reason found, so that we may be pleased with that. Indeed from this, if it were known, truly the standard and the rules of composition of harmonious music would be able to be derived, since evidently there may be established, what put in place, which may please or displease. But not only do these things apply to music, which have been deduced from this source, but all the others things too, which have the same proposed goal, so that they may please further. And thus it may be so widely apparent, that hardly anything may be designated, to which a greater degree of pleasure from these which we seek, may not be able to be acquired from principles, or in general from others, even if scarcely anything useful may be seen to be introduced at first.

7. But regarding metaphysical matters, to which this investigation properly belongs, we seize upon any deliberations that may be pleasing to us, in which we perceive there to be imperfections, and from that where we encounter greater perfection to please us all the more ; truly we are against these things which displease us, in which we see a defect in the perfection or thus an imperfection. It is certain that every perfection gives rise to pleasure, and that it is a property common to us all, both to rejoice in the discovery as well as to enjoy the contemplation of a perfect experience; truly everything is to be avoided, which either is deficient in perfection, or perfection is understood to be lacking. And it will be evident to him, by considering more carefully what pleases; for he will recognise that kind which pleases, and to desire perfection in those things to which he may be adverse.

8. But we understand perfection to be present in some matter, if thus we may put that in place, so that everything conspires in that towards obtaining the proposed goal ; but if some things indeed will be present not relating to the goal, we are aware of a lack of perfection. And, if finally certain things may be adversely impeding the goal being followed by the others, we attribute that to imperfection. Therefore in the first case the matters offered may please us, in the latter case truly displease. We may consider the example of the purpose of a clock, of which the end is to show the parts and divisions of the time : that will please us the most, if from its structure we understand all its parts to be put together thus and interconnected among themselves, so that all agree to indicate the exact time.

9. And from this it follows that in all things, in which perfection shall be present, by necessity all must be present in the same order. For since the order shall be of parts set out following certain established rules, from which it can be known, why some present which are held, shall be present there rather than elsewhere, but in the aforementioned matter of perfection all the parts must be in order, so that they shall be adapted to the goal sought, this same goal will be the rules it may assign, following which the parts of the business at hand are arranged and of which each part may be assigned some location which it maintains. Therefore in turn it may be understood, where there shall be order, there also to be perfection and a law or rule to effect the perfection corresponding to the goal. It is for this reason that a thing will give us pleasure, in which we can grasp the order, and others in which the order is missing will displease us.

10. Moreover we are able to perceive the order in two ways; on the one hand, when the law or rule is known to us, and we may examine the matter according to that; on the other hand, when we do not know the law and from its parts we may enquire about the nature of the matter, for which that shall be the law, which had given rise to that same structure. The above example of the clock brought forth above pertains to the first way; for now the purpose is known or the law of the arrangement of the parts, which is an indication of the time ; and thus we must see clearly that an examination of clocks, or the structure of such shall be just as the aim requires. But if I look at some series such as this 1, 2, 3, 5, 8, 13, 21 etc. and I do not know, what the rule of its progression shall be, then very soon I understand these numbers are related so that any one is the sum of the two preceding numbers, and this I confirm to be the law of these orders.

11. The latter manner of perceiving the order may be considered especially for music ; indeed after hearing a musical harmony [i.e. an opera or chamber music, perhaps] we may understand the order which is maintained both between the notes produced simultaneously as well as the successive ones. Therefore the music may please, if we may perceive the order of the sounds constituting that, and truly it will displease, whenever we do not understand the order, whereby any sound has been moved from its true place; truly there must be more displeasure, when we recognise more often sounds that depart and change from the order that we have judged they are required to hold . Therefore it can happen, that some may notice the order, of which others are not aware, from which the

same thing may give pleasure to some, but displease others. But they can both be deceived; for an order may be present that many do not recognise; and often a certain order itself may seem to be perceived, where there is none, and then different kinds of music arise to be judged.

12. And thus these please, in which we perceive an order present ; but we will be pleased especially, if more of the same kind of thing may be offered, of which we understand the order that they contain ; and we experience the greatest degree of the pleasure, if besides we understand the order which the same things maintain between themselves. But these it is apparent to be less joyful, if we may not perceive the order in certain of these things, and if at that stage we may not discover any order, then for us the proposed item ceases to please. But not only if no order is to be observed, and truly also any certain reasoning present seized upon, which might be present elsewhere, to be troubled, then by which the order will displease us and listening to that may affect us almost painfully.

13. Where we may perceive the order more easily, which is present in the proposed matter, thus we may judge that simpler and more perfect and thus we may be affected with a certain joy and gladness. Truly on the other hand if we may recognised the order with difficulty and on that account may appear less plain and simple, as if we have encountered a certain something as if sadness. Yet in each case, as long as we are aware of the order, the item offered to us may please and in that we may consider some pleasure to be present ; which indeed may be considered to fight with itself, since the same may well be able to please and to have charm, yet which may move us quickly to sadness. But if we consider these musical harmonies and melodies, we recognise everything may be agreeable and must be to please; yet some we see to be adapted to providing joy, others to producing sadness. Wherefore those that please are constituted from two kinds, the one which may makes us happy, and the other sad.

14. Clearly operas of comedies and tragedies are similar to these discussions, of which we find both to be most agreeable; truly the former fill us with joy, but truly the latter by necessity make us feel sad. From which it is understood that to please and give joy are not the same thing, and that there is no further a distinction between pleasing us and making us sad. Truly an account of how these must be prepared in some manner has now been shown ; evidently everything pleases in which we understand order is present, but of these only the ones having a simpler order and easily perceptible make us glad; indeed these which contain a more composite order and manner are accustomed to sadden the mind, so that it may be more difficult to be perceived.

15. There is not a great deal of difference about these things, which philosophers are accustomed to discuss about happiness and sadness ; for they describe happiness thus, as they say that to be a remarkable degree of pleasure ; therefore a greater degree of perfection is required for the excitation of happiness than that which may only please. However their definition of sadness is seen to differ a great deal from that, which we have given; but here we are not attending to speak about that kind of sadness, which is

commonly described among the dispositions, but that which may be agreed to be in the contemplation of imperfection. Nor indeed can this kind of sadness in music not be intended to please, since it is trying to please. And thus the sadness for us rests only in the greater difficulty of perfection put in place or with the perception of the order, and on this account only differs from being a level of happiness.

16. But there are two special properties associated with sounds which can contain order, clearly the low or high pitch associated with a certain level of sound, and the duration of the sound. Therefore on the former account a musical harmony may please, if we can perceive the order which the sounds maintain amongst themselves by their depth and acuteness ; but it may please on the latter account, if we understand the order maintained by the durations of the sounds. Besides these two properties no other is given, which is appropriate for the order by which the sounds are received, except perhaps the sound level or loudness; but even if it is customary for musicians to use music in this way in their harmonies, and that sooner or later loud and soft sounds must be used, yet they do not look for anything pleasing in the perception of the composition or of the order, that these orders of the strengths of the sounds themselves may have by way of pleasure; and for that reason they are neither used to, or able to define the strength of the quantity.

17. Since the order shall be the arrangement of the parts according to some certain rule, anyone who has grasped this rule by inspection will be pleased by the same perceived order. Truly in music the quantities constitute an order; for we may consider either the depth and sharpness of sounds or their duration, as each may be determined by known quantities; evidently the one by the rate of the pulses produced in air, the other truly for the time in which any sound is produced. Therefore anyone who perceives the relation of the rate of the pulses in the sound, understands the order of the sounds and that by itself is a source of pleasure. In a similar manner anyone who knows how to distinguish between the durations of the sounds and how to compare them, also will have observed an order, and may be pleased on this account. But just as we may perceive the order, it is required for that to be set out more clearly, and indeed for each kind on its own.

18. For two proposed sounds we will realize the relation of these, if we understand the ratio, which the number of pulses sent out in the same time may have between each other ; so that if the one may perform 3 pulses in the same time as the other performs 2, and thus we know the relation of these and thus the order on observing this same ratio, of one to one and a half. In a similar manner we understand the mutual relation between more sounds, if we know all the ratios the individual sources maintain between each other, which send out numbers of vibrations of sound in the same time. Also we may capture the joy from sounds of different durations, if we may perceive the ratios which the durations of the individual times have between themselves. From which it will be apparent all the pleasure in music to arise from the perception of ratios, which several numbers hold between each other, since also the times of the duration can be expressed by numbers.

19. Indeed an appreciation of the ratios between sounds is singularly facilitated by this circumstance that we can distinguish many vibrations from each other, and more often we are able to compare these among themselves. Therefore it is much easier to distinguish the ratio of two sounds by listening to them than by looking at the lengths of two lines having the same ratio. But if the ratio of sounds and lines may be judged to be similar, we need receive only two of the individual vibrations for each, and from that we will be able to deduce the ratio of the lines. Moreover, since the sounds for which the frequency is not exceedingly high, many vibrations may be produced in a short time, as from the preceding chapter, where we have treated the number of vibrations of a string made in a second, it is possible to see how an examination of the ratio of sounds becomes easier. On account of which the very complicated ratios used in music compositions, which, if the same were present in lines, would be recognised with more difficulty.

20. Since deeper notes produce fewer vibrations in the same time than higher notes, it is evident the ratio of higher notes between each other can be perceived more readily than that of the deeper notes, if indeed each may endure for an equally long time. Therefore with all else being equal it will be observed by necessity, that the deeper notes themselves last longer and follow each other more slowly than the higher notes, which are able to progress faster. Therefore it is agreed to have observed the rule, so that a longer time may be attributed to deeper notes, shorter for higher notes. Moreover, each therefore is understood to be required to be produced more, where the ratios, which they maintain between themselves, may be considered to be more complicated and more difficult to realize. Thus it can still happen, that higher notes may succeed each other more slowly, while deeper notes may be able to progress faster, if the former may be especially simple, while the latter may maintain complicated ratios.

21. But so that it may be easier to understand the way, by which the order or ratio of two or more notes may be realized, we will try to visualize in how many ways it can happen, and to represent the same by a line figure. Therefore in place of pulses arriving at the ear we will represent these by points on a right line, the distances between which may correspond to the intervals between the pulses, Table I shows several figures of this kind. Whereby on this account equal sounds or those which maintain the same course of deep or acute tones through the whole duration, will be described by a series of equidistant points as in Fig. 1. In which, since everywhere clearly the ratio shall be equal, there is no reason, why the order may not be easily understood. A single tone or, as it is usually called, a unison is required, in the first place and most simply, to be perceived by us to constitute the first and most simple degree of the order, which we will call the first degree of the agreement, and hence maintains the numerical ratio 1 : 1.

Tab. I

Fig. 1.

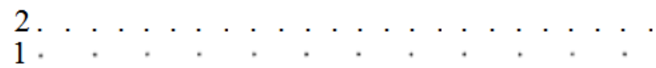

Fig. 2.

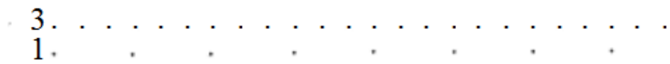

Fig. 3.

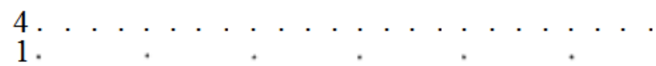

Fig. 4.

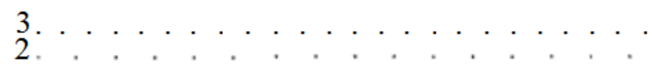

Fig. 5.

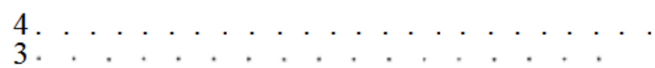

Fig. 6.

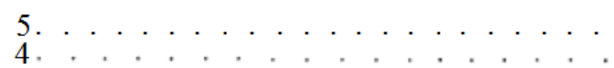

Fig. 7.

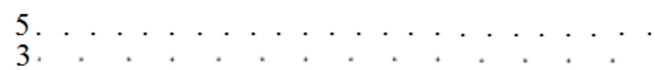

Fig. 8.

Fig. 9.

22. Now there shall be two notes proposed to be heard maintaining a twofold ratio [*i.e.* an octave apart, or frequencies in the ratio of 2:1]; these may be expressed by two series of points, in the second of which the interval of the points will be twice as great as in the first, as in Fig. 2, where the above shows the series of the higher tones, the lower the deeper tones. [These may correspond to the standing waves on a string, in a pipe, etc.] With these considered at the same time also the order is realized easily, as may become apparent by inspection of the figure. Therefore, since after a pure note which is the simplest, we come upon the following degree of pleasure or satisfaction, which thus is maintained by numbers in the ratio 1 : 2. In a similar manner Fig. 3 shows the ratio 1 : 3 and Fig. 4 the ratio 1 : 4. We consider that these two ratios can be understood with the same ease. The first has the advantage of being expressed by numbers smaller than the

second; but which contains the number 4, which is the double of 2, therefore it appears easy enough to construct the ratio 1 : 2, and in this , it also has some advantage over the ratio 1 : 3. Indeed from these two forms we have united these two ratios under the same order of agreement, evidently the third.
[This of course harks back to the Pythagorean experiments, where 1 : 2 was a pleasing ratio of the tones or frequencies of the vibrating strings, other than 1 : 1, as was 1 : 3, 2 : 3, etc., which belonged to the first musical scales produced in a mathematical sense. The inverse ratios of these tones or frequencies were their vibrating lengths, such as 2 : 1, 3 : 1, etc.]

23. Therefore just as the ratio 1 : 1 constitutes the first degree of the agreeable tones, the ratio 1: 2 the second, and likewise 1 : 4 pertains to the third, thus we will refer the ratio 1 : 8 to the fourth degree and 1 : 16 to the fifth , and thus always as in a geometric progression with a common ratio of 2. Hence it is evident the ratio $1:2^n$ pertains to the degree, which is expressed by the number $n+1$. I have adopted this distribution of the degrees all the more because it is based on the ease of recognition of the ratios, thus so that, for example, the fifth may be considered more difficult than the fourth, that therefore may be considered more difficult than the third, and that in turn more difficult than the second. But within these I do not produce any middle degrees, for if n were a fractional number, in this case the ratio would be irrational and in this instance not perceptible [on the diagrams].

24. From these it is apparent, if the number which added to one were composite and has a ratio corresponding to the two sounds , i. e. if it had divisors, then the order of the agreement in the sense of the notation also becomes smaller; just as we see the ratio 1 :4 cannot be regarded to be composite rather than 1:3, since 4 is greater than 3. Therefore on the other hand it is evident the degree of agreement is required to be assumed from the magnitude of the numbers themselves, if they shall be prime; thus the ratio 1 : 5 will be simpler than 1 : 7, although perhaps not simpler than 1 : 8. But from the prime numbers it will now be allowed by induction to put some in place ; for since the ratio 1: 1 will give the first order, 1 : 2 the second order, 1 : 3 the third, we conclude 1 : 5 pertains to the fifth, 1 : 7 to the seventh and generally 1 : p, if indeed p were a prime number, to the order which is indicated by the number p.

25. Again it is deduced from § 23, if the ratio 1 : p refers to the degree, of which the index shall be m, the ratio 1 : 2p pertains to the degree $m+1$, 1 : 4p to the degree $m+2$ and 1 : $2^n p$ to the degree $m+n$. For on multiplying the number p by 2 it is required besides the idea of bisecting or doubling the ratio of 1 : p , so that by the simplest operation the degree of the agreement is increased by one. In a similar manner it is possible to determine the degree of the agreement of the ratio 1 : pq, if p and q were prime numbers ; for the ratio 1 :pq therefore is composed more from 1 : p, than by which 1 : q is composed from 1 : 1. Therefore with the ratio 1 : pq the degree with p, q and 1 must make an arithmetic proportion, from which there will be therefore $p+q-1$.

26. The same reasoning generally remains ; if indeed the ratio 1: P may pertain to the degree p and the ratio 1 : Q to the degree q, on account of the ratios brought forwards the ratio 1 : PQ pertains to $p+q-1$. Evidently each of the ratios are to be added together in turn and one being taken from the sum. And thus of the ratio 1 : pqr (for the positive prime numbers p, q and r), which is composed from 1 : pq and 1 : r and the degrees of these are $p+q-1$ and r, the agreeing degree will be $p+q+r-2$. Similarly the degree of the ratio 1 : $pqrs$ will be $p+q+r+s-3$. And of the ratio 1: $PQRS$ the degree will be $p+q+r+s-3$, if truly the degrees of the ratios 1: P, 1: Q, 1: R and 1: S were p, q, r and s.

27. Therefore from this ratio $1 : p^2$, it is observed that the degree of agreement to be $2p-1$, clearly with the prime number p, and of the ratio $1 : p^3$ the degree will be $3p-2$ and generally the ratio $1 : p^n$ pertains to the degree $np-n+1$. Therefore since $1 : q^m$ shall pertain to the degree $mq-m+1$, the given ratio must be referred according to the rule of the preceding paragraph composed from these $1 : p^n q^m$ to the degree

$$np+mq-n-m+1.$$

And whatever the number P may have been in the ratio 1 : P, the degree will be had, to which it pertains, if this may be resolved into its simple factors and these in turn may be added and the number of factors less by one may be taken from the sum. Thus if the degree may be sought of the ratio 1 : 72, because there is $72 = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3$ and of these the sum of the factors is 12 and the number is 5, 4 is taken from 12; and 8 will be the degree for the ratio 1 : 72.

28. If the ratio were proposed between the three numbers so that $1: p : q$ be understood, where p and q are prime numbers, it will be required in these that both 1 : p and 1 : q are to be understood. But these two ratios may be found with equal ease at the same time and composed from the ratio for these 1 : pq. Therefore the degree which pertains to the ratio 1 : p : q, can be discerned from pq by the rule treated . In the same manner the ratio between the four numbers $1: p : q : r$, where p, q et r again are prime numbers, will be produced from the number pqr. Thus if four tones were proposed to be expressed by these numbers 1: 2 : 3 : 5 , the degree, for which the degree of these may be pertained readily, that they have themselves, must be known from the number 30, which gives the degree eight.

29. But these prime numbers must all be unequal, otherwise the reasoning used is invalid. For the ratio 1 : p : p is seen equally easily to be 1 : p; for the two latter numbers, which have the ratio of equality, can be taken as a single number, nor is this equivalent ratio

$1:p^2$ to be agreed for this. Similarly also, if the numbers p, q, r etc. were not prime, equally reasoning in this manner will not be permitted. So that if the ratio shall be considered $1 : pr : qr : ps$ with p, q, r and s made prime numbers, it will be required to consider only the ratios $1 : p$, $1 : q$, $1 : r$ and $1 : s$, truly without the ratios $1: p$ and $1: r$ taken twice, although they occur twice. On account of which the degree of agreement to be judged will be from the simple composite ratio $1: pqrs$ or from the number $pqrs$.

30. But if we may consider not only the number itself $pqrs$, but also the way, in which it is produced, we understand this number to be the smallest common multiple of the numbers 1, pr, qr and ps or the smallest number, which can be divided by these individual numbers, between which the ratio was proposed to be uncovered. From which we form this universal rule for the known degree of agreement proposed from the ratios of several numbers at once. Truly of these the smallest common divisible number must be sought ; and from this number the degree of the agreement will be defined by the above rule given in § 27. Conforming to this general rule, I have constructed the following table which indicates to what degree each smallest multiple corresponds. I have continued that up to the sixteenth degree, because rarely in music are numbers present which are accustomed to occur to higher degrees.

31. In this table therefore the Roman letters denote the degrees of agreement and the customary numbers there pertaining to all the smallest common divisors:

I.	1;
II.	2;
III.	3, 4;
IV.	6, 8;
V.	5, 9, 12, 16;
VI.	10, 18, 24, 32;
VII.	7, 15, 20, 27, 36, 48, 64;
VIII.	14, 30, 40, 54, 72, 96, 128;
IX.	21, 25, 28, 45, 60, 80, 81, 108, 144, 192, 256;
X.	42, 50, 56, 90, 120, 160, 162, 216, 288, 384, 512;
XI.	11, 35, 63, 75, 84, 100, 112, 135, 180, 240, 243, 320, 324, 432, 576, 768, 1024;
XII.	22, 70, 126, 150, 168, 200, 224, 270, 360, 480, 486, 640, 648, 864, 1152, 1536, 2048;
XIII.	13, 33, 44, 49, 105, 125, 140, 189, 225, 252, 300, 336, 400, 405, 448, 540, 720, 729, 960, 972, 1280, 1296, 1728, 2304, 3072, 4096;
XIV.	26, 66, 88, 98, 210, 250, 280, 378, 450, 504, 600, 672, 800, 810, 896, 1080, 1440, 1458, 1920, 1944, 2560, 2592, 3456, 4608, 6144, 8192;
XV.	39, 52, 55, 99, 132, 147, 175, 176, 196, 315, 375, 420, 500, 560, 567, 675, 756, 900, 1008, 1200, 1215, 1344, 1600, 1620, 1792, 2160, 2187, 2880, 2916, 3840, 3888, 5120, 5184, 6912, 9216, 12288, 16384;
XVI.	78, 104, 110, 198, 264, 294, 350, 352, 392, 630, 750, 840, 1000, 1120, 1134, 1350, 1512, 1800, 2016, 2400, 2430, 2688, 3200, 3240, 3584, 4320, 4374, 5760, 5832, 7680, 7776,

10240, 10368, 13824, 18432, 24576, 32768.

[E.g., for 10, we have $10 = 2^1 \cdot 5^1$, then the order is $1+1\cdot(2-1)+1\cdot(5-1)=6$; again, for 60, we have $60 = 2^2 \cdot 3^1 \cdot 5^1$, then this belongs to the order $1+2\cdot(2-1)+1\cdot(3-1)+1\cdot(5-1)=9$; for 78, we have $78 = 2^1 \cdot 3^1 \cdot 13^1$, which belongs to the order $1+1\cdot(2-1)+1\cdot(3-1)+1\cdot(13-1)=16$. At this point, it is necessary to acknowledge a recent paper by *Josef Sandor*:
Euler and Music; a Forgotten Arithmetic Function by Euler; published in Octagon Mathematical Magazine ; Vol. 17, no. 1. April, 2009,
where the mathematical properties of this function are considered. How Euler came upon this function is not altogether clear, though the diagrams provided had a bearing on this; however, it provides an order for just about any ratio of frequencies which is pleasing. In this paper, the Euler formula is set out for any number n represented by its prime factors : $n = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$, where the p_i are distinct primes and the powers are greater than or equal to one. Thus, the order is given by the formula :

$$E(n) = 1 + \sum_{k=1}^{r} a_k\,(p_k - 1)\,,$$

where it is to be noted that $E(1)=1$, and that the ratio of musical intervals is accommodated by $E(\frac{x}{y}) = E(x \cdot y)$, so that, for example, the octave is represented by $E(\frac{1}{2}) = 2$, the fifth $E(\frac{1}{2}) = \frac{2}{3} = 4$, etc. It is noted here that most of the musical intervals that are found pleasing are to be generated by this formula, and cataloged by the table; all the higher orders are multiples of 2 of inferior orders, as can be observed from the table, apart from the initial values. Finally, we should observe that such a number-theoretic approach was later abandoned in favor of 12 tone scales of the form
$f_i = f_o \cdot 2^{\frac{i}{12}};\ \ 1 \le i \le 12.$ for some f_0, such as $A = 440$ Hz.]

32. But several ways can be used for finding the least common multiple, one of which, which in our situation will present the greatest use, it will be convenient to set out here. Individual proposed numbers are resolved into their simplest factors and these places are observed, in which any of these have a factor of the greatest dimension; then a factor is made with the powers with these maximum dimensions and this will be the smallest common multiple of the numbers given. So that if these numbers were proposed 72, 80, 100, 112, which resolved into simple factors become $2^3 \cdot 3^2$, $2^4 \cdot 5$, $2^2 \cdot 5^2$, $2^4 \cdot 7$, and the simple factors are 2, 3, 5, 7. The first of these, 2, has the maximum power four, the second 3 has the maximum power two, and equally the third , 5, truly in the fourth, 7, only the first power occurs. Whereby the least common multiple is $2^4 \cdot 3^2 \cdot 5^2 \cdot 7$ or 25200, and belongs to the twenty third degree.

33. Therefore for any given numbers we will be able by the precepts that we have established, to know whether it will be easy or difficult to discover their mutual ratio and from what order. Also we will be able to compare and to judge several cases among themselves, whatever may be the easier to grasp. But these numbers constituting the proposed ratio must be rational, whole and as small as possible. Certainly the first of these is easily understood, since the order shall be present with no irrationalities of this kind. But they must be whole, because the minimum common divisor does not include fractions; indeed by the known rules, if which were fractions, they can be changed into whole numbers with the same mutual relation remaining. Besides these must be expressed in ratios in the smallest numbers, thus so that no number other than unity may stand out, by which all these numbers may be divided. But if they shall not be minimal, those present are required to be divided by the greatest common divisor which they have.

34. Therefore in this manner also the degrees of the agreement of the non multiplicative ratios will be determined, such as we have considered at first ; thus the ratio 2: 3, because the smallest common multiple of these is 6, pertains to the fourth degree and equally easy to arrive at the ratio 1 : 6 or 1 : 8 (Fig. 5). Indeed this understanding corresponds to the inspection of the points of this figure, in which indeed the order is easily observed. But just as it may be known of figures of this kind, it becomes more difficult to grasp the ratios pertaining to higher degrees ; if for example the proposed ratio 5: 7, which is referred to the eleventh degree, from the figure of which expressed in this way the order may be hard enough to be seen. In the same manner these things have a place in the following orders, so that, where the order may be expressed by a greater number, there it may be apparent from figures of this kind that the order can be more difficult to be seen .

35. Finally here the manner of requiring the order appears much more general than just for judging the differences of the ratios of low and high notes. For it is also possible for notes of various durations to be accommodated, with the establishment of the notes by numbers proportional to the durations. But in these not only is it allowed to use the order advanced, but also in that case, where the depth and acuteness of the notes may be considered, which occur more often in these pulses and therefore the relation of these is known more easily. Truly the perception of the ratios of several similar sounds of differing duration is to be a consideration of the lines, of which it may be required to understand the mutual relation from a single aspect. Besides also in all these matters, in which decorum and order must be present, this treatment will have a great usefulness, if indeed that, which constitutes the order, can be reduced to quantities and to be expressed in terms of numbers ; just as in architecture, in which it is required to decorate agreeably, so that all the ordinary parts of the building which may be able to be seen, shall be properly ordered.

CAPUT II

DE SUAVITATE ET PRINCIPIIS HARMONIAE

1. Cum hoc capite investigare statuerim, quibus rebus efficiatur, ut eorum, quae in sensus incurrunt, alia nobis placeant, alia displiceant, ante non admodum necessarium arbitror demonstrare esse omnino rationem eius, quare quid placeat vel displiceat, neque temere mentes nostras delectari. Cum enim hoc tempore a plerisque tanquam axioma admittatur nihil sine sufficienti ratione in mundo fieri, neque de hoc erit dubitandum, an eorum, quae placent, detur aliqua ratio. Hoc igitur concesso etiam eorum opinio evanescit, qui musicam a solo hominum arbitrio pendere existimant atque sola consuetudine nostram nobis musicam placere barbaramque, quia nobis sit insolita, displicere.

2. Equidem non nego, et infra ipse probabo, exercitio et crebra auditione fieri posse, ut concentus quispiam nobis placere incipiat, qui primum displicuerit, et vicissim. Attamen hoc principium sufficientis rationis, uti vocatur, non evertitur; non solum enim in ipso obiecto ratio, cur placeat vel displiceat, est quaerenda, sed ad sensus, per quos obiecti imago menti repraesentatur, quoque est respiciendum atque praeterea ad iudicium potissimum, quod ipsa mens de oblata imagine format. Quae res cum in diversis hominibus diversimode evenire possint atque in eodem etiam variis temporibus, mirandum non est eandem rem aliis placere, aliis vero displicere posse.

3. Sed iam video, quale ex hoc contra nos nostrumque institutum deducetur argumentum; nempe harmoniae principia et regulas tradi non posse obiicietur et hanc ob caussam nostrum et omnium eorum, qui musicam legibus includere conati sunt, laborem esse irritum et inanem. Si enim alios alia delectant et haec ipsa, quae delectant, prorsus sunt diversa et opposita, quomodo praecepta tradi poterunt coniungendorum sonorum, ut auditui suavem harmoniam repraesentent? Ac regulae, si quae invenientur, aut nimis erunt universales, ut usum habere nequeant, aut non stabiles nec constantes, sed ad auditorum rationem accommodari debebunt; id quod non solum infinitam industriam requireret, sed omnem certitudinem e musica prorsus tolleret.

4. Sed Musicum similem se gerere oportet Architecto, qui plurimorum perversa de aedificiis iudicia non curans secundum certas et in natura ipsa fundatas leges aedes exstruit; quae etiamsi harum rerum ignaris non placeant, tamen, dum intelligentibus probentur, contentus est. Nam ut in musica ita etiam in architectura tam diversus est diversarum gentium gustus, ut, quae aliis placeant, alii eadem reiiciant. Hanc ob rem ut in omnibus aliis rebus ita etiam in musica eos potissimum sequi oportet, quorum gustus est perfectus et iudicium de rebus sensu perceptis ab omni vitio liberum. Huiusmodi sunt ii, qui non solum a natura auditum acceperunt acutum et purum, sed qui etiam omnia, quae in auditus organo repraesentantur, exacte percipiunt eaque inter se conferentes integrum de iis iudicium ferunt.

5. Cum omnis sonitus, ut capite praecedente ostensum est, nihil aliud sit nisi pulsuum in aëre productorum sese sequentium certus ordo, sonitum distincte percipiemus, si omnes ictus in aurium organa incurrentes sentiemus atque eorum ordinem agnoscemus; et praeterea, quando non omnes ictus sunt aequaliter fortes, si etiam vehementiae singulorum rationem animadvertemus. Huiusmodi igitur requiruntur auditores ad iudicium de rebus musicis ferendum, qui et auditus sensu acuto et singula quaeque percipiente sint praediti et tantum intellectus gradum possideant, ut ordinem, quo ictus aërearum particularum auditus organa percutiunt, percipere de eoque iudicare possint. Hoc enim, ut in sequentibus docebitur, est necessarium ad cognoscendum, an revera suavitas insit in proposito musico opere et quemnam ea teneat gradum.

6. Quamobrem ante omnia operam adhibebimus, ut in quaque re definiamus, quid sit id, cur nobis vel placeat vel displiceat, et quid quamque rem habere oporteat, ut ea oblectemur. Ex hoc enim, si fuerit perspectum, vera norma et regulae componendorum musicorum concentuum derivari poterunt, cum scilicet constiterit, in quo positum sit id, quod placeat displiceatve. Non solum autem quae res ad musicam pertinent, ex hoc fonte sunt deducendae, sed omnes aliae quoque, quae eundem habent scopum propositum, ut placeant. Hocque tam late patet, ut vix quicquam assignari possit, cui non maior suavitatis gradus ex istis, quae quaerimus, principiis possit concillari, aut omnino aliquis, etiamsi vix capax videatur, afferri.

7. Metaphysicos autem, ad quos haec inquisitio proprie pertinet, consulentes deprehendimus omne id nobis placere, in quo perfectionem inesse percipimus, eoque magis nos delectari, quo maiorem perfectionem animadvertimus; contra vero eas res nobis displicere, in quibus perfectionis defectum aut adeo imperfectionem perspicimus. Certum est enim perceptionem perfectionis voluptatem parere hocque omnium spirituum esse proprium, ut perfectionibus detegendis et intuendis delectentur, ea vero omnia, in quibus vel perfectionem deficere vel imperfectionem adesse intelligunt, aversentur. Cuique hoc, qui ea, quae ipsi placent, attentius contemplabitur, erit perspicuum; agnoscet enim perfectionis esse speciem id, quod placet, in iisque, quae aversatur, se perfectionem desiderare.

8. At perfectionem in quapiam re in esse intelligimus, si eam ita constitutam esse deprehendimus, ut omnia in ea ad scopum propositum impetrandum conspirent; sin autem quaedam affuerint ad scopum non pertinentia, perfectionis defectum agnoscimus. Et, si denique quaedam ad vertantur, quae reliqua in scopo assequendo impediant, imperfectionem tribuimus. Primo igitur casu res oblata nobis placet, postremo vero displicet. Contemplemur exempli caussa horologium, cuius finis est temporis partes et divisiones ostendere id maxime nobis placebit, si ex eius structura intelligimus omnes eius partes ita esse confectas et inter se coniunctas, ut omnes ad tempus exacte indicandum concurrant.

9. Ex hisce sequitur, in qua re insit perfectio, in eadem ordinem necessario inesse debere. Nam cum ordo sit partium dispositio secundum certam regulam facta, ex qua cognosci

potest, cur quaeque in eo, quem tenet, loco sit posita potius quam in alio, in re autem perfectione praedita omnes partes ita esse debeant ordinatae, ut ad scopum impetrandum sint accommodatae, iste scopus erit regula, secundum quam partes rei sunt dispositae et quae earum cuique locum, quem tenet, assignat. Vicissim igitur etiam intelligitur, ubi sit ordo, ibi etiam esse perfectionem et legem regulamve ordinis respondere scopo perfectionem efficienti. Hanc ob rem nobis placebunt, in quo ordinem deprehendemus, ordinisque defectus displicebit.

10. Duobus autem modis ordinem percipere possumus; altero, quo lex vel regula nobis iam est cognita, et ad eam rem propositam examinamus; altero, quo legem ante nescimus atque ex ipsa partium rei dispositione inquirimus, quaenam ea sit lex, quae istam structuram produxerit. Exemplum horologii supra allatum ad modum priorem pertinet; iam enim est cognitus scopus seu lex partium dispositionis, quae est temporis indicatio; ideoque horologium examinantes dispicere debemus, an structura talis sit, qualem scopus requirit. Sed si numerorum seriem aliquam ut hanc 1, 2, 3, 5, 8, 13, 21 etc. aspicio nescius, quae eorum progressionis sit lex, tum paullatim eos numeros inter se conferens deprehendo quemlibet esse duorum antecedentium summam hancque esse legem eorum ordinis affirmo.

11. Posterior modus percipiendi ordinis ad musicam praecipue spectat; concentum musicum enim audientes ordinem demum intelligemus, quem inter se tenent soni tum simul tum successive sonantes. Concentus igitur musicus placebit, si ordinem sonorum eum constituentium percipimus, displicebit vero, quando non perspicimus, quare quisque sonus suo loco est dispositus; eo vero magis displicere debebit, quo saepius sonos ab ordine, quem eos tenere oportere iudicamus, recedere et aberrare cognoscemus. Fieri igitur potest, ut alii ordinem animadvertant, quem alii non sentiunt, ex quo eadem res aliis placere, aliis displicere potest. Utrique autem decipi possunt; ordo enim revera inesse potest, quem multi non cognoscunt; et saepe quidam se ordinem percipere videntur, ubi nullus adest, atque hinc tam diversa de rebus musicis oriuntur iudicia.

12. Placent itaque ea, in quibus ordinem, qui inest, percipimus; magis autem delectabimur, si plures eiusmodi res offerantur, quarum quem continent ordinem comprehendimus; atque maximum sentiemus suavitatis gradum, si praeterea ipsarum istarum rerum ordinem, quem inter se tenent, cognoscimus. Ex his apparet, si ordinem in quibusdam earum rerum non percipiamus, minore nos voluptate affici et, si prorsus nullum ordinem animadvertamus, tum etiam nobis rem propositam placere cessare. Sed si non solum ordinem observamus nullum, verum etiam quaedam praeter omnem rationem adesse deprehendimus, quibus ordo, qui alias inesset, turbetur, tum displicebit nobis et fere dolore ea percipientes afficiemur.

13. Quo facilius ordinem, qui in re proposita inest, percipimus, eo simpliciorem ac perfectiorem eum existimamus ideoque gaudio et laetitia quadam afficimur. Contra vero si ordo difficulter cognoscatur isque minus simplex minusque planus videatur, cum quadam quasi tristitia eundem animadvertimus. In utroque tamen casu, dummodo

ordinem sentimus, res oblata nobis placet in eaque suavitatem inesse existimamus; quae quidem inter se pugnare videntur, cum idem possit placere et suavitatem habere, quod animum ad tristitiam concitet. Sed si ipsos musicos concentus et modulationes consideramus, omnes suaves esse et placere debere agnoscimus; interim tamen alias ad laetitiam, àlias ad tristitiam excitandam esse accommodatas videmus. Quamobrem eorum, quae placent, duo constituenda sunt genera, alterum quod laetos, alterum quod tristes faciat animos.
14. Similla haec plane sunt comoediarum et tragoediarum, quarum utraeque suavitate plenae esse debent; illae vero praeterea gaudio animos perfundant, hae vero tristitia afficiant necesse est. Ex quo intelligitur neque idem esse placere et gaudium excitare, neque contraria placere et tristitiam afferre. Horum vero ratio quomodo sit comparata, iam quodammodo est expositum; placent scilicet omnia, in quibus ordinem inesse intelligimus, horum autem ea laetitia tantum afficiunt, quae ordinem habent simpliciorem et facile perceptibilem; illa vero tristes reddere solent animos, quae ordinem continent magis compositum et eiusmodi, ut difficilius possit perspici.

15. Non multum discrepant haec ab iis, quae a philosophis de laetitia et tristitia tradi solent; nam laetitiam ita describunt, ut dicant eam esse notabilem voluptatis gradum; plus igitur perfectionis requiritur ad laetitiam excitandam quam ad id tantum, ut quid placeat. Tristitiae definitio multum quidem differre videtur ab ea, quam dedimus; sed attendendum est nos hic non de ea tristitia loqui, quae inter affectus vulgo describitur, quod constet in imperfectionis contemplatione. Neque ,enim huiusmodi tristitiam musica intendit nec, quia placere conatur, potest. Sicque nobis tristitia tantum in difficiliore perfectionis seu ordinis perceptione ponitur et hanc ob rem a laetitia gradu solum differt.

16. Sunt autem in sonis duae res praecipue, quae ordinem continere possunt, eorum scilicet gravitas vel acumen, in quibus quantitatem sonorum posuimus, et duratio. Ob illam igitur placet musicus concentus, si ordinem, quem soni ratione gravitatis et acuminis inter se tennent, percipimus; sed ob hanc placet, si ordinem, quem durationes sonorum tenent, comprehendimus. Praeter haec duo aliud in sonis non datur, quod ad ordinem recipiendum esset aptum, nisi forte vehementia; sed tametsi et hac musici uti soleant in suis concentibus, ut mox fortes mox debiles effici debeant soni, tamen non in perceptione rationis seu ordinis, quem hi vehementiae gradus inter se habent, suavitatem quaerunt; et hanc ob rem vehementiae quantitatem definire neque solent neque possunt.

17. Cum ordo sit partium dispositio secundum certam quandam legem, is, qui ex inspectione hanc legem cognoscit, idem ordinem percipit eique ipsa perceptio placebit. In musica vero ordinem quantitates constituunt; nam sive gravitatem et acumen sive durationem respiciamus, utrumque quantitatibus determinatur; illud scilicet pulsuum in aëre productorum celeritate, hoc vero tempore, per quod sonus quisque producitur. Qui igitur relationem celeritatum pulsuum in sonis percipit, is ordinem sonorum comprehendit eoque ipse delectatur. Simili modo qui sonorum durationes distinguera et inter se comparare noverit, is etiam ordinem animadvertet et hanc ob rem voluptate

afficietur. Quomodo autem ordinem percipiamus, clarius est exponendum, et quidem de utroque genere seorsim.

18. Duobus sonis propositis percipiemus eorum relationem, si intelligamus rationem, quam pulsuum eodem tempore editorum numeri inter se habent; ut si alter eodem tempore 3 pulsus perficiat, dum alter 2, eorum relationem adeoque ordinem cognoscimus observantes hanc ipsam rationem sesquialteram. Similique modo plurium sonorum mutuam relationem comprehendimus, si omnes rationes, quas singulorum sonorum numeri vibrationum eodem tempore editarum inter se tenent, cognoscemus. Voluptatem etiam ex sonis ,diversarum durationum capimus, si rationes, quas singulorum tempora durationum inter se habent, percipimus. Ex quo apparet omnem in musica voluptatem oriri ex perceptione rationum, quas plures numeri inter se tenent, quia etiam durationum tempora numeris exprimi possunt.

19. Magnum quidem extat in sonorum rationibus percipiendis subsidium, quod singulorum plures ictus percipimus saepiusque eos inter se comparare possumus. Idcirco multo est facilius duorum sonorum rationem discernere audiendo quam duarum linearum eandem rationem habentium intuendo. Similis autem esset ratio sonorum et linearum, si singulorum sonorum duos tantum ictus reciperemus et de relatione eorum intervallorum iudicare cogeremur. Sed cum in sonia non admodum celeribus brevi tempore permulti edantur pulsus, ut ex capite praecedente, ubi de numero vibrationum chordae minuto secundo factarum egimus, videre licet, multo fit facilior rationis sonorum cognitio. Quamobrem in musica perquam compositis uti possunt rationibus, quas, si eaedem in lineis existerent, visus difficillime agnosceret.

20. Cum soni graviores eodem tempore pauciores edant pulsus quam acutiores, perspicuum est acutorum sonorum rationem facilius quam gravium percipi posse, si quidem utrique aeque diu durant. Caeteris igitur paribus oportet, ut soni graviores longius durent tardiusque sese insequantur quam acutiores, qui celerius progredi possunt. Hanc itaque constat observari oportere regulam, ut gravioribus sonis maior tribuatur duratio, acutioribus minor. Utrosque autem eo magis producendos esse intelligitur, quo rationes, quas inter se tenent, magis sunt compositae difficiliusque percipiantur. Fieri ergo tamen potest, ut acutiores tardius incedere debeant, dum graviores celeriter progredi possint, si nimirum hi simplices, illi vero perquam compositas teneant rationes.

21. Quo autem facilius percipi possit modus, quo ordo seu ratio duorum pluriumve sonorum percipitur, conabimur visui, quantum fieri potest, similem repraesentare figuram. Ipsos igitur pulsus in aurem incurrentes exponemus punctis in linea recta positis, quorum distantiae respondeant intervallis pulsuum, cuiusmodi figuras Tab. I plures repraesentat. Hac ergo ratione sonus aequabilis seu qui eundem per totam durationem habet tenorem gravitatis aut acuminis, describetur serie punctorum aequidistantium ut in Fig. 1. In qua, cum ubique ratio aequalitatis conspicua sit, dubium non est, quin ordo facillime intelligatur. Unus igitur sonus vel, ut vocari solet, unisonus primum et

simplicissimum nobis constituat gradum ordinis percipiendi, quem vocabimus primum suavitatis gradum, huncque tenet ratio 1 : 1 in numeris.

22. Sint nunc duo soni auditui propositi tenentes rationem duplam; ii duabus punctorum seriebus exprimentur, in quarum altera intervalla punctorum erunt dupla maiora quam in altera, ut Fig. 2, ubi superior series sonum acutiorem, inferior vero graviorem exhibet. His simul consideratis ordo facile quoque percipitur, quomodo ex figurae inspectione apparet. Hanc igitur, quia post unisonum est simplicissima, facimus gradum suavitatis secundum, qui ideo in numeris ratione 1 : 2 continetur. Simili modo Fig. 3 exhibet rationem 1 : 3 et Fig. 4 rationem 1 : 4; quarum utra sit perceptu facilior, in utramque partem potest disputari. Illa quidem hoc habet, ut minoribus expressa sit numeris, haec vero quadrupla ideo facilius percipi videtur, quod sit rationis duplae dupla hincque non multo difficilius discernatur quam dupla ipsa. Hanc ob rem nos utramque in eundem gradum, scilicet tertium, coniiciemus.

23. Quemadmodum ergo ratio 1 : 1 primum suavitatis gradum constituit et ratio 1: 2 secundum itemque ratio 1 : 4 ad tertium pertinet, ita ad quartum gradum referemus rationem 1 : 8 et ad quintum hanc 1 : 16 et ita porro iuxta progressionem geometricam duplam. Hinc manifestum est rationem 1: 2^n pertinere ad gradum, qui exponitur numero $n+1$. Eo autem libentius istam graduum distributionem assumsi, quod aequaliter in facilitate perceptionis progrediantur, ita ut, quo gradus v. g. quintus difficilius percipitur quam quartus, eo difficilius hic animadvertatur quam tertius, et hic ipse quam secundus. Inter hos autem non facio gradus medios prodeuntes, si n fuerit numerus fractus, quia in hoc casu ratio fit irrationalis et prorsus non perceptibilis.

24. Ex his apparet, si numerus, qui ad unitatem rationem habet respondentem duobus sonis, fuerit compositus, i. e. si habuerit divisores, tum gradum suavitatis propterea etiam fieri minorem; quemadmodum: vidimus rationem 1:4 non pro magis composita esse habendam quam 1:3, quamvis 4 est maior quam 3. Contra ergo manifestum est suavitatis gradum ex magnitudine numerorum ipsa, si sint primi, esse aestimandam; ita ratio 1 : 5 erit simplicior quam

Tab. I

Fig. 1.

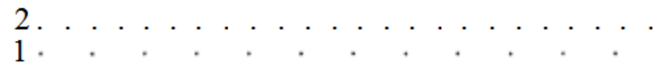

Fig. 2.

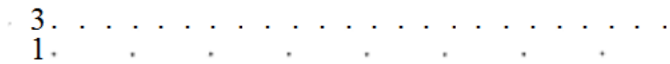

Fig. 3.

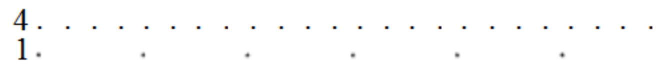

Fig. 4.

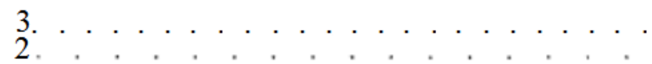

Fig. 5.

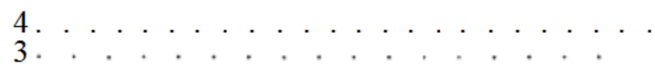

Fig. 6.

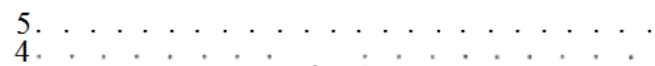

Fig. 7.

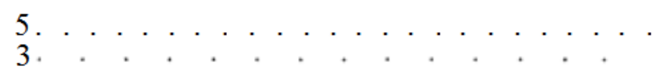

Fig. 8.

Fig. 9.

1 : 7, quamquam forte non simplicior est quam 1 : 8. At de numeris primis iam licebit ex inductione aliquid statuere; cum enim ratio 1: 1 det gradum primum, 1 : 2 gradum secundum, 1 : 3 tertium, concludimus 1 : 5 pertinere ad quintum, 1 : 7 ad septimum et generaliter 1 : *p*, si quidem *p* est numerus primus, ad gradum, qui indicatur numero *p*.

25. Colligitur porro etiam ex § 23, si ratio 1 : *p* ad gradum, cuius index sit *m*, referatur, rationem 1 : 2*p* ad gradum $m+1$ pertinere, 1 : 4*p* ad gradum $m + 2$ et $1\ :\ 2^n p$ ad gradum $m + n$. Multiplicato enim numero *p* per 2 ad rationis perceptionem requiritur praeter perceptionem rationis 1 : *p* bisectio aut duplicatio, qua ut simplicissima operatione gradus suavitatis unitate evehitur. Simili modo determinare licet gradum suavitatis rationis 1 : *pq*, si *p* et *q* fuerint numeri primi; nam ratio 1 :*pq* eo magis est composita quam 1 : *p*, quo 1 : *q* magis est composita quam 1 : 1. Ergo rationis 1 : *pq* gradus cum *p*, *q* et 1 debet proportionem arithmeticam constituere, unde erit igitur $p+q-1$.

26. Idem ratiocinium etiam universaliter subsistit; si enim ratio 1: *P* ad gradum *p* pertineat et ratio 1 ; *Q* ad gradum *q*, pertinebit ob allatas rationes ratio 1 : *PQ* ad gradum $p+q-1$. Scilicet utriusque rationis componentis gradus sunt invicem addendi et unitas a summa subtrahenda. Itaque rationis 1 *:pqr* (positis *p, q* et *r* numeris primis), quae est composita ex 1 : *pq* et 1 : *r* harumque gradus sunt $p+q-1$ et *r*, gradus suavitatis erit $p+q+r-2$. Similiter rationis 1 *:pqrs* gradus erit $p+q+r+s-3$. Et rationis 1: *PQRS* gradus erit $p+q+r+s-3$, si nimirum rationum 1: *P*, 1: *Q*, 1: *R* et 1: *S* gradus fuerint *p, q, r* et *s.*

27. Perspicitur ergo ex his rationis $1 : p^2$ gradum suavitatis esse $2p-1$, posito videlicet *p* numero primo, et rationis $1 : p^3$ gradum esse $3p-2$ atque generaliter rationem $1 : p^n$ ad gradum $np-n+1$ pertinere. Ergo cum $1 : q^m$ pertineat ad gradum $mq-m+1$, referri debet secundum regulam paragraphi praecedentis datam ratio ex his composita $1 : p^n q^m$ ad gradum

$$np+mq-n-m+1.$$

Et quicunque fuerit numerus *P* in ratione 1 : *P*, habebitur gradus, ad quem pertinet, si is resolvatur in omnes suos factores simplices iique invicem addantur et numerus factorum unitate minutus a summa subtrahatur. Sic si quaeratur gradus rationis 1 : 72, quia est $72=2\cdot 2\cdot 2\cdot 3\cdot 3$ horumque factorum summa 12 et numerus 5, subtrahatur 4 a 12; erit 8 gradus suavitatis pro ratione 1 : 72.

28. Si ratio fuerit proposita inter tres numeros ut $1: p:q$, ubi *p* et *q* sunt numeri primi, oportebit in ea et 1 : *p* et 1 : *q* percipere. At hae duae rationes simul aeque facile percipiuntur ac composita ex iis 1 : *pq.* Ergo ad quem gradum pertineat ratio 1 : *p* : *q,* ex numero *pq* dignoscendum est per regulam traditam. Eodem modo ratio inter quatuor numeros $1: p:q:r$, ubi *p, q* et *r* iterum sunt numeri primi, gradus prodibit ex numero *pqr.* Ita si quatuor soni fuerint propositi his numeris 1: 2 : 3 : 5 expressi, gradus, ad quem pertinet facultas ordinem eorum, quem inter se· habent, percipiendi cognosci debet ex numero 30, qui dat gradum octavum.

29. Debent autem hi numeri primi esse omnes inaequales, alioquin ratiocinium adhibitum non valet. Nam ratio 1 : *p* : *p* aeque facile percipitur ac 1 : *p;* duo enim posteriores numeri, qui habent rationem aequalitatis, pro uno haberi possunt neque aequivalens est haec ratio censenda huic $1: p^2$. Similiter etiam, si numeri *p, q, r* etc. non fuerint primi, pariter non hoc modo ratiocinari licebit. Ut si percipienda sit ratio $1 : pr : qr : ps$ positis *p, q, r* et *s* numeris primis, oportebit tantum cognoscere rationes 1 : *p*, 1 : *q*, 1 : *r* et 1 : *s,*

neque vero rationes 1 :p et 1: *r* bis, quanquam bis occurrunt. Quocirca suavitatis gradus aestimandus erit ex ratione ex his simplicibus composita 1: *pqrs* seu ex numero *pqrs.*

30. Si autem non solum ipsum numerum *pqrs*, sed etiam modum, quo prodiit, contemplamur, deprehendimus hune numerum esse minimum communem dividuum numerorum 1, *pr*, *qr* et *ps* seu minimum numerum, qui per hos singulos potest dividi, inter quos rationem detegere erat propositum. Ex quo formamus hanc regulam universalem pro gradu suavitatis cognoscendo in percipienda ratione plurium numerorum simul propositorum. Quaeri nimirum debet eorum omnium minimus communis dividuus; et ex hoc numero per regulam supra datam § 27 gradus suavitatis definietur. Addidi igitur sequentem tabulam, ex qua apparet, ad quem gradum quilibet minimus communis dividuus resultans perducat. Continuavi eam autem non ultra gradum decimum sextum, quia raro numeri ad ulteriores gradus pertinentes occurrere solent. 31. In hac igitur tabula cyphrae Romanae denotant gradus suavitatis et consueti numeri minimos communes dividuos omnes eo pertinentes:

I.	1;
II.	2;
III.	3, 4;
IV.	6, 8;
V.	5, 9, 12, 16;
VI.	10, 18, 24, 32;
VII.	7, 15, 20, 27, 36, 48, 64;
VIII.	14, 30, 40, 54, 72, 96, 128;
IX.	21, 25, 28, 45, 60, 80, 81, 108, 144, 192, 256;
X.	42, 50, 56, 90, 120, 160, 162, 216, 288, 384, 512;
XI.	11, 35, 63, 75, 84, 100, 112, 135, 180, 240, 243, 320, 324, 432, 576, 768, 1024;
XII.	22, 70, 126, 150, 168, 200, 224, 270, 360, 480, 486, 640, 648, 864, 1152, 1536, 2048;
XIII.	13, 33, 44, 49, 105, 125, 140, 189, 225, 252, 300, 336, 400, 405, 448, 540, 720, 729, 960, 972, 1280, 1296, 1728, 2304, 3072, 4096;
XIV.	26, 66, 88, 98, 210, 250, 280, 378, 450, 504, 600, 672, 800, 810, 896, 1080, 1440, 1458, 1920, 1944, 2560, 2592, 3456, 4608, 6144, 8192;
XV.	39, 52, 55, 99, 132, 147, 175, 176, 196, 315, 375, 420, 500, 560, 567, 675, 756, 900, 1008, 1200, 1215, 1344, 1600, 1620, 1792, 2160, 2187, 2880, 2916, 3840, 3888, 5120, 5184, 6912, 9216, 12288, 16384;
XVI.	78, 104, 110, 198, 264, 294, 350, 352, 392, 630, 750, 840, 1000, 1120, 1134, 1350, 1512, 1800, 2016, 2400, 2430, 2688, 3200, 3240, 3584, 4320, 4374, 5760, 5832, 7680, 7776, 10240, 10368, 13824, 18432, 24576, 32768.

32. Habentur autem ad minimum communem dividuum inveniendum plures modi, quorum unum, qui in nostro instituto maximam praestabit utilitatem, hic exponere convenit. Resolvantur singuli numeri propositi in factores suos simplicissimos notenturque ea loca, in quibus quilibet horum factorum maximam habet dimensionem; tum fiat factum ex istis maximarum dimensionum potestatibus hocque erit minimus

communis dividuus datorum numerorum. Ut si fuerint propositi hi numeri 72, 80, 100, 112, qui in factores simplices resoluti fiunt $2^3 \cdot 3^2$, $2^4 \cdot 5$, $2^2 \cdot 5^2$, $2^4 \cdot 7$, suntque simplices factores 2, 3, 5, 7. Horum primus, 2, maximam dimensionem habet quartam, secundi, 3, maxima dimensio est secunda, pariter ac tertii, 5, quarti vero, 7, prima occurrit potestas. Quare minimus communis dividuus est $2^4 \cdot 3^2 \cdot 5^2 \cdot 7$ seu 25200 et pertinet ad gradum vigesimum tertium.

33. Datis igitur quibuscunque numeris poterimus per tradita praecepta cognoscere, utrum facile sit an difficile mutuam eorum rationem et ordinem percipere, et quo gradu. Plures etiam casus poterimus inter se comparare et iudicare, uter facilius possit percipi. Sed numeri hi rationem propositam constituantes debent esse rationales, integri et minimi. Horum quidem primum facile intelligitur, cum in irrationalibus nullus huiusmodi insit ordo. Integri autem esse debent, quia inventio minimi communis dividui non ad fractos pertinet; per notas vero regulas, si qui fuerint fracti, in integras mutari possunt manente omnium eadem mutua relatione. Praeterea in minimis numeris rationes istae debent esse expressae, ita ut nullus extet numerus praeter unitatem, per quem omnes illi numeri dividi possint. Sin autem non sint minimi, eos per maximum, quem habent, communem divisorem ante dividi oportet.

34. Hoc igitur modo etiam rationum non multiplicium, quales initio consideravimus, suavitatis gradus determinabuntur; ita ratio 2: 3, quia minimus communis dividuus est 6, pertinet ad gradum quartum et aeque facile percipitur ac ratio 1 : 6 vel 1 : 8 (Fig. 5) Haec vero perceptio respondet inspectioni huius figurae punctatae, in qua quidem ordo facile perspicitur. At eiusdem modi figuris cognoscetur, quam difficulter rationes ad ulteriores gradus pertinentes percipiantur; sit e. gr. ratio proposita 5: 7, quae ad gradum undecimum refertur, ex cuius figura hoc modo expressa ordo iam satis difficulter perspicietur. Eodem modo se res habet in sequentibus gradibus, ut, quo maiore numero gradus exprimatur, eo difficilius ordinem perspici posse ex huiusmodi figuris appareat.

35. Hic denique modus ordinis perceptionem aestimandi multo patet latius quam ad sonos gravitate acumineve differentes. Accommodari enim etiam potest ad sonos variarum durationum, exponendis sonis per numeros durationibus proportionales. Sed in hisce non tam provectos gradus adhibere licet, quam illo casu, quo sonorum gravitas et acumen spectatur, quia in illis pulsus saepius recurrunt et propterea corum relatio facilius cognoscitur. Perceptio vero rationis plurium sonorum duratione diversorum similis est contemplationi linearum, quarum mutuam relationem ex solo aspectu comprehendere oporteat. Praeterea quoque in omnibus aliis rebus, in quibus decorum et ordo inesse debet, haec tractatio magnam habebit utilitatem, si quidem ea, quae ordinem constituunt, ad quantitates reduci numerisque exprimi possunt; sicut in architectura, in qua decori gratia requiritur, ut omnes aedificii partes ordine, qui percipi possit, sint dispositae.