CHAPTER II

THE MOTION OF WATER IN TUBES OF UNIFORM CROSS-SECTION

PROBLEM 48

27. *If the directrix of some tube of constant cross-section were some curved line, in which the water may be moved by the force of gravity only, to define its motion.*

SOLUTION

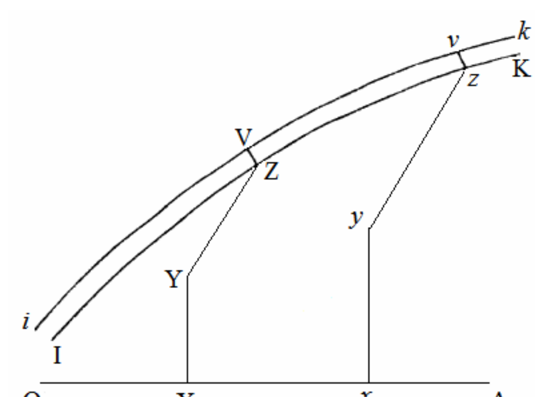


Fig. 38

The directrix *IZzK* of the tube *IiKk* may have some curved figure determined by a twofold equation between the three (Fig. 38) coordinates $Ox = x,\ xy = y,\ yz = z$ and the arc of the directrix $Iz = s$, so that there shall become $ds^2 = dx^2 + dy^2 + dz^2$, so that the quantities *x, y, z* may be able to be considered as functions of *s*. The section of the tube shall be $= \alpha$, and the density of water will be designated by the letter b, so that there shall be $\omega = \alpha$ and $q = b$. Then truly in the elapsed time $= t$ the pressure of the water at $z = p$ and the speed at that place along the direction $zK = \mathfrak{T}$; moreover the forces acting *P, Q, R,* if the applied line *yz* may be placed vertical, are reduced to $P = 0,\ Q = 0$ and $R = -1$. With these in place from problem 46, on account of $q = b$ and $\omega = \alpha$, the motion will be determined by the two following equations:

$$\text{I. } \left(\tfrac{d\mathfrak{T}}{ds}\right) = 0, \ \text{II.} \tfrac{2gdp}{b} = -2gdz - \mathfrak{T}d\mathfrak{T} - ds\left(\tfrac{d\mathfrak{T}}{ds}\right),$$

in the latter with the time *t* taken constant. Therefore from the former it is apparent the speed $\mathfrak{T}$ to be a function of the time only and thus $\mathfrak{T} = \Gamma : t$; from which, since in the other equation *s* may be considered as the only variable, $d\mathfrak{T} = 0$ and $\tfrac{d\mathfrak{T}}{dt} = \Gamma' : t$, and hence there will be had :

$$\tfrac{2gdp}{b} = -2gdz - ds\Gamma' : t$$

and by integrating:

$$\tfrac{2gp}{b} = 2g\left(h - z\right) - s\Gamma' : t + \Delta : t.$$

Therefore for any instant of time the speed of the water in the tube is the same everywhere, but it can be variable in some manner for a different time; on which variability the pressure *p* will not only depend especially, but in addition assumed to be some function of the time : but it is evident these two arbitrary functions must themselves be determined by the external circumstances, by the forces acting on the water.

COROLLARY 1

28. Therefore concerning the motion of water in equally sized tubes this in general cannot be determined further, than that at some instant of time the water may be drawn along the tube with the same speed everywhere, and that the pressure *p* surely will be determined in a certain manner.

COROLLARY 2

29. Therefore whatever function of the time *t* may be taken for the speed $\mathfrak{T}$, it will be allowed to confirm motion of this kind always to be possible in a tube with constant cross-section, provided external forces of this kind may be used, which shall be required to produce similar continuous amounts to that acceleration or retardation.

SCHOLIUM 1

30. In the solution of this problem I have used the prior method established in problem 46, from which it will be agreed also to elicit the solution from the other method of problem 47. Therefore we may put the element of the fluid, which we have considered now after the time *t* at *z*, initially, where $t=0$, to have been at *Z* with there being $OX=X$, $XY=Y$, $YZ=Z$ and with the arc $IZ=S$, and on account of $q=Q=b$ and $\omega=\Omega=\alpha$ the first equation gives $\left(\frac{ds}{dS}\right)=1$ from which it is deduced $s=S+\Gamma:t$ and thus the speed of the water at *z* becomes $\left(\frac{ds}{dS}\right)=\Gamma':t.$ Truly the other equation produces :

$$\frac{2gdp}{b}=-2gdz-ds\cdot\Gamma'':t$$

with the time *t* assumed constant, which integrated gives therefore :

$$\frac{2gp}{b}=2g\left(h-z\right)-s\Gamma'':t+\Delta:t,$$

which at once agrees with the preceding, except that here the speed at *z* after the time *t* shall be expressed by $\Gamma':t,$ since that before were $\Gamma:t$. Only, may be able to object, so that, since *z* and *s* must be considered as functions of the two variables *S* and *t,* in the latter equation indeed the time *t* shall be assumed constant [*i.e.* the steady-state solution], the differentials *dz* and *ds* are not complete but only these parts must be accepted, which arise from the variation of *S* ; and hence in turn the integration, as done, must not be taken to be absolute. So that we may meet this doubt, *z* shall be some function of the two variables *S* and *t,* from which there may become fiat $dz=MdS+Ndt$ *;* and it is certain in that differential equation in place of *dz* there must be written *MdS.* Truly for the same reason, because here the time t is assumed constant, the integral of the same term MdS again is z, to which some function of the time t may be added, but which now is required to be contained in $\Delta:t$; as the same is required to be judged from the integration of the differential ds.

SCHOLION 2

31. Therefore the proposed problem in order that it should be considered to be maximally indeterminate, when whatever external forces may themselves be included, by which the water may be acted on while it is moved in the tube, and thus now at last its motion may be able to be completely determined by these external forces. But while the water is moving in the tube, other external forces may be unable to act on that, unless the mass of water may be compressed by which forces in each end ; evidently a certain mass of water shall be established, which at some moment may occupy a certain length, the length of which is required to be considered as indefinite, and for each to be acted on by certain forces with the aid of plungers; since during the motion neither do we wish for any new mass to enter nor to flow out from the tube anywhere, certainly which cases will be set out specially. Therefore in the first place we may establish a certain mass of water to be moving continually in an infinitely long tube and for each end to be acted on by certain forces; and since the curvature of the tube does not enter into the computation otherwise, except in as much as it may depend on the height z, I shall consider the tube to be taken for the sake of convenience, as if its directrix may be a right line, likewise we are going to designate any point by its height above a fixed horizontal plane.

PROBLEM 49

32. *If a certain mass of water may be moved continually in a tube with a uniform cross-section, and may be acted on by some forces at each end, to determine its motion and pressure at individual points for some time.*

SOLUTION

We will consider a tube curved in some manner (Fig. 39) as if to be extended straight, that only noting the height of each point z above a certain horizontal plane to be

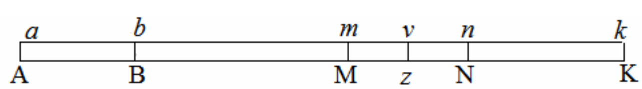


Fig. 39

$= z$, which for the individual points is required to be understood as given. Now in the elapsed time t the mass of water in the tube shall occupy the volume MN, which therefore will be of the constant quantity $= l$, and we may put from some fixed point A of the tube the distances $AM = m$, $AN = n$, so that there shall be $l = n - m$; moreover the elevation above the horizontal plane of the point M shall be $= \mu$, truly of the point N shall be $= \nu$.

But now this water vessel *MN* may be acted on at *M* by the pressure $=M$, at *N* truly by the pressure $=N$, which shall be some given functions of the time, truly the speed of this mass of water shall be $=\mathfrak{T}$, with which it shall be moved towards K, and which is the function of the time sought on putting above $\mathfrak{T}=\Gamma:t$. If now at some middle location *z*, the distance of which from the fixed end *A* shall be $Az=s$, the pressure may be established $=p$, there will be

$$\tfrac{2gp}{b}=2g\left(h-z\right)-s\Gamma':t+\Delta:t.$$

This point *z* initially at *M* may be moved along, then truly at *N,* and since the pressures themselves at these same points, by which we assume the water now at the ends to be acted on, must be equal, hence we elicit the two equations:

$$\tfrac{2gM}{b}=2g\left(h-\mu\right)-m\Gamma':t+\Delta:t$$

and

$$\tfrac{2gN}{b}=2g\left(h-\nu\right)-n\Gamma':t+\Delta:t$$

of which the latter taken from the former leaves:

$$\tfrac{2g}{b}\left(M-N\right)=2g\left(\nu-\mu\right)-\left(n-m\right)\Gamma':t,$$

from which there is deduced

$$\Gamma':t=\tfrac{2g\left(M-N\right)-2gb\left(\nu-\mu\right)}{b\left(n-m\right)}$$

and

$$\Delta:t=\tfrac{2g\left(Mn-Nm\right)}{b\left(n-m\right)}-2gh+\tfrac{2g\left(n\mu-m\nu\right)}{n-m}.$$

But since the speed of the ends *M* and *N* is the same $\mathfrak{T}=\Gamma:t$ and in that time *dt* they progress through the small distances *dm* and *dn*, there will become:

$$\tfrac{dm}{dt}=\mathfrak{T}=\Gamma:t=\tfrac{dn}{dt};$$

from which there becomes $\Gamma':t=\tfrac{ddm}{dt^2}$, thus so that on account of $n-m=l$ with the constant speed of the water *MN* being $=\tfrac{dm}{dt}$, likewise the distance $AM=m$ and from this equation there must be elicited :

$$bl\tfrac{ddm}{dt^2}=2g\left(M-N\right)-2gb\left(\nu-\mu\right)$$

where M and N are given functions of the time t, but μ and ν are variable quantities depending on m and n. Moreover with this equation resolved, by which for the time $=t$ the position of the end M is determined, at first the speed of the water channel $MN=\frac{dm}{dt}$, then truly for any middle point z, as there shall be $Az=s$, the pressure p is defined from this equation:

$$\frac{2gp}{b}=\frac{2g(Mn-Nm)}{bl}+\frac{2g(n\mu-m\nu)}{l}-2gz-\frac{2gs(M-N)}{bl}+\frac{2gs(\nu-\mu)}{l},$$

which therefore gives:

$$p=\frac{Mn-Nm}{l}+\frac{b(n\mu-m\nu)}{l}-bz-\frac{s(M-N)}{l}+\frac{bs(\nu-\mu)}{l},$$

or

$$p=\frac{M(n-s)}{l}+\frac{N(s-m)}{l}+\frac{b\mu(n-s)}{l}-\frac{b\nu(s-m)}{l}-bz.$$

Therefore later for the given time t the position M will have been found in the tube or $AM=m$, from which likewise the speed $\frac{dm}{dt}$ is known, therefore the pressure is defined for any element of the water contained in the volume MN.

COROLLARY 1

33. Clearly if the water column may not be acted by any force at each end, so that there shall be $M=0$ and $N=0$, its motion must be determined from this equation

$$bl\frac{ddm}{dt^2}=-2gb(\nu-\mu)\quad\text{or}\quad\frac{ddm}{dt^2}+\frac{2g}{l}(\nu-\mu)=0,$$

then truly the pressure will be

$$p=\frac{b\mu(n-s)+b\nu(s-m)}{l}-bz.$$

COROLLARY 2

34. Moreover if the two pressures M and N were equal to each other, then the prior differentio-differential equation indeed will remain the same, but truly the latter equation differs by a small amount :

$$p=M+\frac{b\mu(n-s)+b\nu(s-m)}{l}-bz,$$

thus so that this pressure will constantly surpass that by the amount M.

COROLLARY 3

35. The whole matter is reduced to a differential equation of the second order, which, even if the pressures *M* and *N* shall be unequal, yet still will not become more difficult to be solved, but the whole difficulty remains in the heights μ and ν, which evidently are determined by m.

SCHOLIUM

36. The case of the first corollary, were we have made the pressures *M* and *N* to be vanishing, cannot occur except in a vacuum; indeed when the motion is made in air, the water column is pressed on at each end by the weight of the atmosphere, and indeed by an equal force, unless both ends may extend to greatly different heights. Whereby, if the column at each end were free and open, in the second corollary the letter *M* will denote the atmospheric pressure, which since it shall be almost equal to a column of water 33 feet high, if we may write *k* for this height, there will become $M = bk$, and since this has been said always to be concerned with water, it will be agreed for the pressures to be expressed by columns of water, from which we may put in place the density of water *b* equal to unity.
On account of which, we will have the pressure in the water channel:

$$p = k - z + \frac{\mu(n-s)+\nu(s-m)}{l}.$$

But to introduce the atmospheric pressure *k* into the calculation is of the greatest importance, even if it may not be thought so ; indeed with that omitted it may be able to happen often, that the pressure *p* will emerge negative, nor yet may the continuity of the fluid be resolved; so that nevertheless it must arise always, where the pressure actually becomes negative. But in the account given of the atmosphere the motion found can endure, as long as the pressure *p* does not become negative; but to be resolved with certainty with that emerging negative, unless perhaps on account of cohesion or rather for that to be contained by the pressure of the atmosphere.

EXAMPLE 1

37. *If the tube were straight and inclined at some angle to the horizontal, to define the motion of each aperture of the column of water MN in that.*

The inclination of the tube *AK* to the horizontal shall be $= \eta$, and the column of water may be moving upwards evidently from an impulse taken ; but if however we may wish that it may descend, the angle η will be required to be taken negative. Hence therefore the heights above the horizontal plane will be

$$\mu = m\sin\eta,\ \ z = s\sin\eta \ \text{ and } \ v = (l+m)\sin\eta,$$

from which the first equation will become [recall that the acceleration due to gravity is taken to be 2g, in the manner defined by Euler previously]:

$$\tfrac{ddm}{dt^2} + 2g\sin\eta = 0$$

and hence the speed

$$\tfrac{dm}{dt} = 2g(f - t\sin\eta),$$

with the initial speed being $2gf$. But integrated again produces

$$m = g(2ft - tt\sin\eta),$$

if indeed the initial column will have occupied the portion *AB* of the tube. Now at some location z on account of

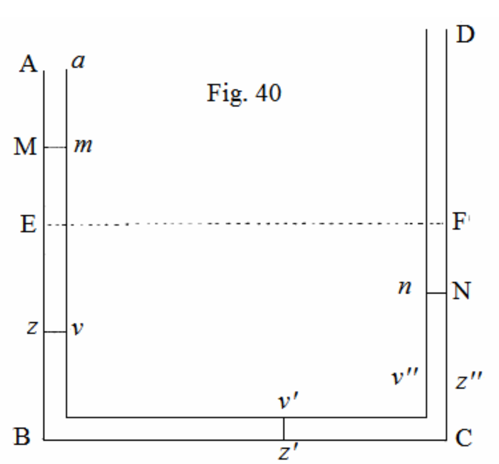
$$\mu(n-s) = m(m+l-s)\sin\eta \ \text{ and } \ v(s-m) = (m+l)(s-m)\sin\eta$$

the pressure will become

$$p = k - z + s\sin\eta = k$$

and thus always and everywhere equal to the atmospheric pressure. Hence it is apparent the water column to be moved in the tube in the same manner as a solid body, either by ascending or descending.

EXAMPLE 2



38. *If the tube* (Fig. 40) *ABCD were bent, so that it may have two vertical branches AB and DC with the middle BC being horizontal, and in that the column of water MBCN may be moved thus, so that its ends M and N always adhere to the vertical parts, to determine this motion.*

In the elapsed time *t* the water column occupies the volume of the tube *MBCN,* so that there shall be

$$MB + BC + CN = l,$$

or with *EF* drawn to the horizontal thus, so that there shall be :

$$BE = CF = \tfrac{1}{2}\left(BM + CN\right),$$

there will become $l = BC + 2BE$. Then on putting $AM = m$
there will become $\mu = BM$ and $v = CN$, hence

$$v - \mu = CN - BM = -2ME.$$

There may be put $AE = e$ and $ME = x$, there will become $m = e - x$, from which the first equation gives

$$-\tfrac{ddx}{dt^2} - \tfrac{4g}{l}x = 0 \text{ or } ddx + \tfrac{4g}{l}xdt^2 = 0,$$

which multiplied by $2dx$ provides requiring to be integrated

$$dx^2 + \tfrac{4g}{l}xxdt^2 = \tfrac{4g}{l}aadt^2$$

and hence

$$2dt\sqrt{\tfrac{g}{l}} = \frac{dx}{\sqrt{(aa-xx)}} \text{ and } 2\left(t+b\right)\sqrt{\tfrac{g}{l}} = \text{Ang sin } \tfrac{x}{a},$$

thus so that now there shall become

$$x = e - m = a\sin 2\left(t+b\right)\sqrt{\tfrac{g}{l}}$$

and the speed at M tending downwards

$$= -\tfrac{2a\sqrt{g}}{\sqrt{l}}\cos 2\left(t+b\right)\sqrt{\tfrac{g}{l}}.$$

Therefore when it arrives at *E*, its speed will be $= \frac{2a\sqrt{g}}{\sqrt{l}}$, if then we may put to become $2\left(t+b\right)\sqrt{\tfrac{g}{l}} = \pi$, from which, since the initial motion may be put in place as desired, we may make $e = a$ and there shall become

$$m = AM = a\left(1 - \cos 2t\sqrt{\tfrac{g}{l}}\right),$$

so that there shall become

$$\tfrac{dm}{dt} = \tfrac{2a\sqrt{g}}{\sqrt{l}}\sin 2t\sqrt{\tfrac{g}{l}},$$

thus so that the initial motion will become $m = 0$ or $EM = EA$ and the speed $= 0$.
Therefore the water column bears an oscillatory motion, of which the maximum departures above and below the horizontal *EF* will become $= a$, and the boundary *M* from the maximum to the minimum height will arise in the time $t = \frac{\pi\sqrt{l}}{2\sqrt{g}}$, and thus here the motion will agree with the oscillations of a simple pendulum, of which the length

$=\frac{1}{2}l = BE + \frac{1}{2}BC$. From these found at some location *z*, so that there shall become $Az = s$, the pressure will be

$$p = k - Bz + \frac{BM(Bz+BC+CN)}{l} + \frac{CN\cdot Mz}{l}$$

or

$$p = k + Mz - \frac{2ME\cdot Mz}{l},$$

where *Mz* in the prior location indicates the depth of the point *z* below *M*, but in the latter location the distance in the tube from the end *M*. Whereby in the horizontal branch the pressure at z'

$$= k + BN - \frac{2ME(MB+Bz')}{l}$$

and at z''

$$= k + BM - Cz'' - \frac{2ME(MB+BC+Cz'')}{l} = k + Nz'' + \frac{2ME\cdot Nz''}{l}.$$

COROLLARY 1

39. Therefore since during the oscillation with regard to the horizontal *EF*, on account of $ME = 0$ the pressure at *z* will be $= k + Ez$, with *Ez* denoting the depth of the point *z* below the line *EF*. Hence if the part of the tube *BC* shall not be straight, but bent upwards, its elevation above *EF* cannot be greater than *k*, since then the pressure there will become negative and the continuity of the fluid will broken up.

COROLLARY 2

40. Therefore if the tube may have the figure (Fig. 41) *ABOCD*, of which the branches *AB* and *DC* shall be vertical, but the middle *BOC* is bent upwards equally on both sides above the horizontal *EF*, so that there shall be $EB + BO = \frac{1}{2}l$, the pressure at the highest point *O* will be

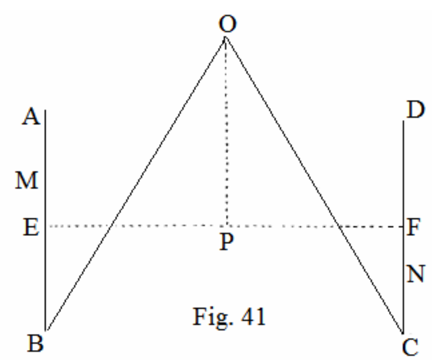


Fig. 41

$$= k - OP + ME - \frac{2ME(OB+MB)}{l}$$

$$= k - OP + ME - \frac{ME(MB+BO)}{EB+BO}$$

$$= k - OP - \frac{ME\cdot ME}{EB+BO}.$$

Therefore lest the continuity of the fluid may be broken up, it does not suffice, that there shall be $OP < k$, for there will be required to be $OP < k - \frac{ME^2}{OB+MB}$.

EXAMPLE 3

41. *A tube* (Fig. 42) *of uniform section may be constructed with the two right branches AB and BC inclined to the horizontal EF in some manner, but the upper tube BC shall be closed at C and with the air evacuated , and in this tube a water column of given length MBN shall move: to define its motion.*

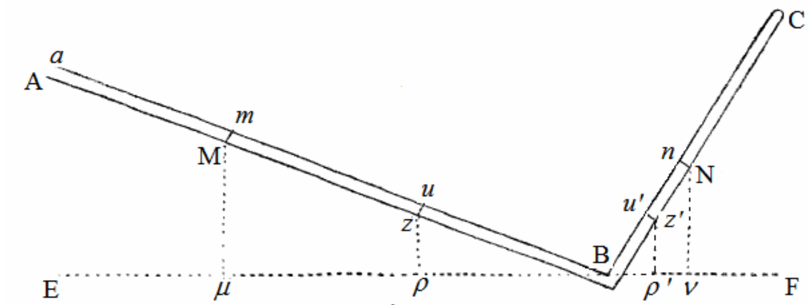


Fig. 42

Let the angle $ABE=\varepsilon$ and the angle $CBF=\xi$, the length of the column $MB+BN=l$ and $AM=m$; at M therefore the water is pressed on by the atmosphere, so that there shall become $M=k$, truly at N the pressure is zero, so that there shall be $N=0$, then truly from the solution of the problem there is $\mu=M\mu$ and $v=Nv$, from which with the density of the water put $=1$ this equation is obtained:

$$l\,\tfrac{ddm}{dt^2}=2gk-2g\left(Nv-M\mu\right),$$

and for some point of the tube z the pressure will be :

$$p=\tfrac{k(l+m-Az)}{l}+\tfrac{M\mu(l+m-Az)}{l}+\tfrac{Nv(Az-m)}{l}-zp$$

or

$$p=\tfrac{k(l-Mz)}{l}+\tfrac{M\mu(l-Mz)}{l}+\tfrac{Nv\cdot Mz}{l}-zp,$$

and for the point z' in the other branch:

$$p=\tfrac{k(l-BM-Bz')}{l}+\tfrac{M\mu(l-MB-Bz')}{l}+\tfrac{Nv(MB+Bz')}{l}-z'p',$$

For this equation requiring to be expedited we may call $BM=x$, so that there shall be $BN=l-x$, and there will become

$$M\mu=x\sin\varepsilon \text{ and } Nv=\left(l-x\right)\sin\zeta;$$

from which the differential equation will be

$$l\tfrac{ddx}{dt^2}+2g\left(k+x\sin\varepsilon-(l-x)\sin\zeta\right)=0,$$

which multiplied by $2dx$ and integrated will produce:

$$l\tfrac{dx^2}{dt^2}+2g\left(2kx+xx\sin\varepsilon+(l-x)^2\sin\zeta\right)=2gff.$$

Whereby the speed of the column $\frac{dm}{dt}=-\frac{dx}{dt}$, if indeed we may be able to carry that towards C, there will be

$$-\tfrac{dx}{dt}=\sqrt{\tfrac{2g}{l}\left(ff-2kx-xx\sin\varepsilon-(l-x)^2\sin\zeta\right)}$$

or

$$-\tfrac{dx}{dt}=\sqrt{\tfrac{2g}{l}\left(ff-ll\sin\zeta-2kx+2lx\sin\zeta-xx(\sin\varepsilon+\sin\zeta)\right)},$$

from which we understand the speed to vanish, when there were

$$x=\frac{-k+l\sin\zeta\pm\sqrt{(kk-2kl\sin\zeta-ll\sin\varepsilon\sin\zeta+ff(\sin\varepsilon+\sin\zeta))}}{\sin\varepsilon+\sin\zeta},$$

moreover the maximum will become, where

$$x=\frac{-k+l\sin\zeta}{\sin\varepsilon+\sin\zeta};$$

and this maximum speed will be

$$=\sqrt{\tfrac{2g}{l}\left(ff+\frac{kk-2kl\sin\zeta-ll\sin\varepsilon\sin\zeta}{\sin\varepsilon+\sin\zeta}\right)}.$$

Truly for the time we will have:

$$dt\sqrt{\tfrac{2g}{l}}=\frac{-dx}{\sqrt{(ff-ll\sin\zeta-2kx+2lx\sin\zeta-xx(\sin\varepsilon+\sin\zeta))}},$$

from which on being integrated we deduce :

$$x=\frac{-k+l\sin\zeta+\cos\lambda t\sqrt{(ff(\sin\varepsilon+\sin\zeta)+kk-2kl\sin\zeta-ll\sin\varepsilon\sin\zeta+)}}{\sin\varepsilon+\sin\zeta},$$

with there being $\lambda=\sqrt{\frac{2g}{l}(\sin\varepsilon+\sin\zeta)}$.

But if now for the point z of the tube we may put $Bz = z$, the pressure there will be

$$p = \frac{(k+x\sin\varepsilon)(l-x+z)}{l} + \frac{(l-x)\sin\zeta}{l}(x-z) - z\sin\varepsilon.$$

But for the point z' in the other branch BC on putting $Bz' = z'$ there will become

$$p' = \frac{(k+x\sin\varepsilon)(l-x-z')}{l} + \frac{(l-x)\sin\zeta}{l}(x+z') - z'\sin\zeta.$$

In that case there will be more succinctly,

$$p = \frac{(k+x(\sin\varepsilon+\sin\zeta))(l-x+z)}{l} - z(\sin\varepsilon+\sin\zeta),$$

and truly this:

$$p' = \frac{(k+x(\sin\varepsilon+\sin\zeta))(l-x-z')}{l}.$$

COROLLARY 1

42. (Fig. 43) shall be with the branch of the tube AB horizontal and with BC vertical and hence $\varepsilon = 0$ and $\zeta = 90°$. From which the speed at M

$$= \sqrt{\tfrac{2g}{l}(ff - ll - 2kx + 2lx - xx)}.$$

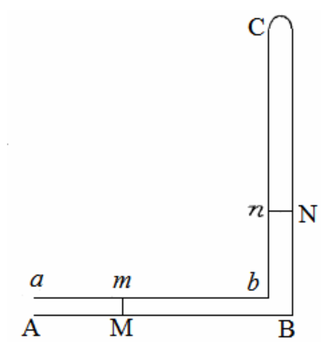


Fig. 43

Initially we may put the whole water column AB to be horizontal and to be at rest there, so that there shall be $AB = l$, therefore by necessity on putting $BM = x = l$ the speed may vanish and there must be taken $ff = 2kl$; from which there it will arise with the column MBN in place, the speed will become

$$= \sqrt{\tfrac{2g}{l}(l-x)(2k-l+x)};$$

and when the whole column arrives in the vertical tube, which happens, if $x = 0$, its speed with which it begins to ascend up at this point will be $= \sqrt{\tfrac{2g}{l}l(2k-l)}$. While therefore the length of the column shall be less than $2k$, the whole column will ascend in the vertical tube, if indeed the vertical tube were greater than $2k$.

COROLLARY 2

43. But on taking $ff = 2kl$ and $\lambda = \sqrt{\frac{2g}{l}}$, the equation integrated twice becomes:

$$x = -k + l + k\cos(\lambda t + \gamma).$$

From which if there were initially $x = l$, the constant angle γ vanishes, so that there shall become

$$x = l - k\left(1 - \cos\lambda t\right).$$

Therefore the time, where the whole column enters into the vertical tube, must be defined hence:

$$1 - \cos\lambda t = \tfrac{l}{k} \quad \text{or} \quad \lambda t = \text{Ang cos}\left(1 - \tfrac{l}{k}\right).$$

Whereby if $l = k$, there will be $t = \frac{\pi}{2\lambda} = \frac{\pi\sqrt{l}}{2\sqrt{2g}}$, but since there shall be $l = 2k$, there becomes $t = \frac{\pi\sqrt{l}}{\sqrt{2g}}$.

SCHOLIUM

44. Nothing stands in the way (Fig. 44), whereby we may substitute mercury for water, and then k will be the height of mercury in a barometer; and hence we will be able to define the oscillations of mercury in a barometer, if beyond K the horizontal tube may be added BA of the same cross-section. Therefore we may put in the equilibrium state the height $BK = k$ and $BE = e$, so that the total column of mercury $l = e + k$. Now with a certain agitation made, the column shall be in the state MBN with there being $BM = x$ and $EM = x - e$, and the speed of the mercury in the ascending tube will be :

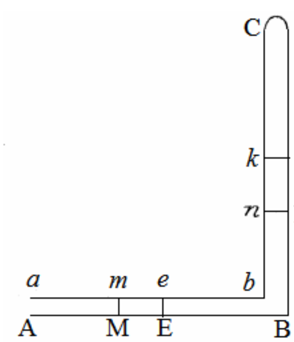


Fig. 44

$$-\tfrac{dx}{dt} = \sqrt{\tfrac{2g}{l}\left(ff - ll - 2kx + 2lx + xx\right)},$$

which we may put to be vanishing on putting $x = a = BA$, thus so that there may be put $ff = \left(l - a\right)^2 + 2ak$, with which done the speed will be

$$= \sqrt{\tfrac{2g}{l}\left(a - x\right)\left(2k - 2l + a + x\right)}.$$

We may put $EA = c$, $EM = y$, so that there shall be $a = c + e$, $x = e + y$, and on account of $l = e + k$ this speed will be :

$$-\frac{dy}{dt} = \sqrt{\frac{2g}{e+k}(c-y)(c+y)} = \sqrt{\frac{2g(cc-yy)}{(e+k)}},$$

from which we gather

$$\frac{dt\sqrt{2g}}{\sqrt{(e+k)}} = -\frac{dy}{\sqrt{(cc-yy)}}$$

and by integrating,

$$\frac{t\sqrt{2g}}{\sqrt{(e+k)}} = \text{Ang}\cos\frac{y}{c} \quad \text{or} \quad y = c\cos\frac{t\sqrt{2g}}{\sqrt{(e+k)}}.$$

Whereby the mercury in the barometer will perform its oscillations about the state of equilibrium Kk, with the time of each oscillation being $\frac{\pi\sqrt{(e+k)}}{\sqrt{2g}}$, or these will be isochronous with a pendulum, the length of which $= e+k$.

PROBLEM 50

45. *If water* (Fig. 45) *may be moving into a tube at a constant rate, so that it may flow out from the other end on being compressed by some force on ascending, to determine the speed of the efflux and the pressure at the individual elements of the water.*

SOLUTION

Whatever shape the tube may have, it may be considered as if extended in the direction $AaOo$; to which may be added the scale of the height $\alpha\omega$, of which the applied lines $z\pi$ show the height of each point of the tube z above a given horizontal plane .

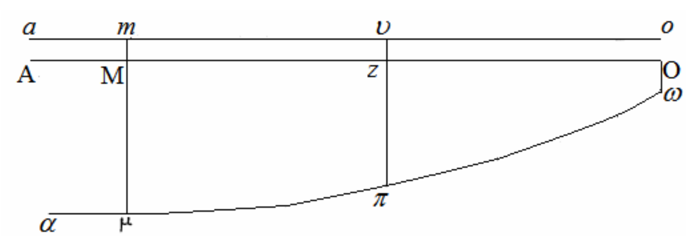


Fig. 45

Now in the elapsed time t water will flow out through the opening of the tube Oo with the speed $\mathfrak{T}$, by which likewise the whole mass of the fluid shall follow, which is in the tube at this stage. But now the water present in the part of the tube MO and at Mm may be acted on by a force expressed by the height $= M$. But at O, where the water flows out into the air, no other pressure can occur, except that of the atmosphere, which may be put in place equal to a column of water of height $= k$. Initially the water reaches as far as to A and the length may be put $AO = a$, and now there shall be $AM = m$, a function of the time t, from which the speed $\mathfrak{T} = \frac{dm}{dt}$ is defined. Now some element of the water may be considered at z , the height of which above the horizontal plane shall be $z\pi = z$, and with the density of the water put $=1$ and with the pressure at $z=p$, then truly the distance $Az=s$, from problem 48 we arrive at this equation:

$$2gp=2g(h-z)-s\Gamma':t+\Delta:t,$$

where there is

$$\Gamma:t=\mathfrak{T}=\tfrac{dm}{dt} \text{ and thus } \Gamma':t=\tfrac{ddm}{dt^2};$$

indeed the distance $AM=m$ and the speed $\mathfrak{T}$ is a function of the time *t* only. Now we will transfer the first point *z* to *M,* where since the pressure shall be given $=M$, on account of $s=m$ we will have :

$$2gM=2g(h-M\mu)-m\Gamma':t+\Delta:t;$$

thereafter we will transfer the point *z* to the opening *O* by putting $s=a$, where since equally the pressure shall be known $=k$, there will be

$$2gk=2g(h-O\omega)-a\Gamma':t+\Delta:t.$$

From these equations we deduce initially

$$2g(M-k)=2g(O\omega-M\mu)+(a-m)\Gamma':t$$

and thus $\Gamma':t=\tfrac{ddm}{dt^2}=\tfrac{2g(M-k+M\mu-O\omega)}{a-m}$, then truly, since there shall be

$$2g(M-p)=2g(z-M\mu)+(s-m)\Gamma':t,$$

there will become

$$M-p=z-M\mu+\tfrac{(s-m)(M-k+M\mu-Ow)}{a-m}$$

or

$$p=\tfrac{M(a-s)+k(s-m)+M\mu(a-s)+O\omega(s-m)}{a-m}-z,$$

or

$$p=\tfrac{(M+M\mu)(a-s)+(k+O\omega)(s-m)}{a-m}-z$$

or also

$$p=\tfrac{Oz(M+M\mu)+Mz(k+O\omega)}{MO}-z.$$

Therefore the whole problem will depend on that second order differential equation.

COROLLARY 1

46. If the pressure at M were either constant or depending on the distance $AM = m$, since the height $M\mu$ will depend on the same, and $O\omega$ is constant, the differential of the differential equation multiplied by $2dm$ becomes the return of the integral :

$$\frac{dm^2}{dt^2} = 4g\int \frac{M-k+M\mu-O\omega}{a-m} dm,$$

from which form the square of the speed may be expressed.

COROLLARY 2

47. If the efflux were into a volume with the air removed, it is seen how it must be written in our form ; and if at M the water may sustain no other force besides the atmospheric pressure, there will become $M = k$. Whereby, if each end may appear to the air, there will be

$$M = k \text{ and } \frac{dm^2}{dt^2} = 4g\int \frac{M\mu-O\omega}{a-m} dm$$

and for the pressure at z there will become

$$p = k - z + \frac{Oz\cdot M\mu + Mz\cdot O\omega}{MO}.$$

EXAMPLE 1

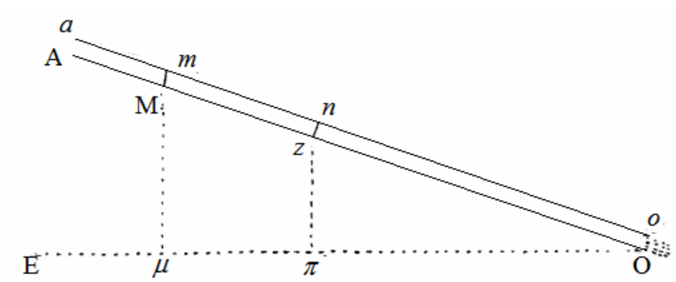


Fig. 46

48. *AO shall be* (Fig. 46) *a right tube inclined at some angle to the horizontal, which initially will have been filled with water from O as far as to A, and thence may flow out through the opening Oo, to determine this motion.*

With the angle put in place $AOE = \varepsilon$, on account of $MO = a - m$ the height $M\mu = (a-m)\sin\varepsilon$ and the height $O\omega = 0$. Whereby since there shall be $M = k$, there will become

$$\frac{dm^2}{dt^2} = 4g\int dm\sin\varepsilon = 4gm\sin\varepsilon,$$

since on making $AM = m = 0$ the motion is put to have started from rest. Hence therefore again there becomes

$$\frac{dm}{\sqrt{m}} = 2dt\sqrt{g}\sin\varepsilon \text{ and on being integrated } \sqrt{m} = t\sqrt{g}\sin\varepsilon;$$

from which we conclude all the water to be going to flow out in the time $=\frac{\sqrt{a}}{\sqrt{g}\sin\varepsilon}$; which time agrees with that, with which a heavy body may be going to descent upon an inclined plane AO .

Truly at the point z the pressure is $p=k-z\pi+\frac{Oz\cdot M\mu}{MO}$, but since there shall be $MO:M\mu=Oz:z\pi$, there becomes $p=k$, or through the whole path MO the pressure evidently the same as that of the atmosphere, which generally would not be in doubt.

EXAMPLE 2

49. *If, as before, water may flow from the inclined right tube AO not into air, but into a vacuum space, to determine this motion.*

With the angle remaining $AOE=\varepsilon$, and $AM=m$, $AC=a$, there becomes

$$M\mu=(a-m)sin\,\varepsilon \text{ and } O\omega=0,$$

then truly $M=k$ and what before is k here is $=0$, and thus we will have:

$$\frac{dm^2}{dt^2}=4g\int\frac{k+(a-m)\sin\varepsilon}{a-m}dm=4gkl\frac{a}{a-m}+4gm\sin\varepsilon,$$

so that evidently on putting $m=0$ the motion will begin from rest: from which it follows on making $m=a$ the end droplet is going to be expelled with an infinite speed, which is not to be considered as absurd, since it must be understood to be spread infinitely thin at the end, and to which at once the smallest thickness must be attributed, the speed certainly to become moderate. But hence not unless the time itself may be defined approximately, since there shall be

$$2t\sqrt{g}=\int\frac{dm}{\sqrt{\left(kl\frac{a}{a-m}+m\sin\varepsilon\right)}}$$

neither in any case shall the inclination s vanish, nor the angle may become right.
Then truly on taking $Az=s$ the pressure will become :

$$p=\frac{k\cdot Oz}{MO}+\frac{Oz\cdot M\mu}{MO}-z\pi=\frac{k(a-s)}{a-m}.$$

But for the case, in which the tube AO maintains a horizontal position and $\varepsilon=0$, if we may put $l\frac{a}{a-m}=\frac{x}{a}$, there becomes $m=a\left(1-e^{-\frac{x}{a}}\right)$ and hence

$$\frac{2t\sqrt{gk}}{\sqrt{a}}=\int e^{-\frac{x}{a}}\frac{dx}{\sqrt{x}},$$

from which there is deduced approximately:

$$\frac{t\sqrt{gk}}{\sqrt{ax}}=e^{-\frac{x}{a}}\left(1+\frac{2}{3}\cdot\frac{x}{a}+\frac{2}{3}\cdot\frac{2}{5}\cdot\frac{xx}{aa}+\frac{2}{3}\cdot\frac{2}{5}\cdot\frac{2}{7}\cdot\frac{x^3}{a^3}+\text{etc.}\right),$$

therefore for the initial motion, where x is very small and

$$m=x-\frac{xx}{2a}+\frac{x^3}{6aa}-\text{etc.},$$

there becomes

$$\frac{t\sqrt{gk}}{\sqrt{ax}}=\left(1-\frac{x}{a}+\frac{x^2}{2aa}-\text{etc.}\right)\left(1+\frac{2}{3}\cdot\frac{x}{a}+\frac{2}{3}\cdot\frac{2}{5}\cdot\frac{xx}{aa}+\text{etc.}\right),$$

or

$$t=\frac{\sqrt{ax}}{\sqrt{gk}}\left(1-\frac{1}{3}\cdot\frac{x}{a}+\frac{1}{10}\cdot\frac{xx}{aa}+\text{etc.}\right).$$

EXAMPLE 3

50. *A tube may be established* (Fig. 47) *equally wide ABO with the two right branches, with the one to be horizontal from AB, the other BO turned downwards, which since initially it were full of water, water will begin to flow out through the opening Oo, to determine its motion and pressure at individual places.*

The length of the horizontal branch shall be $AB=b$, and of the vertical $BO=c$ and thus $a=b+c$. Therefore since at the time t water will flow forwards from A in M with there being $AM=m$, on account of the pressure at $M=k$, and thence also at O, the square of the speed at M will become :

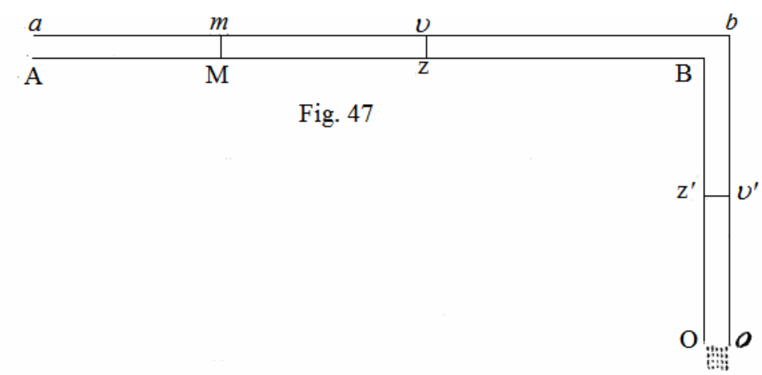


Fig. 47

$$\frac{dm^2}{dt^2}=4g\int\frac{cdm}{a-m}=4gcl\frac{a}{a-m},$$

from which there becomes:

$$2dt\sqrt{gc}=\frac{dm}{\sqrt{l\frac{a}{a-m}}},$$

where the same difficulty of the integration occurs as in the previous example. But the pressure at some point z of the horizontal tube will be

$$p = k - c + \tfrac{(c+Bz)c}{a-m} = k - \tfrac{c\cdot Mz}{a-m} = k - \tfrac{BO\cdot Mz}{BM+BO},$$

and thus the pressure at the angle B will become a minimum $= k - \tfrac{BO\cdot Mz}{BM+BO}$. But by taking z' in the vertical tube the pressure there will be

$$p' = k - Oz' + \tfrac{Oz'\cdot BO}{BM+BO} = k - \tfrac{Oz'\cdot BM}{BM+BO}.$$

Moreover the initial motion at *A* will correspond to an accelerative force $= \tfrac{c}{a}$ with gravity expressed by unity.

COROLLARY 1

51. Therefore the initial motion, while both the tubes were full, the pressure at *B* is a minimum everywhere; and thus it can become negative, if each branch shall be greater than *k;* which if it may have happened, the continuity at *B* is broken and the water will descend faster through the vertical tube, than the rest is able to follow through the horizontal tube.

EXAMPLE 4

52. *A right vertical tube* (Fig. 48) *shall be hermetically sealed at A beyond the aperture, but with a pressure greater than the atmospheric pressure k, which if initially it were full, to define the descent of the fluid.*

In the time *t* the fluid will have descended through $AM = m$, with the height $AO = a > k$, and since there will be a vacuum above *Mm*, there will become $M = 0$, from which the speed squared of descent at *M* becomes

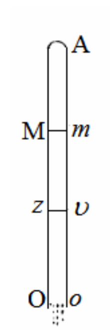


Fig. 48

$$\tfrac{dm^2}{dt^2} = 4g\int \tfrac{-k+a-m}{a-m}dm = 4g\left(m - kl\tfrac{a}{a-m}\right).$$

Whereby since the initial speed, where $m = 0$, were zero, that will become a maximum, when $m = a - k$ or $OM = k$, in which case there will be

$$\tfrac{dm}{dt} = 2\sqrt{g\left(a - k - kl\tfrac{a}{k}\right)},$$

truly again afterwards it will decrease and vanish, when there becomes $kl\tfrac{a}{a-m} = m$. We may put the height *a* to exceed *k* by a very small amount and to become $a = k + \omega$, and the maximum speed will become

$$= 2\sqrt{g\left(\omega - kl\left(1+\tfrac{\omega}{k}\right)\right)} = 2\omega\sqrt{\tfrac{g}{2k}}$$

corresponding to the interval $AM = \omega$. But the speed will vanish again, when there will become

$$k\left(\tfrac{m}{a} + \tfrac{mm}{2aa}\right) = m \text{ or } m = 2\omega,$$

truly there is found closer:

$$m = 2\omega - \tfrac{2\omega^2}{3k} + \tfrac{4\omega^3}{9kk} - \tfrac{44\omega^4}{135k^2} + \text{etc.}$$

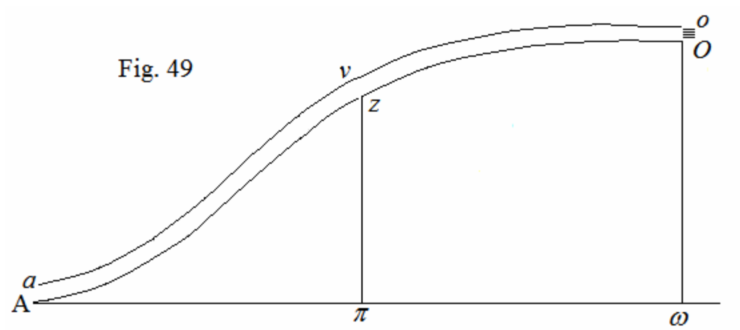
Finally the pressure a some location *z* will become

$$p = \tfrac{Oz\cdot MO + Mz\cdot k}{MO} - Oz = \tfrac{k\cdot Mz}{MO}.$$

PROBLEM 51

53. *If water* (Fig. 49) *may be moving thus in a tube with an equally large cross-section, so that it may flow out at its latter end Oo, truly at the other end Aa it will continually flow in propelled by a given force, to define this propulsion motion of the water through the tube.*

SOLUTION



Here I consider a tube *AO* of any curvature, into the opening of which *Aa* water is entering continually by some force expressed by the height *L*, which can be considered either as constant or expresses by a function of the time *t*, the continual intrusion of water and its propulsion is accustomed to be effected with the help of pumps, by the force of which water is raised from the lower level *A* and streams forth there at the higher level *O*. For this account I take the end *A* at the bottom position, from which the water is led in, from which *Aw* drawn to the horizontal, I estimate the heights above of the individual points of the tube *z*, thus so that the height of the upper opening *Oo,* where the water is expelled, shall be sit *Oω.* Therefore for some point *z* with the length of the tube $Az = s$ and with the height $\pi z = z$, the speed of the water in the tube $= \mathfrak{T}$ in the elapsed time $= t$, by which likewise it will flow out at *Oo*, and which is a function of the time *t,* as I have put in problem 48, $\mathfrak{T} = \Gamma : t$, then truly with p denoting the pressure at *z,* as also we shall be referring to water, so that there shall become $b = 1$, we find this equation:

$$2gp = 2g\left(h - z\right) - s\Gamma' : t + \Delta : t,$$

while clearly we have assumed the water in the tube to have progressed from the end *A* to the end *O*, and an account of the motion is contained in this general equation. Therefore it is required to adapt that to the present case considered by implementing these conditions, and so that at *A* the given pressure shall be $=L$ and at O, $=k$, with k denoting the height of the column of water of equal weight to the atmosphere. We may move the indefinite point *z* first to the opening *A*, where there becomes $s=0$, $z=0$ and $p=L$, and thus:

$$2gL = 2gh + \Delta : t,$$

Then with the same transferred to the opening *O*, where there becomes $s = AO$, $z = O\omega$ and $p = k$, we will have:

$$2gk = 2g(h - O\omega) - AO \cdot \Gamma' : t + \Delta : t,$$

from which we deduce

$$2g\left(L-k\right) = 2g \cdot O\omega + AO \cdot \Gamma' : t,$$

so that there shall become:

$$\Gamma' : t = \tfrac{d\mathfrak{T}}{dt} = \tfrac{2g\left(L-k-O\omega\right)}{AO}.$$

Then truly for the pressure at some indefinite place *z* there will become

$$2g(L-p) = 2gz + \tfrac{2gs\left(L-k-O\omega\right)}{AO}$$

or

$$p = L - z - \tfrac{s\left(L-k-O\omega\right)}{AO}.$$

Now we may put the whole length of the tube $AzO = l$ and the height of the orifice *Oo* to be $O\omega = a$, and in the first place we have obtained for the speed $\mathfrak{T}$

$$\tfrac{d\mathfrak{T}}{dt} = \tfrac{2g\left(L-k-a\right)}{l},$$

and thus on integrating

$$\mathfrak{T} = \tfrac{2g}{l}\left(\int Ldt - \left(a+k\right)t\right),$$

then truly, for the pressure at some point *z* of the tube, there will be

$$p = L - z - \tfrac{s\left(L-k-a\right)}{l}.$$

COROLLARY 1

54. But if therefore the propelling force L were constant and $= a+k$, the acceleration of the water in the tube vanishes and thus its flow through the tube will be uniform, but how great its speed is going to become, cannot be defined from these principles, but must be inferred from the nature of the impelling forces.

COROLLARY 2

55. Moreover, if the propelling force L shall be always greater than $a+k$, the speed of the water propelled through the tube will be increased continually, but if it were smaller, it will be diminished continually. Nor therefore can the state of the outflow of the water with regard to the speed of the water hence be given with any certainty at a given time.

SCHOLIUM 1

56. Thus indeed in whatever way this paradox may appear to be contrary to experience, yet we have established from the hypothesis, the pressure at A always to propel the water with the same force, arising from which it is required to be agreed, whatever its speed were, and if it were allowed to apply such forces, there is no reason to doubt, why this effect should not actually be going to follow here also. Whereby since this arises less in practice, it is required to judge the forces, which are accustomed to be used for the propulsion of the water, by no means to be of this nature, that they may produce the same pressure, with whatever speed the water may be emerging. Moreover in addition it may be agreed well enough all the forces, which are accustomed to be demanded from men, animals, and the wind, for the flow of water to be prepared thus, so that with the speed increased the forces may be weakened and finally vanish. Indeed however great a force of this kind may be applied to an object at rest, and this object is moved at once, that may go forth less, whereby it is not permitted to define such forces absolutely, but the magnitude of these itself must be determined from some level of the speed, by which they act. Thus if we may put in place machines, by which water is driven through the tube, a force of this kind to be applied which, while it is used with the speed $= c$, shall be equal to the weight of the water, of which the volume shall be $= V$: and the machine to be constructed thus, so that it may perform always with this speed $= c$, which can be done with the aid of rotation always. Now since in our case the pressure at A may be expressed by the height $= L$, if we may put the cross-section of the $= \omega$, that will be equal to the weight of a volume of water $= L\omega$, which so that by that force V it may be produced with the motion c, its speed hence may be determined, evidently if we may assume the force $L\omega$ may propel the water with the speed $\mathfrak{T}$, it shall be required for $L\omega\mathfrak{T} = Vc$, and hence $\mathfrak{T} = \frac{Vc}{L\omega}$. But so that the water may be propelled uniformly by this motion, we see there must become $L = a+k$, where indeed we can ignore the pressure of the atmosphere k, which is present also in the force at A, thus so that it may suffice to put in place $L = a$, from which it is seen the water is going to be propelled from the tube with the speed $\mathfrak{T} = \frac{Vc}{a\omega}$.

SCHOLIUM 2

57. Since here no principal force is acting besides *V,* but into the speed by which it acts, may be introduced into the calculation : this product *Vc*, since it must be considered especially in determining the effectiveness of all machines, deserves a special mention, and therefore is going to be called by me the *action*, thus so that the *action* shall be the product of each force by the speed, by which it acts, where it is required to be observed initially, provided that in machines the forces may be increased or decreased, the speed to be changed always in the inverse ratio, so that the *action* may remain the same. Thus if by a machine the principle force *V* may be changed into V' transferred to another place, the speed, by which this is performed, will be $=\frac{Vc}{V'}$, and if then the speed of the action shall be $=c'$, the force will be $V'=\frac{Vc}{c'}$. Clearly this is consistent with the use of particular machines, so that the same *action* is preserved by the forces acting, where the force or the speed may be changed as it pleases. Thus in the case of problems, in which there was a need of the force $=a\omega$ for the water to be required to be propelled through the tube *AO*, if the principal force moving the machine shall be $=V$ taken jointly with the speed $=c$, the machine will require to be constructed thus, so that in the translation of the force to the location *A,* where the water is introduced into the tube. the force may become $=a\omega$, and since then by necessity its speed shall become $\mathfrak{T}=\frac{Vc}{a\omega}$, hence the speed of the water propelled by the tube may be determined at once. If perhaps on account of the structure of the machine the driving force at *A,* which we have put $=L\omega$, may arise greater than $a\omega$, the speed of the action will be reduced in the same ratio, truly on account of $L>a$ the motion of the water will be accelerated: then truly the principal force will obtain a greater speed, and hence its magnitude *V* may appear to be diminished, from which, just as its *action* may increase or decrease, in turn, since the motion will have been induced for uniformity, the speed of the water propelled through the tube must be defined.

SCHOLIUM 3

58. Moreover of all the forces, which are accustomed to be used acting on machines, an account must be prepared, so that, while they may act on an object at rest and therefore they may act with no effect, they will exert the maximum force, which shall be $=F$, then truly with a continually increased speed a smaller force will be exert, in the end clearly nothing, since they must act with a certain speed $=e$. Truly therefore the speed for this case, this force truly vanishes, for each end the *action* is zero. If now with some speed smaller than *e,* which shall be $=u$, the same force will act, and its magnitude can be estimated $=F\left(1-\frac{u}{e}\right)^2$, therefore its *action* is $=Fu\left(1-\frac{u}{e}\right)^2$, since at any rate both the case $u=0$ as well as $u-e$ vanish, therefore the maximum will emerge , if $u=\frac{1}{3}e$, and then there will be $=\frac{4}{27}Fe$. Whereby always it may be agreed thus machines to be constructed, so that the *action* of the forces, which will be used, may be returned a maximum, unless which rule will be observed, a much less efficient machine will be

produced, than from the same forces acting , it may be able to be obtained if duly constructed. Therefore with such a force used we may recall the preceding problem to determine the solution.

PROBLEM 52

59. *If in the case of the preceding problem water may enter into the tube at AO with a force, which at rest may exert a strength* $=F$ *, but moved with the speed* $=e$ *it loses all its strength, to define, how machines shall be required to be adapted for this force, so that it may deliver the maximum effect or, the greatest abundance of water may be drawn off in a given time.*

SOLUTION

We may put this power of the machine to be working with the speed $=u$, so that the force which it may exert shall be $F=\left(1-\frac{u}{e}\right)^2$, but the machine must be constructed thus, so that for the water required to be propelled through the tube that force will be multiplied in the ratio 1: n , therefore it shall act there with the speed $=\frac{u}{n}$, with which therefore the water now may be moved forwards through the tube, by which generally this motion itself shall be impressed, since indeed we may regard this to be true for continued motion. Therefore now there will become $\mathfrak{T}=\frac{u}{n}$, and on putting the cross-section of the tube $=\omega$, the force propelling the water in the tube $nF\left(1-\frac{u}{e}\right)^2=L\omega$, thus so that there shall become

$$L=\frac{NF}{\omega}\left(1-\frac{u}{e}\right)^2.$$

Therefore since we will have found $\frac{d\mathfrak{T}}{dt}=\frac{2g(L-a)}{l}$, where we have omitted the atmospheric pressure k at the opening Oo, since the propelling force itself is helped by the same amount, now if there shall become $L>a$ or if $L<a$, in each case the motion soon will be led to uniformity, thus so that there shall become $L=a$ and thus $1-\frac{u}{e}=\sqrt{\frac{a\omega}{nF}}$: and thus from the power applied, as we have assumed, on account of $u=e\left(1-\sqrt{\frac{a\omega}{nF}}\right)$, the water will be propelled through the tube with the speed $\mathfrak{T}=\frac{e}{n}\left(1-\sqrt{\frac{a\omega}{nF}}\right)$, thus so that in the individual seconds the volume of water to be ejected by the opening $=\mathfrak{T}\omega$. Here it is apparent in the first place, if there were $\frac{a\omega}{nF}>1$ or $nF<a\omega$, clearly no motion can be produced. Moreover, the maximum effect will be obtained, if $u=\frac{1}{3}e$, and hence $\frac{4}{9}=\frac{a\omega}{nF}$, from which the machine must be constructed thus, so that there may become, $n=\frac{9a\omega}{4F}$, then truly $\mathfrak{T}=\frac{4}{27}\cdot\frac{Fe}{a\omega}$, and the quantity of water ejected in one second $=\frac{4}{27}\cdot\frac{Fe}{a}$,

where the force F must be expressed by the weight returned to the volume of the equivalent mass of water, thus so that F will denote a certain volume.

COROLLARY 1

60. Therefore so that if the height *a,* to which the water must be raised, as well as the speed *e,* or the distance in feet that it is required to run through in one second, truly the volume F may be expressed in cubic feet, then the formula $\frac{4}{27}\cdot\frac{Fe}{a}$ will give the volume of water expressed likewise in cubic feet, so that in individual seconds it may be raised to a height of a feet.

COROLLARY 2

61. Therefore from the source of the power, which at rest exerts a force $=F$, moreover with the speed of the motion $=e$ all the force is lost, and the supply of water cannot be raised to a height a greater than $\frac{4}{27}\cdot\frac{Fe}{a}$. Nor truly will this effect be obtained, unless the machine may be constructed thus, so that the moving force applied to that in the transportation requiring for propelling the water may be increased in the ratio $1:n=1:\frac{9a\omega}{4F}$.

SCHOLIUM 1

62. So that this may be understood more clearly, we may put the strength of a man to be used, which at rest may be estimated to be 70 pounds or about equal to one cubic foot of water, so that there shall become $F=1$; but the maximum speed, by which no greater force may prevail to be thrust forwards, to be $7\frac{1}{2}$ feet per second or $e=7\frac{1}{2}$. Therefore this man, if his labor may be expended in the most gainful manner, will be able to raise a volume of water $=\frac{10}{9a}$ cubic ft. to a height of a feet per second, and this shall be, if the machine may be constructed thus, so that it may be able to be operated with a speed $=2\frac{1}{2}$ ft. per second; and then his action is $=\frac{4}{27}Fe=\frac{10}{9}$, thus always so that the action expressed in this manner, if it may be divided by the height a, will produce the amount of water to be raised per second. But in addition in the construction of the machine it is required to consider the area of the tube ω, since the force moving through the transportation must be increased in the ratio $1:\frac{9a\omega}{4F}$; which counter ratio cannot depend on the speed $=e$. Then since the maximum action of one man shall be $=\frac{10}{9}$, if λ men may be working together, the action of these will be $=\frac{10}{9}\lambda$, the effect of which is proportional always. If it shall be required to use horses and in the unmoving case of one horse the force may be considered three times greater than of a man and the maximum speed also three times greater, the new action of this will become nine times greater, or one horse will prevail to surpass as many as nine men.

SCHOLIUM 2

63. If we may wish to use the flow of a river for setting a machine in motion, the rotation of its blades will give rise to the motion, the determination of the effect by raising water must be established in this manner. The surface shall be ff , which the thrust of the water may strike normally, and e may denote the speed of the river, from which the height, by which a heavy weight will acquire the same speed by falling, will be $=\frac{ee}{4g}$: therefore the force of the river on this stationary surface will be $=\frac{eeff}{4g}$, evidently equal to the weight of such a volume of water, which will be required to write in place of the letter F ; then truly, because the impulse vanishes at once and the blade will move with that speed of the flow $=e$, this is that speed , which before we have indicated by the letter e. Therefore the greatest action arises, when the surface ff may be moved with the speed $=\frac{1}{3}e$ [see § 58], and this action $=\frac{e^3ff}{27g}$ and thus is proportional to the cube of the speed of the river. And thus by this action a quantity of water $=\frac{e^3ff}{27ga}$ will be raised to the height $=a$ in each second ; therefore since a quantity $=\frac{10}{9a}$ cubic ft. may be raised by one man, the effect of the water will be equivalent to λ men with there being $\lambda=\frac{e^3ff}{30g}$, while e and f are expressed in feet, where it is required to be observed that $g=15\frac{1}{2}$ ft. and thus $\lambda=\frac{e^3ff}{465}$. So that if $ff=1$ sq.ft. and the running water can complete $7\frac{1}{2}$ ft. per second, a single man will produce the same effect. Indeed here we have assumed the tube AC to be of the same width everywhere, truly the matter will be resolved in the same manner, even if its width were variable, which case we will set out in the following chapter; but it is required always to consider the tube as being very narrow.

CAPUT II

DE MOTU AQUAE IN TUBIS AEQUALITER UBIQUE AMPLIS

PROBLEMA 48

27. *Si tubi aequaliter ampli directrix fuerit linea curva quaecunque, in quo aqua a sola gravitate animata moveatur, eius motum definire.*

SOLUTIO

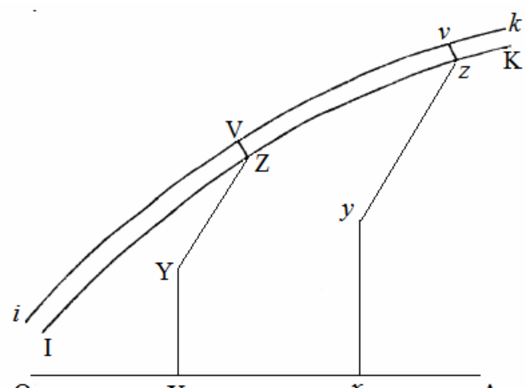


Fig. 38

Habeat (Fig. 38) tubi *IiKk* directrix *IZzK* figuram quamcunque curvam aequatione duplici inter ternas coordinatas $Ox = x,\ xy = y,\ yz = z$ determinatam ponaturque directricis arcus $Iz = s$, ut sit $ds^2 = dx^2 + dy^2 + dz^2$, ut quantitates *x, y, z* ut functiones ipsius *s* spectari queant. Sit tubi amplitudo $= \alpha x$, et densitas aquae littera *b* designetur, ut sit $w = \alpha$ et $q = b$. Tum vero elapso tempore $= t$ sit pressio aquae in $z = p$ et celeritas ibidem secundum directionem $zK = \mathfrak{T}$; vires autem sollicitantes *P, Q, R,* si applicata *yz* statuatur verticalis, reducuntur ad $P = 0,\ Q = 0$ et $R = -1$. His positis ex problemate 46, ob $q = b$ et $\omega = \alpha$, motus duabus sequentibus aequationibus determinabitur:

$$\text{I. } \left(\tfrac{d\mathfrak{T}}{ds}\right) = 0,\ \text{II.} \tfrac{2gdp}{b} = -2gdz - \mathfrak{T}d\mathfrak{T} - ds\left(\tfrac{d\mathfrak{T}}{ds}\right),$$

in posteriori sumto tempore *t* constante. Ex priori ergo patet celeritatem $\mathfrak{T}$ functionem esse temporis *t* tantum ideoque $\mathfrak{T} = \Gamma : t$; unde cum in altera sola *s* ut variabilis spectetur, fiet $d\mathfrak{T} = 0$ et $\frac{d\mathfrak{T}}{dt} = \Gamma' : t$, hincque habebitur:

$$\tfrac{2gdp}{b} = -2gdz - ds\Gamma' :\ t$$

et integrando:

$$\tfrac{2gp}{b} = 2g\left(h - z\right) - s\Gamma' : t + \Delta : t.$$

Quovis ergo temporis instante celeritas aquae in tubo ubique est eadem, diverso autem tempore utcunque variabilis esse potest; a qua variabilitate pressio *p* non solum maxime pendet, sed insuper functionem temporis quamcunque assumit: has autem duas functiones in se arbitrarias ex circumstantiis et viribus extrinsecus aquam urgentibus determinari debere per se est perspicuum.

COROLLARIUM 1

28. Circa motum ergo aquae in tubis aeque amplis hic in genere plus non determinatur, quam quod quovis temporis momento aqua ubique pari celeritate secundum tubi tractum moveatur et quod pressio p certo quodam modo determinetur.

COROLLARIUM 2

29. Quaecunque igitur functio temporis t pro celeritate $\mathfrak{T}$ accipiatur, semper affirmare licet eiusmodi motum aquae in tubo aequaliter amplo esse possibilem, dummodo eiusmodi vires externae adhibeantur, quae illi continuae accelerationi seu retardationi producendae sint pares.

SCHOLION 1

30. In huius problematis solutione usus sum methodo priore in problemate 46 exposita, unde conveniet quoque solutionem ex altera methodo problematis 47 elicere. Ponamus ergo fluidi elementum, quod nunc post tempus t in z consideravimus, initio, ubi $t=0$, fuisse in Z existente $OX=X$, $XY=Y$, $YZ=Z$ et arcu $IZ=S$, atque ob $q=Q=b$ et $\omega=\Omega=\alpha$ prior aequatio dat $\left(\frac{ds}{dS}\right)=1$ unde colligitur $s=S+\Gamma:t$ ideoque celeritas aquae in z fit $\left(\frac{ds}{dS}\right)=\Gamma':t.$ Altera vero aequatio praebet:

$$\frac{2gdp}{b}=-2gdz-ds\cdot\Gamma'':t$$

tempore t sumto constante, quae ergo integrata dat:

$$\frac{2gp}{b}=2g\left(h-z\right)-s\Gamma'':t+\Delta:t,$$

quae cum praecedente prorsus convenit, nisi quod hic celeritas in z post tempus t exprimatur per $\Gamma':t$, cum ea ante esset $\Gamma:t$. Id tantum obiici posset, quod, cum z et s spectari debeant ut functiones binarum variabilium S et t, in posteriori vero aequatione tempus t constans sit assumtum, differentialia dz et ds non completa sed eae tantum partes accipi debeant, quae ex sola variabilitate ipsius S oriuntur; hincque vicissim integralia non absolute, uti est factum, capi debere. Cui dubio ut occurramus, sit z functio quaecunque binarum variabilium S et t, unde fiat $dz=MdS+Ndt$; atque certum est in illa aequatione differentiali loco dz scribi debere MdS. Verum ob eandem rationem, quod hic tempus t sumitur constans, membri MdS integrale iterum est z, cui quidem functio ipsius t adiungi posset, quae autem iam in $\Delta:t$ contineri est censenda; quod idem de integratione differentialis ds est iudicandum.

SCHOLION 2

31. Problema igitur propositum ut maxime indeterminatum spectari debet, cum vires quascunque externas, quibus aqua, dum in tubo movetur, sollicitari potest, in se complectatur, ideoque nunc demum verus eius motus ex his viribus externis penitus determinari debeat. Dum autem aqua in tubo versatur, aliae vires externae in eam agere nequeant, nisi quibus massa aquae in utroque termino prematur; massa scilicet aquae certa, quae quovis momento in tubo, cuius extensio ut indefinita est consideranda, certam longitudinem occupet, est statuenda, quae utrinque ope pistillorum certis viribus sollicitetur; quandoquidem durante motu neque novam aquae molem accedere neque usquam effluxum ex tubo concedere velimus, quippe qui casus peculiarem requirunt evolutionem. Primo ergo statuamus in tubo infinitae longitudinis certam aquae molem continuo moveri et utroque termino iugiter certis viribus sollicitari; et quia tubi curvatura non aliter in computum venit, nisi quatenus inde altitudo *z* pendet, commoditatis gratia tubum, quasi eius directrix esset linea recta, contemplabor, simul pro quovis puncto eius altitudinem super plano horizontali fixo assignaturus.

PROBLEMA 49

32. *Si certa aquae massa in tubo aequaliter amplo continuo moveatur et utrinque a viribus quibuscunque urgeatur, eius motum et pressionem in singulis punctis ad quodvis tempus determinare.*

SOLUTIO

Tubum quomodocunque curvum (Fig. 39) tanquam in rectum extensum consideremus, id tantum annotantes puncti cuiusque *z* altitudinem supra certum planum horizontale esse

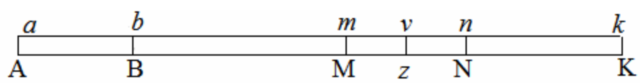


Fig. 39

$= z$, quae pro singulis punctis ut data est concipienda. Iam elapso tempore *t* aquae massa in tubo occupet spatium *MN,*quod ergo erit constantis quantitatis $= l$, ponamusque a puncto tubi fixo *A* distantia $AM = m$, $AN = n$, ut sit $l = n - m$; elevatio autem super planum horizontale sit puncti $M = \mu$, puncti vero $N = v$. At iam haec vena aquea *MN* in *M* urgeatur pressione $= M$, in *N* vero pressione $= N$, quae sint functiones temporis quacunque datae, celeritas vero huius massae aqueae sit $= \mathfrak{T}$, qua versus *K* promoveatur, et quae est functio temporis quaesita supra posita $\mathfrak{T} = \Gamma : t$. Si nunc in loco *z* quovis medio, cuius distantia a termino fixo *A* sit $Az = s$, pressio statuatur $= p$, erit

$$\frac{2gp}{b} = 2g\left(h - z\right) - s\Gamma' : t + \Delta : t.$$

Transferatur hoc punctum *z* primo in *M,* tum vero in *N,* et quia in his punctis pressiones illis ipsis, quibus nunc aqua in his terminis urgeri assumimus, aequales esse debent, hinc duas elicimus aequationes:

$$\frac{2gM}{b} = 2g\left(h - \mu\right) - m\Gamma' : t + \Delta : t$$

et

$$\tfrac{2gN}{b} = 2g\left(h-\nu\right) - n\Gamma' : t + \Delta : t$$

quarum haec ab illa subtracta relinquit:

$$\tfrac{2g}{b}\left(M-N\right) = 2g\left(\nu-\mu\right) - \left(n-m\right)\Gamma' : t,$$

unde colligitur

$$\Gamma' : t = \tfrac{2g\left(M-N\right)-2gb\left(\nu-\mu\right)}{b\left(n-m\right)}$$

et

$$\Delta : t = \tfrac{2g\left(Mn-Nm\right)}{b\left(n-m\right)} - 2gh + \tfrac{2g\left(n\mu-m\nu\right)}{n-m}.$$

Quia autem terminorum *M* et *N* eadem est celeritas $\mathfrak{T} = \Gamma : t$ iique tempore *dt* per spatiola *dm* et *dn* progrediuntur, erit

$$\tfrac{dm}{dt} = \mathfrak{T} = \Gamma : t = \tfrac{dn}{dt};$$

unde fit $\Gamma' : t = \tfrac{ddm}{dt^2}$, ita ut ob $n - m = l$ constanti celeritas aquae $MN = \tfrac{dm}{dt}$ perinde ac distantia $AM = m$ ex hac aequatione elici debeat:

$$bl\tfrac{ddm}{dt^2} = 2g\left(M-N\right) - 2gb\left(\nu-\mu\right)$$

ubi *Met N* sunt functiones datae temporis *t,* at μ et ν quantitates variabiles ab *m* et *n* pendentes. Resoluta autem hac aequatione, qua ad tempus $= t$ locus termini *M* determinatur, habebitur primo celeritas aqueae venae $MN = \tfrac{dm}{dt}$, tum vero pro quocunque puncto medio *z,* ut sit $Az = s$, pressio *p* ex hac aequatione definitur:

$$\tfrac{2gp}{b} = \tfrac{2g(Mn-Nm)}{bl} + \tfrac{2g(n\mu-m\nu)}{l} - 2gz - \tfrac{2gs(M-N)}{bl} + \tfrac{2gs(\nu-\mu)}{l},$$

quae ergo praebet:

$$p = \tfrac{Mn-Nm}{l} + \tfrac{b(n\mu-m\nu)}{l} - bz - \tfrac{s(M-N)}{l} + \tfrac{bs(\nu-\mu)}{l},$$

seu

$$p = \tfrac{M\left(n-s\right)}{l} + \tfrac{N\left(s-m\right)}{l} + \tfrac{b\mu(n-s)}{l} - \tfrac{b\nu(s-m)}{l} - bz.$$

Postquam ergo pro dato tempore *t* inventus fuerit locus *M* in tubo seu $AM = m$, unde simul celeritas $\tfrac{dm}{dt}$ innotescit, inde pro quovis elemento aquae in spatio *MN* contento pressio definitur.

COROLLARIUM 1

33. Si venam aqueam utrinque nulla plane vis urgeat, ut sit $M=0$ et $N=0$, eius motus ex hac aequatione debet determinari

$$bl\tfrac{ddm}{dt^2}=-2gb(\nu-\mu) \quad \text{seu} \quad \tfrac{ddm}{dt^2}+\tfrac{2g}{l}(\nu-\mu)=0,$$

tum vero pressio erit

$$p=\tfrac{b\mu(n-s)+b\nu(s-m)}{l}-bz.$$

COROLLARIUM 2

34. Sin autem binae pressiones *M* et *N* inter se fuerint aequales, tum prior quidem aequatio differentio-differentialis manet eadem, altera vero parumper discrepat:

$$p=M+\tfrac{b\mu(n-s)+b\nu(s-m)}{l}-bz,$$

ita ut haec pressio illam constanter superet quantitate *M.*

COROLLARIUM 3

35. Totum negotium ad aequationem differentialem secundi gradus reducitur, quae, etiamsi pressiones *M* et *N* sint inaequales, ideo tamen non fit solutu difficilior, sed tota difficultas residet in altitudinibus μ et ν, quippe quae per *m* determinantur.

SCHOLION

36. Casus corollarii primi, quo pressiones *M* et *N* evanescentes fecimus, nonnisi in vacuo locum habere potest; quando enim motus fit in aëre, vena aquea in utroque termino a pondere atmosphaerae premitur et quidem pari vi, nisi ambo termini ad altitudines maxime differentes pertingant. Quare, si vena in utroque termino fuerit libera et aperta, in corollario secundo littera *M* denotabit pressionem atmosphaerae, quae cum fere aequivaleat columnae aqueae 33 pedes altae, si pro hac altitudine scribamus *k,* erit $M=bk$, et quia hic perpetuo de aqua est sermo, pressiones etiam per columnas aqueas exprimi conveniet, unde densitatem aquae *b* unitati aequalem statuamus.
Quocirca pressionem in aquae vena habebimus:

$$p=k-z+\tfrac{\mu(n-s)+\nu(s-m)}{l}.$$

Maximi autem momenti est pressionem atmosphaerae *k* in calculum introducere, etiamsi non sentiatur; ea enim omissa saepe fieri posset, ut pressio *p* evaderet negativa, neque tamen continuitas fluidi solvatur; quod tamen semper evenire debet, ubi pressio revera fit negativa. Atmosphaerae autem ratione habita motus inventus durare potest, quamdiu pressio *p* tum non fit negativa; ea autem negativa evadente continuitas certe solvitur, nisi forte ob cohaesionem seu potius aetheris pressionem contineatur.

EXEMPLUM 1

37. *Si tubus fuerit rectus et ad horizontem utcunque inclinatus, motum venae aqueae MN utrinque apertae in eo definire.*

Sit inclinatio tubi *AK* ad horizontem $=\eta$, et vena aquea in eo sursum moveatur impetu scilicet accepto; sin autem velimus, ut descendat, angulum η negativum capi oportet. Hinc ergo erunt altitudines super plano horizontali

$$\mu = m\sin\eta,\ \ z = s\sin\eta \ \text{ et } \ v = (l+m)\sin\eta,$$

unde prima aequatio fit

$$\frac{ddm}{dt^2} + 2g\sin\eta = 0$$

hincque celeritas

$$\frac{dm}{dt} = 2g(f - t\sin\eta),$$

existente $2gf$ celeritate initiali. At porro prodit

$$m = g(2ft - tt\sin\eta),$$

siquidem initio vena occupaverit tubi portionem *AB*. Iam in loco quovis *z* ob

$$\mu(n-s) = m(m+l-s)\sin\eta \ \text{ et } \ v(s-m) = (m+l)(s-m)\sin\eta$$

erit pressio

$$p = k - z + s\sin\eta = k$$

ideoque semper et ubique pressioni atmosphaerae aequalis. Hinc patet venam aqueam in tubo perinde moveri ac corpus solidum sive ascendendo sive descendendo.

EXEMPLUM 2

38. *Si* (Fig. 40) *tubus ABCD fuerit recurvatus, ut duo habeat brachia verticalia AB et DC medio BC existente horizontali, in eoque vena aquea MBCN ita moveatur, ut eius termini M et N in brachiis verticalibus semper haereant, hunc motum determinare.*

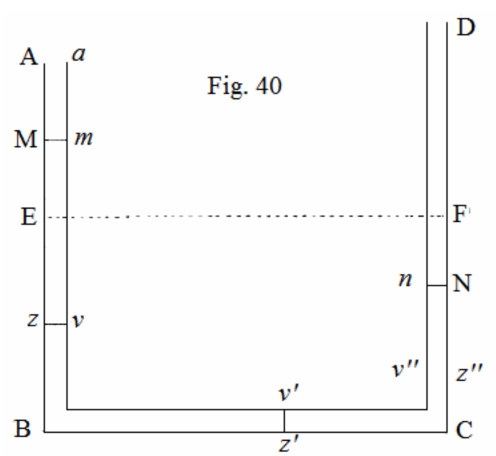


Fig. 40

Elapso tempore *t* occupet vena aquea spatium tubi *MBCN,* ut sit

$$MB + BC + CN = l,$$

seu ducta horizontali *EF* ita, ut sit

$$BE = CF = \tfrac{1}{2}\left(BM + CN\right),$$

erit $l = BC + 2BE$. Tum posito $AM = m$
erit $\mu = BM$ et $v = CN$, hinc
$v - \mu = CN - BM = -2ME.$
Statuatur $AE = e$ et $ME = x$, erit $m = e - x$, unde prima aequatio dat

$$-\tfrac{ddx}{dt^2} - \tfrac{4g}{l}x = 0 \text{ seu } ddx + \tfrac{4g}{l}xdt^2 = 0,$$

quae per $2dx$ multiplicata praebet integrando

$$dx^2 + \tfrac{4g}{l}xxdt^2 = \tfrac{4g}{l}aadt^2$$

hincque

$$2dt\sqrt{\tfrac{g}{l}} = \tfrac{dx}{\sqrt{(aa-xx)}} \text{ et } 2\left(t+b\right)\sqrt{\tfrac{g}{l}} = \text{Ang sin } \tfrac{x}{a},$$

ita ut iam sit

$$x = e - m = a\sin 2\left(t+b\right)\sqrt{\tfrac{g}{l}}$$

et celeritas in *M* deorsum tendens

$$= -\tfrac{2a\sqrt{g}}{\sqrt{l}}\cos 2\left(t+b\right)\sqrt{\tfrac{g}{l}}.$$

Cum ergo in *E* pervenit, erit eius celeritas $= \tfrac{2a\sqrt{g}}{\sqrt{l}}$, si tum fieri ponamus $2\left(t+b\right)\sqrt{\tfrac{g}{l}} = \pi$, unde cum motus initium, ubi lubuerit, constitui queat, faciamius $e = a$ sitque

$$m = AM = a\left(1 - \cos 2t\sqrt{\tfrac{g}{l}}\right),$$

ut sit

$$\tfrac{dm}{dt} = \tfrac{2a\sqrt{g}}{\sqrt{l}}\sin 2t\sqrt{\tfrac{g}{l}},$$

ita ut motus initio fuerit $m = 0$ seu $EM = EA$ et celeritas $= 0$. Vena aquea ergo motu oscillatorio feretur, cuius excursiones maximae supra et infra horizontalem *EF* erunt $= a$, et terminus *M* ab altitudine maxima ad minimam perveniet tempore $t = \tfrac{\pi\sqrt{l}}{2\sqrt{g}}$, sicque hic motus conveniet cum oscillationibus penduli simplicis, cuius longitudo $= \tfrac{1}{2}l = BE + \tfrac{1}{2}BC.$ His inventis in loco quocunque *z,* ut sit $Az = s$, pressio erit

$$p = k - Bz + \tfrac{BM\,(Bz+BC+CN)}{l} + \tfrac{CN\cdot Mz}{l}$$

seu

$$p = k + Mz - \tfrac{2ME\cdot Mz}{l},$$

ubi *Mz* in priori loco denotat profunditatem puncti *z* infra *M,* at in posteriori loco distantiam in tubo a termino *M.* Quare in brachio horizontali erit pressio
in z'

$$= k + BN - \tfrac{2ME(MB+Bz')}{l}$$

et
in z''

$$= k + BM - Cz'' - \tfrac{2ME(MB+BC+Cz'')}{l} = k + Nz'' + \tfrac{2ME \cdot Nz''}{l}.$$

COROLLARIUM 1

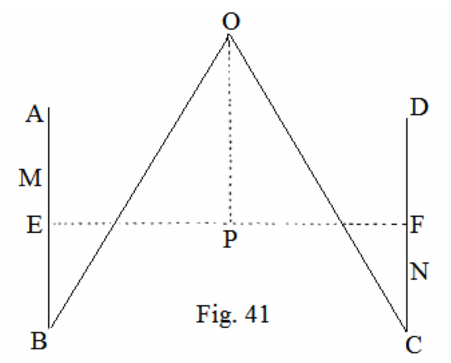


Fig. 41

39. Cum ergo fluidum inter oscillandum in horizontalem *EF* pertingit, ob $ME = 0$ pressio in *z* erit $= k + Ez$, denotante *Ez* profunditatem puncti *z* infra lineam *EF.* Hinc si tubi pars *BC* non sit recta, sed sursum inflexa, eius elevatio super *EF* maior esse nequit quam *k,* quia tum pressio ibi fieret negativa et fluidi continuitas solveretur.

COROLLARIUM 2

40. Si ergo tubus habeat figuram (Fig. 41) *ABOCD,* cuius brachia *AB* et *DC* sint verticalia, medium autem *BOC* sursum supra horizontalem *EF* inflexum aequaliter utrinque,
ut sit $EB + BO = \tfrac{1}{2}l$, pressio in summo puncto *O* erit

$$= k - OP + ME - \tfrac{2ME(OB+MB)}{l}$$

$$= k - OP + ME - \tfrac{ME(MB+BO)}{EB+BO}$$

$$= k - OP - \tfrac{ME \cdot ME}{EB+BO}.$$

Ne ergo continuitas fluidi solvatur, non sufficit, ut sit $OP < k$, sed oportet esse $OP < k - \tfrac{ME^2}{OB+MB}$.

EXEMPLUM 3

41. *Constet* (Fig. 42) *tubus aequaliter amplus duobus ramis rectis AB et BC ad horizontem EF utcunque inclinatis, ramus autem BC superne in C sit clausus et aëre vacuus, in hocque tubo moveatur vena aquae MBN datae longitudinis, eius motum definire.*

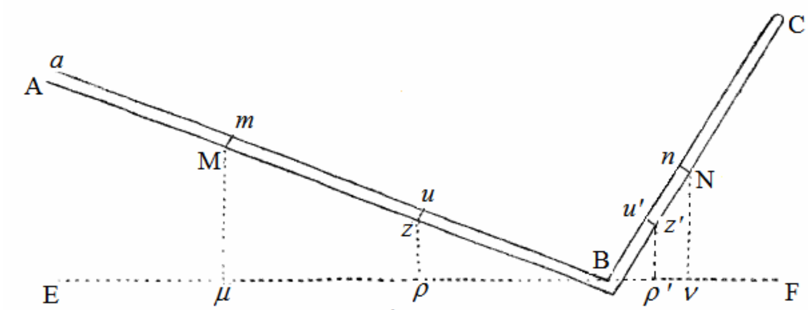


Fig. 42

Sit angulus $ABE=\varepsilon$ et angulus $CBF=\xi$, longitudo venae $MB+BN=l$ et $AM=m$; in M ergo aqua premitur ab atmosphaera, ut sit $M=k$, in N vero nulla est pressio, ut sit $N=0$, tum vero ex solutione problematis est $\mu=M\mu$ et $v=Nv$, unde densitate aquae posita $=1$ habetur haec aequatio

$$l\frac{ddm}{dt^2}=2gk-2g\left(Nv-M\mu\right),$$

ac pro tubi loco quocunque z erit pressio

$$p=\frac{k(l+m-Az)}{l}+\frac{M\mu(l+m-Az)}{l}+\frac{Nv(Az-m)}{l}-zp$$

seu

$$p=\frac{k(l-Mz)}{l}+\frac{M\mu(l-Mz)}{l}+\frac{Nv\cdot Mz}{l}-zp,$$

at pro puncto z' in altero ramo

$$p=\frac{k(l-BM-Bz')}{l}+\frac{M\mu(l-MB-Bz')}{l}+\frac{Nv(MB+Bz')}{l}-z'p',$$

Ad hunc calculum expediendum vocemus $BM=x$, ut sit $BN=l-x$, eritque

$$M\mu=x\sin\varepsilon \text{ et } Nv=(l-x)\sin\zeta;$$

unde aequatio differentialis erit

$$l\frac{ddx}{dt^2}+2g\left(k+x\sin\varepsilon-(l-x)\sin\zeta\right)=0,$$

quae per $2dx$ multiplicata et integrata praebet:

$$l\frac{dx^2}{dt^2}+2g\left(2kx+xx\sin\varepsilon+(l-x)^2\sin\zeta\right)=2gff.$$

Quare celeritas venae $\frac{dm}{dt} = -\frac{dx}{dt}$, si quidem eam versus *C* ferri ponamus, erit

$$-\frac{dx}{dt} = \sqrt{\frac{2g}{l}\left(ff - 2kx - xx\sin\varepsilon - (l-x)^2\sin\zeta\right)}$$

seu

$$-\frac{dx}{dt} = \sqrt{\frac{2g}{l}\left(ff - ll\sin\zeta - 2kx + 2lx\sin\zeta - xx(\sin\varepsilon + \sin\zeta)\right)},$$

unde intelligimus celeritatem evanescere, cum fuerit

$$x = \frac{-k + l\sin\zeta \pm \sqrt{\left(kk - 2kl\sin\zeta - ll\sin\varepsilon\sin\zeta + ff(\sin\varepsilon + \sin\zeta)\right)}}{\sin\varepsilon + \sin\zeta},$$

maxima autem fiet, ubi

$$x = \frac{-k + l\sin\zeta}{\sin\varepsilon + \sin\zeta};$$

haecque celeritas maxima erit

$$= \sqrt{\frac{2g}{l}\left(ff + \frac{kk - 2kl\sin\zeta - ll\sin\varepsilon\sin\zeta}{\sin\varepsilon + \sin\zeta}\right)}.$$

Pro tempore vero habebimus:

$$dt\sqrt{\frac{2g}{l}} = \frac{-dx}{\sqrt{\left(ff - ll\sin\zeta - 2kx + 2lx\sin\zeta - xx(\sin\varepsilon + \sin\zeta)\right)}},$$

unde integrando colligimus:

$$x = \frac{-k + l\sin\zeta + \cos\lambda t\sqrt{\left(ff(\sin\varepsilon + \sin\zeta) + kk - 2kl\sin\zeta - ll\sin\varepsilon\sin\zeta +\right)}}{\sin\varepsilon + \sin\zeta},$$

with there being $\lambda = \sqrt{\frac{2g}{l}(\sin\varepsilon + \sin\zeta)}$.
Quodsi iam pro tubi puncto z ponamus $Bz = z$, erit pressio ibidem

$$p = \frac{(k + x\sin\varepsilon)(l - x + z)}{l} + \frac{(l-x)\sin\zeta}{l}(x - z) - z\sin\varepsilon.$$

At pro puncto z' in altero ramo *BC* ponendo $B\,z' = z'$ erit

$$p' = \frac{(k + x\sin\varepsilon)(l - x - z')}{l} + \frac{(l-x)\sin\zeta}{l}(x + z') - z'\sin\zeta.$$

Illo casu erit succinctius

$$p = \frac{(k+x(\sin\varepsilon+\sin\zeta))(l-x+z)}{l} - z(\sin\varepsilon+\sin\zeta),$$

hoc vero

$$p' = \frac{(k+x(\sin\varepsilon+\sin\zeta))(l-x-z')}{l}.$$

COROLLARIUM 1

42. Sit (Fig. 43) tubi brachium *AB* horizontale et *BC* verticale hincque $\varepsilon = 0$ et $\zeta = 90°$. Unde fit celeritas in *M*

$$=\sqrt{\tfrac{2g}{l}(ff - ll - 2kx + 2lx - xx)}.$$

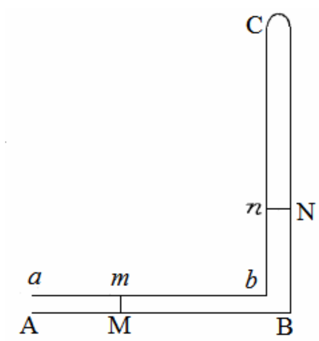


Fig. 43

Ponamus initio totam venam tubum horizontalem *AB* occupasse ibique quievisse, ut sit $AB = l$, necesse ergo est, ut posito $BM = x = l$ celeritas evanescat sumique debeat $ff = 2kl$; unde cum vena in situm *MBN* pervenerit, erit celeritas

$$=\sqrt{\tfrac{2g}{l}(l-x)(2k-l+x)};$$

et quando tota vena in tubum verticalem pervenit, quod fit, si $x = 0$, eius celeritas qua ascendere perget, erit adhuc $=\sqrt{\tfrac{2g}{l}l(2k-l)}$. Dum ergo longitudo venae minor sit quam *2k,* tota vena in tubum verticalem ascendit, siquidem fuerit altior quam 2k.

COROLLARIUM 2

43. Sumto autem $ff = 2kl$ et $\lambda = \sqrt{\tfrac{2g}{l}}$, aequatio bis integrata fit:

$$x = -k + l + k\ cos(\lambda t+\gamma).$$

Unde cum initio fuerit $x = l$, angulus constans γ evanescit, ut sit

$$x = l - k(1-\cos\lambda t).$$

Tempus ergo, quo tota vena in tubum verticalem intrat, hinc definiri debet

$$1-\cos\lambda t = \tfrac{l}{k} \text{ seu } \lambda t = \text{Ang cos } \left(1-\tfrac{l}{k}\right).$$

Quare si $l = k$, erit $t = \frac{\pi}{2\lambda} = \frac{\pi\sqrt{l}}{2\sqrt{2g}}$, sin autem sit $l = 2k$, fit $t = \frac{\pi\sqrt{l}}{\sqrt{2g}}$

SCHOLION

44. Nihil impedit (Fig. 44), quominus pro aqua mercurium substituamus, ac tum k erit altitudo mercurii in barometro; atque hinc oscillationes mercurii in barometro definire poterimus, si ipsi infra K adiunctus sit tubus horizontalis BA eiusdem amplitudinis. Ponamus ergo in statu aequilibrii altitudinem $BK = k$ et $BE = e$, ut sit tota vena mercurialis $l = e+k$. Facta iam quadam agitatione, sit vena in statu MBN existente $BM = x$ et $EM = x-e$, atque celeritas mercurii in tubo ascendentis erit:

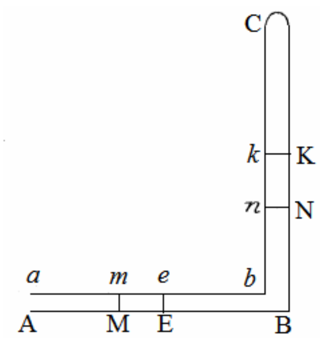


Fig. 44

$$-\frac{dx}{dt} = \sqrt{\frac{2g}{l}\left(ff - ll - 2kx + 2lx + xx\right)}$$

quam posito $x = a = BA$ evanuisse ponamus, ita ut statui debeat $ff = (l-a)^2 + 2ak$, quo facto erit celeritas

$$= \sqrt{\frac{2g}{l}(a-x)(2k-2l+a+x)}\,.$$

Statuamus $EA = c$, $EM = y$, ut sit $a = c+e$, $x = e+y$, et ob $l = e+k$ erit haec celeritas:

$$-\frac{dy}{dt} = \sqrt{\frac{2g}{e+k}(c-y)(c+y)} = \sqrt{\frac{2g(cc-yy)}{(e+k)}},$$

unde colligimus

$$\frac{dt\sqrt{2g}}{\sqrt{(e+k)}} = -\frac{dy}{\sqrt{(cc-yy)}}$$

et integrando

$$\frac{t\sqrt{2g}}{\sqrt{(e+k)}} = \text{Ang}\cos\frac{y}{c} \quad \text{seu} \quad y = c\cos\frac{t\sqrt{2g}}{\sqrt{(e+k)}}.$$

Quare mercurius in barometro circa statum aequilibrii Kk oscillationes peraget, tempore cuiusque existente $\frac{\pi\sqrt{(e+k)}}{\sqrt{2g}}$, seu eae erunt isochronae pendulo, cuius longitudo est $= e+k$.

PROBLEMA 50

45. *Si* (Fig. 45) *aqua in tubo aequaliter amplo ita moveatur, ut in altero termino effluat, in altero vero continuo succedente prematur a vi quacunque, hunc motum effluxus et pressionem in singulis elementis aquae determinare.*

SOLUTIO

Quamcunque tubus habuerit figuram, is tanquam in directum extensus *AaOo* consideretur; cui adiungatur scala altitudinum $\alpha\omega$, cuius applicatae $z\pi$ exhibent cuiusque tubi puncti z altitudinem super dato plano horizontali.

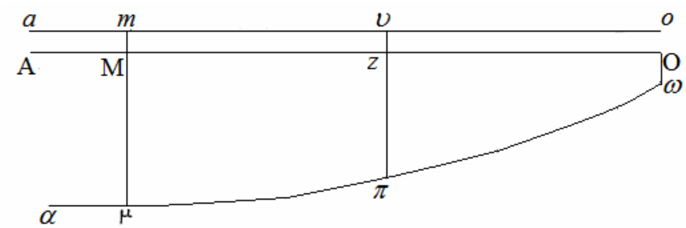


Fig. 45

Iam elapso tempore t aqua effluat per tubi orificium *Oo* celeritate $\mathfrak{T}$, qua simul tota fluidi massa, quae adhuc est in tubo, succedat. Occupet autem iam aqua tubi partem *MO* et in *Mm* urgeatur vi exprimenda per altitudinem $= M$. In O autem, ubi aqua effluit in aërem, alia pressio locum habete nequit, nisi atmosphaerae, quae aequivalens statuatur columnae aqueae altitudinis $= k$. Pertigerit initio aquaus que ad A et ponatur longitudo $AO = a$, atque nunc sit $AM = m$, functio temporis t, ex qua definitur celeritas $\mathfrak{T} = \frac{dm}{dt}$.
Iam quodvis aquae elementum in z consideretur, cuius altitudo supra planum horizontale sit $z\pi = z$, et posita densitate aquae $= 1$ et pressione in $z = p$, tum vero distantia $Az = s$, ex problemate 48 hanc nanciscimur aequationem

$$2gp = 2g(h-z) - s\ \Gamma' : t + \Delta : t,$$

ubi est

$$\Gamma : t = \mathfrak{T} = \tfrac{dm}{dt} \quad \text{ideoque} \quad \Gamma' : t = \tfrac{ddm}{dt^2};$$

est enim distantia $AM = m$ et celeritas $\mathfrak{T}$ functio temporis t tantum. Transferamus nunc primo punctum z in M, ubi cum pressio sit data $= M$, ob $s = m$ habebimus:

$$2gM = 2g(h - M\mu) - m\Gamma' : t + \Delta : t;$$

deinde transferamus punctum z in orificium O ponendo $s = a$, ubi cum pressio pariter sit cognita $= k$, erit

$$2gk = 2g(h - O\omega) - a\Gamma' : t + \Delta : t.$$

Ex his aequationibus colligimus primo

$$2g(M-k) = 2g(O\omega - M\mu) + (a-m)\Gamma' : t$$

ideoque $\Gamma' : t = \frac{ddm}{dt^2} = \frac{2g(M-k+M\mu-O\omega)}{a-m}$, tum vero, cum sit

$$2g(M-p)=2g(z-M\mu)+(s-m)\Gamma':t,$$

erit

$$M-p=z-M\mu+\tfrac{(s-m)(M-k+M\mu-Ow)}{a-m}$$

seu

$$p=\tfrac{M(a-s)+k(s-m)+M\mu(a-s)+O\omega(s-m)}{a-m}-z$$

vel

$$p=\tfrac{(M+M\mu)(a-s)+(k+O\omega)(s-m)}{a-m}-z$$

vel etiam

$$p=\tfrac{Oz(M+M\mu)+Mz(k+O\omega)}{MO}-z.$$

Totum ergo negotium ab illa aequatione differentio-differentiali pendet.

COROLLARIUM 1

46. Si pressio in M fuerit vel constans vel a spatio $AM=m$ pendens, quoniam altitudo $M\mu$ ab eodem pendet et $O\omega$ est constans, aequatio differentio-differentialis per $2dm$ multiplicata fit integrabilis reddens:

$$\tfrac{dm^2}{dt^2}=4g\int\tfrac{M-k+M\mu-O\omega}{a-m}dm,$$

qua forma quadratum celeritatis exprimitur.

COROLLARIUM 2

47. Si effluxus fieret in spatium ab aëre vacuum, perspicuum est in nostris formis scribi debere ; ac si in M aqua nullam aliam vim praeter pressionem atmosphaerae sustineat, erit $M=k$. Quare, si aqua utrinque aëri pateat, erit

$$M=k \quad \text{et} \quad \tfrac{dm^2}{dt^2}=4g\int\tfrac{M\mu-O\omega}{a-m}dm$$

et pro pressione in z fiet

$$p=k-z+\tfrac{Oz\cdot M\mu+Mz\cdot O\omega}{MO}.$$

EXEMPLUM 1

48. *Sit* (Fig. 46) *tubus rectus AO utcunque inclinatus ad*

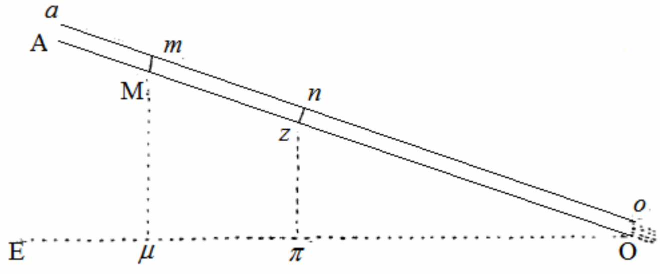


Fig. 46

horizontem, qui initio ab O ad A usque fuerit aqua plenus, indeque per orificium Oo effluat, hunc motum determinare.

Posito angulo $AOE=\varepsilon$, ob $MO=a-m$ erit altitudo $M\mu=(a-m)\sin\varepsilon$ et $O\omega=0$. Quare cum sit $M=k$, fiet

$$\tfrac{dm^2}{dt^2}=4g\int dm\sin\varepsilon = 4gm\sin\varepsilon,$$

quia facto $AM=m=0$ motus a quiete incepisse ponitur. Hinc igitur porro fit

$$\tfrac{dm}{\sqrt{m}}=2dt\sqrt{g}\ \sin\varepsilon \text{ et integrando } \sqrt{m}=t\sqrt{g}\ \sin\varepsilon;$$

unde concludimus aquam omnem e tubo effluxuram esse tempore $=\frac{\sqrt{a}}{\sqrt{g}\ \sin\varepsilon}$;
quod tempus convenit cum eo, quo corpus grave super plano inclinato *AO* esset descensurum.

In puncto *z* vero est pressio $p=k-z\pi+\frac{Oz\cdot M\mu}{MO}$, cum autem sit $MO:M\mu=Oz:z\pi$, fit $p=k$, seu per totam venam *MO* pressio est eadem scilicet atmosphaerae, quae vulgo nulla reputatur.

EXEMPLUM 2

49. *Si ut ante ex tubo inclinato recto AO aqua non in aërem, sed in spatium vacuum effluat, hunc motum determinare.*

Manente angulo $AOE=\varepsilon$, et $AM=m$, $AC=a$, est

$$M\mu=(a-m)\ sin\,\varepsilon \text{ et } O\omega=0,$$

tum vero $M=k$ et quod ante est *k* hic est $=0$, sicque habebimus:

$$\tfrac{dm^2}{dt^2}=4g\int\tfrac{k+(a-m)\sin\varepsilon}{a-m}dm=4gkl\tfrac{a}{a-m}+4gm\sin\varepsilon,$$

ut scilicet posito $m=0$ motus a quiete inceperit: ex quo sequitur facto $m=a$ extremam guttulam celeritate infinita expulsum iri, quod non adeo absurdum est putandum, cum de ultimo quasi strato infinite tenui intelligi debeat, cui statim atque minima crassities tribuitur, celeritas admodum fit modica. Ipsum autem tempus hinc nonnisi appropinquando definiri potest, cum sit

$$2t\sqrt{g}=\int\frac{dm}{\sqrt{\left(kl\frac{a}{a-m}+m\sin\varepsilon\right)}}$$

neque ullo casu sive inclinatio *s* evanescat, sive in angulum rectum abeat.

Deinde vero sumto $Az = s$ erit pressio

$$p = \frac{k \cdot Oz}{MO} + \frac{Oz \cdot M\mu}{MO} - z\pi = \frac{k(a-s)}{a-m}.$$

Pro casu autem, quo tubus AO situm tenet horizontalem et $\varepsilon = 0$, si ponamus $l\frac{a}{a-m} = \frac{x}{a}$, fit $m = a\left(1 - e^{-\frac{x}{a}}\right)$ hincque

$$\frac{2t\sqrt{gk}}{\sqrt{a}} = \int e^{-\frac{x}{a}} \frac{dx}{\sqrt{x}}\ ,$$

unde approximando colligitur:

$$\frac{t\sqrt{gk}}{\sqrt{ax}} = e^{-\frac{x}{a}}\left(1 + \frac{2}{3}\cdot\frac{x}{a} + \frac{2}{3}\cdot\frac{2}{5}\cdot\frac{xx}{aa} + \frac{2}{3}\cdot\frac{2}{5}\cdot\frac{2}{7}\cdot\frac{x^3}{a^3} + \text{etc.}\right)\ ,$$

pro motus ergo initio, ubi x valde parvum et

$$m = x - \frac{xx}{2a} + \frac{x^3}{6aa} - \text{etc.},$$

fit

$$\frac{t\sqrt{gk}}{\sqrt{ax}} = \left(1 - \frac{x}{a} + \frac{x^2}{2aa} - \text{etc.}\right)\left(1 + \frac{2}{3}\cdot\frac{x}{a} + \frac{2}{3}\cdot\frac{2}{5}\cdot\frac{xx}{aa} + \text{etc.}\right)\ ,$$

seu

$$t = \frac{\sqrt{ax}}{\sqrt{gk}}\left(1 - \frac{1}{3}\cdot\frac{x}{a} + \frac{1}{10}\cdot\frac{xx}{aa} + \text{etc.}\right).\ t$$

EXEMPLUM 3

50. *Constet* (Fig. 47) *tubus aequaliter amplus ABO duobus brachiis rectis, altero horizontali a AB, altero verticali BO deorsum verso, qui cum initio fuisset plenus, aqua per orificium Oo effluere coeperit, eius motum determinare et pressionem in singulis locis.*

Sit longitudo brachii horizontalis $AB = b$ et verticalis $BO = c$ ideoque $a = b + c$. *Cum* igitur tempore t aqua ex A in M profluxerit existente $AM = m$, ob pressionem in $M = k$, perinde ac in O, erit celeritatis in M quadratum:

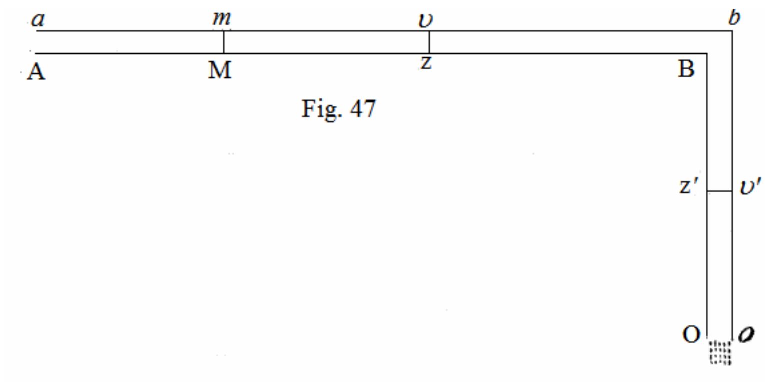


Fig. 47

$$\frac{dm^2}{dt^2} = 4g\int\frac{cdm}{a-m} = 4gcl\frac{a}{a-m},$$

unde fit

$$2dt\sqrt{gc} = \frac{dm}{\sqrt{l\frac{a}{a-m}}},$$

ubi eadem occurrit difficultas integrationis atque in exemplo praecedente. Pressio autem in puncto quovis z tubi horizontalis erit

$$p = k - c + \frac{(c+Bz)c}{a-m} = k - \frac{c\cdot Mz}{a-m} = k - \frac{BO\cdot Mz}{BM+BO},$$

sicque in angulo B erit pressio $= k - \frac{BO\cdot Mz}{BM+BO}$ minima. At sumpto z' in tubo ,verticali pressio ibi erit

$$p' = k - Oz' + \frac{Oz'\cdot BO}{BM+BO} = k - \frac{Oz'\cdot BM}{BM+BO}.$$

Motus autem initium in A respondet vi acceleratrici $= \frac{c}{a}$ gravitate per unitatem expressa.

COROLLARIUM 1

51. Initio ergo motus, dum ambo tubi erant pleni, pressio in B est omnium minima; atque adeo negativa fieri potest, si uterque ramus maior quam k; quod si evenerit, continuitas in B rumpitur et aqua per tubum verticalem celerius descendit, quam reliqua per tubum horizontalem sequi potest.

EXEMPLUM 4

52. *Sit* (Fig. 48) *tubus* rectus *verticalis supra in A hermetice clausus infra apertus, at altior pressione atmosphaerae k, qui si initio fuerit plenus, descensum fluidi definire.*

Tempore t descenderit fluidum per $AM = m$, existente altitudine $AO = a > k$, et quia supra Mm erit vacuum, fiet $M = 0$, unde celeritatis descensus in M quadratum fit

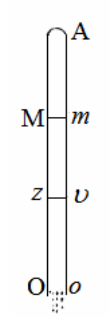


Fig. 48

$$\frac{dm^2}{dt^2} = 4g\int \frac{-k+a-m}{a-m}dm = 4g\left(m - kl\frac{a}{a-m}\right).$$

Quare cum celeritas initio, ubi $m = 0$, fuerit nulla, ea maxima fiet, ubi $m = a - k$ seu $OM = k$, quo casu erit

$$\frac{dm}{dt} = 2\sqrt{g\left(a - k - kl\frac{a}{k}\right)},$$

dehinc vero iterum decrescet et evanescet, ubi fit $kl\frac{a}{a-m} = m.$. Ponamus altitudinem a valde parum excedere k esseque $a = k + \omega$, et celeritas maxima fiet

$$= 2\sqrt{g\left(\omega - kl\left(1+\frac{\omega}{k}\right)\right)} = 2\omega\sqrt{\frac{g}{2k}}$$

respondens spatio $AM = \omega$. Iterum autmn celeritas evanescet, ubi erit

$$k\left(\frac{m}{a} + \frac{mm}{2aa}\right) = m \text{ seu } m = 2\omega,$$

propius vero reperitur:

$$m = 2\omega - \frac{2\omega^2}{3k} + \frac{4\omega^3}{9kk} - \frac{44\omega^4}{135k^2} + \text{etc.}$$

Pressio tandem in quovis loco z erit

$$p = \frac{Oz\cdot MO + Mz\cdot k}{MO} - Oz = \frac{k\cdot Mz}{MO}.$$

PROBLEMA 51

53. *Si aqua* (Fig. 49) *in tubo aequaliter amplo ita moveatur, ut in altera eius termino Oo effluat, in altera vero Aa continue aliunde affluat data vi propulsa, hunc aquae per tubum propulsae motum definire.*

SOLUTIO

Tubum igitur utcunque curvatum AO hic considero, in cuius orificium Aa aqua continuo intruditur vi quacunque per altitudinem L expressa, quae vel ut constans vel

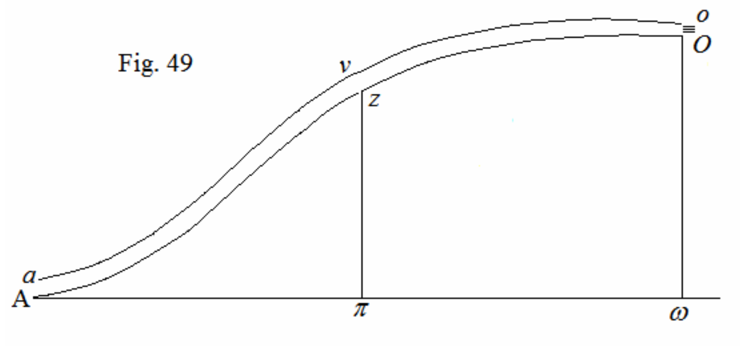


Fig. 49

functio temporis *t* spectari potest, cuiusmodi continua aquae intrusio et propulsio ope antliarum effici solet, quarum vi ex loco inferiori *A* in al*w* tiorem *O* elevatur ibique effunditur. Quamobrem tubi terminum *A* in imo loco positum sumo, a quo ducta horizontali *Aw,* singulorum punctorum tubi *z* altitudines super ea aestimo, ita ut supremi orificii *Oo,* ubi aqua expellitur, altitudo sit *Oω.* Posita ergo pro puncto quovis *z* longitudine tubi $Az = s$ et altitudine $\pi z = z$, sit elapso tempore $= t$ celeritas aquae in tubo $= \mathfrak{T}$, qua simul in *Oo* effluit, et quae est functio temporis *t,* quam in problemate 48 posui $\mathfrak{T} = \Gamma : t$, tum vero denotante *p* pressionem in *z,* quam etiam ad aquam referamus, ut sit $b = 1$, hanc invenimus aequationem

$$2gp = 2g\left(h - z\right) - s\Gamma' : t + \Delta : t,$$

dum scilicet sumimus aquam in tubo a termino *A* ad terminum *O pro*gredi, atque in hac aequatione universa motus ratio continetur. Eam ergo ad casum oblatum accommodari oportet has conditiones implendo, ut et in *A* sit pressio data $= L$ et in $O = k$, denotante *k* altitudinem columnae aqueae atmosphaerae aequiponderantis. Punctum *z* indefinitum primum ad orificium *A* transferamus, quo fit $s = 0$, $z = 0$ et $p = L$, ideoque

$$2gL = 2gh + \Delta : t,$$

Deinde eodem ad orificium *O* translato, ubi fit $s = AO$, $z = O\omega$ et $p = k$, habebimus:

$$2gk = 2g(h - O\omega) - AO \cdot \Gamma' : t + \Delta : t,$$

unde colligimus

$$2g\left(L - k\right) = 2g \cdot O\omega + AO \cdot \Gamma' : t,$$

ita ut sit:

$$\Gamma' : t = \frac{d\mathfrak{T}}{dt} = \frac{2g\left(L - k - O\omega\right)}{AO}.$$

Tum vero pro pressione in loco indefinito fiet

$$2g(L - p) = 2gz + \frac{2gs\left(L - k - O\omega\right)}{AO}$$

seu

$$p = L - z - \frac{s\left(L - k - O\omega\right)}{AO}.$$

Statuamus nunc totam tubi longitudinem $AzO = l$ et orificii *Oo* altitudinem $O\omega = a$, ac primo pro celeritate $\mathfrak{T}$ obtinuimus

$$\frac{d\mathfrak{T}}{dt} = \frac{2g(L-k-a)}{l},$$

sicque integrando

$$\mathfrak{T} = \frac{2g}{l}\left(\int Ldt - (a+k)t\right),$$

deinde vero pro pressione in quovis loco tubi *z* erit

$$p = L - z - \frac{s(L-k-a)}{l}.$$

COROLLARIUM 1

54. Quodsi ergo vis propellens *L* fuerit constans et $= a+k$, acceleratio aquae in tubo evanescit ideoque eius fluxus per tubum erit uniformis, quanta autem sit futura eius celeritas, ex his principiis non definitur, sed ex natura virium impellentium concludi debet.

COROLLARIUM 2

55. Sin autem vis propellens *L* perpetuo maior esset quam $a + k$, aquae per tubum propulsae celeritas continuo augeretur, sin autem minor esset, continuo diminueretur. Neque ergo hinc quicquam certi circa aquae celeritatem dato tempore effusam statui potest.

SCHOLION 1

56. Quantumvis hoc paradoxum atque adeo experientiae contrarium videatur, tamen hypothesi, qua statuimus pressionem in *A* perpetuo eadem vi aquam propulsare, quaecunque fuerit eius celeritas, prorsus est consentanea, ac si tales vires applicare liceret, nullum est dubium, quin etiam hic effectus revera sit secuturus. Quare cum hoc in praxi minus eveniat, iudicandum est vires, quae ad aquam propulsandam adhiberi solent, neutiquam eius esse indolis, ut eadem pressione agant, quacunque celeritate aqua progrediatur. Satis autem superque constat omnes vires, quae ab hominibus, animalibus, aquae fluxu et vento peti solent, ita esse comparatas, ut aucta celeritate debilitentur ac tandem evanescant. Quantacunque enim sit huiusmodi vis obiecto quiescenti applicata, statim atque hoc obiectum movetur, ea minor evadit, quare tales vires non absolute definire licet, sed earum quantitas pro quovis celeritatis gradu, quo agunt, seorsim debet determinari. Ita si ponamus machinae, qua aqua per tubum propellitur, eiusmodi vim esse applicatam, quae, dum celeritate $= c$ operatur, aequalis sit ponderi aquae, cuius volumen sit $= V$: atque machinam ita esse instructam, ut perpetuo hac celeritate $= c$ agat, id quod semper ope rotarum fieri potest. Cum iam in nostro casu pressio in *A* altitudine $= L$ exprimatur, si amplitudinem tubi statuamus $= \omega$, aequabitur ea ponderi voluminis aquae $= L\omega$, quae ut a vi illa *V* celeritate *c* mota producatur, illius celeritas hinc determinatur,

scilicet si vim $L\omega$ celeritate $\mathfrak{T}$ aquam propellere sumamus, oportet sit $L\omega\mathfrak{T} = Vc$, hincque $\mathfrak{T} = \frac{Vc}{L\omega}$. Ut autem aqua hoc motu uniformiter propellatur, vidimus esse debere $L = a+k$, ubi quidem pressionem atmosphaerae k omittere possumus, qui eadem quoque vim in A comitatur, ita ut sufficiat statui $L = a$, ex quo perspicuum est aquam per tubum propulsum iri celeritate $\mathfrak{T} = \frac{Vc}{a\omega}$.

SCHOLION 2

57. Cum hic non vis principalis sollicitans sola *V,* sed in celeritatem *c,* qua agit, ducta in computum ingrediatur, hoc productum *Vc,* quod in omnium machinarum effectu determinando maxime debet spectari, peculiarem denominationem meretur, et propterea *actio* a me est vocatum, ita ut *actio* sit productum cuiusque vis per celeritatem, qua agit, multiplicata, ubi imprimis est observandum, dum in machinis vires vel intenduntur vel minuuntur, celeritatem semper in ratione inversa mutari, ut *actio* eadem maneat. Sic si per machinam vis principalis *V* in alium locum translata abeat in *V',* celeritas, qua haec operatur, erit $= \frac{Vc}{V'}$, ac si tum celeritas actionis sit $= c'$, vis erit $V' = \frac{Vc}{c'}$. Machinarum scilicet usus praecipuus in hoc consistit, ut servata eadem *actione* vis sollicitantis, vel vis vel celeritas ad lubitum immutetur. Ita in casu problematis, quo opus erat vi $= a\omega$ ad aquam per tubum *AO* propellendam, si vis principalis machinam movens sit *V* cum celeritate $= c$ coniuncta, machinam ita instructam esse oportet, ut in translatione vis ad locum *A,* ubi aqua in tubum intruditur, vis fiat $= a\omega$, et quia tum eius celeritas necessario fit $\mathfrak{T} = \frac{Vc}{a\omega}$, hinc celeritas aquae per tubum propulsae sponte determinatur. Si forte ob machinae structuram vis urgens in *A,* quam posuimus $= L\omega$, maior extaret quam $a\omega$, celeritas actionis in eadem ratione imminueretur, verum ob $L > a$ motus aquae accelleraretur: tum ergo vis principalis maiorem obtineret celeritatem, hincque eius quantitas ipsa *V* diminutionem pateretur, ex quo, prout eius *actio* increscat vel decrescat, deinceps, cum motus ad uniformitatem fuerit perductus, celeritas aquae per tubum propulsae definiri debet.

SCHOLION 3

58. Omnium autem virium, quae ad machinas agitandas adhiberi solent, ratio ita est comparata, ut, dum obiectum quiescens urgent celeritateque propterea nulla agunt, maximam vim exerant, quae sit $= F$, tum vero aucta celeritate continuo minorem exerant vim, tandemque plane nullam, cum certa celeritate, quae sit $= e$, agere debeant. Quia ergo illo casu celeritas, hoc vero vis evanescit, utroque *actio* est nulla. Si iam celeritate quacunque minore quam *e,* quae sit $= u$, eadem vis agat, eius quantitas aestimari potest $= F\left(1-\frac{u}{e}\right)^2$, cuius ergo *actio* est $= Fu\left(1-\frac{u}{e}\right)^2$, quae utique tam casu $u = 0$ quam $u - e$ evanescit, maxima ergo evadit, si $u = \frac{1}{3}e$, ac tum erit $= \frac{4}{27}Fe$. Quare semper machinas ita instrui conveniet, ut virium, quae adhibentur, *actio* reddatur maxima, quae regula nisi observetur, machina multo minorem effectum praestabit, quam ab iisdem viribus agitata, si debite instrueretur, obtineri posset. Tali ergo vi adhibita problema praecedens ad solutionem determinatam revocemus.

PROBLEMA 52

59. *Si in casu praecedentis problematis aqua in tubum AO intrudatur a potentia, quae in quiete exerat vim* $=F$ *, mota autem celeritate* $=e$ *omni vi destituatur, definire, quomodo machina ad hanc vim sit accommodanda, ut effectus maximus reddatur seu, maxima aquae copia dato tempore eliciatur.*

SOLUTIO

Ponamus hanc potentiam machinae applicatam celeritate $=u$ operari, ut sit vis, quam exerat, $F=\left(1-\frac{u}{e}\right)^2$, machinam autem ita esse instructam, ut ad aquam per tubum propulsandam ea vis in ratione 1: *n* multiplicetur, ibi igitur agat celeritate $=\frac{u}{n}$ qua propterea aqua iam per tubum promoveatur, undecunque ipsi hic motus sit impressus, quandoquidem hic ad motus continuationem spectamus. Erit ergo nunc $\mathfrak{T}=\frac{u}{n}$, et posita tubi amplitudine $=\omega$, vis aquam in tubo propellens $nF\left(1-\frac{u}{e}\right)^2=L\omega$, ita ut sit

$$L=\frac{NF}{\omega}\left(1-\frac{u}{e}\right)^2.$$

Quare cum invenerimus $\frac{d\mathfrak{T}}{dt}=\frac{2g(L-a)}{l}$, ubi pressionem atmospaerae *k* in orificio *Oo* omittimus, quia pari pressione ipsa vis propellens adiuvatur, iam sive it $L>a$ sive $L<a$, utroque casu motus mox ita ad uniformitatem perducetur, ut fiat $L=a$ ideoque $1-\frac{u}{e}=\sqrt{\frac{a\omega}{nF}}$: sicque a potentia ita applicata, uti assumimus, ob $u=e\left(1-\sqrt{\frac{a\omega}{nF}}\right)$ aqua per tubum propelletur celeritate $\mathfrak{T}=\frac{e}{n}\left(1-\sqrt{\frac{a\omega}{nF}}\right)$, ita ut singulis minutis secundis aquae volumen $=\mathfrak{T}\omega$ per orificium eiiciatur. Hic primo patet, sit fuerit $\frac{a\omega}{nF}>1$ seu $nF<a\omega$, nullum plane motum produci posse. Maximus autem effectus obtinebitur, si $u=\frac{1}{3}e$, hincque $\frac{4}{9}=\frac{a\omega}{nF}$, unde machina ita instrui debet, ut fiat $n=\frac{9a\omega}{4F}$, tum vero $\mathfrak{T}=\frac{4}{27}\cdot\frac{Fe}{a\omega}$, et quantitas aquae uno minuto secundo eiectae $=\frac{4}{27}\cdot\frac{Fe}{a}$, ubi vis *F* ad pondus reducta per volumen massae aqueae aequilibrantis exprimi debet, ita ut *F* denotet certum volumen.

COROLLARIUM 1

60. Si ergo tam altitudo *a,* ad quam aqua debet elevari, quam celeritas *e* seu spatium ea percurrendum uno minuto secundo in pedibus, volumen *F* vero in pedibus cubicis exprimatur, tum formula $\frac{4}{27}\cdot\frac{Fe}{a}$, dabit volumen aquae itidem in pedibus cubicis expressum, quod singulis minutis secundis ad altitudinem *a* pedum elevari poterit.

COROLLARIUM 2

61. A potentia ergo, quae in quiete exerit vim $=F$, celeritate autem motus $=e$ omnem vim amittit, maior aquae copia ad altitudinem a elevari nequit, quam $\frac{4}{27}\cdot\frac{Fe}{a}$. Neque vero hic effectus obtinebitur, nisi machina ita sit instructa, ut vis movens ei applicata in translatione ad aquam propellendam augeatur in ratione $1:n=1:\frac{9a\omega}{4F}$.

SCHOLION 1

62. Quo haec clarius perspiciantur, ponamus vi hominis esse utendum, quae in quiete aestimetur 70 librarum seu unius pedis cubici aquae, ut sit $F=1$; maximam autem celeritatem, qua nullam amplius vim exerere valeat, esse $7\frac{1}{2}$ pedum seu $e=7\frac{1}{2}$. Hic ergo homo, si eius opera modo maxime lucroso impendatur, singulis minutis secundis ad altitudinem a pedum elevare poterit volumen aquae $=\frac{10}{9a}$ ped. cub. hocque fit, si machina ita sit instructa, ut operari possit celeritate $=2\frac{1}{2}$ ped.; ac tum eius actio est $=\frac{4}{27}Fe=\frac{10}{9}$, ita ut semper actio hoc modo expressa, si per altitudinem a dividatur, praebeat quantitatem aquae singulis minutis secundis elevandae. In machinae autem constructione insuper ad amplitudinem tubi ω est spectandum, quoniam vis movens per translationem augeri debet in ratione $1:\frac{9a\omega}{4F}$; quae ratio contra non a celeritate $=e$ pendet. Deinde cum unius hominis actio maxima sit $=\frac{10}{9}$, si λ homines operi admoveantur, eorum actio erit $=\frac{10}{9}\lambda$, cui semper effectus est proportionalis. Si equis sit utendum et in quiete unius equi vis triplo maior censeatur quam hominis celeritasque maxima etiam triplo maior, eius actio novies fiet maior, seu unus equus tantum praestare valebit, quantum novem homines.

SCHOLION 2

63. Si cursu fluminis ad machinam agitandam uti velimus, cuius impulsu palmulae rotae ad motum incitentur, determinatio effectus in aqua elevanda hoc modo institui debet. Sit ff superficies, quae aquae impulsum normaliter excipiat, et e denotet celeritatem fluminis, unde altitudo, ex qua grave eandem celeritatem lapsu acquirit, erit $=\frac{ee}{4g}$: vis ergo fluminis in hanc superficiem quietam erit $=\frac{eeff}{4g}$, ponderi scilicet tanti voluminis aquae, quam loco litterae F scribi oportet; tum vero, quia impulsus evanescit statim ac palmula ipsa fluminis celeritate $=e$ movetur, haec est illa celeritas, quam ante littera e notavimus. Actio ergo maxima evadet, cum superficies ff celeritate $=\frac{1}{3}e$ movetur, eritque haec actio $=\frac{e^3ff}{27g}$ ideoque cubo celeritatis fluminis proportionalis. Hac itaque actione ad altitudinem $=a$ singulis minutis secundis elevabitur aquae quantitas $=\frac{e^3ff}{27ga}$; cum ergo ab uno homine elevetur quantitas $=\frac{10}{9a}$ ped. cub., effectus aquae aequivalebit λ

hominibus existente $\lambda = \frac{e^3 ff}{30g}$, dum *e* et *f* in pedibus exprimuntur, ubi notandum est esse $g = 15\frac{1}{2}$ ped. ideoque $\lambda = \frac{e^3 ff}{465}$. Quodsi $ff = 1$ ped. quadr. et fluvius conficiat spatium $7\frac{1}{2}$ ped. uno minuto secundo, unus homo eundem effectum producet. Hic quidem assumsimus tubum *AC* eiusdem ubique esse amplitudinis, verum res pari modo se habet, etiamsi eius amplitudo fuerit variabilis, quem casum sequenti capite expendamus; semper autem tenendum est tubum ut angustissimum consideri.