CHAPTER IV

CONCERNING THE MINIMAL DISTURBANCE OF AIR FLOWING IN TUBES VARYING IN WIDTH

PROBLEM 81

76. *While air may be moving in some manner in a tube of varying width, its motion to be recalled according to analytical formulas, the determination of which may be put in place for some time.*

SOLUTION

We may consider the initial state of the air as known [see Ch. 1, § 1 for the horizontal case]; therefore then for some location *S* of the tube the distance may be called $AS = S$; we may put the density to become $= Q$ and the speed along the direction of the tube AB, to be $= \Upsilon$, thus so that *Q* and Υ shall be given functions of the quantity *S;* of which also the function $= \Omega$ shall be the size [*i.e.* cross-sectional area] of the tube at this location, and likewise the height of this, which shall be $= Z$ above a certain fixed horizontal plane, if indeed we may wish to have to take gravity into account in the motion of the air. Now in the elapsed time $= t$ we may put that particle of air to have arrived at *s* [from *S*], so that the interval $As = s$ and the extent [i.e. area of cross-section] of the tube shall be ω, and its height above that horizontal plane $= z$, which quantities ω and *z* are functions of this given *s*, truly this quantity *s* is a function of the two variables *S* and *t;* now indeed at this location *s* the density of the air shall be $= q$, the speed towards $B = \mathfrak{T}$, which is $= \left(\frac{ds}{dt}\right)$, and the pressure $= p$, as we have got to know through the density *q* thus to be determined, so that there shall become $p = \frac{aq}{b}$; [Recall that Euler assumes the initial pressure *a* and density *b* are in proportion isothermally; also 2*g* corresponds to the acceleration of gravity.]; the density *q* is some function of the two variables *S* et *t.* For this case these two equations are presented from the principles established above:

$$q\omega\left(\frac{ds}{dS}\right) = Q\Omega \text{ and } \frac{2gdp}{q} = -2gdz - ds\left(\frac{dds}{dt^2}\right) = \frac{2gadq}{bq},$$

in which the time *t* of the latter equation may be considered as constant, which therefore can be shown thus [*i.e.* where *S* is the only variable]:

$$\frac{2ga}{bq}\left(\frac{dq}{dS}\right) + 2g\left(\frac{dz}{dS}\right) + \left(\frac{ds}{dS}\right)\left(\frac{dds}{dt^2}\right) = 0.$$

Moreover, from the first equation we will have :

$$lq = lQ + l\Omega - l\omega - \left(\tfrac{ds}{dS}\right),$$

from which we deduce on assuming S alone to be variable:

$$\tfrac{1}{q}\left(\tfrac{dq}{dS}\right) = \tfrac{dQ}{QdS} + \tfrac{d\Omega}{\Omega dS} - \tfrac{1}{\omega}\left(\tfrac{d\omega}{dS}\right) - \frac{\left(\tfrac{dds}{dS^2}\right)}{\left(\tfrac{ds}{dS}\right)}.$$

With which value substituted we will have this equation, independent of q:

$$\tfrac{2ga}{bq}\left(\tfrac{dQ}{QdS} + \tfrac{d\Omega}{\Omega dS}\right)\left(\tfrac{ds}{dS}\right) - \tfrac{2ga}{b\omega}\left(\tfrac{d\omega}{dS}\right)\left(\tfrac{ds}{dS}\right) - \tfrac{2ga}{b}\left(\tfrac{dds}{dS^2}\right)$$
$$+2g\left(\tfrac{dz}{dS}\right)\left(\tfrac{ds}{dS}\right) + \left(\tfrac{ds}{dS}\right)^2\left(\tfrac{dds}{dt^2}\right) = 0,$$

where Q and Ω are given functions of S, truly the quantities ω and z are functions of s, which itself is a function of the two variables S and t, and thus the nature of this equation will have to be determined. Therefore so that if there may be put $d\omega = uds$ and $dz = rds$, in order that u and r shall be given functions of s, our equation adopts this form :

$$\tfrac{2ga}{b}\left(\tfrac{dQ}{QdS} + \tfrac{d\Omega}{\Omega dS}\right)\left(\tfrac{ds}{dS}\right) + 2g\left(r - \tfrac{au}{b\omega}\right)\left(\tfrac{ds}{dS}\right)^2 - \tfrac{2ga}{b}\left(\tfrac{dds}{dS^2}\right) + \left(\tfrac{ds}{dS}\right)^2\left(\tfrac{dds}{dt^2}\right) = 0,$$

from which, s shall be such a function of S and s, it will be required to be investigated, but there the speed found at once is known $\mathfrak{T} = \left(\tfrac{ds}{dt}\right)$, then truly

the density $q = \dfrac{Q\Omega}{\omega\left(\tfrac{ds}{dS}\right)}$ and likewise the pressure $p = \tfrac{aq}{b}$.

SCHOLIUM

77. From this general equation, which includes all motions within itself, to which the air in any tubes are able to be subjected, it is clear enough, to what degree even now we may be removed from its perfect solution, and that on account of defective analysis alone. Indeed since the function s, which is sought, besides the differential formulas may be involved also with the letters ω, u and r, this is the reason for so much complexity, which in no way would we wish in the general solution of our problem. Whereby, since now we have observed everything above, which will be able to be established here, restricted to the minimum motion only, which are the generation and propagation of sound of any kind, while meanwhile the cases, which otherwise may be considered most simply, just as if the air in the tube is being compressed or relaxed, evidently we may bring together to be left intact. Therefore it is a wonder, that these cases, which at first have been considered to be especially hard and scarcely undertaken by authors, now are allowed to be our own to be investigated, with everything else excluded. On account of which I will

adapt the general equation found here to that case, where the disturbance of the air can be considered as a minimum.

PROBLEM 82

78. *If the disturbance of the air excited in a tube varying in width were as a minimum, to find the equation, by which this motion may be contained continually.*

SOLUTION

All the denominations may remain, as have been established in the preceding problem, and since there was there $u=\frac{d\omega}{ds}$ and $r=\frac{dz}{ds}$, these values may be restored, and in addition for the sake of brevity there may be put $\frac{2ga}{b}=cc$, so that *c* will denote the distance, through which sound may be propagated in one second ; with which done the equation found will adopt this form:

$$\frac{1}{c^2}\left(\frac{ds}{dS}\right)^2\left(\frac{dds}{dt^2}\right)=\left(\frac{dds}{dS^2}\right)+\frac{d\omega}{\omega ds}\left(\frac{ds}{dS}\right)^2-\frac{d\Omega}{\Omega dS}\left(\frac{ds}{dS}\right)-\frac{bdz}{ads}\left(\frac{ds}{dS}\right)^2-\left(\frac{dQ}{QdS}\right)\left(\frac{ds}{dS}\right),$$

for which resolved again there will be had for the elapsed time $=t$:

$$\text{the density } q=\frac{Q\Omega}{\omega\left(\frac{ds}{dS}\right)}, \text{ the speed } \mathfrak{T}=\left(\frac{ds}{dt}\right) \text{ and the pressure } p=\frac{aq}{b}.$$

Now we may put $s=S+\upsilon$, so that it shall become a function of *S* and *t* themselves of this kind, which shall vanish with the time put $t=0$, since then there must become $s=S$, $q=Q$, $\omega=\Omega$ and $\mathfrak{T}=\Upsilon$, and here we may consider υ as a very small quantity besides *S*, or at least such, that $\left(\frac{d\upsilon}{ds}\right)$ may be able to be ignored besides. Therefore since there shall be

$$\left(\frac{ds}{dt}\right)=\left(\frac{d\upsilon}{dt}\right),\ \left(\frac{dds}{dt^2}\right)=\left(\frac{dd\upsilon}{dt^2}\right),\ \left(\frac{ds}{dS}\right)=1+\left(\frac{d\upsilon}{dS}\right) \text{ and } \frac{dds}{dS^2}=\left(\frac{dd\upsilon}{dS^2}\right);$$

where $\left(\frac{ds}{dS}\right)$ occurs in our equation in, in its place it will be allowed to write unity, as well as in the two terms

$$\frac{d\omega}{\omega ds}\left(\frac{ds}{dS}\right)^2-\frac{d\Omega}{\Omega dS}\left(\frac{ds}{dS}\right),$$

since here on account of the minimum difference between *s* and *S* there is almost

$$\frac{d\omega}{\omega ds}=\frac{dQ}{\Omega dS},$$

and thus besides the very small $\left(\frac{d\upsilon}{ds}\right)$, no further differential can be considered as vanishing. With which circumstances observed we will have

$$\frac{1}{c^2}\left(\frac{dd\upsilon}{dt^2}\right)=\left(\frac{dd\upsilon}{dS^2}\right)+\frac{d\omega}{\omega ds}-\frac{d\Omega}{\Omega dS}+\frac{d\Omega}{\Omega dS}\left(\frac{d\upsilon}{dS}\right)-\frac{bdz}{ads}-\left(\frac{dQ}{QdS}\right),$$

where, since the points *S* and *s* shall be close together, it will be allowed to write $\frac{dZ}{dS}$ in place of $\frac{dz}{ds}$, a function of *S* only. Then on account of the same reasoning, since the formula $\frac{d\omega}{\omega ds}$ arises from the formula $\frac{d\Omega}{\Omega dS}$, if here in place of *S* there may be written $S+\upsilon$, there will become

$$\frac{d\omega}{\omega ds}=\frac{d\Omega}{\Omega dS}+\frac{\upsilon}{dS}d\cdot\frac{d\Omega}{\Omega dS}.$$

Therefore for the sake of brevity we may put $\frac{d\Omega}{\Omega dS}=U$, which will be a given function of *S*, from which the amplitude will become

$$\Omega=Ae^{\int Uds},$$

and our equation will become:

$$\frac{1}{c^2}\left(\frac{dd\upsilon}{dt^2}\right)=\left(\frac{dd\upsilon}{dS^2}\right)+U\left(\frac{d\upsilon}{dS}\right)+\frac{\upsilon dU}{dS}-\frac{bdZ}{ads}-\frac{dQ}{QdS},$$

with which resolved there will become from the determination of the motion:

$$q=\frac{Q\Omega}{\omega\left(1+\left(\frac{d\upsilon}{dS}\right)\right)}=\frac{Q}{(1+U\upsilon)\left(1+\left(\frac{d\upsilon}{dS}\right)\right)}=Q\left(1-U\upsilon-\left(\frac{d\upsilon}{dS}\right)\right),$$

since there becomes :

$$\omega=\Omega+\frac{\upsilon d\Omega}{dS}=Ae^{\int UdS}\left(1+U\upsilon\right),$$

the speed $\mathfrak{T}=\left(\frac{ds}{dt}\right)$ and the pressure $p=\frac{aq}{b}$.

COROLLARY 1

79. Therefore the whole matter returns to this, so that it may be investigated from the differential equation of the second degree found, such a function shall be the quantity υ of the two variables *S* and *t*, where certainly I note the two latter terms $-\frac{bdZ}{ads}-\frac{dQ}{QdS}$ do not impair the resolution, since they constitute a function of the variable *S* only.

COROLLARY 2

80. But the conditions, under which it will be allowed to elicit the complete integral from the equations found with the aid of the methods known at this stage, will depend on the nature of the function *U*, where that is defined by the variable *S*. And it can happen in accordance with the nature of that, sometimes the integration may succeed, sometimes it may outdo the strengths of the calculation.

SCHOLIUM

81. Here therefore it is required to have recourse to that more sublime analysis, which depends on the arts of functions of two variables, with which I have shown how to treat equations of this kind

$$\frac{1}{cc}\left(\frac{dd\upsilon}{dt^2}\right)=\left(\frac{dd\upsilon}{dS^2}\right)+U\left(\frac{d\upsilon}{dS}\right)+\upsilon T \ ;$$

where in the case of the functions *U* and *T*, which are assumed to involve the variable *S* only, it will be required to enquire, by which complete integral it will be allowed to be shown. But before everything here it will be required to be observed, as often as this integration succeeds by certain known methods, the integral of this kind always to be expressed by known methods, so that there shall become

$$\upsilon = Lf:(S+ct)+Mf':(S\pm ct)+Nf'':(S\pm ct)\ +\text{ etc.},$$

where this is required to be held concerning the signs of the functions : if there were

$$f:u=V,\ \text{ to be }\ f':u=\frac{dV}{du},\quad f'':u=\frac{ddV}{du^2}\ \text{ etc.}$$

Hence it is evident an infinitude of solutions can have a place, provided a progression of this form may be continued. Therefore I will investigate the first solution or the first integrable case from the first term only of this progression, then I may elicit the second case of integrability on calling in two terms of that to help, and thus again it will be allowed to ascend higher, by accepting more terms continually.

PROBLEM 83

82. *To find the ratio of the amplitudes of the tube, in which the air performs the minimal disturbances, so that the determination of the motion requiring to be resolved may succeed by the first method,*

SOLUTION

Since with the amplitude of the tube put in place $\Omega = Ae^{\int Uds}$, this equation must be able to be integrated:

$$\tfrac{1}{c^2}\left(\tfrac{dd\upsilon}{dt^2}\right) = \left(\tfrac{dd\upsilon}{dS^2}\right) + U\left(\tfrac{d\upsilon}{dS}\right) + \tfrac{\upsilon dU}{dS} - \tfrac{bdZ}{ads} - \tfrac{dQ}{QdS},$$

we may put $\upsilon = Lf:(S+ct)+O$, where L and O shall be functions of S only, and with the substitution made we will obtain:

$$\begin{aligned}
\left(\tfrac{dd\upsilon}{dS^2}\right) &= Lf'':(S+ct) + \tfrac{2dL}{dS}f':(S+ct) + \tfrac{2ddL}{dS^2}f:(S+ct) + \tfrac{ddO}{dS^2}\\
U\left(\tfrac{d\upsilon}{dS}\right) &= \qquad\qquad +ULf':(S+ct) + \tfrac{UdL}{dS}f:(S+ct) + \tfrac{UdO}{dS}\\
\tfrac{\upsilon dU}{dS} &= \qquad\qquad\qquad\qquad + \tfrac{LdU}{dS}f:(S+ct) + \tfrac{OdU}{dS}\\
-\tfrac{bdZ}{adS} - \tfrac{dQ}{QdS} &= \qquad\qquad\qquad\qquad\qquad - \tfrac{bdZ}{ads} - \tfrac{dQ}{QdS},
\end{aligned}$$

which will be required to be taken together with $\tfrac{1}{c^2}\left(\tfrac{dd\upsilon}{dt^2}\right) = Lf'':(S+ct)$ itself : from which these equations arise :

I. $2\tfrac{dL}{dS} + LU = 0$

II. $\tfrac{ddL}{dS^2} + \tfrac{UdL+LdU}{dS} = 0$

III. $\tfrac{ddO}{dS^2} + \tfrac{UdO+OdU}{dS} - \tfrac{bdZ}{adS} - \tfrac{dQ}{QdS} = 0,$

the second of which integrated gives :

$$\tfrac{dL}{dS} + LU = C\ [= -\alpha],$$

which together with the first produces $-\tfrac{dL}{dS} = C,$ and hence

$$L = \alpha S + \beta \quad \text{and} \quad U = \tfrac{-2\alpha}{\alpha S+\beta} = \tfrac{d\Omega}{\Omega dS}.$$

Whereby, since there shall become:

$$\tfrac{d\Omega}{\Omega} = \tfrac{-2\alpha dS+\beta}{\alpha S+\beta}$$

the amplitude of the tube

$$\Omega = \frac{ff}{(\alpha S+\beta)^2},$$

allowing the solution with the aid of the first method. Truly the third equation integrated gives :

$$\tfrac{dO}{dS} + UO - \tfrac{b}{\alpha}Z - lQ = C$$

or on account of $U = \frac{-2\alpha}{\alpha S+\beta}$

$$dO - \tfrac{2\alpha OdS}{\alpha S+\beta} - \tfrac{b}{\alpha}ZdS - dSlQ = CdS,$$

which divided by $(\alpha S+\beta)^2$ and integrated provides

$$\frac{O}{(\alpha S+\beta)^2} = \int \frac{dS\left(\frac{b}{a}Z + lQ + C\right)}{(\alpha S+\beta)^2}.$$

Whereby, if the amplitude of the tube shall be variable thus, so that for the length $AS = S$ the amplitude may correspond $\Omega = \frac{ff}{(\alpha S+\beta)^2}$, then the complete integral of the determined motion will be

$$\upsilon = (\alpha S+\beta)^2 \int \frac{dS\left(C + \frac{b}{a}Z + lQ\right)}{(\alpha S+\beta)^2} + (\alpha S+\beta)\Gamma : (S+ct) + (\alpha S+\beta)\Delta : (S-ct),$$

since the function assumed $f:(S+ct)$ may be allowed to be doubled by introducing both $-c$ as well as $+c$.

COROLLARY 1

83. Likewise in this solution, the case of tubes of constant equal sizes thus is required to be defined at once, so that there shall be $\alpha = 0$ and $\beta = 1$. But this solution extends out much wider, since with its aid disturbances of the air in tubes of this kind with unequal amplitudes also may be able to be defined, the amplitude of which is held in this formula,

$$\Omega = \frac{ff}{(\alpha S+\beta)^2}.$$

COROLLARY 2

84. Moreover with this function υ found in the elapsed time *t* the air, which was at *S* initially, will be translated through the interval $Ss = \upsilon$, then truly its density on account of $U = \frac{-2\alpha}{\alpha S+\beta}$ will become:

$$q=Q\left(1+\tfrac{2\alpha\upsilon}{\alpha S+\beta}-\left(\tfrac{d\upsilon}{dS}\right)\right)\text{ and the speed }\mathfrak{T}=\left(\tfrac{d\upsilon}{dt}\right).$$

COROLLARY 3

85. Moreover, since on differentiating there shall become :

$$\left(\tfrac{d\upsilon}{dS}\right)=2\alpha\left(\alpha S+\beta\right)\int\frac{dS\left(C+\frac{b}{a}Z+lQ\right)}{\left(\alpha S+\beta\right)^{2}}+C+\tfrac{b}{\alpha}Z+lQ+\alpha\Gamma:\left(S+ct\right)+\alpha\Delta:\left(S-ct\right)$$
$$+\left(\alpha S+\beta\right)\Gamma':\left(S+ct\right)+\left(\alpha S+\beta\right)\Delta':\left(S-ct\right),$$

there will be had for the density:

$$\tfrac{q}{Q}=1-C-\tfrac{b}{a}Z-lQ+\alpha\Gamma:\left(S+ct\right)+\delta\Delta:(S-ct)$$
$$-\left(\alpha S+\beta\right)\Gamma':\left(S+ct\right)-(\alpha S+\beta)\Delta':(S-ct)$$

and for the speed :

$$\mathfrak{T}=c\left(\alpha S+\beta\right)\Gamma':\left(S+ct\right)-c(\alpha S+\beta)\Delta':(S-ct).$$

SCHOLIUM

85*a*. If the tubes, to which this solution is adapted, we may establish as round, so that all the sections made normally to its direction shall be circles, the shape of those (Fig. 89) is a conoidal hyperbola arising from the rotation of the equilateral hyperbola *KCM* around the other asymptote *IB*. Indeed since in this hyperbola there shall be $SM\cdot IS=a$, the amplitude at S shall be $=\pi SM^{2}=\frac{\pi aa}{IS^{2}}$, therefore with the interval taken $IA=\frac{\beta}{\alpha}$ and on putting $AS=S$, on account of $IS=\frac{\alpha S+\beta}{\alpha}$ the amplitude will become $\Omega=\frac{\pi\alpha\alpha aa}{(\alpha S+\beta)^{2}}$, and thus $ff=\pi\alpha\alpha aa$. Therefore as often as the tube will have had a hyperbolic conoidal shape of this kind, the disturbances of the air in tubes of this kind, while they shall be minimal, will be able to be defined likewise, and in tubes of equal size. Yet the motion itself requiring to be determined will be somewhat harder. Yet here it will be agreed to have considered the weight of the air, which we have ignored in the above chapters, plainly not disturbing the investigation: that which also with the other forces is required to be considered, which may disturb the air strongly.

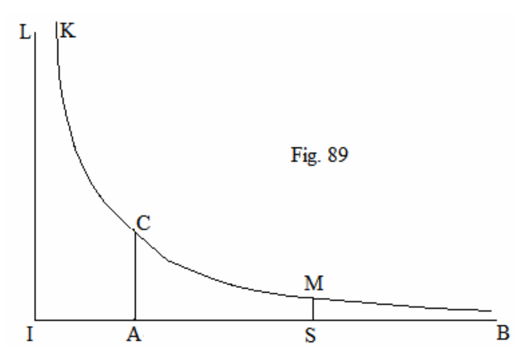


Fig. 89

PROBLEM 84

86. *To examine the reasoning, by which the size of the tube must be prepared, so that it shall be able to show the complete integral of the differentio-differential equation in the following form.*

SOLUTION

Evidently here the nature of a function U is sought depending on the variable S alone, so that the complete integral of our equation may be able to be expressed in the following form of this kind:

$$\upsilon = O + Mf' : (S+ct) + Lf : (S+ct),$$

where L, M, O shall be functions of the variable S only. Therefore we may make the substitution in our equation and we will find :

$$\begin{array}{llll}
0 = \left(\frac{ddO}{dS^2}\right) & +Mf''' : (S+ct) & +\frac{2dM}{dS} f'' : (S+ct) & +\frac{2ddM}{dS^2} f' : (S+ct) \\
+U\frac{dO}{dS} & +L & +\frac{2dL}{dS} & +\frac{ddL}{dS^2} f : (S+ct) \\
+\frac{OdU}{dS} & +UM & +\frac{UdM}{dS} & \\
-\frac{bdZ}{adS} & & +UL & +\frac{UdL}{dS} \\
-\frac{dQ}{QdS} - Mf''' : (S+ct) - Lf'' : (S+ct) & & +\frac{MdU}{dS} & +\frac{LdU}{dS},
\end{array}$$

the individual terms of which, as far as the different functions may be included, will be required to be returned separately to zero: from which the following four equations arise:

$$\text{I.} \quad 2\frac{dM}{dS} + UM = 0$$

$$\text{II.} \quad \frac{ddM}{dS^2} + 2\frac{dL}{dS} + \frac{UdM + MdU}{dS} + UL = 0$$

$$\text{III.} \quad \frac{ddL}{dS^2} + \frac{UdL + LdU}{dS} = 0$$

$$\text{IV.} \quad \frac{ddO}{dS} + UdO + OdU - \frac{b}{a}dZ - \frac{dQ}{Q} = 0.$$

The third integrated gives $\frac{dL}{dS} + UL = A,$ which combined with the first by eliminating U presents $MdL - 2LdM = AMdS,$ from which there is deduced

$$L = AMM\int \frac{dS}{MM} \text{ and } U = -\frac{2dM}{MdS}.$$

With these values substituted into the second equation this equation is come upon:

$$-\frac{ddM}{dS^2}+2A+\frac{2AMdM}{dS}\int\frac{dS}{MM}=0.$$

For this requiring to be resolved so that there may be $\int\frac{dS}{MM}=R$, and hence $dS=MMdR$, and since there becomes $\frac{ddM}{dS}=d\cdot\frac{dM}{dS}=d\cdot\frac{dM}{MMdR}$, that equation multiplied by dS presents

$$-d\cdot\frac{dM}{MMdR}+2AMMdR+2AMRdM=0,$$

with which resolved there will be had

$$dS=MMdR,\ L=AMMR \text{ and } U=-\frac{2dM}{M^3dR}.$$

Truly with that same equation multiplied by R is returned with the integral, since there shall become

$$\int Rd\frac{dM}{MMdR}=\frac{RdM}{MMdR}+\frac{1}{M},$$

and thus we will have:

$$\frac{RdM}{MMdR}+\frac{1}{M}=AMMRR+B=+\frac{MdR+RdM}{MMdR}.$$

Finally there may be put $MR=x$ or $M=\frac{x}{R}$, there will become

$$Axx+B=\frac{RRdx}{xxdR}$$

and hence

$$\frac{dR}{RR}=\frac{dx}{xx(Axx+B)} \text{ or } \frac{BdR}{RR}=\frac{dx}{xx}-\frac{Adx}{Axx+B}$$

and thus

$$\frac{B}{R}=\frac{1}{x}+\int\frac{Adx}{Axx+B},$$

from which R by x, then on account of $M=\frac{x}{R}$ all the remaining quantities will be given by x : moreover there will become

$$dS=\frac{xxdR}{RR}=\frac{dx}{Axx+B} \text{ and } UdS=\frac{d\Omega}{\Omega}=-\frac{2dM}{M},$$

thus so that the amplitude shall become

$$\Omega=\frac{C}{MM}=\frac{CRR}{xx}.$$

Hence we shall evolve the case, in which the amplitude Ω is defined algebraically by the variable *S*, which happens, if the constant $B=0$, then indeed there will become

$$\frac{dR}{RR}=\frac{dx}{Ax^4} \text{ and thus } \frac{1}{R}=\frac{1+Dx^3}{3Ax^3} \text{ and } R=\frac{3Ax^3}{1+Dx^3} \text{ and also } M=\frac{1+Dx^3}{3Axx}.$$

But since there shall be $dS=\frac{dx}{Axx+B}$, there will be $\frac{1}{x}=-AS-E$, or $x=-\frac{1}{AS+E}$
and hence

$$M=\frac{(AS+\mathrm{E})^3-D}{3A(AS+E)} \text{ and } R=-\frac{3A}{(AS+\mathrm{E})^3-D};$$

from which again we deduce

$$\Omega=\frac{9AAC(AS+\mathrm{E})^2}{\left((AS+E)^3-D\right)^2} \text{ and } L=\frac{-(AS+E)^3+D}{(AS+E)^2}.$$

Finally for the quantity *O* we obtain :

$$\frac{dO}{dS}+UO-\frac{b}{a}Z-lQ+lF=0$$

or

$$dO+\frac{Od\Omega}{\Omega}-\frac{b}{a}ZdS-dS\,l\frac{Q}{F}=0.$$

Therefore

$$O=\frac{1}{\Omega}\int\Omega dS\left(\frac{b}{a}Z+l\frac{Q}{F}\right).$$

We may change the constants a little, and when the amplitude of the tube Q thus will have been prepared with the abscissa $AS=S$, so that there shall become

$$Q=\frac{ff(\alpha S+\beta)^2}{\left((\alpha S+\beta)^3-\gamma\right)^2},$$

then on taking

$$O=\frac{1}{\Omega}\int\Omega dS\left(\frac{b}{a}Z+l\frac{Q}{B}\right),$$

$$M=\frac{(\alpha S+\beta)^3-\gamma}{3\alpha(\alpha S+\beta)} \text{ and } L=\frac{-(\alpha S+\beta)^3+\gamma}{3(\alpha S+\beta)^2},$$

the complete integral of our equation will become,

$$\upsilon=O+M\Gamma':(S+ct)+L\Gamma:(S-ct)$$
$$+M\Delta':(S-ct)+L\Delta:(\mathrm{S}-ct),$$

from which again there is defined :

the density $q = Q\left(1 - \frac{\upsilon d\Omega}{\Omega dS}\right)$, and the speed $\mathfrak{T} = \left(\frac{d\upsilon}{dt}\right)$.

COROLLARY 1

87. Here in the first place it is required to be observed the quantities found for *L* and *M* can be multiplied by some constant quantity, as they are agreed to include indefinite functions, hence it will be able to assume

$$M = \frac{(\alpha S+\beta)^3 - \gamma}{3\alpha\delta(\alpha S+\beta)} \text{ and } L = \frac{-(\alpha S+\beta)^3 + \gamma}{\delta(\alpha S+\beta)^2},$$

so that it is required to be considered, if perhaps these values may become infinite.

COROLLARY 2

88. Indeed if there may be taken $\gamma = \infty$ and $f = n\gamma$, so that the amplitude may become

$$\Omega = nn(\alpha S + \beta)^2,$$

also there may be assumed $\delta = \gamma$, and there will become

$$M = \frac{-1}{\alpha(\alpha S+\beta)} \text{ and } L = \frac{1}{(\alpha S+\beta)^2}.$$

Moreover in this case, conical or pyramidal tubes will be obtained, in which equally therefore, the minimum disturbances of the air will be allowed to be defined.

COROLLARY 3

89. If there may be taken $\gamma = 0$, the tube thus will be formed so that there shall become

$$\Omega = \frac{ff}{(\alpha S+\beta)^4}$$

or the amplitudes will be inversely as the fourth powers of the abscissas : moreover, there becomes then:

$$M = \frac{1}{\alpha}(\alpha S + \beta)^2 \text{ and } L = -(\alpha S + \beta).$$

COROLLARY 4

90. Moreover this solution can be applied in general to conoidal tubes of this kind (Fig. 90), which arise from a conversion of this kind of hyperbolic curve about the axis *IB*, for which with the abscissa AS put $= x$ and the applied line $SM = y$, there shall become $y = \frac{a^3x}{x^3+b^3}$: which curve cutting the axis at *A* has the two asymptotes *GH* and *KL* normal to each other.

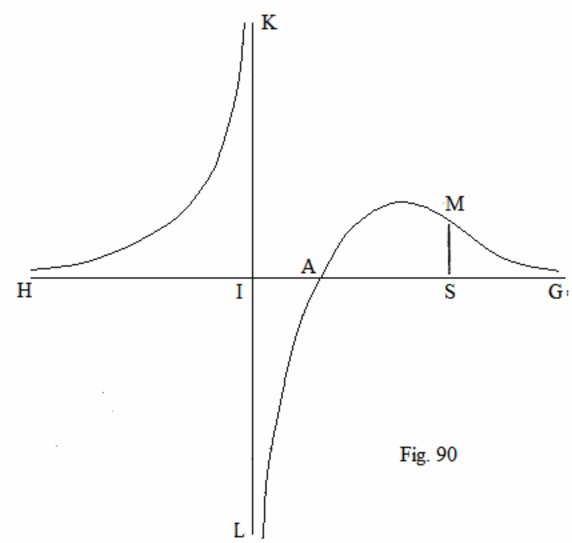


Fig. 90

SCHOLIUM 1

91. Also if we may wish to know the shapes of the transcending tubes, we may make $B = Amm$ and $A = \frac{\alpha}{m}$, and the equation $dS = \frac{dx}{Axx+B}$ or $\alpha dS = \frac{mdx}{xx+mm}$ will give $x = m\text{tang}(\alpha S+\beta)$, from which again there is deduced

$$\frac{\alpha mm}{R} = \alpha S + \gamma + \cot\left(\alpha S + \beta\right) \text{ or } R = \frac{\alpha mm}{\alpha S+\gamma+\cot(\alpha S+\beta)}$$

and hence

$$M = \frac{1+(\alpha S+\gamma)\text{tang}(\alpha S+\beta)}{\alpha m} \quad \text{and} \quad L = \frac{1+(\alpha S+\gamma)\text{tang}(\alpha S+\beta)}{m\cot(\alpha S+\beta)}.$$

Then the amplitude becomes

$$\Omega = \frac{C}{1+(\alpha S+\gamma)\text{tang}(\alpha S+\beta)},$$

truly the quantity *O* hence is defined as before :

$$O = \frac{1}{\Omega}\int \Omega dS\left(\frac{b}{a}Z + l\frac{Q}{B}\right),$$

Moreover from these forms we deduce the equations requiring to be resolved by a much simpler method. For since the first equation elicited may give $U = -\frac{2dM}{MdS}$, truly the third equation integrated gives $\frac{dL}{dS} + UL = A$, there will become

$$dL - \frac{2LdM}{M} = AdS,$$

we may put $L = My$, and there becomes $Mdy - ydM = AdS$. Now the second, which can occur, integrated gives

$$\tfrac{dM}{dS}+2L+UM+\int ULdS=0,$$

which on account of

$$UM=-\tfrac{2dM}{dS} \text{ and } \int ULdS=-2\int \tfrac{LdM}{M}=-2\int ydM$$

becomes :

$$\tfrac{dM}{dS}+2My-\tfrac{2dM}{dS}-2\int ydM=0,$$

which contracts into this : $2\int Mdy-\tfrac{dM}{dS}=0.$ But from that above there becomes

$$2Mdy=AdS+ydM+Mdy,$$

thus so that there shall become

$$2\int Mdy=AS+My+B=\tfrac{dM}{dS},$$

hence, since it may be allowed to deduce equally, we will consider these two equations:

$$Mdy-ydM=AdS \quad \text{and} \quad 2\int Mdy=\tfrac{dM}{dS},$$

which both we will differentiate on assuming the element dS constant

$$Mddy-yddM=0 \quad \text{and} \quad 2Mdy=\tfrac{ddM}{dS},$$

from which there becomes

$$\tfrac{ddM}{M}=\tfrac{ddy}{y}=2dydS \quad \text{or} \quad ddy=2ydydS,$$

the integral of which is $dy=dS\left(yy+mm\right),$ and hence again

$$\text{Ang tang}\,\tfrac{y}{m}=mS+n \quad \text{and on being converted} \quad y=m\,\text{tang}(mS+n).$$

Now from the equation

$$\frac{Mdy-ydM}{yy}=\frac{AdS}{yy}=\frac{AdS}{mm\,\text{tang}^2(mS+n)}$$

there becomes

$$-\frac{M}{y}=-\frac{M}{m\text{tang}^2(mS+n)}=\frac{A}{mm}\int\frac{dS\cos^2(mS+n)}{\sin^2(mS+n)}=\frac{A}{mm}\int\frac{dS}{\sin^2(mS+n)}-\frac{AS}{mm}$$
$$=-\frac{A}{m^3\text{tang}(mS+n)}-\frac{AS}{mm}-\frac{Ak}{m^3},$$

from which there becomes :

$$M = \tfrac{A}{mm}\left(1+\left(mS+k\right)\operatorname{tang}\left(mS+n\right)\right),$$

and hence again

$$L = \frac{A\left(1+\left(mS+k\right)\operatorname{tang}\left(mS+n\right)\right)}{m\cot\left(mS+n\right)}$$

and finally the amplitude is produced :

$$\varOmega = \tfrac{\text{Const.}}{MM} \quad \text{or} \quad \varOmega = \frac{ff}{\left(1+\left(mS+k\right)\operatorname{tang}^2\left(mS+n\right)\right)}$$

SCHOLIUM *2*

92. Yet this resolution will be allowed to be put in place easier in this manner : since from the first equating there shall become $U = -\tfrac{2dM}{MdS}$, at once there may be put $L = My$, and there will become

$$MU = -2\tfrac{dM}{dS} \text{ and } LU = -\tfrac{2ydM}{dS},$$

from which the second equation will become

$$\tfrac{ddM}{dS} + 2d\cdot My - 2d\cdot\tfrac{dM}{dS} - 2ydM = 0$$

or

$$-\tfrac{ddM}{dS} + 2Mdy = 0 \quad \text{or} \quad \tfrac{ddM}{dS} = 2dydS,$$

in a similar manner the third equation will go into this form:

$$\tfrac{dd\cdot My}{dS} - 2d\cdot\tfrac{ydM}{dS} = 0$$

and with the factors expanded out

$$Mddy + 2dMdy + yddM - 2dydM - 2yddM = 0,$$

that is

$$Mddy - ddM = 0 \text{ or } \tfrac{ddM}{M} = \tfrac{ddy}{y}.$$

From these taken together there becomes as before $ddy = 2ydydS$. But if we may wish hence at once to elicit the algebraic tube forms, the above constant may be taken to be $m = 0$, so that there shall become $dy = yydS$ and thus

$$\tfrac{1}{y} = -S - \tfrac{\beta}{\alpha} \text{ or } y = \tfrac{-\alpha}{\alpha S+\beta}.$$

Then the equation $Mdy - ydM = AdS$ gives

$$-\tfrac{M}{y} = \tfrac{A}{\alpha\alpha}\int dS\left(\alpha S+\beta\right)^2 = \tfrac{A}{3\alpha^3}\left(\left(\alpha S+\beta\right)^3 - \gamma\right)$$

and thus

$$M = \frac{A\left(\left(\alpha S+\beta\right)^3-\gamma\right)}{3\alpha\alpha\left(\alpha S+\beta\right)} \quad \text{and} \quad L = -\frac{A\left(\left(\alpha S+\beta\right)^3-\gamma\right)}{3\alpha\left(\alpha S+\beta\right)^2},$$

and finally the amplitude

$$\Omega = \frac{C\left(\alpha S+\beta\right)^2}{\left(\left(\alpha S+\beta\right)^3-\gamma\right)^2}$$

absolutely as before.

SCHOLIUM 3

93. But besides these two cases, for which the form of the tube either is algebraic or depends on the circle, it is agreed not to omit the third case, where that is determined by exponential quantities. But for that the integral of the equation $ddy = 2ydydS$ is assumed $dy = dS(yy - mm),$ from which there becomes

$$2mS+2n = l\,\frac{y-m}{y+m},$$

and from which

$$y = \frac{1+e^{2mS+2n}}{1-e^{2mS+2n}}m,$$

then truly again,

$$-\frac{M}{y} = \frac{A}{mm}\int \frac{dS\left(1-e^{2mS+2n}\right)^2}{\left(1+e^{2mS+2n}\right)^2} = \frac{A}{m^3}\cdot\frac{e^{2mS+2n}\left(mS+k-1\right)+mS+k+1}{1+e^{2mS+2n}}$$

and thus

$$M = -\frac{A}{mm}\cdot\frac{e^{2mS+2n}\left(mS+k-1\right)+mS+k+1}{1-e^{2mS+2n}}$$

and

$$L = -\frac{A}{mm}\cdot\frac{e^{2mS+2n}\left(mS+k-1\right)+mS+k+1}{\left(1-e^{2mS+2n}\right)^2}\left(1+e^{2mS+2n}\right)$$

and the amplitude

$$\Omega = \frac{C\left(1-e^{2mS+2n}\right)^2}{\left(e^{2mS+2n}\left(mS+k-1\right)+mS+k+1\right)^2}.$$

From this case it is certain, and likewise for the above case depending on the circle, to be produced algebraically, if there may be put $m = 0$, which is less obvious how to be established and not worth the industry it demands of the Analyst.

PROBLEM 85

92. *To investigate the case for the amplitude of the tube, so that a differentio-differential equation may be able to be integrated by operations of the third order, or its complete integral may be able to be shown by the third form explained above.*

SOLUTION

Therefore in this case the integral sought must have such a form:

$$\upsilon = O + Nf'':(S+ct) + Mf':(S+ct) + Lf:(S+ct),$$

and since now it is evident the quantity O going to be defined as before, only we may observe kinds of functions:

$f'''':(S+ct)$	$f''':(S+ct)$	$f'':(S+ct)$	$f':(S+ct)$	$f:(S+ct)$
$+N$	$+\frac{2dN}{dS}$	$+\frac{ddN}{dS^2}$		
	$+M$	$+\frac{2dM}{dS}$	$+\frac{ddM}{dS^2}$	
		$+L$	$+\frac{2dL}{dS}$	$+\frac{ddL}{dS^2}$
	$+UN$	$+\frac{UN}{dS}$		
		$+UM$	$+\frac{UM}{dS}$	
			$+UL$	$+\frac{UL}{dS}$
		$+\frac{NdU}{dS}$	$+\frac{MdU}{dS}$	$+\frac{LdU}{dS}$
$-N$	$-M$	$-L$		

from which we deduce the four following equations:

$$\text{I. } \frac{2dN}{dS} + UN = 0$$

$$\text{II. } \frac{ddN}{dS^2} + \frac{2dM}{dS} + \frac{UdN+NdU}{dS} + UM = 0$$

$$\text{III. } \frac{ddM}{dS^2} + \frac{2dL}{dS} + \frac{UdM+MdU}{dS} + UL = 0$$

$$\text{IV. } \frac{ddL}{dS^2} + \frac{UdL+LdU}{dS} = 0.$$

Now since from the first there shall be $U = \frac{-2dN}{NdS}$, we may put $M = Ny$ and $L = Nz$, so that there may become

$$UN = \frac{-2dN}{dS},\ UM = \frac{-2ydN}{dS} \text{ and } UL = \frac{-2zdN}{dS}$$

with which values requiring to be substituted, equation II. becomes

$$\frac{ddN}{dS^2}+\frac{2Ndy+2ydN}{dS}-\frac{2ddN}{dS^2}-\frac{2ydN}{dS}=0,$$

which contracts into this form:

$$\text{II.}\quad \frac{-ddN}{dS}+2Ndy=0.$$

Hence the third equation truly emerges:

$$\frac{Nddy+2dNdy+yddN}{dS^2}+\frac{2Ndz+2zdN}{dS}-\frac{2yddN-2dydN}{dS^2}-\frac{2zdN}{dS}=0,$$

which is contracted into this form:

$$\text{III.}\quad \frac{Nddy-yddN}{dS}+2Ndz=0.\ \left[\text{giving}: \ \frac{Nddy-yddN}{N}+2dzdS=0 \text{ hence } \frac{ddy+2dzdS}{y}=\frac{ddN}{N};\right]$$

Finally the fourth is established thus:

$$\frac{Nddz+2dNdz+zddN}{dS^2}-\frac{2zddN-2dzdN}{dS^2}=0,$$

which therefore provides this form:

$$\text{IV.}\quad \frac{Nddz-zddN}{dS}=0.\ \left[\textit{i.e.}\ \frac{ddz}{z}=\frac{ddN}{N}\right]$$

From these three there becomes only :

$$\frac{ddN}{N}=2dydS=\frac{ddy+2dzdS}{y}=\frac{ddz}{z}$$

from which these two equations requiring to be resolved are proposed:

$$2ydydS=ddy+2dzdS \quad\text{and}\quad yddz-zddy=2zdzdS\,,$$

of which the latter two equations integrated gives at once

$$ydz-zdy=dS\left(zz+A\right),$$

truly the prior two equations equally integrated provides :

$$dy+2zdS=dS(yy+B)$$

or $dy = dS\left(yy-2z+B\right)$ now that equation divided by this provides this equation free from the element dS

$$\frac{ydz-zdy}{dy}=\frac{zz+A}{yy-2z+B};$$

let there be $z=uy$ and this equation becomes :

$$\frac{yydu}{dy}=\frac{uuyy+A}{yy-2uy+B}$$

or $y^4du-2uy^3du+Byydu-uuyydy-Ady=0.$

Here it is considered necessary, that we may make $A=0$ and $B=0$; with which done we will have:

$$yydu-2uydu-uudy=0 \text{ or } yydu=d\cdot uuy \;;$$

we may divide by per u^4y^2, there will become $\frac{du}{u^4}=\frac{d\cdot uuy}{u^4yy}$, and the integral of this :

$$\frac{1}{3u^3}=\frac{1}{uuy}-\frac{1}{3m^3} \text{ or } y=\frac{3m^3u}{m^3+u^3} \text{ and } z=\frac{3m^3uu}{m^3+u^3},$$

from which there becomes

$$dS=\frac{dy}{yy-2uy}=\frac{ydz-zdy}{zz}$$

or

$$dS=\frac{du}{uu}=-d\cdot\frac{y}{z},$$

thus so that there shall be

$$\frac{1}{u}=-S-\frac{\beta}{\alpha} \text{ and } u=\frac{-\alpha}{\alpha S+\beta}.$$

Therefore now it terms of S we will have the values :

$$y=\frac{-3\alpha m^3\left(\alpha S+\beta\right)^2}{m^3\left(\alpha S+\beta\right)^3-\alpha^3} \text{ and } z=\frac{3\alpha^2m^3\left(\alpha S+\beta\right)}{m^3\left(\alpha S+\beta\right)^3-\alpha^3}.$$

For the sake of brevity we may put $\frac{\alpha^3}{m^3}=\gamma^3$, so that there shall become

$$y=\frac{-3\alpha\left(\alpha S+\beta\right)^3}{\left(\alpha S+\beta\right)^3-\gamma^3} \text{ and } z=\frac{3\alpha\alpha\left(\alpha S+\beta\right)}{\left(\alpha S+\beta\right)^3-\gamma^3}.$$

Now again we deduce from equation IV :

$$Ndz - zdN = AdS = \frac{Adu}{uu},$$

which on account of

$z = \frac{3\alpha^3 uu}{\alpha^3+\gamma^3 u^3}$ divided by zz gives

$$-d.\frac{N}{z} = \frac{Bdu}{u^6}\left(\alpha^3+\gamma^3 u^3\right)^2,$$

and on integrating

$$-\frac{N}{z} = C+B\left(\gamma^6 u - \frac{\alpha^3\gamma^3}{u^2} - \frac{\alpha^6}{5u^5}\right),$$

hence $N = \frac{Bz}{5u^5}\left(\alpha^6+5\alpha^3\gamma^3u^3 - 5\gamma^6u^6\right) - Cz$, and by expanding out

$$N = \frac{A(\alpha S+\beta)^6 - 5A\gamma^3(\alpha S+\beta)^3 - 5A\gamma^6 + B(\alpha S+\beta)}{(\alpha S+\beta)^3-\gamma^3}$$

clearly with the constants changed, from which the amplitude of the tube $\Omega = \frac{C}{NN}$ is defined at once, then truly there will become

$$M = \frac{-3\alpha(\alpha S+\beta)^2}{(\alpha S+\beta)^3-\gamma^3}N \quad \text{and} \quad L = \frac{3\alpha\alpha(\alpha S+\beta)}{(\alpha S+\beta)^3-\gamma^3}N.$$

Moreover, for the quantity O as before there is

$$O = \frac{1}{\Omega}\int \Omega ds\left(\frac{b}{a}z + l\frac{Q}{B}\right).$$

Therefore whenever the amplitude Ω is defined in this manner, the complete solution of our problem will be found thus, so that there shall be:

$$\begin{aligned} \upsilon = O + N\Gamma'' : (S+ct) + M\Gamma' : (S+ct) + L\Gamma : (S+ct) \\ + N\Delta'' : (S-ct) + M\Delta' : (S-ct) + L\Delta : (S-ct). \end{aligned}$$

Truly thereon from that the density is

$q = Q\left(1 - \frac{\upsilon d\Omega}{\Omega dS} - \left(\frac{d\upsilon}{ds}\right)\right)$ and the speed $\mathfrak{T} = \left(\frac{d\upsilon}{dt}\right)$.

COROLLARY 1

95. This is a particular determination of the amplitude of the tube allowing the proposed resolution in a twofold manner, while we have put two arbitrary constants into the equation between y and z found equal to zero; yet meanwhile the form elicited for Ω

includes several arbitrary constants at this point, and thus may appear more general than the previous case in each extension accepted.

COROLLARY 2

96. Simpler cases arise contained in this solution, if there may become $\gamma = 0$, then indeed there will be:

$$N = \frac{A(\alpha S+\beta)^6 + B(\alpha S+\beta)}{(\alpha S+\beta)^3};$$

therefore just as there were either $A = 0$ or $B = 0$, there will be found

$$\text{either } N = (\alpha S+\beta)^{-2} \text{ or } N = (\alpha S+\beta)^{3};$$

therefore in the first case the amplitude is defined thus, so that there shall be $\Omega = C(\alpha S+\beta)^4$, truly in the second case $\Omega = C(\alpha S+\beta)^{-6}$.

COROLLARY 3

97. The solution found by the operation of the first order also will give these two simpler cases:

$$\Omega = C(\alpha S+\beta)^0 \text{ and } \Omega = C(\alpha S+\beta)^{-2},$$

truly the operation of the second order will have provided these two cases:

$$\Omega = C(\alpha S+\beta)^2 \text{ and } \Omega = C(\alpha S+\beta)^{-4}$$

now this of the third order supplies:

$$\Omega = C(\alpha S+\beta)^4 \text{ and } \Omega = C(\alpha S+\beta)^{-6}$$

from which it is allowed to conclude this method can be applied to all tubes contained by this formula $\Omega = C(\alpha S+\beta)^{\pm 2i}$, with odd exponents excluded.

SCHOLIUM 1

98. Clearly all the forms of tubes allowing a resolution in this manner must be taken from this equation :

$$y^4du - 2uy^3du - uuyydy + Byydu - Ady = 0,$$

where *A* and *B* denote constants depending on our choice, yet both of which I have forced to vanish in the preceding solutions, so that the integration may succeed. Moreover now I note, provided there shall be $A=0$ and thus

$$dS=\frac{ydz-zdy}{zz}=\frac{du}{uu},$$

even now the integration can be performed, whatever constant may be assumed for *B*. Indeed since then there shall become

$$yydu-2uydu-uudy+Bdu=0,$$

and we may put $uuy=x$ or $y=\frac{x}{uu}$, there will become

$$\frac{xxdu}{u^4}-dx+Bdu=0 \quad \text{or} \quad d\cdot\frac{1}{x}+Bdu\left(\frac{1}{x}\right)^2+\frac{du}{u^4}=0,$$

which is the integrable case of the Riccati equation, the form of which satisfies

$\frac{1}{x}=\frac{\alpha}{u}+\frac{\beta}{uu}$; for there shall become

$$-\frac{\alpha}{uu}-\frac{2\beta}{u^3}+\frac{B\alpha\alpha}{uu}+\frac{2B\alpha\beta}{u^3}+\frac{B\beta\beta}{u^4}+\frac{1}{u^4}=0,$$

from which it must be assumed

$$B=-\frac{1}{\beta\beta} \quad \text{et} \quad \alpha=\frac{1}{B}=-\beta\beta,$$

thus so that there must become

$$\frac{1}{x}=\frac{-\beta\beta}{u}+\frac{\beta}{uu}.$$

Therefore there may be put

$$\frac{1}{x}=\frac{-\beta\beta}{u}+\frac{\beta}{uu}+\frac{1}{p},$$

so that we may elicit the complete integral of the equation

$$d\cdot\frac{1}{x}-\frac{du}{\beta\beta}\left(\frac{1}{x}\right)^2+\frac{du}{u^4}=0,$$

and there will become :

$$dp-\frac{2pdu}{u}+\frac{2pdu}{\beta uu}+\frac{du}{\beta\beta}=0,$$

which multiplied by $\frac{e^{-\frac{2}{\beta u}}}{uu}$ and integrated provides :

$$\frac{e^{-\frac{2}{\beta u}}p}{uu}+\frac{e^{-\frac{2}{\beta u}}}{2\beta}=\frac{C}{2\beta} \text{ or } p=\frac{Ce^{\frac{2}{\beta u}}uu}{2\beta}-\frac{uu}{\beta\beta},$$

thus so that there shall become

$$\frac{1}{x}=-\frac{\beta\beta}{u}+\frac{\beta}{uu}+\frac{2\beta}{uu\left(Ce^{\frac{2}{\beta u}}-1\right)}$$

and hence

$$x=\frac{uu\left(Ce^{\frac{2}{\beta u}}-1\right)}{\beta(1+\beta u)+C\beta e^{\frac{2}{\beta u}}(1-\beta u)}=uuy.$$

Therefore $y=\frac{Ce^{\frac{2}{\beta u}}-1}{\beta(1+\beta u)+C\beta e^{\frac{2}{\beta u}}(1-\beta u)}$ and $z=\frac{u\left(Ce^{\frac{2}{\beta u}}-1\right)}{\beta(1+\beta u)+C\beta e^{\frac{2}{\beta u}}(1-\beta u)}$

with $u=-\frac{m}{mS+n}$; then truly there is $Ndz-zdN=DdS=\frac{Ddu}{uu}$, from which divided by zz N is found and thence the rest.

Here if we may make $\beta=\infty$, the preceding solution will be found.

SCHOLIUM 2

99. If it may please to progress further in this manner, the calculation becomes more laborious, but yet here it will be allowed to overcome the special difficulties by the use of special methods. Just as if for the form of the integral of the fourth order we may put :

$$\upsilon=O+Nf''':(S+ct)+Mf'':(S+ct)+Lf':(S+ct)+Kf:(S+ct)$$

and on account of $UdS=\frac{dQ}{Q}=\frac{-2dN}{N}$ and thus $Q=\frac{ff}{NN}$ there may be put

$$M=Nx,\ L=Ny \text{ and } K=Nz$$

and these equations may be arrived at :

$$\frac{ddN}{N}=2dxdS=\frac{ddx+2dydS}{x}=\frac{ddy+2dzdS}{y}=\frac{ddz}{z},$$

from which we deduce:

I. $2xdxdS=ddx+2dydS$

II. $yddx-xddy+2ydydS-2xdzdS=0$

III. $ddz-2zdxdS=0$

IV. $zddy-yddz+2zdzdS=0$

and hence on integrating:

$$\text{I. } xxdS = dx + 2ydS + \text{Const.}dS$$
$$\text{II.} + \text{III.} \quad ydx - xdy + dz + yydS - 2xzdS = \text{Const.}dS$$
$$\text{IV. } zdy - ydz + zzdS = \text{Const.}dS$$

We may take all these three constants equal to zero and finally there is produced $\frac{y}{z} = -S$, thus so that there shall become $y = -zS$, were there is no need for the addition of a constant, since it will be allowed to write $S + \frac{\beta}{\alpha}$ in place of S as above. Therefore on putting $y = -zS$ from the first equation there becomes $dx = dS\left(xx + 2zS\right)$ and from the second

$$-Szdx + Sxdz - xzdS + dz + SSzzdS = 0,$$

which divided by zz and integrated provides

$$-\tfrac{Sx}{z} - \tfrac{1}{z} + \tfrac{1}{3}S^3 + \tfrac{1}{3}A = 0 \text{ and thus } z = \tfrac{3(1+Sx)}{A+S^3}$$

and with this value substituted into the first equation there becomes

$$dx = dS\left(xx + \tfrac{6S(1+Sx)}{A+S^3}\right),$$

since this may be satisfied by $x = -\frac{1}{S}$, there may be put $x = -\frac{1}{S} + \frac{1}{\upsilon}$ and there becomes:

$$0 = d\upsilon - \tfrac{2\upsilon dS}{S} + \tfrac{6\upsilon SSdS}{A+S^3} + dS,$$

the integral of which

$$\upsilon = \frac{BSS + ASS - AS^4 - \frac{1}{5}S^7}{\left(A+S^3\right)^2}$$

provides

$$x = \frac{-B + 3ASS + \frac{6}{5}S^5}{BS + AA - AS^3 - \frac{1}{5}S^6},$$

and hence again:

$$z = \frac{3(A+S^3)}{BS + AA - AS^3 - \frac{1}{5}S^6} \text{ and } y = \frac{-3S\left(A+S^3\right)}{BS + AA - AS^3 - \frac{1}{5}S^6}.$$

There remains $zdN - Ndz = CdS$ and thus $\frac{N}{z} = C\int \frac{dS}{zz}$, from which the remainder is easily established.

SCHOLION 3

100. By progressing continually in this manner more shapes of the tubes are found, in which the minimal disturbance of the air will be allowed to be defined, and indeed any procedure adopted to supply the needs of the problem becomes infinitely greater than the preceding. Yet meanwhile an infinite number of tubes remain excluded, for which even now the motion of the air cannot be determined. Therefore after the figure of the cylinder, which happily we have been able to set out successfully, the conoidal hyperbolas scarcely demanding a more difficult solution than that, which shall need to be supplied likewise by the same first operation, thus so that these two figures shall be required to be referred to the first order. From the second order the conical figure is especially noteworthy contained by the equation $\Omega = Ass$, then truly also the hyperbola $\Omega = \frac{A}{s^4}$, as well as this appearing much more widely algebraically $\Omega = \frac{Ass}{\left(s^3+B\right)^2}$, besides which innumerable other transcendent equations have been elicited ; but for the higher orders the determination of the motion becomes continually more difficult, so that there shall be no reward for undertaking such labors. And thus I am going to submit only the most simple kinds for examination.

CAPUT IV

DE AGITATIONE MINIMA AERIS IN TUBIS INAEQUALITER AMPLIS

PROBLEMA 81

76. *Dum aër quomodocunque in tubo inaequaliter amplo movetur, eius motum ad formulas analyticas revocare, quibus eius determinatio ad quodvis tempus contineatur.*

SOLUTIO

Initio statum aëris tanquam cognitum spectamus; tum igitur pro loco tubi quocunque *S* vocata distantia $AS = S$ ponamus fuisse densitatem $= Q$ et celeritatem secundum tubi directionem $AB = \Upsilon$, ita ut *Q* et Υ sint functiones datae quantitatis *S;* cuius etiam functio erit amplitudo tubi in hoc loco, quae sit $= \Omega$, perinde atque eius altitudo super plano quodam horizontali fixo, quae sit $= Z$, siquidem et gravitatis rationem in motu aëris habere velimus. Iam elapso tempore $= t$ illam aëris particulam ponamus pervenisse in *s*, ut sit spatium $As = s$ et tubi amplitudo ω eiusque altitudo super illo plano horizontali $= z$, quae quantitates ω et *z* sunt functiones datae ipsius *s*, ipsa vero haec quantitas *s* est functio duarum variabilium *S* et *t;* nunc vero in hoc loco *s* sit aëris densitas $= q$, celeritas versus $B = \mathfrak{T}$, quae est $= \left(\frac{ds}{dt}\right)$, et pressio $= p$, quam per densitatem *q* ita determinari novimus, ut sit $p = \frac{aq}{b}$; densitas *q* est quoque functio binarum variabilium *S* et *t.* His positis principia supra stabilita pro hoc casu praebent has duas aequationes:

$$q\omega\left(\frac{ds}{dS}\right) = Q\Omega \text{ et } \frac{2gdp}{q} = -2gdz - ds\left(\frac{dds}{dt^2}\right) = \frac{2gadq}{bq},$$

in qua posteriori tempus *t* ut constans spectatur, quae propterea ita exhiberi potest

$$\frac{2ga}{bq}\left(\frac{dq}{dS}\right) + 2g\left(\frac{dz}{dS}\right) + \left(\frac{ds}{dS}\right)\left(\frac{dds}{dt^2}\right) = 0.$$

Ex priori autem habemus

$$lq = lQ + l\Omega - l\omega - \left(\frac{ds}{dS}\right),$$

unde solam *S* variabilem sumendo colligimus:

$$\frac{1}{q}\left(\frac{dq}{dS}\right) = \frac{dQ}{QdS} + \frac{d\Omega}{\Omega dS} - \frac{1}{\omega}\left(\frac{d\omega}{dS}\right) - \frac{\left(\frac{dds}{dS^2}\right)}{\left(\frac{ds}{dS}\right)}.$$

Quo valore substituto aequationem habebimus a *q* liberam hanc:

$$\frac{2ga}{bq}\left(\frac{dQ}{QdS}+\frac{d\Omega}{\Omega dS}\right)\left(\frac{ds}{dS}\right)-\frac{2ga}{b\omega}\left(\frac{d\omega}{dS}\right)\left(\frac{ds}{dS}\right)-\frac{2ga}{b}\left(\frac{dds}{dS^2}\right)$$
$$+2g\left(\frac{dz}{dS}\right)\left(\frac{ds}{dS}\right)+\left(\frac{ds}{dS}\right)^2\left(\frac{dds}{dt^2}\right)=0,$$

ubi Q et Ω sunt functiones datae ipsius S, quantitates vero ω et z functiones ipsius s, quae ipsa est functio binarum variabilium S et t, eiusque natura hinc determinari debet. Quodsi ergo ponatur $d\omega = uds$ et $dz = rds$, ut u et r sint functiones ipsius s datae, aequatio nostra hanc induet formam:

$$\frac{2ga}{b}\left(\frac{dQ}{QdS}+\frac{d\Omega}{\Omega dS}\right)\left(\frac{ds}{dS}\right)+2g\left(r-\frac{au}{b\omega}\right)\left(\frac{ds}{dS}\right)^2-\frac{2ga}{b}\left(\frac{dds}{dS^2}\right)+\left(\frac{ds}{dS}\right)^2\left(\frac{dds}{dt^2}\right)=0,$$

unde, qualis s sit functio ipsarum S et s, investigari oportet, ea autem inventa statim cognoscitur celeritas $\mathfrak{T}=\left(\frac{ds}{dt}\right)$, tum vero

densitas $q=\dfrac{Q\Omega}{\omega\left(\frac{ds}{dS}\right)}$ simulque pressio $p=\frac{aq}{b}$.

SCHOLION

77. Ex hac aequatione generali, quae omnes motus, qui in aërem in tubis quibuscunque cadere possunt, in se complectitur, satis liquet, quantopere etiamnunc ab eius solutione perfecta simus remoti, idquae ob solum Analyseos defectum. Cum enim functio s, quae quaeritur, praeter formulas differentiales etiam in litteris ω, u et r involvatur, haec tanta complexio in causa est, quod generalem nostri problematis solutionem nullo modo sperare queamus. Quare, ut supra iam observavimus, omnia, quae hic praestare licet, ad solos motus minimos restringuntur, cuiusmodi sunt generatio et propagatio soni, dum interea casus, qui alias simplicissimi videantur, veluti si aër in tubo comprimitur vel relaxatur, prorsus intactos relinquere cogimur. Mirum igitur est, quod ii casus, qui primo intuitu maxime ardui sunt visi et ab Auctoribus vix suscepti, nunc soli nostrae investigationi permittuntur, reliquis omnibus exclusis. Quocirca aequationem generalem hic inventam ad eum casum accommodabo, quo agitatio aëris ut minima spectari potest.

PROBLEMA 82

78. *Si aëris agitatio in tubo inaequaliter amplo excitata fuerit quam minima, aequationem invenire, qua huius motus continuatio continetur.*

SOLUTIO

Maneant omnes denominationes, uti in praecedente problemate sunt constitutae, et quia ibi erat $u=\frac{d\omega}{ds}$ et $r=\frac{dz}{ds}$, restituantur hi valores, insuperque brevitatis gratia ponatur

$\frac{2ga}{b}=cc$, ut c denotet spatium, per quod sonus uno minuto secundo propagatur; quo facto aequatio inventa hanc induet formam:

$$\frac{1}{c^2}\left(\frac{ds}{dS}\right)^2\left(\frac{dds}{dt^2}\right)=\left(\frac{dds}{dS^2}\right)+\frac{d\omega}{\omega ds}\left(\frac{ds}{dS}\right)^2-\frac{d\Omega}{\Omega dS}\left(\frac{ds}{dS}\right)-\frac{bdz}{ads}\left(\frac{ds}{dS}\right)^2-\left(\frac{dQ}{QdS}\right)\left(\frac{ds}{dS}\right),$$

qua resoluta porro habebitur pro tempore elapso $=t$:

celeritas , tum vero

$$\text{densitas}\quad q=\frac{Q\Omega}{\omega\left(\frac{ds}{dS}\right)},\quad \text{celeritas}\quad \mathfrak{T}=\left(\frac{ds}{dt}\right)\ \text{et pressio}\quad p=\frac{aq}{b}.$$

Iam ponamus $s=S+\upsilon$, ut sit eiusmodi functio ipsarum S et t, quae evanescat posito tempore $t=0$, quandoquidem tum fieri debet $s=S$, $q=Q$, $\omega=\Omega$ et $\mathfrak{T}=\Upsilon$, hicque spectamus υ ut quantitatem valde parvam prae S, vel saltem talem, ut $\left(\frac{d\upsilon}{ds}\right)$ prae unitate negligi queat. Cum igitur sit

$$\left(\frac{ds}{dt}\right)=\left(\frac{d\upsilon}{dt}\right),\ \left(\frac{dds}{dt^2}\right)=\left(\frac{dd\upsilon}{dt^2}\right),\ \left(\frac{ds}{dS}\right)=1+\left(\frac{d\upsilon}{dS}\right)\ \text{et}\ \frac{dds}{dS^2}=\left(\frac{dd\upsilon}{dS^2}\right);$$

ubi in nostra aequatione occurrit $\left(\frac{ds}{dS}\right)$, eius loco unitatem scribere licet, praeterquam in duobus terminis

$$\frac{d\omega}{\omega ds}\left(\frac{ds}{dS}\right)^2-\frac{d\Omega}{\Omega dS}\left(\frac{ds}{dS}\right),$$

quia hic ob differentiam inter s et S minimam fere est

$$\frac{d\omega}{\omega ds}=\frac{dQ}{\Omega dS},$$

ideoque prae differentia non amplius particula $\left(\frac{d\upsilon}{ds}\right)$ ut evanescens spectari potest. Qua circumstantia observata habebimus

$$\frac{1}{c^2}\left(\frac{dd\upsilon}{dt^2}\right)=\left(\frac{dd\upsilon}{dS^2}\right)+\frac{d\omega}{\omega ds}-\frac{d\Omega}{\Omega dS}+\frac{d\Omega}{\Omega dS}\left(\frac{d\upsilon}{dS}\right)-\frac{bdz}{ads}-\left(\frac{dQ}{QdS}\right),$$

ubi, cum puncta S et s sibi sint proxima, loco $\frac{dz}{ds}$ scribere licet $\frac{dZ}{dS}$ functionem ipsius S tantum. Deinde ob eandem rationem, quia formula $\frac{d\omega}{\omega ds}$ nascitur ex formula $\frac{d\Omega}{\Omega dS}$, si hic loco S scribatur $S+\upsilon$, erit

$$\frac{d\omega}{\omega ds}=\frac{d\Omega}{\Omega dS}+\frac{\upsilon}{dS}d\cdot\frac{d\Omega}{\Omega dS}.$$

Ponamus ergo brevitatis gratia $\frac{d\Omega}{\Omega dS} = U$, quae erit functio data ipsius *S*, unde fit amplitudo

$$\Omega = Ae^{\int Uds},$$

fietque nostra aequatio:

$$\frac{1}{c^2}\left(\frac{dd\upsilon}{dt^2}\right) = \left(\frac{dd\upsilon}{dS^2}\right) + U\left(\frac{d\upsilon}{dS}\right) + \frac{\upsilon dU}{dS} - \frac{bdZ}{ads} - \frac{dQ}{QdS},$$

qua resoluta erit pro motus determinatione:

$$q = \frac{Q\Omega}{\omega\left(1+\left(\frac{d\upsilon}{dS}\right)\right)} = \frac{Q}{(1+U\upsilon)\left(1+\left(\frac{d\upsilon}{dS}\right)\right)} = Q\left(1 - U\upsilon - \left(\frac{d\upsilon}{dS}\right)\right),$$

quia est

$$\omega = \Omega + \frac{\upsilon d\Omega}{dS} = Ae^{\int UdS}\left(1+U\upsilon\right),$$

celeritas $\mathfrak{T} = \left(\frac{ds}{dt}\right)$ et pressio $p = \frac{aq}{b}$.

COROLLARIUM 1

79. Totum ergo negotium huc redit, ut ex aequatione differentiali secundi gradus inventa investigetur, qualis functio sit quantitas υ binarum variabilium *S* et *t*, ubi quidem observo binos terminos postremos $-\frac{bdZ}{ads} - \frac{dQ}{QdS}$ resolutionem non impedire, quoniam functionem solius variabilis *S* continent.

COROLLARIUM 2

80. Conditiones autem, sub quibus aequationis inventae integrale completum eruere licet ope methodorum quidem adhuc cognitarum, ab indole functionis *U*, qua ea per variabilem *S* definitur, pendent. Pro eiusque natura fieri potest, ut integratio modo succedat, modo calculi vires superet.

SCHOLION

81. Hic igitur confugiendum est ad ea analyseos sublimioris, quae circa functiones duarum variabilium versatur, artificia, quibus huiusmodi aequationes

$$\frac{1}{cc}\left(\frac{dd\upsilon}{dt^2}\right) = \left(\frac{dd\upsilon}{dS^2}\right) + U\left(\frac{d\upsilon}{dS}\right) + \upsilon T$$

tractare docui; ubi in eos casus functionum *U* et *T*, quae solam variabilem *S* involvere assumuntur, inquiri oportet, quibus integrale completum exhibere licet. Ante omnia autem hic observari oportet, quoties haec integratio succedit per methodos quidem cognitas, integrale semper huiusmodi forma exprimi, ut sit

$$\upsilon = Lf:(S+ct) + Mf':(S\pm ct) + Nf'':(S\pm ct) + \text{etc.},$$

ubi circa haec functionum signa tenendum est: si fuerit

$$f:u = V, \quad \text{esse} \quad f':u = \tfrac{dV}{du}, \quad f'':u = \tfrac{ddV}{du^2} \quad \text{etc.}$$

Hinc patet infinitas solutiones locum habere posse, prout huius formae progressio ulterius continuetur. Primam ergo solutionem seu primos integrabilitatis casus ex solo primo termino huius progressionis investigabo, deinde duos eius terminos in subsidium vocando secundos integrabilitatis casus eliciam, sicque porro ad altiores ascendere licebit, plures continuo terminos accipiendo.

PROBLEMA 83

82. *Invenire rationem amplitudinis tubi, in quo aër minimas peragit agitationes, ut motus determinatio per primam solvendi methodum succedat.*

SOLUTIO

Cum posita tubi amplitudine $\Omega = Ae^{\int Uds}$, haec aequatio integrari debeat:

$$\tfrac{1}{c^2}\left(\tfrac{dd\upsilon}{dt^2}\right) = \left(\tfrac{dd\upsilon}{dS^2}\right) + U\left(\tfrac{d\upsilon}{dS}\right) + \tfrac{\upsilon dU}{dS} - \tfrac{bdZ}{ads} - \tfrac{dQ}{QdS},$$

statuamus $\upsilon = L\,f:(S+ct)+O$, ubi L et O tantum sint functiones ipsius S, et facta substitutione obtinebimus:

$$\begin{aligned}
\left(\tfrac{dd\upsilon}{dS^2}\right) &= Lf'':(S+ct) + \tfrac{2dL}{dS}f':(S+ct) + \tfrac{2ddL}{dS^2}f:(S+ct) + \tfrac{ddO}{dS^2}\\
U\left(\tfrac{d\upsilon}{dS}\right) &= \qquad\qquad +ULf':(S+ct) + \tfrac{UdL}{dS}f:(S+ct) + \tfrac{UdO}{dS}\\
\tfrac{\upsilon dU}{dS} &= \qquad\qquad\qquad\qquad + \tfrac{LdU}{dS}f:(S+ct) + \tfrac{OdU}{dS}\\
-\tfrac{bdZ}{adS} - \tfrac{dQ}{QdS} &= \qquad\qquad\qquad\qquad\qquad - \tfrac{bdZ}{ads} - \tfrac{dQ}{QdS},
\end{aligned}$$

quae iunctim sumta ipsi $\tfrac{1}{c^2}\left(\tfrac{dd\upsilon}{dt^2}\right) = Lf'':(S+ct)$ aequari oportet: unde nascuntur hae aequationes:

$$\text{I.}\quad 2\tfrac{dL}{dS}+LU=0$$
$$\text{II.}\quad \tfrac{ddL}{dS^2}+\tfrac{UdL+LdU}{dS}=0$$
$$\text{III.}\quad \tfrac{ddO}{dS^2}+\tfrac{UdO+OdU}{dS}-\tfrac{bdZ}{adS}-\tfrac{dQ}{QdS}=0$$

quarum secunda integrata dat

$$\tfrac{dL}{dS}+LU=C,$$

quae cum prima collata praebet $-\frac{dL}{dS}=C,$ hincque

$$L=\alpha S+\beta \quad \text{et} \quad U=\tfrac{-2\alpha dS}{\alpha S+\beta}=\tfrac{d\Omega}{\Omega dS}.$$

Quare, cum sit

$$\tfrac{d\Omega}{\Omega}=\tfrac{-2\alpha dS+\beta}{\alpha S+\beta}$$

fiet amplitudo tubi

$$\Omega=\frac{ff}{(\alpha S+\beta)^2},$$

solutionem ope primae methodi admittens. Tertia vero aequatio integrata dat:

$$\tfrac{dO}{dS}+UO-\tfrac{b}{\alpha}Z-lQ=C$$

seu ob $U=\frac{-2\alpha}{\alpha S+\beta}$

$$dO-\tfrac{2\alpha OdS}{\alpha S+\beta}-\tfrac{b}{a}ZdS-dSlQ=CdS,$$

quae per $(\alpha S+\beta)^2$ divisa et integrata producit

$$\frac{O}{(\alpha S+\beta)^2}=\int\frac{dS\left(\frac{b}{a}Z+lQ+C\right)}{(\alpha S+\beta)^2}.$$

Quare, si amplitudo tubi ita sit variabilis, ut longitudini $AS=S$ respondeat amplitudo $\Omega=\frac{ff}{(\alpha S+\beta)^2},$ tum aequationis motum determinantis integrale completum erit

$$\upsilon=(\alpha S+\beta)^2\int\frac{dS\left(\frac{b}{a}Z+lQ+C\right)}{(\alpha S+\beta)^2}+(\alpha S+\beta)\Gamma:(S+ct)+(\alpha S+\beta)\Delta:(S-ct),$$

quandoquidem functionem assumtam $f:(S+ct)$ geminare licet introducendo tam $-c$ quam $+c$.

COROLLARIUM 1

83. In hac solutione statim continetur casus tuborum aeque amplorum constantes ita definiendo, ut sit $\alpha = 0$ et $\beta = 1$. Haec autem solutio multo latius patet, cum eius ope agitationes aëris in tubis eiusmodi inaequaliter amplis quoque definiri queant, quorum amplitudo in hac formula continetur

$$\Omega = \frac{ff}{(\alpha S+\beta)^2}.$$

COROLLARIUM 2

84. Inventa autem hac functione υ elapso tempore *t* aër, qui initio erat ad *S*, translatus erit per intervallum $Ss = \upsilon,$ tum vero eius densitas ob $U = \frac{-2\alpha}{\alpha S+\beta}$ erit

$$q = Q\left(1+\tfrac{2\alpha\upsilon}{\alpha S+\beta}-\left(\tfrac{d\upsilon}{dS}\right)\right) \text{ et celeritas } \mathfrak{T} = \left(\tfrac{d\upsilon}{dt}\right).$$

COROLLARIUM 3

85. Cum autem differentiando sit:

$$\upsilon = 2\alpha\left(\alpha S+\beta\right)\int\frac{dS\left(\frac{b}{\alpha}Z+lQ+C\right)}{(\alpha S+\beta)^2} +C+\tfrac{b}{\alpha}Z+lQ+\alpha\Gamma:\left(S+ct\right)+\alpha\varDelta:\left(S-ct\right)$$
$$+\left(\alpha S+\beta\right)\Gamma':\left(S+ct\right)+\left(\alpha S+\beta\right)\varDelta':\left(S-ct\right),$$

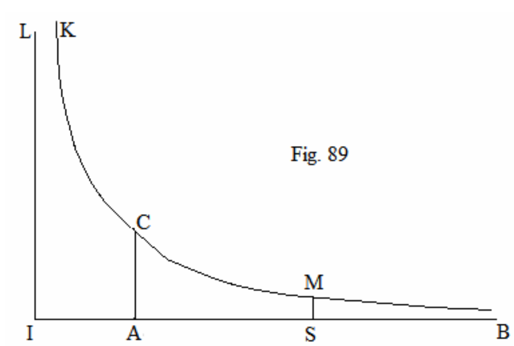
habebitur pro densitate:

$$\tfrac{q}{Q} = 1-C-\tfrac{b}{a}Z-lQ+\ \alpha\Gamma:\left(S+ct\right)+\delta\varDelta:(S-ct)$$
$$-\left(\alpha S+\beta\right)\Gamma':\left(S+ct\right)-(\alpha S+\beta)\varDelta':(S-ct)$$

et pro celeritate:

$$\mathfrak{T} = c\left(\alpha S+\beta\right)\Gamma':\left(S+ct\right)-c(\alpha S+\beta)\varDelta':(S-ct).$$

SCHOLION



85. Si tubos, ad quos haec solutio est accommodata, rotundos statuamus, ut omnes sectiones ad eius directionem normaliter factae Fig. 89 sint circuli, eorum figura (Fig.89) est conoidica hyperbolica conversione hyperbolae aequilaterae *KCM* circa alteram asymtotam *IB* nata. Cum enim in hac hyperbola sit $SM \cdot IS = a,$ erit amplitudo in

$S=\pi SM^2=\frac{\pi aa}{IS^2}$, sumto ergo intervallo $IA=\frac{\beta}{\alpha}$ et posito $AS=S$, ob $IS=\frac{\alpha S+\beta}{\alpha}$ fiet amplitudo $\Omega=\frac{\pi\alpha\alpha aa}{(\alpha S+\beta)^2}$, ideoque $ff=\pi\alpha\alpha aa.$ Quoties ergo tubus habuerit huiusmodi figuram conoidicam hyperbolicam, aëris agitationes in huiusmodi tubis, dummodo sint minimae, perinde definiri poterunt, atque in tubis aequaliter amplis.
Interim tamen ipsa motus determinatio aliquanto erit operosior. Ceterum hic observari convenit aëris gravitatem, quam in superioribus capitibus neglegimus, investigationem plane non turbare: id quod etiam de aliis viribus, quae forte aërem sollicitarent, est tenendum.

PROBLEMA 84

86. *Investigare rationem, qua tubi amplitudo debet esse comparata, ut aequationis differentiodifferentialis integrale completum per secundam formam exhiberi possit.*

SOLUTIO

Hic scilicet quaeritur indoles functionis *U* a sola variabili *S* pendentis, ut aequationis nostrae integrale completum huiusmodi forma exprimi queat:

$$\upsilon=O+Mf':(S+ct)+L\,f:(S+ct),$$

ubi *L*, *M*, *O* sint functiones solius variabilis *S*. Faciamus ergo substitutionem in nostra aequatione ac reperiemus:

$$\begin{array}{llll}
0=\left(\frac{ddO}{dS^2}\right)+Mf''':(S+ct)+ & \frac{2dM}{dS}\,f'':(S+ct)+ & \frac{2ddM}{dS^2}\,f':(S+ct) & \\
+U\frac{dO}{dS} & +L & +\frac{2dL}{dS} & +\frac{ddL}{dS^2}\,f:(S+ct) \\
+\frac{OdU}{dS} & +UM & +\frac{UdM}{dS} & \\
-\frac{bdZ}{ads} & & +UL & +\frac{UdL}{dS} \\
-\frac{dQ}{QdS}-Mf''':(S+ct)-Lf'':(S+ct) & & +\frac{MdU}{dS} & +\frac{LdU}{dS},
\end{array}$$

cuius singula membra, quatenus diversas functiones complectuntur, seorsim ad nihilum redigi oportet: ex quo sequentes quatuor aequationes nascuntur:

$$\text{I.}\quad 2\frac{dM}{dS}+UM=0$$

$$\text{II.}\quad \frac{ddM}{dS^2}+2\frac{dL}{dS}+\frac{UdM+MdU}{dS}+UL=0$$

$$\text{III.}\quad \frac{ddL}{dS^2}+\frac{UdL+LdU}{dS}=0$$

$$\text{IV.}\quad \frac{ddO}{dS}+UdO+OdU-\frac{b}{a}dZ-\frac{dQ}{Q}=0.$$

Tertia integrata dat $\frac{dL}{dS}+UL=A,$ quae cum prima combinata eliminando U praebet $MdL-2LdM=AMdS,$ unde colligitur

$$L=AMM\int\frac{dS}{MM}\ \text{ et }\ U=-\frac{2dM}{MdS}.$$

His valoribus in secunda aequatione substitutis pervenitur ad hanc aequationem

$$-\frac{ddM}{dS^2}+2A+\frac{2AMdM}{dS}\int\frac{dS}{MM}=0.$$

Ad hanc resolvendam sit $\int\frac{dS}{MM}=R$, hincque $dS=MMdR$, et quia est $\frac{ddM}{dS}=d\cdot\frac{dM}{dS}=d\cdot\frac{dM}{MMdR}$, illa aequatio per dS multiplicata praebet

$$-d\cdot\frac{dM}{MMdR}+2AMMdR+2AMRdM=0,$$

qua resoluta habebitur

$$dS=MMdR,\ L=AMMR\ \text{ et }\ U=-\frac{2dM}{M^3dR}.$$

Verum ista aequatio per R multiplicata integrabilis redditur, cum sit

$$\int Rd\,\frac{dM}{MMdR}=\frac{RdM}{MMdR}+\frac{1}{M},$$

ideoque habebimus:

$$\frac{RdM}{MMdR}+\frac{1}{M}=AMMRR+B=+\frac{MdR+RdM}{MMdR}.$$

Ponatur denique $MR=x$ seu $M=\frac{x}{R}$, erit

$$Axx+B=\frac{RRdx}{xxdR}$$

hincque

$$\frac{dR}{RR}=\frac{dx}{xx(Axx+B)}\ \text{ seu }\ \frac{BdR}{RR}=\frac{dx}{xx}-\frac{Adx}{Axx+B}$$

ideoque

$$\frac{B}{R}=\frac{1}{x}+\int\frac{Adx}{Axx+B},$$

unde R datur per x, tum ob $M=\frac{x}{R}$ reliquae quantitates omnes per x dabuntur: fit autem

$$dS=\frac{xxdR}{RR}=\frac{dx}{Axx+B}\ \text{ et }\ UdS=\frac{d\Omega}{\Omega}=-\frac{2dM}{M},$$

ita ut sit amplitudo

$$\Omega=\frac{C}{MM}=\frac{CRR}{xx}.$$

Evolvamus hinc casus, quibus amplitudo Ω per variabilem *S* algebraice definitur, quod fit, si constans $B = 0$, tum enim erit

$$\frac{dR}{RR} = \frac{dx}{Ax^4} \text{ ideoque } \frac{1}{R} = \frac{1+Dx^3}{3Ax^3} \text{ et } R = \frac{3Ax^3}{1+Dx^3} \text{ atque } M = \frac{1+Dx^3}{3Axx}.$$

Cum autem sit $dS = \frac{dx}{Axx+B}$ erit $\frac{1}{x} = -AS - E$, seu $x = -\frac{1}{AS+E}$
hincque

$$M = \frac{(AS+\mathrm{E})^3 - D}{3A(AS+E)} \text{ et } R = -\frac{3A}{(AS+\mathrm{E})^3 - D};$$

unde porro colligimus

$$\Omega = \frac{9AAC(AS+\mathrm{E})^2}{\left((AS+E)^3 - D\right)^2} \text{ et } L = \frac{-(AS+E)^3 + D}{(AS+E)^2}.$$

Denique pro quantitate *O* obtinemus:

$$\frac{dO}{dS} + UO - \frac{b}{a}Z - lQ + lF = 0$$

seu

$$dO + \frac{Od\Omega}{\Omega} - \frac{b}{a}ZdS - dS\, l\frac{Q}{F} = 0.$$

Ergo

$$O = \frac{1}{\Omega}\int \Omega dS\left(\frac{b}{a}Z + l\frac{Q}{F}\right).$$

Immutemus parumper constantes, et quando amplitudo tubi *Q* pro abscissa $AS = S$ ita fuerit comparata, ut sit

$$Q = \frac{ff(\alpha S+\beta)^2}{\left((\alpha S+\beta)^3 - \gamma\right)^2},$$

tum sumtis

$$O = \frac{1}{\Omega}\int \Omega dS\left(\frac{b}{a}Z + l\frac{Q}{B}\right),$$

$$M = \frac{(\alpha S+\beta)^3 - \gamma}{3\alpha(\alpha S+\beta)} \text{ et } L = \frac{-(\alpha S+\beta)^3 + \gamma}{3(\alpha S+\beta)^2},$$

erit nostrae aequationis integrale completum,

$$\upsilon = O + M\Gamma' : (S+ct) + L\Gamma : (S-ct)$$
$$+ M\Delta' : (S-ct) + L\Delta : (\mathrm{S}-ct),$$

unde porro definitur

$$\text{densitas } q = Q\left(1 - \frac{\upsilon d\Omega}{\Omega dS}\right) \text{ et celeritas } \mathfrak{T} = \left(\frac{d\upsilon}{dt}\right).$$

COROLLARIUM 1

87. Hic primum observandum est quantitates pro *L* et *M* inventas per quantitatem quamcunque constantem multiplicari posse, quam functiones indefinitae complecti sunt censendae, hinc sumi poterit

$$M = \frac{(\alpha S+\beta)^3-\gamma}{3\alpha(\alpha S+\beta)} \text{ et } L = \frac{-(\alpha S+\beta)^3+\gamma}{3(\alpha S+\beta)^2},$$

quod tenendum est, si forte illi valores fiant infiniti.

COROLLARIUM 2

88. Si enim capiatur $\gamma = \infty$ et $f = n\gamma$, ut amplitudo fiat

$$\Omega = nn(\alpha S+\beta)^2,$$

sumatur quoque $\delta = \gamma$, eritque

$$M = \frac{-1}{\alpha(\alpha S+\beta)} \text{ et } L = \frac{1}{(\alpha S+\beta)^2}.$$

Hoc autem casu tubi habebuntur conici vel pyramidici, in quibus ergo pariter agitationes aëris minimas definire licebit.

COROLLARIUM 3

89. Si sumatur $\gamma = 0$, tubus ita erit formatus ut sit

$$\Omega = \frac{ff}{(\alpha S+\beta)^4}$$

seu amplitudines erunt reciproce ut biquadrata abscissarum: tum autem fit

$$M = \tfrac{1}{\alpha}(\alpha S+\beta)^2 \text{ et } L = -(\alpha S+\beta).$$

COROLLARIUM 4

90. In genere autem haec solutio ad eiusmodi tubos conoidicos (Fig. 90) applicari potest, qui oriuntur ex conversione huiusmodi curvae hyperbolicae circa axem *IB*, pro qua posita abscissa $AS = x$ et applicata $SM = y$, sit $y = \frac{a^3x}{x^3+b^3}$: quae curva axem in *A* secans duas habet asymtotas *GH* et *KL* inter se normales.

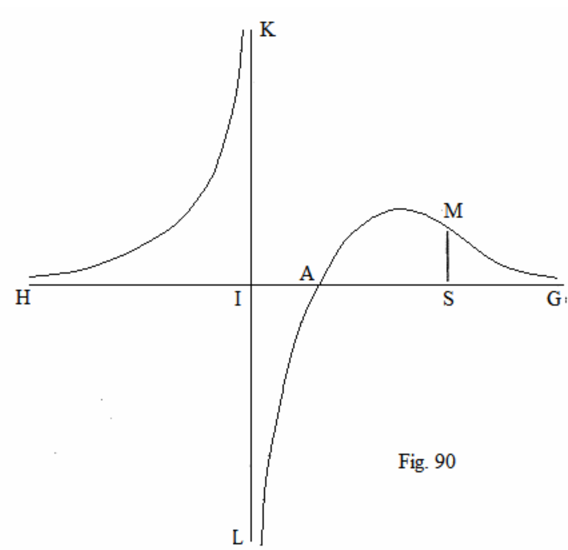


Fig. 90

SCHOLION 1

91. Si etiam figuras tubi transcendentes cognoscere velimus, faciamus $B = Amm$ et $A = \frac{\alpha}{m}$, et aequatio $dS = \frac{dx}{Axx+B}$ seu $\alpha dS = \frac{mdx}{xx+mm}$ dabit $x = m\text{tang}(\alpha S + \beta)$, unde porro colligitur

$$\frac{\alpha mm}{R} = \alpha S + \gamma + \cot\left(\alpha S + \beta\right) \text{ seu } R = \frac{\alpha mm}{\alpha S+\gamma+\cot\left(\alpha S+\beta\right)}$$

hincque

$$M = \frac{1+\left(\alpha S+\gamma\right)\text{tang}\left(\alpha S+\beta\right)}{\alpha m} \quad \text{et} \quad L = \frac{1+\left(\alpha S+\gamma\right)\text{tang}\left(\alpha S+\beta\right)}{m\cot\left(\alpha S+\beta\right)}.$$

Deinde fit amplitudo

$$\Omega = \frac{C}{1+\left(\alpha S+\gamma\right)\text{tang}\left(\alpha S+\beta\right)},$$

quantitas vero O hinc definitur ut ante

$$O = \frac{1}{\Omega}\int \Omega dS\left(\frac{b}{a}Z + l\frac{Q}{B}\right),$$

Ex his autem formis colligimus methodum aequationes primum erutas multo facilius resolvendi. Cum enim prima det $U = -\frac{2dM}{MdS}$, tertia vero integrata $\frac{dL}{dS} + UL = A$, erit

$$dL - \frac{2LdM}{M} = AdS,$$

ponamus $L = My$, fietque $Mdy - ydM = AdS$. Nunc secunda, quod fieri potest, integrata praebet

$$\frac{dM}{dS} + 2L + UM + \int ULdS = 0,$$

quae ob

$$UM = -\frac{2dM}{dS} \text{ et } \int ULdS = -2\int\frac{LdM}{M} = -2\int ydM$$

transit in:

$$\frac{dM}{dS} + 2My - \frac{2dM}{dS} - 2\int ydM = 0,$$

quae contrahitur in hanc $2\int Mdy - \frac{dM}{dS} = 0.$ Ex illa autem fit

$$2Mdy = AdS + ydM + Mdy,$$

ita ut sit

$$2\int Mdy = AS + My + B = \tfrac{dM}{dS},$$

unde, cum parum colligere liceat, consideremus has duas aequationes:

$$Mdy - ydM = AdS \quad \text{et} \quad 2\int Mdy = \tfrac{dM}{dS},$$

quas ambas differentiemus sumto elemento dS constante

$$Mddy\ yddM = 0 \quad \text{et} \quad 2Mdy = \tfrac{ddM}{dS},$$

unde fit

$$\tfrac{ddM}{M} = \tfrac{ddy}{y} = 2dydS \quad \text{seu} \quad ddy = 2ydydS,$$

cuius integrale est $dy = dS\left(yy + mm\right)$, hincque porro

$$\text{Ang tang}\,\tfrac{y}{m} = mS + n \quad \text{et convertendo} \quad y = m\text{tang}(mS+n).$$

Iam ex aequatione

$$\tfrac{Mdy - ydM}{yy} = \tfrac{AdS}{yy} = \tfrac{AdS}{mm\,\text{tang}^2\left(mS+n\right)}$$

fit

$$-\tfrac{M}{y} = -\tfrac{M}{m\text{tang}^2(mS+n)} = \tfrac{A}{mm}\int \tfrac{dScos^2(mS+n)}{\sin^2(mS+n)} = \tfrac{A}{mm}\int \tfrac{dS}{\sin^2(mS+n)} - \tfrac{AS}{mm}$$
$$= -\tfrac{A}{m^3\text{tang}(mS+n)} - \tfrac{AS}{mm} - \tfrac{Ak}{m^3},$$

unde fit

$$M = \tfrac{A}{mm}\Big(1 + \left(mS + k\right)\text{tang}(mS+n)\Big),$$

hincque porro

$$L = \tfrac{A\left(1 + \left(mS+k\right)\text{tang}(mS+n)\right)}{m\cot(mS+n)}$$

ac denique prodit amplitudo

$$\Omega = \tfrac{\text{Const.}}{MM} \quad \text{seu} \quad \Omega = \tfrac{ff}{\left(1 + \left(mS+k\right)\text{tang}^2\left(mS+n\right)\right)}.$$

SCHOLION *2*

92. Adhuc facilius hanc resolutionem instituere licet hoc modo: cum ex prima aequatione sit $U = -\tfrac{2dM}{MdS}$, ponatur statim $L = My$, eritque

$$MU = -2\tfrac{dM}{dS} \quad \text{et} \quad LU = -\tfrac{2ydM}{dS},$$

ex quo secunda aequatio fiet

$$\frac{ddM}{dS}+2d\cdot My-2d\cdot\frac{dM}{dS}-2ydM=0$$

seu

$$-\frac{ddM}{dS}+2Mdy=0 \quad \text{seu} \quad \frac{ddM}{dS}=2dydS,$$

simili modo tertia abit in hanc formam:

$$\frac{dd\cdot My}{dS}-2d\cdot\frac{ydM}{dS}=0$$

factaque evolutione

$$Mddy+2dMdy+yddM-2dydM-2yddM=0,$$

hoc est

$$Mddy-yddM=0 \text{ seu } \frac{ddM}{M}=\frac{ddy}{y}.$$

Ex his coniunctis fit ut ante $ddy=2ydydS$. Quodsi hinc statim tubi formas algebraicas elicere velimus, sumatur superior constans $m=0$, ut sit $dy=yydS$ ideoque

$$\frac{1}{y}=-S-\frac{\beta}{\alpha} \text{ seu } y=\frac{-\alpha}{\alpha S+\beta}.$$

Tum aequatio $Mdy-ydM=AdS$ dat

$$-\frac{M}{y}=\frac{A}{\alpha\alpha}\int dS\left(\alpha S+\beta\right)^2=\frac{A}{3\alpha^3}\left(\left(\alpha S+\beta\right)^3-\gamma\right)$$

ideoque

$$M=\frac{A}{3\alpha\alpha(\alpha S+\beta)} \quad \text{et } L=-\frac{A\left((\alpha S+\beta)^3-\gamma\right)}{3\alpha(\alpha S+\beta)^2},$$

ac denique amplitudo

$$\Omega=\frac{C(\alpha S+\beta)^2}{\left((\alpha S+\beta)^3-\gamma\right)^2}$$

prorsus ut ante.

SCHOLION 3

93. Praeter hos duos casus autem, quibus forma tubi vel est algebraica vel a circulo pendet, tertium non omitti convenit, quo ea per quantitates exponentiales determinatur. Pro eo autem aequationis $ddy=2ydydS$ integrale sumitur $dy=dS(yy-mm)$, unde fit

$$2mS+2n=l\,\frac{y-m}{y+m},$$

unde fit

$$y=\frac{1+e^{2mS+2n}}{1-e^{2mS+2n}}m,$$

tum vero porro

$$-\frac{M}{y}=\frac{A}{mm}\int\frac{dS\left(1-e^{2mS+2n}\right)^2}{\left(1+e^{2mS+2n}\right)^2}=\frac{A}{m^3}\cdot\frac{e^{2mS+2n}(mS+k+1)+mS+k+1}{1+e^{2mS+2n}}$$

ideoque

$$M=-\frac{A}{mm}\cdot\frac{e^{2mS+2n}(mS+k-1)+mS+k+1}{1-e^{2mS+2n}}$$

et

$$L=-\frac{A}{mm}\cdot\frac{e^{2mS+2n}(mS+k-1)+mS+k+1}{\left(1-e^{2mS+2n}\right)^2}\left(1+e^{2mS+2n}\right)$$

atque amplitudo

$$\Omega=\frac{C\left(1-e^{2mS+2n}\right)^2}{\left(e^{2mS+2n}(mS+k-1)+mS+k+1\right)^2}.$$

Certum est ex hoc casu perinde atque ex priori a circulo pendente casum algebraicum prodire debere, si statuatur $m=0$, quae tamen evolutio minus est obvia et haud exiguam industriam Analystae postulat.

PROBLEMA 85

92. *Investigare casus pro tubi amplitudine, ut aequatio differentio-differentialis inventa per operationes tertii ordinis integrari queat vel eius integrale completum per tertiam formam supra explicatam exhiberi queat.*

SOLUTIO

Hoc ergo casu integrale quaesitum talem formam habere debebit:

$$\upsilon=O+Nf'':(S+ct)+Mf':(S+ct)+Lf:(S+ct),$$

et quia iam patet quantitatem *O* ut ante definitum iri, tantum diversas functionum species notemus:

$f'''':(S+ct)$	$f''':(S+ct)$	$f'':(S+ct)$	$f':(S+ct)$	$f:(S+ct)$
$+N$	$+\frac{2dN}{dS}$	$+\frac{ddN}{dS^2}$		
	$+M$	$+\frac{2dM}{dS}$	$+\frac{ddM}{dS^2}$	
		$+L$	$+\frac{2dL}{dS}$	$+\frac{ddL}{dS^2}$
	$+UN$	$+\frac{UN}{dS}$		
		$+UM$	$+\frac{UM}{dS}$	
			$+UL$	$+\frac{UL}{dS}$
		$+\frac{NdU}{dS}$	$+\frac{MdU}{dS}$	$+\frac{LdU}{dS}$
$-N$	$-M$	$-L$		

unde deducimus quatuor sequentes aequationes:

$$\text{I.}\quad \frac{2dN}{dS}+UN=0$$

$$\text{II.}\quad \frac{ddN}{dS^2}+\frac{2dM}{dS}+\frac{UdN+NdU}{dS}+UM=0$$

$$\text{III.}\quad \frac{ddM}{dS^2}+\frac{2dL}{dS}+\frac{UdM+MdU}{dS}+UL=0$$

$$\text{IV.}\quad \frac{ddL}{dS^2}+\frac{UdL+\mathrm{L}dU}{dS}=0.$$

Cum nunc ex prima sit $U=\frac{-2dN}{NdS}$, ponamus $M=Ny$ et $L=Nz$, ut fiat $UN=\frac{-2dN}{dS}$, $UM=\frac{-2ydN}{dS}$ et $UL=\frac{-2zdN}{dS}$

quibus valoribus substituendis aequatio II. fit

$$\frac{ddN}{dS^2}+\frac{2Ndy+2ydN}{dS}-\frac{2ddN}{dS^2}-\frac{2ydN}{dS}=0,$$

quae contrahitur in hanc formam:

$$\text{II.}\quad \frac{-ddN}{dS}+2Ndy=0.$$

Tertia aequatio vero hinc evadit:

$$\frac{Nddy+2dNdy+yddN}{dS^2}+\frac{2Ndz+2zdN}{dS}-\frac{2yddN-2dydN}{dS^2}-\frac{2zdN}{dS}=0,$$

quae contrahitur in hanc formam:

$$\text{III.}\quad \frac{Nddy-yddN}{dS}+2Ndz=0.$$

Quarta denique ita evolvitur:

$$\frac{Nddz+2dNdz+zddN}{dS^2}-\frac{2zddN-2dzdN}{dS^2}=0,$$

quae ergo praebet hanc formam:

$$\text{IV.}\quad \frac{Nddz-zddN}{dS}=0.$$

Ex his triplici modo fit:

$$\frac{ddN}{N}=2dydS=\frac{ddy+2dzdS}{y}=\frac{ddz}{z}$$

unde hae duae aequationes resolvendae proponuntur:

$$2ydydS = ddy + 2dzdS \text{ et } yddz - zddy = 2zdzdS,$$

quarum posterior integrata statim dat

$$ydz - zdy = dS\left(zz + A\right),$$

prior vero pariter integrata praebet:

$$dy + 2zdS = dS(yy+B)$$

seu $dy = dS\left(yy - 2z+B\right)$ iam illa per hanc divisa suppeditat hanc aequationem ab elemento *dS* liberam

$$\frac{ydz - zdy}{dy} = \frac{zz+A}{yy-2z+B};$$

sit $z = uy$ et haec aequatio fit

$$\frac{yydu}{dy} = \frac{uuyy + A}{yy-2uy+B}$$

seu $y^4du - 2uy^3du + Byydu - uuyydy - Ady = 0.$

Necessitas hic urget, ut faciamus $A = 0$ et $B = 0$; quo facto habebimus:

$$yydu - 2uydu - uudy = 0 \text{ seu } yydu = d\cdot uuy;$$

dividamus per u^4y^2, erit $\frac{du}{u^4} = \frac{d\cdot uuy}{u^4yy}$ eiusque integrale:

$$\frac{1}{3u^3} = \frac{1}{uuy} - \frac{1}{3m^3} \text{ seu } y = \frac{3m^3u}{m^3+u^3} \text{ et } z = \frac{3m^3uu}{m^3+u^3},$$

unde fit

$$dS = \frac{dy}{yy-2uy} = \frac{ydz-zdy}{zz}$$

sive

$$dS = \frac{du}{uu} = -d\cdot\frac{y}{z},$$

ita ut sit

$$\frac{1}{u} = -S - \frac{\beta}{\alpha} \text{ et } u = \frac{-\alpha}{\alpha S+\beta}.$$

Ergo iam per *S* habebimus hos valores:

$$y = \frac{-3\alpha m^3\left(\alpha S+\beta\right)^2}{m^3\left(\alpha S+\beta\right)^3-\alpha^3} \text{ et } z = \frac{3\alpha^2m^3\left(\alpha S+\beta\right)}{m^3\left(\alpha S+\beta\right)^3-\alpha^3}.$$

Ponamus brevitatis gratia $\frac{\alpha^3}{m^3}=\gamma^3$, ut fiat

$$y=\frac{-3\alpha(\alpha S+\beta)^2}{(\alpha S+\beta)^3-\gamma^3} \quad \text{et} \quad z=\frac{3\alpha\alpha(\alpha S+\beta)}{(\alpha S+\beta)^3-\gamma^3}.$$

Nunc porro ex aequatione IV colligimus:

$$Ndz-zdN=AdS=\frac{Adu}{uu},$$

quae ob

$$z=\frac{3\alpha^3uu}{\alpha^3+\gamma^3u^3} \text{ divisa per } zz \text{ dat}$$

$$-d.\frac{N}{z}=\frac{Bdu}{u^6}\left(\alpha^3+\gamma^3u^3\right)^2,$$

et integrando

$$-\frac{N}{z}=C+B\left(\gamma^6u-\frac{\alpha^3\gamma^3}{u^2}-\frac{\alpha^6}{5u^5}\right),$$

hinc $N=\frac{Bz}{5u^5}\left(\alpha^6+5\alpha^3\gamma^3u^3-5\gamma^3u^6\right)-Cz$ et evolvendo

$$N=\frac{A(\alpha S+\beta)^6-5A\gamma^3(\alpha S+\beta)^3-5A\gamma^6+B(\alpha S+\beta)}{(\alpha S+\beta)^3-\gamma^3}$$

mutatis scilicet constantibus, unde statim definitur amplitudo tubi $\Omega=\frac{C}{NN}$, tum vero erit

$$M=\frac{-3\alpha(\alpha S+\beta)^2}{(\alpha S+\beta)^3-\gamma^3}N \quad \text{et} \quad L=\frac{3\alpha\alpha(\alpha S+\beta)}{(\alpha S+\beta)^3-\gamma^3}N.$$

Pro quantitate autem O ut ante est

$$O=\frac{1}{\Omega}\int\Omega ds\left(\frac{b}{a}z+l\frac{Q}{B}\right).$$

Quoties ergo amplitudo Ω hoc modo definitur, solutio completa nostri problematis ita se habebit, ut sit:

$$\upsilon=O+N\Gamma'':(S+ct)+M\Gamma':(S+ct)+L\Gamma:(S+ct)$$
$$+\ N\Delta'':(S-ct)+M\Delta':(S-ct)\ +L\Delta:(S-ct).$$

Deinde vero est

densitas $q=Q\left(1-\frac{\upsilon d\Omega}{\Omega dS}-\left(\frac{d\upsilon}{ds}\right)\right)$ et celeritas $\mathfrak{T}=\left(\frac{d\upsilon}{dt}\right)$.

COROLLARIUM 1

95. Determinatio haec amplitudinis tubi resolutionem propositam admittens duplici modo est particularis, dum duas constantes arbitrarias in aequatione inter *y* et *z* inventa nihilo aequales posuimus; interim tamen forma pro Ω eruta plures adhuc constantes arbitrarias complectitur ideoque latius patet quam casus praecedens in omni extensione acceptus.

COROLLARIUM 2

96. Casus simpliciores in hac solutione contenti oriuntur, si fiat $\gamma = 0$, tum enim erit:

$$N = \frac{A(\alpha S+\beta)^6 + B(\alpha S+\beta)}{(\alpha S+\beta)^3};$$

prout igitur fuerit vel $A = 0$ vel $B = 0$, reperitur

$$\text{vel } N = (\alpha S+\beta)^{-2} \text{ vel } N = (\alpha S+\beta)^3;$$

illo igitur casu amplitudo ita definitur, ut sit $\Omega = C(\alpha S+\beta)^4$, hoc vero $\Omega = C(\alpha S+\beta)^{-6}$.

COROLLARIUM 3

97. Solutio per operationem primi ordinis inventa etiam duos casus simpliciores dederat hos:

$$\Omega = C(\alpha S+\beta)^0 \text{ et } \Omega = C(\alpha S+\beta)^{-2},$$

operatio vero secundi ordinis praebuerat hos duos

$$\Omega = C(\alpha S+\beta)^2 \text{ et } \Omega = C(\alpha S+\beta)^{-4}$$

nunc haec tertii ordinis suppeditat:

$$\Omega = C(\alpha S+\beta)^4 \text{ et } \Omega = C(\alpha S+\beta)^{-6}$$

unde concludere licet hanc methodum ad omnes tubos applicari posse hac formula $\Omega = C(\alpha S+\beta)^{\pm 2i}$ contentos, exclusis exponentibus imparibus.

SCHOLION 1

98. Omnes plane tuborum formae hoc modo resolutionem admittentes repeti debent ex hac aequatione

$$y^4du-2uy^3du-uuyydy+Byydu-Ady=0,$$

ubi A et B constantes ab arbitrio nostro pendentes denotant, quas tamen ambas in praecedente solutione evanescentes facere sum coactus, ut integratio succederet. Nunc autem observo, dummodo sit $A=0$ ideoque

$$dS=\tfrac{ydz-zdy}{zz}=\tfrac{du}{uu},$$

integrationem etiamnunc administrari posse, quaecunque constans pro B assumatur. Cum enim tum sit

$$yydu-2uydu-uudy+Bdu=0,$$

ponamus $uuy=x$ seu $y=\tfrac{x}{uu}$, erit

$$\tfrac{xxdu}{u^4}-dx+Bdu=0 \quad \text{seu} \quad d\cdot\tfrac{1}{x}+Bdu\left(\tfrac{1}{x}\right)^2+\tfrac{du}{u^4}=0,$$

qui est casus integrabilis aequationis RICCATIANAE, cui satisfacit forma

$\tfrac{1}{x}=\tfrac{\alpha}{u}+\tfrac{\beta}{uu}$; fit enim

$$-\tfrac{\alpha}{uu}-\tfrac{2\beta}{u^3}+\tfrac{B\alpha\alpha}{uu}+\tfrac{2B\alpha\beta}{u^3}+\tfrac{B\beta\beta}{u^4}+\tfrac{1}{u^4}=0,$$

unde sumi debet

$$B=-\tfrac{1}{\beta\beta} \quad \text{et} \quad \alpha=\tfrac{1}{B}=-\beta\beta,$$

ita ut sit

$$\tfrac{1}{x}=\tfrac{-\beta\beta}{u}+\tfrac{\beta}{uu}.$$

Ponatur ergo

$$\tfrac{1}{x}=\tfrac{-\beta\beta}{u}+\tfrac{\beta}{uu}+\tfrac{1}{p},$$

ut aequationis

$$d\cdot\tfrac{1}{x}-\tfrac{du}{\beta\beta}\left(\tfrac{1}{x}\right)^2+\tfrac{du}{u^4}=0$$

integrale completum eruamus, orieturque:

$$dp-\tfrac{2pdu}{u}+\tfrac{2pdu}{\beta uu}+\tfrac{du}{\beta\beta}=0,$$

quae per $\frac{e^{-\frac{2}{\beta u}}}{uu}$ multiplicata et integrata praebet:

$$\frac{e^{-\frac{2}{\beta u}}p}{uu}+\frac{e^{-\frac{2}{\beta u}}}{2\beta}=\frac{C}{2\beta} \text{ seu } p=\frac{Ce^{\frac{2}{\beta u}}uu}{2\beta}-\frac{uu}{\beta\beta},$$

ita ut sit

$$\frac{1}{x}=-\frac{\beta\beta}{u}+\frac{\beta}{uu}+\frac{2\beta}{uu\left(Ce^{\frac{2}{\beta u}}-1\right)}$$

hincque

$$x=\frac{uu\left(Ce^{\frac{2}{\beta u}}-1\right)}{\beta(1+\beta u)+C\beta e^{\frac{2}{\beta u}}(1-\beta u)}=uuy.$$

Ergo $y=\frac{Ce^{\frac{2}{\beta u}}-1}{\beta(1+\beta u)+C\beta e^{\frac{2}{\beta u}}(1-\beta u)}$ et $z=\frac{u\left(Ce^{\frac{2}{\beta u}}-1\right)}{\beta(1+\beta u)+C\beta e^{\frac{2}{\beta u}}(1-\beta u)}$
existente $u=-\frac{m}{mS+n}$; tum vero est $Ndz-zdN=DdS=\frac{Ddu}{uu}$, ex qua per *zz* divisa reperitur *N* indeque reliqua.

Si hic faceremus $\beta=\infty$, praecedens solutio resultaret.

SCHOLION 2

99. Si hoc modo ulterius progredi placeat, calculus quidem fit laboriosior, sed tamen artificio hic usitato praecipuas difficultates superare licebit. Veluti si pro integralis forma quarti ordinis ponamus:

$$\upsilon=O+Nf''':(S+ct)+Mf'':(S+ct)+Lf':(S+ct)+Kf:(S+ct)$$

et ob $UdS=\frac{dQ}{Q}=\frac{-2dN}{N}$ ideoque $Q=\frac{ff}{NN}$ ponatur

$$M=Nx,\ L=Ny \text{ et } K=Nz$$

perveniturque ad has aequationes:

$$\frac{ddN}{N}=2dxdS=\frac{ddx+2dydS}{x}=\frac{ddy+2dzdS}{y}=\frac{ddz}{z},$$

unde deducimus:

$$\text{I. } 2xdxdS = ddx + 2dydS$$
$$\text{II.} yddx - xddy + 2ydydS - 2xdzdS = 0$$
$$\text{III.} ddz - 2zdxdS = 0$$
$$\text{IV.} zddy - yddz + 2zdzdS = 0$$

hincque integrando

$$\text{I. } xxdS = dx + 2ydS + \text{Const.}dS$$
$$\text{II.}+\text{III.} \quad ydx - xdy + dz + yydS - 2xzdS = \text{Const.}dS$$
$$\text{IV. } zdy - ydz + zzdS = \text{Const.}dS$$

Omnes has tres constantes nihilo sumamus aequales et ultima praebet $\frac{y}{z} = -S$, ita ut sit $y = -zS$, ubi constantis additione non est opus, quia loco S scriptum concipere licet $S + \frac{\beta}{\alpha}$ uti supra. Posito igitur $y = -zS$ ex prima fit $dx = dS\left(xx + 2zS\right)$ et ex secunda

$$-Szdx + Sxdz - xzdS + dz + SSzzdS = 0,$$

quae per zz divisa et integrata edit

$$-\frac{Sx}{z} - \frac{1}{z} + \frac{1}{3}S^3 + \frac{1}{3}A = 0 \text{ ideoque } z = \frac{3(1+Sx)}{A+S^3}$$

hocque valore in prima substituto fit

$$dx = dS\left(xx + \frac{6S(1+Sx)}{A+S^3}\right),$$

cui cum satisfaciat $x = -\frac{1}{S}$, ponatur $x = -\frac{1}{S} + \frac{1}{\upsilon}$ eritque

$$0 = d\upsilon - \frac{2\upsilon dS}{S} + \frac{6\upsilon SSdS}{A+S^3} + dS,$$

cuius integrale

$$\upsilon = \frac{BSS + ASS - AS^4 - \frac{1}{5}S^7}{\left(A+S^3\right)^2}$$

suppeditat

$$x = \frac{-B + 3ASS + \frac{6}{5}S^5}{BS + AA - AS^3 - \frac{1}{5}S^6},$$

hincque porro

$$z = \frac{3(A+S^3)}{BS + AA - AS^3 - \frac{1}{5}S^6} \text{ et } y = \frac{-3S\left(A+S^3\right)}{BS + AA - AS^3 - \frac{1}{5}S^6}.$$

Superest $zdN - Ndz = CdS$ ideoque $\frac{N}{z} = C\int\frac{dS}{zz}$, unde reliqua facile expediuntur.

SCHOLION 3

100. Hoc modo progrediendo continuo plures reperiuntur tuborum figurae, in quibus aëris agitationes minimas definire licebit, et quaelibet adeo operatio infinities maiorem multitudinem suppeditat quam praecedens. Interim tamen infinitae tuborum figurae manent exclusae, pro quibus etiamnum aëris motum determinari non licet. Post figuram ergo cylindricam, quam felici successu expedivimus, sequitur conoidica hyperbolica, vix difficiliorem solutionem postulans quam illa, cum ab eadem operatione prima sit suppeditata, ita ut hae duae figurae ad primum ordinem sint referendae. Ex ordine secundo imprimis notatu est digna figura conica aequatione $\Omega = Ass$ contenta, tum vero etiam hyperbolica $\Omega = \frac{A}{s^4}$, nec non haec algebraica multo latius patens $\Omega = \frac{Ass}{\left(s^3+B\right)^2}$, praeter quas innumerabiles aliae transcendantes sunt erutae; pro altioribus autem ordinibus motus determinatio continuo fit operosior, ut operae non sit pretium tantos labores suscipere. Species itaque tantum simplicissimas examini sum subiecturus.