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QUADRATURE OF THE CIRCLE

BOOK IV : THE ELLIPSE: Part I.

ARGUMENT

It has been a pleasing task to divide the subject matter into six parts, and therein the distinctive properties of the ellipse and their natures are to be proposed methodically.

Section one: Indeed initially, the essential division of these parts has arisen from the section of the cone, and for the necessary remaining fundamental qualities to arise henceforth. **[I-XLI :Pages 1-30]**.

Section two: Here the ellipse is divided in some manner, and a comparison of the sectors and segments of these ellipses is undertaken. [**XLII-LXX**:**Pages 31-46].**

Section three: The consideration is to be undertaken here of both equal, as well as unequal axes and diameters. And indeed in the first place the powers of these to be considered : these being the lines which connect the ends of the diameters together. [**LXXI-CXIX**:**Pages 47-78].**

Section four: The poles and the understanding of these to be designated by the shortest distance from a given point on the axis to a line through the periphery. [**CXX-CXLVII**:**Pages 79-98].**

Section five: various kinds of ellipse may arise, such as from lines, as well as from circles, or even from that contained by an ellipse itself. [**CXLVIII-CLXIII**:**Pages 99-109].**

Section six: The ellipse may be compared with the circle, in which an order may be maintained here also, as in the first place of the proportions and powers of lines, secondly segments & the sections of these, then the figures from each; to be brought together and inscribed amongst themselves. [**CLXIV-CCVI**:**Pages 110-138].**

Several of the remaining propositions of this book, and of the two following books, are those of Apollonius, but demonstrated in another longer way by me, with a few exceptions, which nevertheless may be seen to be similar to the others in this work, which the more studious reader of geometry may not desire pertaining to the teaching of conics. All the other propositions, and which in particular constitute by far the greater part of the work, have been found and demonstrated by us. Whereby if anyone indeed may find certain theorems in the more recent books of geometry which may agree with us, I shall wish it to be understood that the authors of these books shall have emerged into the light, and have now been published for many years before being discovered by me. With which few matters, I have wished to inform my reader, not that I may be detracted on account of anything found, but rather that I may remove the suspicion of plagiarism from myself.

THE ELLIPSE.

DEFINITIONS.

I.

A diameter of an ellipse is a right line drawn within the ellipse, which bisects all lines drawn parallel to a certain line within the same, and if indeed it may cut these at right angles, it shall be called an axis: moreover there shall be two axes in any ellipse, and indeed with these taken together (which are called the extreme diameters), i.e. bisecting each other mutually at right angles, which will become apparent from their position.

II.

Symmetry is said to apply to the diameter associated with each of the sets of parallel lines, and to the bisection of the same by division.

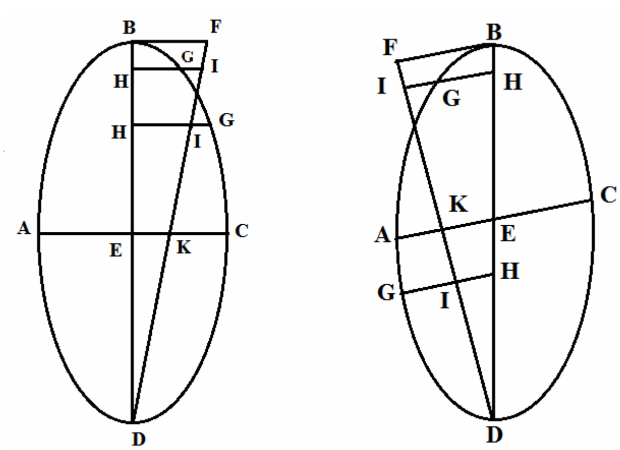
III.

The centre of an ellipse is the point which bisects a diameter.
Moreover, we will show in Prop. 7 of this book, that all lines drawn in the ellipse through the centre will be bisected.

IV.

Diameters which mutually bisect each other are said to be conjugate.

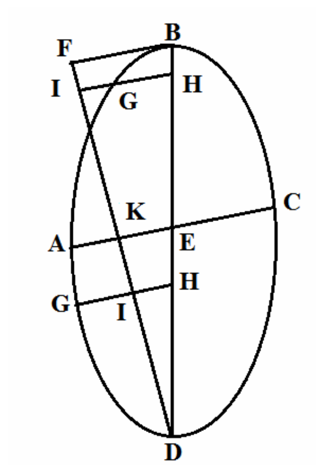
V.



I call the latus rectum such a line, that it is able to put the applied lines on the diameter in order ; or as if the latus rectum acts as a measure of the powers of the ordered lines put in place coupled together on the diameter. [The latus rectum becomes identified as the ordinate of a special point on a diameter, or along an axis ; the forerunner of x and y coordinates.]

The reasoning may be made clearer by an example: the diameter of the ellipse ABC shall be BD, of which FB will represent the latus rectum : and with FD joined, some point H may be assumed on the diameter, and HG shall be normal to the diameter BD, crossing FD at I : therefore the individual squares of the normal lines HG taken in order along the axis will be equal to the individual rectangles BHI, (as we will demonstrate in Prop. 2 of this section) which shall be smaller than the rectangle FBH, but similar to the rectangle FBD itself.

And then besides the latus rectum of Apollonius, at this point others are required to be set out to follow that special one. Truly for me it is seen to be necessary at least that the latus rectum may be applied at right angles to the diameter : [and for other angles to a diameter] and in place of the squares and rectangles, rhombi and rhomboidal figures to be compared with each other. And thus to the old kinds of latus rectum accepted, I add another new line of this kind which shall be required for the diameter of ellipses, whatever the ordering of the right lines GH put in place shall become: and indeed the latus rectum BF may be put parallel to the ordinates of the applied lines : the ordering of the individual rhombuses HG put in place at the angles IHB will be equal to the angles of the individual rhombuses IHB, and for a certain latus rectum FB, with the ordered applied lines put parallel: the order of the individual positions of the rhomboids with the same angles, which are diminished from the rhomboid FBH by the similar rhomboids FBD. See Prop. 12 of this book for the demonstration.



Again it has been found there from the deliberations by the ancients, that some of the certainty will have to be noted, by which the properties of the remaining sections may be understood properly and themselves observed may be able to be rendered more clearly: as in the individual conic sections these plainly are diverse, and thus different kinds of latus rectum will receive attention in the different cases ; and the rectangles with right sides and with the parts between the vertex of the same and the points by which the parts of the diameters between the vertex of the same and where the intercepts are cut held different for a long time in the individual cases, have a proportion to the ordered squares of the positions; indeed with the ellipse these squares become smaller to these similar figures which the right and transverse lines contain from the preceding rectangle ; in the parabola with these the same, are equal ; in the hyperbola truly they exceed with a similar figure to these, &c. From which the individual figures sort out their nomenclature.

Other powers of the ordinate positions for different diameters are different also, thus for the individual diameters, they are to be designated by a single and proper latus rectum: which in all respects, as with the right sides found, you will find demonstrated in its own place.

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VI.

The figure is a rectangle because it is contained by a right side and by a transverse side (that is by the diameter, for that also is accustomed to be called transverse).

VII.

Poles or elliptic foci, are the points (which Apollonius thus calls from the comparison made) in which the axis shows division of the rectangle under the axis with segments held equal to the fourth part of the figure: by which they are enacted according to their location.

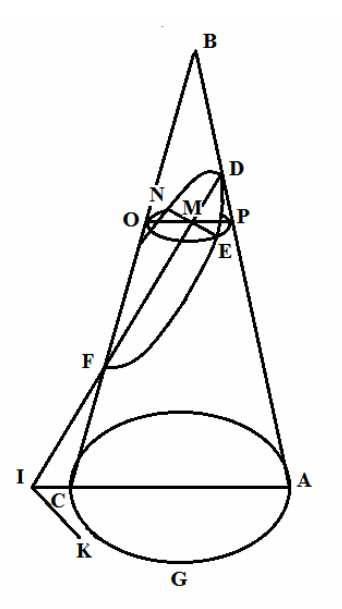
VIII.

The section below opposite is when the cone is cut by a plane through the axis with a triangle being producing , again otherwise it may be cut by a plane, which indeed may mark out a similar triangle (to the triangle produced), but thus put in place so that the angles which are in each triangle are equal, but the sides shall be different.

THE ELLIPSE

FIRST PART

The section to be drawn from a cone, and initially the essential properties of the same is shown.



PROPOSITION I.

The right cone AGCB shall be cut by a plane through the axis producing the triangle ABC. Then it shall be cut by another plane not parallel to the base of the cone AGC, meeting each side of the triangle at D and F: from which (the section produced) shall be the figure DEFN in the cone, but the common section of that cutting plane with triangle ABC shall be the line DFI; truly the common section of the same plane with the plane in which the base of the cone is AGC, shall be the right line IK, as it will be required to be perpendicular to the diameter AC of the base of the cone, or it may be put in place at right angles to the direction which is maintained by the diameter AC .

I say the figure DEFN not to be a circle.

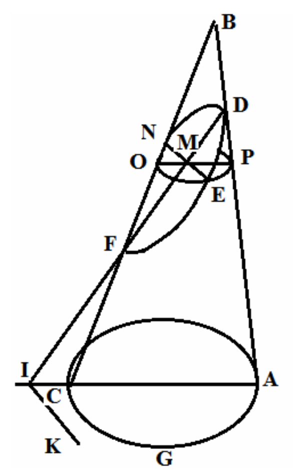
Demonstration.

Through some point M of the right line DF, NE is drawn parallel to IK, in plane of the figure DEFN: and through that same point M, the right line OP is drawn in the plane of the triangle ABC parallel to the right line ACI, but a plane is acting through the lines NE, OP. This will be parallel to the base AGC [§15.book 11], and thence the circle OEPN will be produced [§16.prolog.], of which the diameter will be OP.

Therefore since OP is parallel to AC, the triangles BOP , BCA are similar, but BCA is isosceles, and therefore BOP is isosceles. Therefore [§34.lines] the rectangle FMD is larger than the rectangle OMP ; but the rectangle OMP [§35.book 3] is equal to the rectangle NME. Therefore the rectangle FMD is greater than the rectangle NME; therefore it is apparent from §35.book 3 the figure DEFN not to be a circle. Q.e.d.

PROPOSITION II.

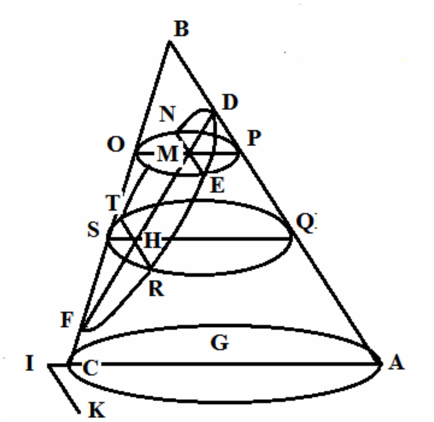
Now a scalene cone ABC shall be given, and the cutting plane which produces the figure DEFN in the cone, and neither shall the figure be parallel to the base of the cone AGC, nor arranged below the axis NE in the opposite manner. Truly all the rest may be put in place and shall become the same as in the first proposition.



Again I say the figure DEFN not to be a circle.

Demonstration.

Since OP is parallel to AC and that shall cut FD in M, but not contrariwise below, that is the angle BFD not being put in place equal to the angle BAC, it is apparent from § 36 of our first book that the rectangle FMD to be unequal to the rectangle OMP; but the rectangle OMP is equal to the rectangle NME. Therefore the rectangle FMD also is unequal to the rectangle NME, therefore it is clear from §35 of our third book that the figure DEFN not to be a circle. Q.e.d.



PROPOSITION III.

Some cone shall be given, either right or scalene, and the remaining items may be put place and they will become the same as above:
I say the right line NE to be cut into equal parts by the right line DF at M.

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Demonstration.

From the hypothesis the right line PM is parallel to the right line AC, and ME parallel to IK. Whereby PM and EM define equal angles [with AI and KI] ; and the angle AIK from the hypothesis is right, for the common section IK were placed perpendicular to ACI, therefore from the first proposition also PME is right. And thus since the section ONPE shall be a circle [§.26 prolog.], and its diameter OP, it is evident EMN [§.3], to which it is normal, to be bisected at M by the diameter of the circle OP. Moreover, from the hypothesis, the point M is common to the three right lines OP, NE, DF. Therefore NE to be bisected at M by DF. Q.e.d.

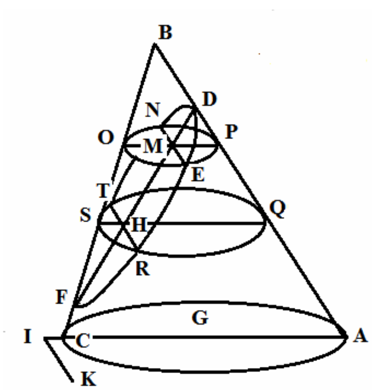
Corollary.

Hence it is apparent, if some number of right lines may be drawn parallel to IK or NE, all are to be bisected by DF: indeed the same is demonstrated in all cases. From which it shall be evident further of the section DEFN ; (which henceforth we will call an ellipse) the diameter to be the line DF [Def.1], the right line truly NE [Def.2], and the remaining lines parallel to this to be the ordered applied lines for the diameter DF.

PROPOSITION IV.

With the same in place, some line RT of the ellipse DEFN may be drawn, parallel to EN or IK, cutting the diameter DF at the point H.

I say, the rectangle DMF to be to the rectangle DHF, as the square EM to the square RH.



Demonstration.

The right line QHS may be drawn through the point H parallel to the right line OP meeting with the sides of the triangle ABC at Q and S. Then a plane is acting through the right lines QS, TR , this will be parallel to the base AGC and hence the section will produce the circle QRS; now truly the ratio of the rectangle DMF to the rectangle DHF is composed from the ratio DM to DH, (that is, since PM, QH are parallel from the construction from the ratio PM to QH) and from the ratio MF to HF (that is, since MO, HS are parallel from the construction, from the ratio MO to HS.) And the ratio of the rectangle PMO to the rectangle QHS is composed also from the ratios PM to QH, and MO to HS. Therefore the rectangle DMF is to the DHF as the rectangle PMO to the rectangle QHS; that is since the sections PEO, ORS are circles, so that the rectangle EMN to the rectangle DHF shall be as the rectangle to the rectangle

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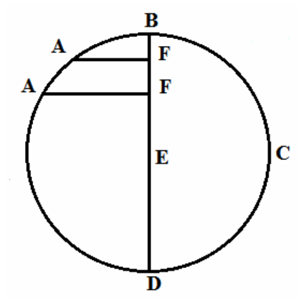
RHT, that is, since EN, RT are bisected by the diameter DF in M and H, as the square EM to the square RH. Q.e.d.

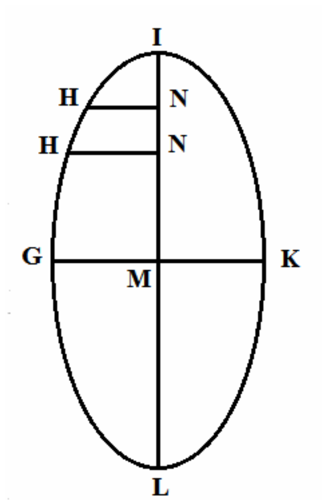
$$[\text{rect.DMF} : \text{rect.DHF} = \text{DM.MF} : \text{DH.HF} = (\text{DM} : \text{DH}) \times (\text{MF} : \text{HF}) = (\text{PM} : \text{QH}) \times (\text{MF} : \text{HF})$$
$$= (\text{PM} : \text{QH}) \times (\text{MO} : \text{HS}); \text{ But rect.PMO} : \text{rect.QHS} = (\text{PM} : \text{QH}) \times (\text{MO} : \text{HS});$$
$$\therefore \text{rect.DMF} : \text{rect.DHF} = \text{rect.PMO} : \text{rect.QHS} = \text{rect.EMN} : \text{rect.RHT} = \text{EM}^2 : \text{RH}^2 .]$$

Scholium.



We have shown by this proposition the proportion of the rectangles to the squares, which are established from the segments of the diameter of the ellipse, to be ordered according to the same diameter of the applied lines : which indeed is a primary essential property of the ellipse : truly thus since this belongs to the ellipse, so that also it may be found in its own way in the circle, I have thought it worthwhile for me, if briefly I may show the difference, between that and the ellipse in the nearby diagram.

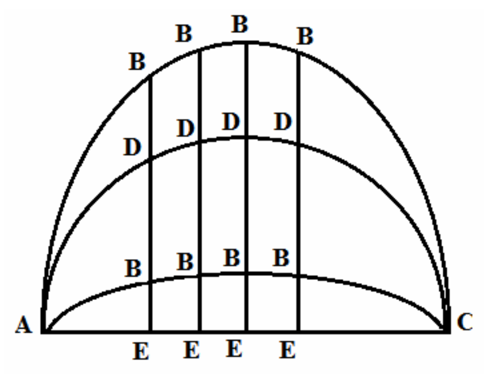
Let BD be the diameter of the circle ABC; the centre E: and the normal AF to the diameter: moreover IL shall be some diameter of the ellipse HIK , that HN may cut in order: truly M shall be the centre of the section. Therefore since in the circle, the right lines AF are normal to the diameter BD, the rectangles BFD will be equal to the squares AF, and hence the square AF is to the square AF as the rectangle BFD to the rectangle BFD: in the same manner in the case of the ellipse since the right lines HN shall be placed in order to the diameter IL, the square HN will be to the square HN, as the rectangle INL to the rectangle INL: therefore that is agreed for each section; that there be a proportion between the squares put in order, which is of the rectangles, under the diameters of the segments to which they are put in order: truly they differ in this respect, since in the circle the proportions of the rectangles under the segments of the diameter shall be equal to the squares of the positions in order; truly of the inequalities of the ellipse (if the case of the equality of the diameters taken together may be excepted, concerning which several with their own position), which we make the first and second of this plane.



From which it follows in the first place, in the axis of the ellipse, one axis is to be greater than the other, which can be shown thus : in the ellipse HIK there shall be some axis IL which HN shall cut in order: the right line GK is acting through the centre M parallel to HN itself: I say these axes to be unequal: indeed so that as the rectangle INL to the rectangle IML, thus the square HN to the square GM, and on interchanging, as the rectangle INL to the square HN thus the rectangle IML to the square GM; but the rectangle INL is unequal to the square HN ; and therefore the rectangle IML, (that is the square IM) is unequal to the square GM: therefore the right line GM is unequal to the right line IM, therefore the whole IL, truly the axis, is unequal to the whole GK, that is, to the other axis. Which was proposed.

Then if the axes shall be equal in a given ellipse, now the ellipse will not differ from a circle: therefore so that the rectangles under the axis of the segment shall be equal to the squares of the positions in order.

Secondly it follows, if above the axis AC of the ellipse ABC, the semicircle ADC may be described , and the lines BE may be drawn in order crossing the semicircle at D, so that BE shall be to BE, as DE is to DE; indeed it is the case both in the ellipse as well as in the semicircle, that the rectangle AEC to be to the rectangle AEC, thus as the square BE to the square BE, and the square DE to the square DE; and also as the square BE to the square BE thus the square DE to the square DE.



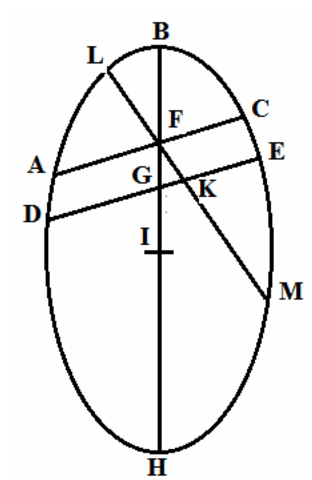
PROPOSITION V.

To find the diameter of a given ellipse.

Construction & demonstration.

The parallel lines AC, DE are drawn within the ellipse, which are bisected at the points F, G by the right line BH ; and the right line BH is drawn through F and G.

I say this to be the diameter.



The demonstration is clear, if indeed BH were not the diameter, then it shall become LFM, cutting DE in K. Therefore since LM is put to be the diameter, and from the parallel lines it shall bisect the one AC, at F, and it shall bisect the other DE in K. Which cannot happen since from the construction, DE shall be bisected at G . Therefore neither is LM nor any other line drawn through F to be the diameter besides that, which also passes through G, that is besides BH itself. Therefore we have found the diameter in the given ellipse. Q.e.f.

PROPOSITION VI.

To find the centre of a given ellipse.

Construction & demonstration.

To seek the diameter of the ellipse BH by the preceding, as bisected in L. From the third definition it is apparent I to be the centre.

Corollary.

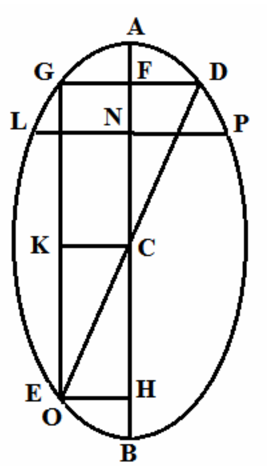
It is apparent from this proposition every diameter to pass through the centre. And from which you deduce the other easily, without doubt all the lines passing through the centre to be diameters.

PROPOSITION VII.

The ellipse ADB shall be given, of which the diameter shall be AB, truly the right line LP, shall be one of these, which we have demonstrated in proposition three of this section to be bisected by the diameter: C shall be the centre of the ellipse C.

I say all the lines drawn through the centre to be bisected in the centre.

Demonstration.

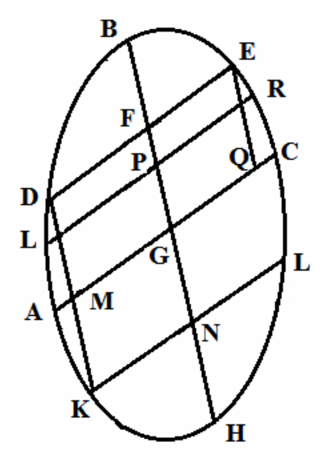


Indeed some right line DO may be drawn through the centre C; DFG shall be drawn from D parallel to LP, GE parallel to AB, & EH, CK parallel to GD, or to LP. Therefore since FGEH is a parallelogram, GF, EH will be equal : and hence the squares GF, EH are equal. And as the square GF is to the square EH, thus the rectangle AFB is to the rectangle AHB, therefore the rectangles AFB, AHB are equal, therefore as AF to AH, thus so BH to BF : therefore on dividing, thus AF to FH and BH to HF are equal ; therefore AF, BH are equal. Whereby since also the whole diameter AB shall be bisected at C, as is apparent from the definition of the centre, also the remainder FH is bisected at C. Therefore since KC is parallel to GD, EH, themselves to be parallel to LP itself, also the right line GE shall be bisected at K, and truly DG bisected at F, as itself being parallel to LP. Therefore there shall be as DG to GF, that is as DG to CK, thus GE to KE. Therefore the points DCE are on a line ; but also the points DCO are on a line, since from the hypothesis DCO shall be a right line. Therefore DCE & DCO are one and the same right line. And DCE is bisected at C, (since indeed from the construction: GE, FC shall be parallel, therefore as DF to FG, thus DC to CE.) Therefore also DCO is bisected at C. Q.e.d.

PROPOSITION VIII.

The ellipse ABCH shall be given, of which the diameter shall be BH; truly in order for the applied diameter to be LPR: the centre of the ellipse to become G. Moreover the right line AGC will be drawn through the centre G, parallel to the applied right lines in order.
I say BH, AC to be conjugate diameters.

Demonstration.



Some point M may be taken on AG, through which KD shall be drawn parallel to the diameter BH, meeting the ellipse at the points D and K ; from which there may be drawn DFE, KNL parallel to LR itself. Therefore since DK, NF is a parallelogram, thus the right lines DF, KN are equal, and the squares DF, KN also are equal. Whereby since the rectangle BFH shall be to the rectangle BNH as the square DF to the square KN, the rectangles BFH, BNH also are equal, and hence, as we have shown in the preceding, BF and NH are equal. Truly BG and HG are equal. And therefore the remaining FG, NG are equal, or FN is bisected at G. Therefore KD parallel to the diameter BH shall be bisected at M for AC. Similarly I may show any other parallel diameter to AC to be bisected by BH. Whereby since also BH bisects DE, LR and all the remaining which are put in order to BH, and from the hypothesis parallel to AC itself; it is clear from the definition BH, AC to be conjugate diameters .

Corollary.

An axis given through the centre of the ellipse drawn perpendicular to a given axis is said to be the conjugate to this axis.
From the discussion now brought forwards it will be easy for the reader himself to elicit a demonstration of this.

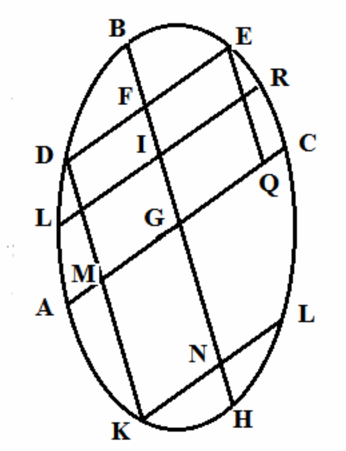
PROPOSITION IX.

An ellipse shall be given and its diameter BH.
It shall be required to show the diameter conjugate to BH.

Construction and demonstration.

Some other right line KD may be drawn parallel to BH; DK, BH each bisected at M and G, and AC may be drawn through M and G .

I say AC, BH to be conjugate diameters.



And indeed in the first place it is apparent from Prop. 5 of this section that the right line AC to be a diameter, and BH to be a diameter from the hypothesis; therefore both are diameters. But which I shall now show to be conjugate. Because AC is a diameter and it bisects KD, KD to be applied as the ordinate to AC , that is, these parallel lines are going to be bisected by the diameter AC. Therefore the rest to be parallel to KD itself. But DK, from the construction with the lines parallel to it, is parallel to the diameter BH. Therefore the diameter AC shall bisect the lines parallel to the diameter BH. Thereupon the line DE may be drawn, parallel to the diameter AC and EQ, parallel to the right line DK; therefore MDEQ is a parallelogram, in which since FG is parallel to DM itself, EQ is drawn parallel, therefore DF to FE, as MG to GQ; also the sides of the parallelogram DM, EQ to be equal, and the squares DM, EQ will be equal. Now truly since DM, EQ, are placed in order to the diameter AC, the ratio between the rectangles AMC, AQC will be the same as between the squares DM; EQ, that is, of equality : and hence as is apparent from the demonstration in Prop. 7 of this section, AM, QC will be equal. Whereby since all the AG,CG shall be equal (indeed since G is the centre of the ellipse ; since it bisects the diameter BH) also the remaining MG, QG, are equal; therefore since there is MG to GQ, thus DF to FE, also DF, FE are equal, that is DE is bisected at F. Therefore from that it is defined, DE is put in order to the diameter BH, and therefore the remaining are themselves placed parallel to BH in order. And from the construction DE, with the other diameter AC parallel to itself. Therefore the diameter BH bisects the parallel diameter AC. Whereby since also I shall have shown earlier AC to bisect the parallels to BH, thus BH, AC to be conjugate diameters. Therefore what was demanded has been done.

First Corollary.

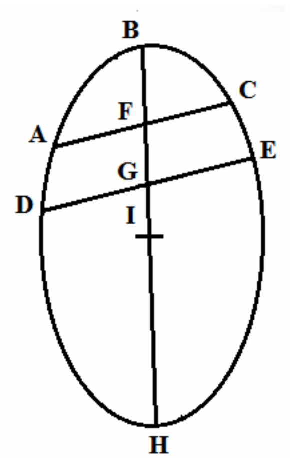
For a given axis of the ellipse, you will find the conjugate axis, if you will draw a right line through the centre of the ellipse perpendicular to the given axis, the matter to be evident from the corollary of Prop. eight.

Second Corollary.

From this problem it shall be evident how, from a given point D on the ellipse, a right line must be adjoined in order to the diameter BH. Indeed the diameter AC may be found conjugate to the diameter BH; and from the given point D, DFE may be drawn parallel to AC itself.

I say DFE may be placed in order to the diameter BH. The demonstration to be evident from the proposition.

PROPOSITION X.



Of the diameters placed in order (AC, DE, &c.) that is greater which is closer to the centre (I).

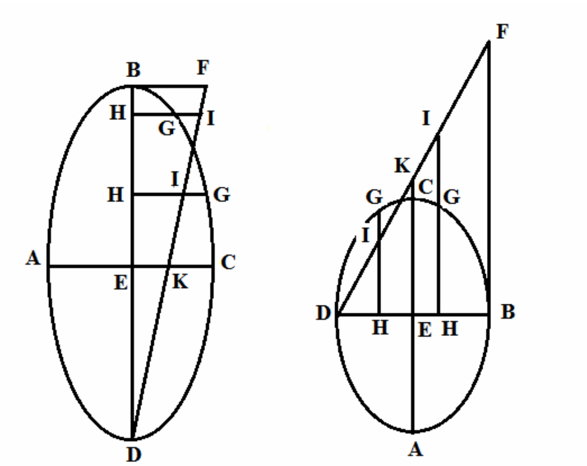
Demonstration.

The rectangle BGH is greater than the rectangle BFH as is apparent from the second part of Prop. five. And the square EG is to the square CF as the rectangle BGH to the rectangle BFH. Therefore the square EG is greater than the square CF. And therefore the ordinate for the position EG is greater than for the position CF. Q.e.d

PROPOSITION XI.

BD shall be the axis of the ellipse ABC ; it is required to show the latus rectum of this ellipse.

Construction & demonstration.



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AC is connected to the axis BD through the centre E , and the continued proportions BD, AC, BF arise [coroll.9 of this section]: I say that FB to be the latus rectum ; indeed BF to be parallel to AC itself: and the ordinate FD of the line GH may be drawn which cross each other at I, truly FD will cut the line AC at K. Since EC, GH are put in place in order on the axis BD, as the square GH will be to the square EC, thus as the rectangle BHD shall be to the rectangle BED: but as the rectangle BHD to the rectangular BED, the rectangle IHB is to the rectangle KEB (because evidently they have been composed from the same ratio BH to BE, and from HD to ED, that is, HI to EK.) therefore as the square GH to the square CE, thus as the rectangle IHB is to the rectangle KEB, and by permutation and inverted so that as the rectangle KEB to the square CE, thus the rectangle IHB is to the square GH: but since the square AC shall be equal to the rectangle FBBD (since from the construction: BD, AC, BF are three lengths in continued proportion) EC will be a square equal to the rectangle KEB, (without doubt the fourth part of the square AC, is indeed the square of AC bisected at E) equal to the fourth part of the rectangle KEB on FBBD. And therefore the square HG is equal to the rectangle IHB: therefore HG can be the distance which may be added to the length FB having HB, being deficient by the rectangle FBH, similar to the figure from the rectangle BFD : whereby we have shown the latus rectum FB [def.6] is …etc. Q.e.f.

Corollary.

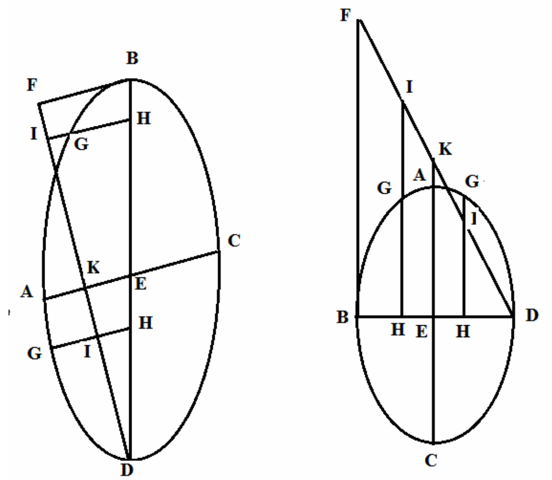
Hence it follows the first four lines, without doubt the latus rectum of the minor axis, the major axis, the minor axis, and the latus rectum of the major axis to be in continued proportion.

In the second place it follows, with the latus rectum both of the minor and major axis given, whereby an ellipse will be show, so that between the two lengths given, the two axes lengths will be found in the middle.

PROPOSITION XII.

Let BD be the diameter of some ellipse ABC, it will be required to show its latus rectum.

Construction & demonstration.

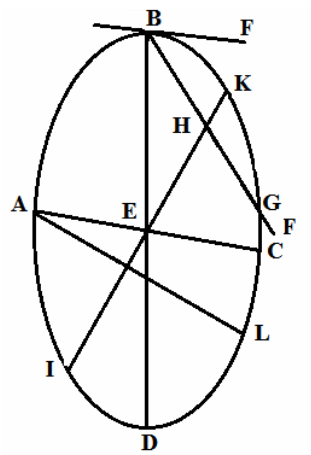


The diameter AC may be drawn through E conjugate to BD itself [§.9], and the continued proportions BD, AC, BF, and BF shall arise parallel to the diameter AC : and the line FD may be drawn, which will cut the right line AC at K, and the ordered lines GH may be drawn, which cross the line FD at I. Since BD, AC , FB are lines in continued proportion, the square AC shall be equal to the rectangle on FB, BD; and therefore the square AE (without doubt equal to the fourth part of the square AC, indeed AC bisected at E) equal to the rectangle on KE [$=\frac{1}{2}\cdot \mathrm{AE}$], EB[$=\frac{1}{2}\cdot \mathrm{BD}$], the fourth part of the rectangle FBD: whereby, as we have shown in the preceding, thus also we have shown the square HG to be equal to the rectangle IHHB, and for the rhombus IH at the angle IHB is equal to the rhomboidal IHB at the same angle IHB , as the square IH to the rectangle IHB, (for the ratios of the rhombus to the rhomboid and of the square to the rectangle are composed from the same ratios, truly from IH to IH and IH to HB,) therefore since the square IH shall be equal to the rectangle IHB, also the rhombus IH will be equal to the rhomboid IHB ; therefore the right line IH can be the rhomboid at the angle IHB of the applied ordinate, which (since it is shown easily) applied to the rhomboid FBH at the same angle, is smaller by a similar rhomboid to that which shall be at the same angle for the diameter DB and the right line BF. Therefore FB is the latus rectum. Which was sought.

PROPOSITION XIII.

Every right line (BF,) which is drawn through the end of the diameter (BD) parallel to the ordinate (AC) of the applied line, is a tangent to the ellipse.
And that which is drawn parallel to the tangent, is the ordinate [or the ordered line] for the applied diameter.

Demonstration.

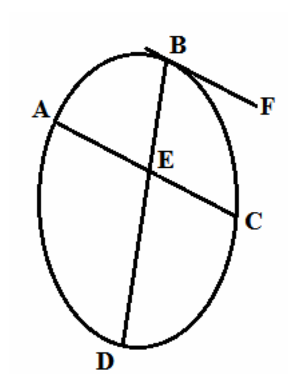


Indeed if the right line BF may not be a tangent of the ellipse, it may cut that at G; and with BG bisected at H, KI passes through H and E, crossing both sides of the perimeter at I and K. Since, from the construction, BG is parallel to AC, moreover each will bisected by the right line IK, and IK will be a diameter and AC, BG the ordinates put in place for that; but, from the construction, the right line AC also cuts the diameter BD, therefore one and the same right line ordinate AC shall cut the two diameters BD, IK. Which cannot happen; otherwise indeed each diameter BD, KI also must be drawn parallel to AC itself, and hence may be bisected at two points, therefore it is clear the line FB, to be tangential to the cut: as it was in the first place. So that if some line AC may be drawn parallel to the tangent BF crossing the diameter at E, it will be an ordinate put in place to the diameter. If the ordinate AL may not be drawn from A : AL will be parallel to the tangent BF; and whereby it shall be parallel to AC itself, which cannot happen, since the same will be cut at A : therefore AL cannot be an ordinate put in place nor any other besides AC, which was the other. Therefore the truth of the proposition is evident.

PROPOSITION XIV.

To draw a tangent through a given point on the periphery.

Construction & demonstration.



ABC shall be an ellipse and B a given point on the periphery, it is required to draw a right line through B which section shall be a tangent at B; to find the centre and through this draw the diameter BD, to which there may be put some ordinate line AC, to which there may act the parallel line BF through B, it is evident therefore BF to be the tangent; therefore through a given point on the periphery….etc. Q.e.f.

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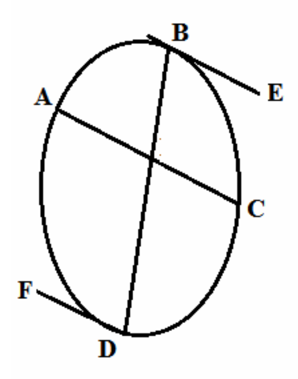
PROPOSITION XV.

Draw lines through the ends of a diameter, parallel to each other, being tangents to the ellipse.

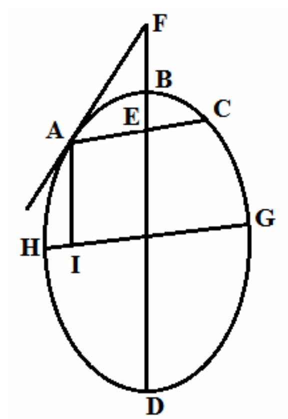


Demonstration.

Indeed it is evident from §.13 in this section, some ordinate AC drawn to the diameter BD, both the tangents BE as well as DF to be parallel to that ordinate, and thus to each other. Q.f.d.

PROPOSITION XVI.

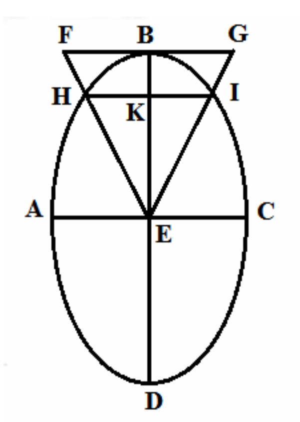
A tangent drawn through an ordinate end meets the diameter beyond the section.



Demonstration

BD shall be the diameter of the ellipse ABC, and the ordinate AEC put in place, and the tangent AF is acting through A; I say that tangent meets the diameter at F. For the diameter HG is found in conjunction with its diameter BD, from A the line AI may be sent parallel to BD, therefore since AI, BD are parallel, and AF shall meet the right line AI, it is evident DB produced also to meet with BD. Q.f.d.

PROPOSITION XVII.



And on taking the ellipse ABC, the axis of which shall be BD, the line FG shall be a tangent at B, with the equal parts FB, BG tangents together; I shall make the two diameters FE, GE to depart from F and G crossing the ellipse at H and I. I say the connected line HI to be parallel to FG.

Demonstration.

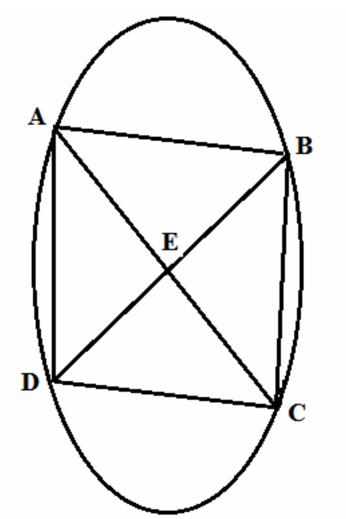
HK may be put parallel to FG, which produced crosses EG at I; and thus HK will be equal to KI. Therefore, since the rectangle BKD shall have the same ratio to BED, as the square HK to the square AE [§.4], there will be also the rectangle BKD to the rectangle BED, as the square IK to the square EC: from which the point I taken to be a point on the ellipse, and HI to intersect with the same point I of the perimeter BIC, and the right line EG intersects at the same point I. Q.f.d.

PROPOSITION XVIII.

With the same figure remaining, the axes of the ellipse ABC shall be AC, BD, and HE some diameter, it will be required to deduce the diameter from E towards C, to be equal to HE itself.

Construction & demonstration.

The angle BEH shall be made equal to the angle BEI, I say the right line be required to be satisfied, indeed the lines HE, EI produced cross the line of the tangent acting through B at F and G. The points H, I may be joined; since the angles BEH, BEI may be put equal, moreover the angles EBF, EBG to be right, and BE the common line, it is clear the triangles FBE, GBE, and thus the sides FB, BG to be equal to each other, and HI shall be parallel to FG, and thus as FE to GE, there shall be HE to IE, whereby HE, IE to be equal lines; therefore we may deduce the diameter from E….etc. Q.e.f.



PROPOSITION XIX.

The lines connecting the ends of any diameters in an ellipse are equal and parallel to each other.

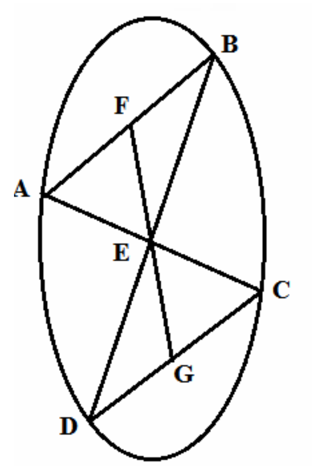
Demonstration.

Any two diameters AC, BD shall cut the ellipse ABC ; I say the joined lines AB, CD, likewise AD, BC, to be equal and parallel to each other: since DB, AC shall be bisected at E, as DE to EB, thus there shall be CE to AE , and on permutation there are, as DE to CE, thus BE to AE, and truly the angles E are equal, therefore the triangles DEC, AER are similar ; therefore as DE to EB, there shall be DC ad AB; whereby since DE: EB are equal, also AB, DC are equal. Similarly, we may show AD, BC to be equal. Q.e.d.

PROPOSITION XX.

Lines which are put parallel to the ends of a diameter, also are equal to each other.

Demonstration.



Some diameter BD shall cut the ellipse ABC, and the parallel lines AB, CD shall be drawn from B and D within the section. I say these are to be equal to each other. With the centre E found, and AB bisected at F, join FE, and produce to G, and since the diameter EF shall bisect AB, also it shall bisect DC, parallel to AB itself. Then since the triangles FEB, DEG are similar; DE will be to DG, as EB to BF: and by interchanging, as DE to EB there shall be DG to BF; but DE, EB are equal, and therefore are equal BF, DC, so that now shown to be half of these AB, DC. Therefore the whole lengths AB, DC are equal, Q.e.d.

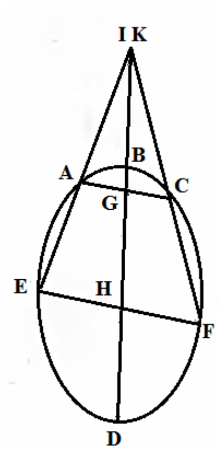
Corollary.

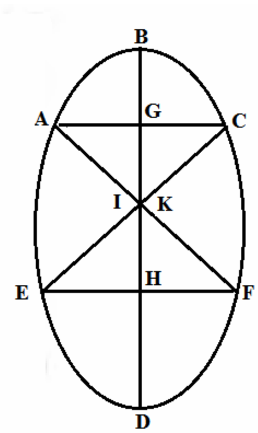
Hence if follows the joined lines AE, EC to be collinear: indeed since the sides AF, FE shall be equal to the two sides CG, GE, and the angles contained by equal sides are equal, it is clear AFE, CGE to be triangles equal to each other, and the angle AEF to be equal to the angle CEG, and thus AE, EG to be collinear.

PROPOSITION XXI.

The lines through the ends of two unequal parallel lines drawn in the ellipse, meet in the same point with the diameter, to which the ordinates put in place are parallel.

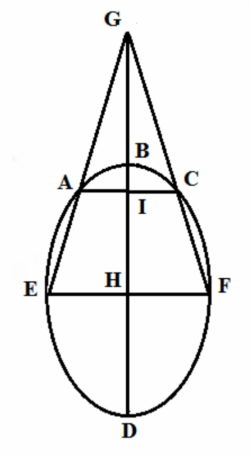
Demonstration.





Any two unequal parallel ordinate lines AC, EF placed in order for the diameter DB, cut the ellipse ECD. I say EA, FC come together and cut the diameter BD at the same ordinate point. Since the lines AC, EF are ordinates for the diameter BD, both ordinate lines are bisected at G and H, from which AG to GC shall be as EH to HF, and on interchanging as AG ad EH, thus GC to HR, now EA concurs with the diameter at Ii truly the other FC at K, therefore there will be as IG to IH, thus IA to IE, that is, as before, as AG to EH, that is as IG to IH, therefore the points I and K are the same; therefore the point I is common to the intersection of the lines EI, FI, HI. Q.e.d.

PROPOSITION XXII.

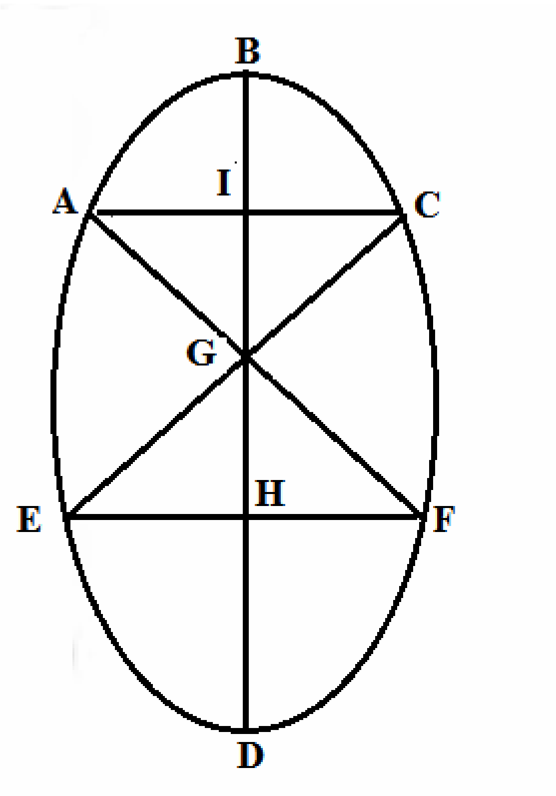


BD shall be the diameter of the ellipse ABC to which the ordinate EF shall be put in place, and from E and F lines may be drawn crossing with the diameter at the point G, truly with the ellipse at A and C.
I say the joined line AC, to be parallel to EF.

Demonstration.

AI may be drawn parallel to EF and produced shall cross the line FG at C, therefore since EH is equal to HF, and AI itself will be equal to IC; but since AI shall be parallel to EF, the rectangle BID will be to the rectangle BHD, as the square AI to the square EH. Also therefore, as the rectangle BID to the rectangle BHD, thus the square IC to the square HF; from which the point C is the common intersection of the right lines FG, AI with the perimeter BCF of the ellipse; and therefore AC joins the points A, C, parallel to EF. Q.e.d.

PROPOSITION XXIII.



The parallel lines AC, EF shall be drawn in the ellipse, through the ends of which EC, AF may be drawn through joined together at G and GIH may be drawn through G shall bisect AC at I.
I say the other line also to be bisected.

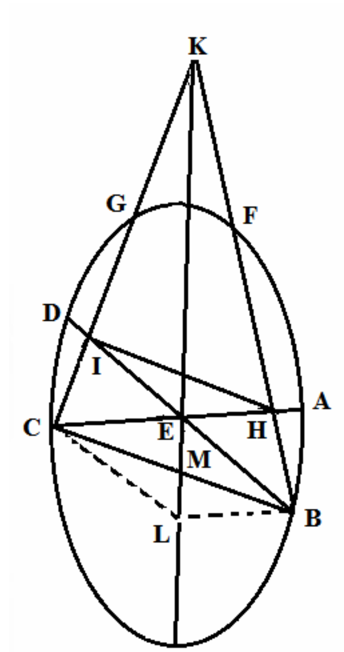
Demonstration.

As HG to IG, thus EH to AI, and as HG to IG, thus FH to CI; therefore EH to AI, as HF to IC; therefore on interchanging, EH to HF as AI to IC; but AI, IC are equal, and therefore EH, HF are equal, and thus both EF as well as AC are bisected; therefore GIH is a diameter est. Q.e.d.

PROPOSITION XXIV.

The two diameter AC, BD cut the ellipse ABC, and BC may be joined, the diameter KL shall act through the centre E, bisecting BC in M, and from B and C the right lines BF, CG, are drawn to the same point K of the diameter cutting the lines AC, BD at H and I.

I say the rectangle AHC to be to the rectangle DIB as the square AC to the square DB.



Demonstration.

The line CL may be put from C parallel to BD, crossing the diameter KL at L, and with BL joined, since CL shall be parallel to DE, EMB, CML will be triangles similar to each other : truly since CM, MB are equal, also the triangle CML, EMB will be equal and the side LM equal to the side EM ; therefore in the triangles BML, CME, the two sides CM, ME are equal to the two sides BM, ML, and moreover the angles BML, EMC contained by these are equal. Therefore the bases of the angles LBM, ECB are equal; therefore BL, CEA are parallel. Therefore BH to HK shall be as LE to EK, that is, (since from the construction BI, CL are parallel), as CI to IK. Therefore IH shall be parallel to CB, and so that HE is to EC, thus as IE to ER, and on putting together and interchanging, as EC to EB, thus HC to BI, but as CE to BE, thus AC is to BD, since each shall be bisected in the centre; therefore as AC to BD, thus HC to BI. Therefore also as AC to DB, thus AH to dL. Whereby since the rectangle AHC shall have the ratio to the rectangle DIB composed from the sides by the ratios AH to DI, and HC to BI, which both are shown to be the same with the ratio AC to BD, the ratio of the rectangles will be double the ratio AC to BD, that is the same as the squares of AC, BD. Q.e.d.

PROPOSITION XXV.

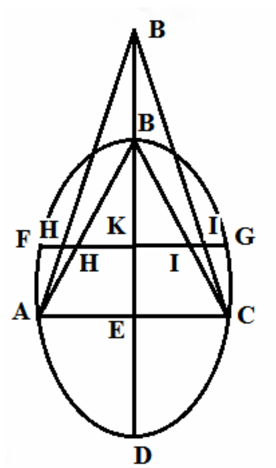
The two lines CG, BF drawn within the ellipse cross the diameter of the ellipse MK at the same point K. Then two other diameters BD, CA which thus will be cut by the right lines CG, BF so that the rectangles BID, CHA shall be proportional to the squares BD, AC .

I say the joined lines IH, CB to be parallel.

Demonstration.

Since it is, as the square BD to the square CA, that is, as the square ED to the square EA, thus the rectangle BID to the rectangle CHA ; there will become on interchanging, so that the square ED, (that is the rectangle BID with the square EI to the rectangle BID is to the square EI) to the rectangle BID, as the square EA, (that is the rectangle CHA with the square EH) to the rectangle CHA to the square EH. Therefore on interchanging, the rectangle BID is to the rectangle CHA as the square EI to the square EH, but also the rectangle BID is to the rectangle CHA as the square BD to the square CA; that is as the square ED to the square EA. And thus the square EI is to the square EH as the square EH is to the square EA: and thus the right line EI to the right line ED, that is EB, as the right line EH to the right line EA, that is EC, therefore IH, CB are parallel. Q.e.d.

PROPOSITIO XXVI.



Let BD be the diameter of the ellipse ABC, for which the position of the ordinate shall be the right line AC : and with the right lines drawn from A and C, which shall cut the diameter at the same point B, FG shall be drawn parallel to AC, crossing AB, CB at H and I, truly with the diameter BD at K.

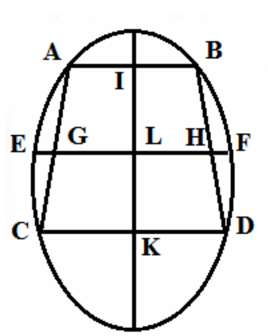
I say FH, GI to be equal lines.

Demonstration.

Since FG shall be an ordinate placed parallel to AC, and also the ordinate put in place for the diameter BD, and thus bisected at K ; but also HI is bisected at K , just as AC in E, therefore with the equalities HK, IK, removed, the remaining FH, IG are equal. Q.e.d.

PROPOSITION XXVII.

Any two parallels AB, CD drawn may cut the ellipse ABC, and with AC, BD joined, EF may be drawn parallel AB, cutting the lines AC, BD at G and H.
I say the right lines EG, FG to be equal.



Demonstration.

The line IK is acting through the points I and K with AB, CD bisected at I and K; IK will be that diameter, and the line EF parallel to the line AB parallel will be bisected at L; moreover HG is bisected at L as is CD at K; or AB at I, therefore with the equal lines GL, LH removed, the remaining lines EG, FH remain equal. Q.e.d.

PROPOSITION XXVIII.

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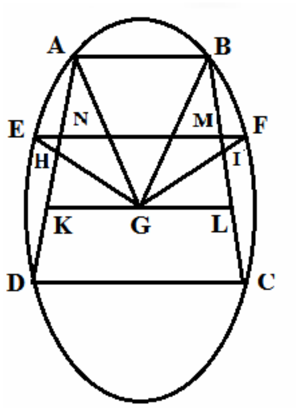
Any two parallel lines AB, CD may cut the ellipse ABC, and with AD, BC joined, ENMF may be drawn parallel to AB, and from E and F, the semi-diameters EG, FG may be put in place which shall bisect the lines AD, BC at H and I.
I say EG, FG to be divided proportionally at H and I.



Demonstration.

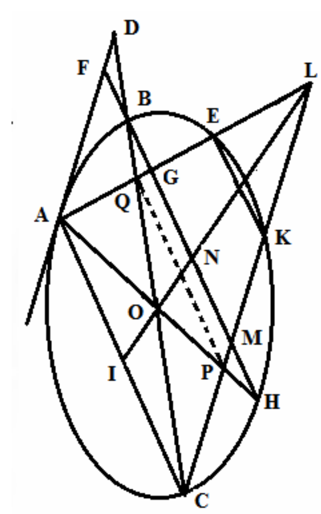
KL may be drawn through G, parallel to AB , crossing AD, BC, at K and L. Since EF, KL are parallel, there will be: as EN to KG, thus EH to HG; and as FI to IG, thus FM to LG; but as EN to KG, thus FM is to LG, (since EN, FM likewise KG, LG, shall be equal,) therefore as EH to HG, there shall be FI to IG. Q.f.d.

Corollary.

Hence it is clear the join HI to be equidistant from DC, and thus the lines AD, BC to be divided proportionally at H and I.

PROPOSITION XXIX.

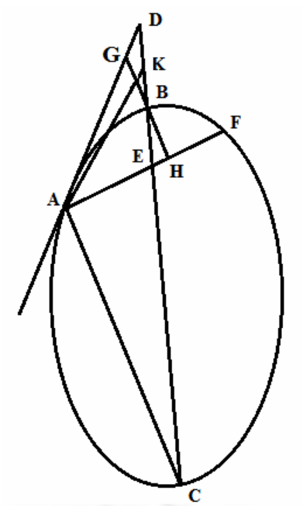
AD shall be a tangent to the ellipse, of which a diameter is BC and the centre O, meeting the diameter at D, and with the ordinate AQE drawn from the point A, and AC joined, the right line FBG may be drawn through B parallel to the right line AC.
I say FB, BG to be equal.
Demonstration.



FG shall meet the ellipse at H, and HO, AO which are lie on the same line may be joined [§.20] Then with AC bisected at I, the diameter IOL may be drawn through I meeting the right line AE at L and the points L, C may be joined by the right line LC, crossing the ellipse at K, and the right line FG in M; truly the right line HO at P.
Since from the construction the ordinate AC has been put in place for the diameter IL, and the two right lines through A and C meet the diameter at the same point L [§.25] , EK will be parallel to AC. And truly BH parallel to AC from the hypothesis: and the radii OB, OH shall cut AE, CK at Q and P, therefore QP shall be parallel to BH [§.28.cor.] ; therefore the three right lines are parallel: AC, QP, EK. Whereby since from the hypothesis AE shall be bisected at Q, and CK bisected in P, and hence placed to be an ordinate to the

diameter AH. And thus CK shall be parallel to the tangent AD. Moreover FM is parallel to AC by the hypothesis, therefore FM is equal to AC, but also BH is equal to AC. Therefore FM, BH are equal; whereby with the common part BM removed FB, HM are equal. Also GB, HM are equal, and thus FB, GB are equal. Q.e.d.

PROPOSITION XXX



The right line AD shall be a tangent to the ellipse ABC at A meeting the diameter at D : and the ordinate line AF shall be drawn from A to the diameter BD.
I say the right line DC to be divided in the extreme and mean ratio of the proportions at B and E, that is, as CD is to BD, thus as CH is to HB: if it were divided at B and E in the extreme and mean ratio of the proportions AF to BC, and if the ordinate line were acted on in the mean ratio of the proportions through E : I say the join AD to be a tangent to the section.
Demonstration.

Join AC, the line GH is acting through B parallel to the line AC crossing the line AF at H and the tangent AD at G. Since AC, BH shall be parallel lines, there shall be as AC to BH, thus CE to EH, and thus CE to EB: but as AC to BH, thus there is AC to GB, (since GB, BH are equal [§.29]) therefore as AC to GB, thus there is CE to BE: moreover as AC to GB, thus CD to DB (since GB, AC are parallel) therefore as CD to DB, thus CE is to BE. Since in the first place there were now as CD to BD, thus CH to HB, if the ordinate AF were acting through E: I say the nearby line AD to be a tangent to the section at A. For if AD were not a tangent, suppose a tangent may be put through A which shall cross the diameter BD at K, therefore there will be, as CE to EB, thus CK to KB, but as CE is to EB, thus CD to DB, and therefore CK to KB, which cannot happen, since the point K shall fall either above or below D. Therefore AK is not a tangent nor any other besides AD. Q.f.d.

Corollary.

Propositions 29 and 30 also are true for the circle, although moreover there may be more cases where there may be a tangent for the circle than we have shown in this book of the ellipse, yet I make no mention of the circle except that it must be assumed for the following demonstrations.

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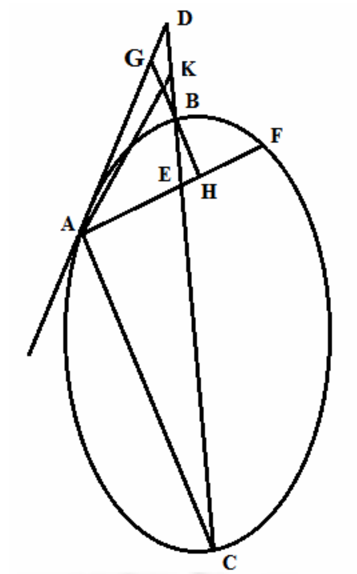
PROPOSITION XXXI

With the same figure proposed remaining, to deduce the tangent for a given point D beyond the section.

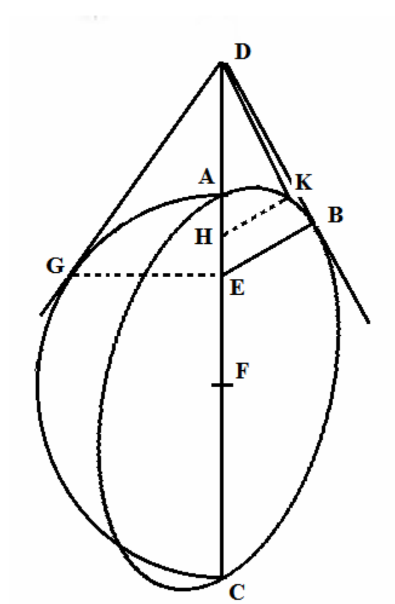
Construction and Demonstration.

The diameter DBC may be drawn from the point D beyond the section, and there shall become as CD to DB, thus CB to EB, and through E to BC, the ordinate AF may be put in place, and AD may be joined, it is apparent from the preceding the line AD to be a tangent to the section at A; therefore for a given point beyond the ellipse, etc. Q.e.d.



PROPOSITION XXXII

The right line BD shall be a tangent to the ellipse ABC, of which the diameter shall be AC, meeting the diameter at D: and from B the ordinate BE may be drawn to the diameter AC: moreover the centre of the section shall be F.
I say FE, FA, FD to be lines in continued proportion and if FE, FA, FD were continued in proportional, and the right line EB were acting through the ordinate E, I say the section to contain the joined line BD. A proposition of Apollonius.

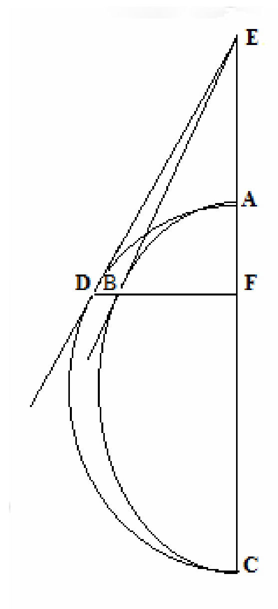


Demonstration.

The circle AGC is described with centre F and with the radius FA , then from the point E the normal may be drawn to the diameter AC crossing the circle at G: and the right line GD may be drawn, since the right line ordinate EB is put in place for the diameter AC and the tangent acting through B it meets the same diameter at D, there will be as CD to DA [§.30] thus CE to EA ; moreover in the circle, with the right line EG normal to the diameter AC [§.30 cor.], and therefore the right line GD is a tangent to the circle at G; [§.30 cor.] whereby in the circle the lines FE, FA, FD will be in continued proportional : but the same lines are common to the ellipse, and therefore in the ellipse FE, FA, FD will be in continued proportion. Because if FE, FA, FD shall be in continued proportion, and through E the ordinate EB may be drawn, I say the junction BD to be a tangent of the ellipse at B; truly if the right line DK may be drawn from D touching the ellipse at K ; and from K the ordinate KH may be put in place ; therefore by the first part of this section, FH to FA, shall be as FA to FD; but also by the hypothesis : FE is to FA, as FA to FD. Therefore FE is to FA, as FH is to FA, which cannot be done, since FG shall be greater or less than FE. According to which DK is not going to be a tangent, but rather DB. Q.f.d.

PROPOSITION XXXIII

Let AC be the axis of the ellipse ABC, so that upon which with the diameter AC the semicircle ADC, and with the assumed point F found on the axis which shall not be the centre, from FD crosses the ellipse at B.
I say the tangents acting through B and D, occur at one and the same point on the axis AC.



Demonstration.

The tangent BE is acting through B, meeting the axis at E, and ED may be joined; since the ordinate FB has been put in place to the axis and the tangent BE to the section drawn, there will become CF to FA, as CE to EA: and from which with ED the tangent to the circle; therefore the tangents acting through B and D, are agreed to meet the axis in one and the same point. Q.f.d.

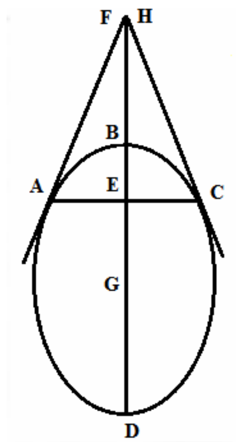
Corollary.

Hence we will show easily, if the two tangents meet the normal FD at the same point in the diameter FD, which if it may become a single point of contact D, to go through the other tangent as well.

PROPOSITION XXXIV

Let BD be the diameter of the ellipse ABC, for which AC may be put for the ordinate; the tangents through A and C.
I say this diameter to be present at one and the same point.

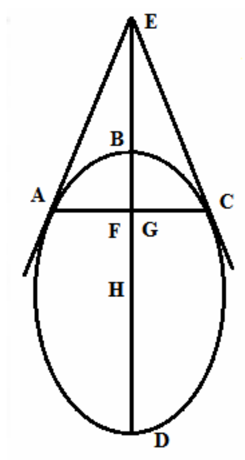
Demonstration.



By §16. above, it is evident the individual tangents drawn through A and C intersect on the diameter ; therefore if they may not meet at the same point, the tangent AF shall meet the diameter at F, and CH at H: Since the tangent AF concurs with the diameter at F, to that there will become DE to EB, thus as DF to FB; thus as DH to BH, and on dividing as DB to BF, thus DB to BH, which cannot happen; whereby the tangents do not cross the diameter at different points: therefore at the same point. Q.e.d.

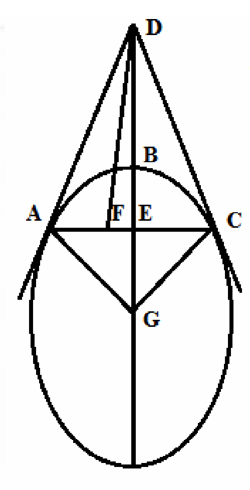
PROPOSITION XXXV

Let ABC be an ellipse, the diameter BD produced to some point E, and from E the lines EA, EC shall be tangents to the section at A & C. I say the joined line AC to be the ordinate put in place for the diameter BD.



Demonstration.

AF shall be put as an ordinate for BD and H shall be the centre of the ellipse ; therefore [§.32 above] the line EH will be divided at B and F into three continued proportionals, some ordinate CG will be dropped down to BD and again EH shall be divided at B and G into the three lines in continued proportion; therefore F and G are the same points ; whereby the right line AFC is the ordinate put in place for the diameter BD. Q.f.d.



Corollary.

Hence it follows if the ellipse ABC shall have tangents at A and C, the two right lines AE, CE meeting at D; and AC may be joined, which will be bisected at E; the right line DE to pass through the centre or the joined line DE to be the diameter of the section; indeed if ED shall not be the diameter, the diameter DF may be drawn from D; meeting the line AC at F; therefore by the preceding the line AC is bisected at F, and thus the point F, to be the same as the point E, from which the right line DF to be the same as the line DE: which is contrary to the supposition; whereby DE is the diameter of the section. Q.e.d.

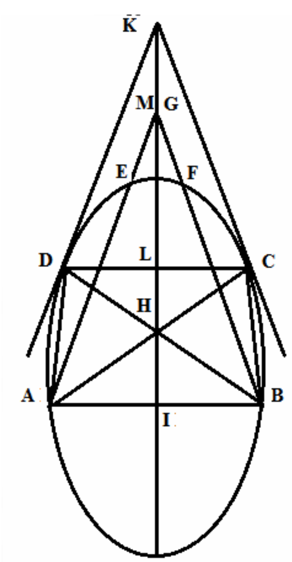
PROPOSITION XXXVI.

If the two right lines meeting at D may be tangents to the ellipse, and from the centre there may be drawn GA, GC, GD.
I say the triangles GCD, GAD to be equal.

Demonstration.

The right line AC shall join the points of contact, since AC is bisected at E, the triangles GAE, GEC, likewise DEC, DEA will be equal : and thus the two triangles DEC, DEA, that is the whole triangle DCG, will be equal to the two triangles DEA, EAG, that is, to the whole triangle GAD. Q.e.d.

PROPOSITION XXXVII.



AC and DB shall cut the ellipse : whatever tangents are acting through the tangents C and D, which by §.34 above meet the diameter HK at the same point.

I say the lines AG, BG drawn from A and B, to be parallel with DK, CK themselves, to intersect the diameter HK at the same single point.

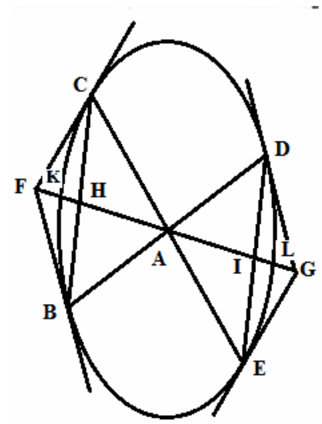
Demonstration.

AE may be put to cross the diameter at G and BF at M; then DC, AB, DA, CB may be joined. Since DC connects the tangents DK, CK, it shall be bisected by the diameter HK at L [§.35], truly there is AB parallel to DC[§.19] ; and therefore this shall be bisected by the diameter at I; whereby since the whole lengths DC, AB are equal, therefore since also DL, AI, will be parallel, which are joined by these parallel lines DA, IL, therefore this figure AGDK is a parallelogram; therefore the figure AGKD is a parallelogram, and on that account DA is equal to GK; in a similar manner we show that BC is equal to MK. Whereby since DA, BC, shall be equal, also KG, KM will be equal, therefore G and M are one and the same point by which, with the lines DC, CK drawn parallel to the tangents from the points A, B, cross the diameter. Q.f.d.

PROPOSITION XXXVIII.

Any two diameters BD, CE shall cut the ellipse of which the centre is A and with BC, DE joined, the diameter FG may be put in place, which shall bisect BC at H, and moreover ED shall be bisected at I, then the tangents are acting through C and B, likewise the tangents through D and E, which will cross the diameter FG at the same points F and G.

I say the following triangles to be equal to each other; in the first place the triangles ACF, ABF, secondly the triangles ACF, ADG, thirdly the triangles CBF, DEG.



Demonstration.

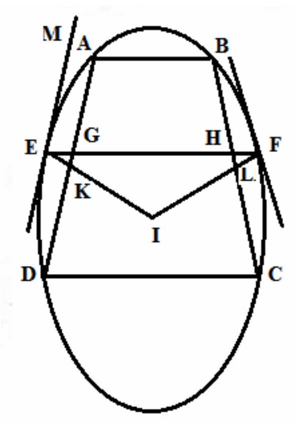
The diameter FG shall meet the ellipse at K and L. Therefore since from the hypothesis, BC shall be bisected at H, there will be both the triangle ACH equal to the triangle AHB, as well as the triangle HCF equal to the triangle HFB, from which the whole triangle ACF is equal to the whole triangle AFB, which was the first to be shown. Again since BC, DE shall be parallel [§.19] and BC shall be bisected at H by the diameter FG, and likewise ED shall be bisected at I; whereby since CB, DE shall be equal, the halves of these HC, DI are equal, whereby also the right lines CD, HI, the right lines of which may be joined, also are parallel; therefore the triangles ACH, ADI are between the same parallel lines. Moreover the bases AH, AI are equal, indeed AK is equal to AL, and the bases KH, AI are equal, (indeed AK is equal to IL): therefore the triangle ACH shall be equal to the triangle ADI. Now truly AH, AK, AF, and likewise AI, AL, AG, are in continued proportion ; whereby the ratio AH to AF, is the twofold ratio of AH to AK; and the ratio AI to AG is the twofold [i.e. the square] of the ratio AI to AL. Therefore since the ratios AH to AK, AI to AL, shall be the same (for AH is equal to AI and AK is equal to AL), and the ratios AH to AF, AI to AG shall be the same, and the duplicate ratios of the same ratios will be the same between themselves : and hence also the triangle AHC is to the triangle AEC, as the triangle AID to the triangle AGD. Whereby since the triangles ACH, ADI may be shown to be equal, also AFC, AGD will be equal, which was the other equality of the triangles. From which it is now apparent, if you may add FCH, GDI, for which to be equal if you may add FBH, GEI, which we have shown clearly to be equal by the same discussion, thus the whole triangles CBF, DEG, will be equal. Which was required to be demonstrated in the third place.

Corollary.

Hence it is apparent the quadrilaterals CFBA, DGEA, to be equal, indeed by the same discourse we will prove the triangles ABF, AEG to be equal in the same manner as we have proved the triangles ACF, ADG to be equal.

PROPOSITION XXXIX.

Any two parallel lines AB, CD may cut the ellipse and with AD, CB joined, the right line EM parallel to AD itself, may be a tangent to the section at E, and from E there may be drawn EF, parallel to AB, cutting the lines AD, CB at G and H.
I say the tangent line drawn through F to be parallel to BC itself BC.

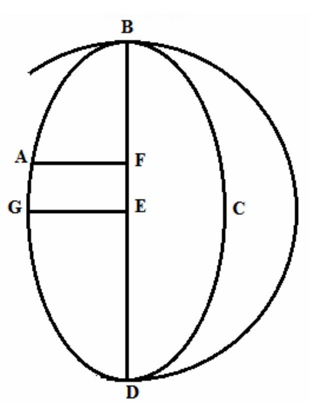


Demonstration.

The lines IE, IF may be drawn from the centre I crossing AD, CB at K and L : therefore since the tangent EM parallel to AD, and the diameter IE drawn to the tangent will cut the line AD at K, AD will be bisected at K [§.13]; moreover the right line BC is divided at L just as AD at K, indeed the right line KL joining the points K, L is parallel to AB, DC. Therefore BC is bisected at L by the diameter IF; and thus is parallel to the tangent drawn through F. Q.f.d.

PROPOSITION XL.

The circle described on the major axes as diameter exterior to the ellipse will meet the ellipse in two points only.



Demonstration.

BD shall be the major axis of the ellipse ABC, and from the centre E of this ellipse, and with the radius EB a circle is described, I say to cross that ellipse only at the two points B and D. For if it were possible to happen in addition at the point A, and the ordinate AF to the axis shall be acting through A, and the minor axis GE may be drawn, therefore there will become : as the rectangle BFD to the square FA thus the rectangle BED to the square EG: but the rectangle BFD in the circle is equal to the square FA; and therefore the rectangle BED, that is the square BE is equal to the square GE, which cannot happen since BE shall be a greater line GE, therefore the circle cannot meet the ellipse at A: and not in any other point, besides B and D. Q.f.d.

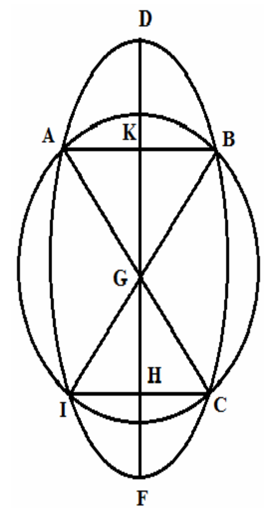
Corollary.

We will demonstrate by a similar discussion a circle described about the minor axis of the ellipse only to meet at the two furthest points of the ellipse, and the whole circle to lie inside the ellipse.

PROPOSITION XLI.

A circle is described from the centre of the ellipse, if it may cut the ellipse, then it will be cut at four points.

Demonstration.



There shall be a circle described from the centre G of the ellipse, and indeed you may cut the ellipse at B, the axis FD may be drawn, and the right line BGI; then the ordinate BKA shall be applied crossing the ellipse at A, likewise the right lines AGC, IC may be drawn; in the triangles BKG, AKG, BK, AK shall be equal, and KG is common, and the angle at K to be right; therefore GB, GA are equal ; whereby since the point B shall be on the circle, and moreover the same point A also is a point on the ellipse, therefore the circle shall cut the ellipse at A. Then AB, IC are parallel, and thus since the angle AKH shall be right, also IHK will be a right angle IHK; and hence the ordinate IC is placed on the axis at DF, and is bisected at H, moreover the whole lengths AB, IC are equal, therefore AK, IH the halves of these also are equal. Therefore in the triangles GKA, GHI, AK is equal to IH, and KG itself is equal to HG, truly the angles AKG, IHG also are equal; therefore GA, GI are equal; whereby since the point A and also the point I shall be on the circle, and also the point I is on the ellipse. Therefore the circle shall cut the ellipse at I. Similarly we may show the circle to meet the ellipse at C. Therefore the circle cuts the ellipse at the four points. Q.e.d.

Corollary.

But so that the circle may not cut the ellipse in more than four points, may be easily deduced from the demonstration now put in place.

THE ELLIPSE: Part II.

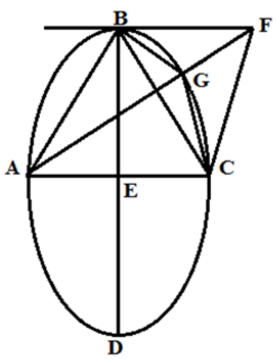
Concerning the Sectors and Segments of the Ellipse.

PROPOSITION XLII.

BD shall be the diameter of the ellipse ABC, for which the ordinate AEC may be put in place; and ABC may be joined.

I say ABC to be the maximum triangle of these which are able to be inscribed in the segment ABC.

Demonstration.

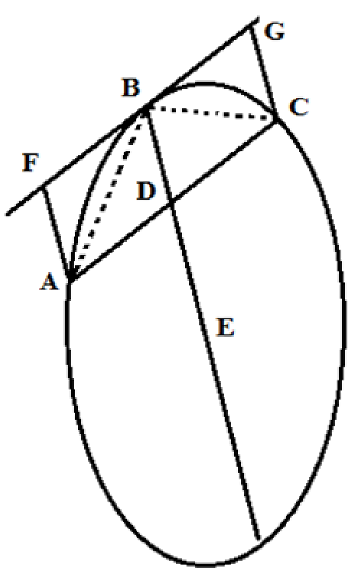


With the tangent BF acting through B, from A some right line AF may be drawn crossing the ellipse at G, the tangent at F and the lines GC, FC may be joined : Since the tangent FG falls above G, therefore the triangle AFC is greater than the triangle AGC; but the triangle AFC is equal to the triangle ABC on account of AC, BF being parallel lines; and therefore ABC is greater than the triangle AGC: from which since the same may be shown from all the other triangles ; to be apparent the triangle ABC, to be the maximum of those which are able to be described within the segment ABC. Q.e.d.

Corollary.

Hence the practise is readily deduced for inscribing the maximum triangle for any segment: without doubt by raising to the diameter BD, and by joining the points AB, BC. The demonstration apparent from the previous.

PROPOSITION XLIII.



The maximum triangle inscribed in any segment cannot to be greater than half of the inscribed ellipse, but to be greater than half of this same segment.

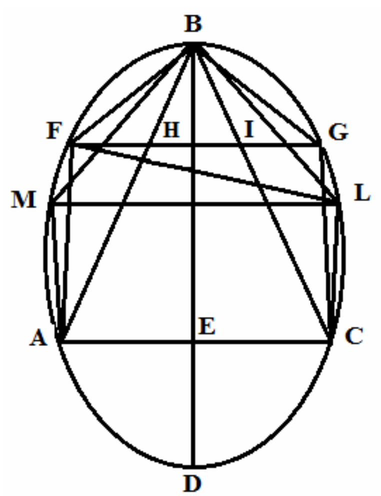
Demonstration.

Let the segment of the ellipse be ABC, the maximum triangle ABC inscribed not to exceed half of the ellipse. I say that to be the greater than half of the segment ABC: indeed with the diameter BE drawn, which divides the chord AC equally at D, the lines AF, CG may be erected from A and C parallel to the diameter BE, which meet the tangent line FG drawn through B, at F and G ; and AB, CB may be joined: the triangle ABC [§.41, Ch.1] is half of the parallelogram AG. And the parallelogram AG is greater than the segment of the ellipse ABC, since the lines AF, CG, FG fall outside the ellipse; and therefore the triangle ABC is greater than half of this same segment. Q.f.d.

PROPOSITION XLIV.

The diameter BD shall cut the ellipse ABC, to which the ordinate AEC may be put in place thus: with AB, CB joined, the maximum triangle AFB may be inscribed for the segment AFB, and from F, FG may be placed parallel to AC, and BG, GC may be joined. I say the maximum triangle BGC to belong to these which can be inscribed for the segment ABC, and if the triangles were the largest, I say FG to be parallel to AC.

Demonstration.



Because FH, GI are equal lines [§.26], the triangles FBH, GBI having the same altitude, will be equal, similarly the triangles FAH, GIC set up between the parallels FG, AC, will be equal and hence the whole triangles BFA, BGC will be equal. Therefore, if BGC were not be the maximum, BGC may be replaced by another greater triangle BLC, and from L, LM may be drawn parallel to AC; therefore so that the first triangle BLC will be equal to the triangle AMB, that is AFB to be greater than the triangle BGC, which is contrary to the supposition, since BFA shall be put to be the maximum. Therefore BGC is the maximum triangle of these which can be described by the segment BGC, which establishes the first part.

Since the triangles BFA, BGC shall be the greatest, we will demonstrate AC to be parallel to the joined line FG: for if it were not parallel, there shall be another line either above or below FG itself parallel to AC; without doubt to be the right line FL; and BL, CL may be joined. Therefore, by the first part of this Prop., the triangle BLC would be the greatest, which cannot occur, since from the hypothesis, FGC shall be the maximum. Therefore neither FL, nor any other line besides FG, is parallel to AC. Q.e.d.

First Corollary.

Hence it follows, that if the triangles BFA, BGC shall have been the greatest of these which shall be described by the segments, then they are to be equal. For by the second part of this Prop., FG is parallel to AC. From which FH, GI are equal, and also the triangles FBH, BIG, and FAK, GCI are equal; and thus the whole triangles BFA, BGC are equal.

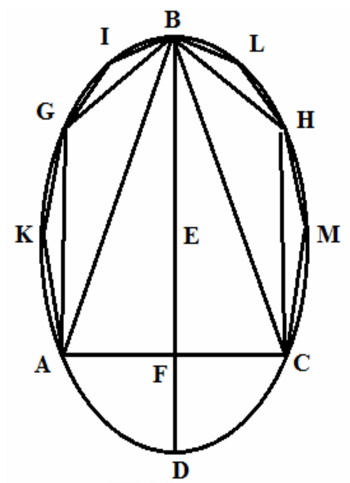
Second Corollary.

So that if the two lines AC, FG were put in place to be the ordinate lines for the diameter, the greatest triangles are inscribed by the segments AF, CG, also to be equal to each other, plainly we will demonstrate by the same discussion, which we have used in the proposition and in the first corollary, with nothing else changed, just as in place of Prop. 26 here it will be required to assume Prop. 27.

PROPOSITION XLV.

BD shall be some diameter of the ellipse ABC for which an ordinate line AFC may be put in place.
I say the segment AGBF to be equal to the segment CHBF.

Demonstration.



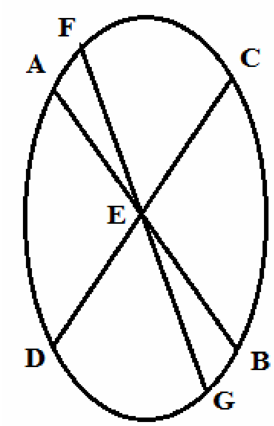
With the remaining segments joined AB, CB, the greatest triangles AGB, CHB will be inscribed [§.42, Ch.1] equal to each other, by the first Cor. of the preceding Proposition, and [§.43, Ch.1] from the halves of the greater segments ABG, CBH: then the greatest triangles AKG, GIB, CMH, BLH will be inscribed as before, and the greatest triangles will be inscribed on each side of the segment, by the remaining triangles AKG, GIB, will be as part of the first triangle by the first cor., part by the following equal triangles CMH, BLH, and with half of the greater segments therefore since that inscribed always shall be able to be continued in each segment AGB, BHC, and the parts removed on both sides shall be equal to each other, and with the greater halves of the segments from which they are removed, it is agreed [§.216, Progress.] the segment AGB to be equal to the segment CHB. Whereby with the equal triangles ABF, CBF added, the whole segments AGBF, CHBF will be equal.

Q.e.d.
Corollary.

Hence it follows the ellipse to be cut into two equal parts by any : for BD shall be any diameter, and the ordinate AFC may be drawn through any point of this, by the proposition now demonstrated, the segment ABF is equal to the segment BCF, again from the same proposition the segment ADF shall be equal to the segment CDF; therefore the segments ABF, ADF, that is the whole segment DAB, are equal to the segments BCF, CDF, that is, to the whole segment DCB; therefore the ellipse is divided into two equal parts by the diameter BD.

Translated from Latin by Ian Bruce; 24/7/2021.Free download at 17centurymaths.com.

PROPOSITION XVI.



Two conjugate diameters divide the ellipse into four parts, and the four diameters dividing the ellipse, are conjugate to each other.

Demonstration.

AB, CD shall be two conjugate diameters in the ellipse ABC ; I say these to divide the ellipse into four parts; and if AB, CD shall divide the ellipse into four parts, I say these to be conjugate. Since AB, CD are conjugate diameters ; AB will be the ordinate put in place for the diameter DC, so that now AEC, CEB as well as AED, BED shall be equal parts ; moreover AEC, AED are equal sectors on account of the same ratio; therefore the four sectors are equal to each other AEC, CEB, BED, DEA, are equal to each other, and the lines AB, CD divide the ellipse into four parts: Which was the first part.

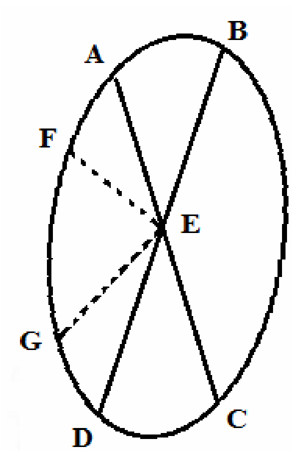
Now if the ellipse shall be divided into four parts, I say AB, CD to be conjugate diameters; truly if CD itself may not be drawn, conjugate to FG, therefore FEC shall be the fourth sector of the ellipse through its first part. And the sector AEC from the hypothesis also is the fourth part of the ellipse, therefore the sectors FEC, AEC are equal, both the part and the whole, which cannot happen; therefore the diameter FG is not the conjugate of CD itself, nor any other besides AB. Q.e.d.

Corollary.

Hence to be apparent, the sectors of whatever conjugations, to be equal with the individual sectors of whatever the kind of the other singular conjugation, and if the sectors shall be equal and the sides of one shall be conjugate, also the sides of the other to be conjugate.

PROPOSITION XLVII.

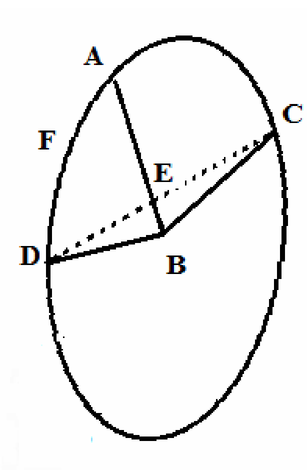
Sectors to the opposite vertex are equal to each other.



Demonstration.

The diameters cut the ellipse ABC in some manner AC, BD: I say the sectors opposite the vertex to be equal to each other. Indeed two diameters EF, EG may be drawn from the centre E, and EF the conjugate of EB itself: EG truly the conjugate of AE itself. Therefore since the sectors BEF, AEG, [§.46, Ch.1] are equal, with the common sector AEF removed, the sector AEB will be equal to the sector FEG: again since the sectors SED, GEC shall be equal, with the common sector DEG removed, the sector DEC will be equal to the sector FEG, that is AEB to be equal to the opposite vertex. In the same manner it will be shown the sectors AED, BEC to be equal. Therefore, &c. Q.f.d.

PROPOSITION XLVIII.



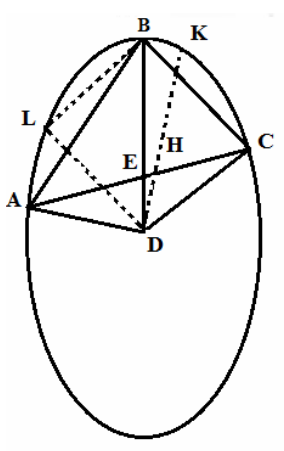
ABC shall be some sector in the ellipse ADC, it is required to draw a right line from B to the periphery, which may constitute with the line AB a sector equal to the given ABC.

Construction & demonstration.

The ordinate may be drawn from C to the diameter AB, with the right line CED, and BD may be joined; I say what is required to be done, indeed the segment AED [§.45, Ch.1] is equal to the segment AEC, and since CE, ED shall be equal ; the triangle DEB to be equal to the triangle CEB, and therefore the sector ABD to be equal to the sector ABC. Therefore we have shown, etc. Q.f.f.

PROPOSITION XLIX.

Equal sectors ADB, CDB may have a common line BD, and AB, CB may be joined. I say the segments of the lines AB, CB removed to be equal to each other, and if the segments were equal, I say the sectors also to be equal.
Demonstration.

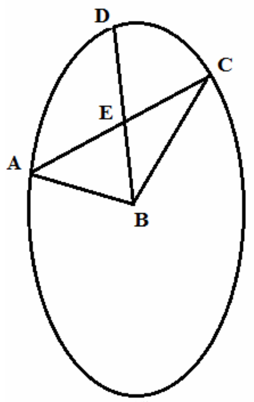


A,C may be joined and AC crosses the line for the diameter BD at E, then if AC may not be bisected at E: it shall be bisected at H, and the diameter HK acts through H. Because AC has been drawn the ordinate line to the diameter DK, the segments AHK, CHK will be equal; moreover AHD and CHD are equal triangles, therefore the sectors ADK, CDK are equal to each other, therefore the sector CDK is equal to a part of the whole sector CDB. Which is absurd; whereby the line AC is not bisected at H ; nor in another point except at E: and therefore thus AC, is placed to be the ordinate for the diameter BD. From which the segment AEB is equal to the segment CEB; but AEB, CEB are equal triangles, therefore the remaining segment AB, is equal to the segment CB; which was the first to be shown.

Now AB, CB shall be equal segments, and from the points A, B, C there may be placed the diameters AD, BD, CD, I say the sectors ADB, CDB, to be equal to each other; but if truly the sector CDB shall be equal to the sector BDL, and the points LB may be joined, therefore the segment LB is equal to the segment CB, that is to AB by the hypothesis, and thus a part is equal to the whole.

Since this cannot happen; therefore the sector BDL [§.48 above] is not equal to the sector CDB: nor otherwise to any sector besides the sector ADB. Q.e.d.

PROPOSITION L.



ABC shall be some sector.

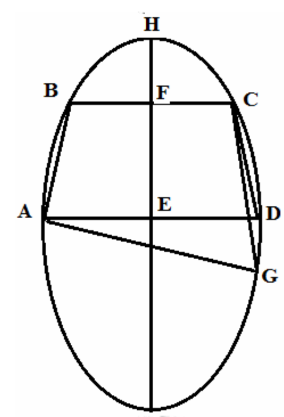
I say a line drawn from the centre, which bisects the chord AC, also cuts the sector into two equal parts.

Demonstration.

The diameter BD may be drawn from the centre B; bisecting the line AC at E: I say the sectors ABD, CBD to be equal: Indeed since AC shall be bisected at E, ABE, CBE shall be equal triangles, but, since the ordinate AC has been put in place for the diameter BD, the segments AED, CED also are equal [§.45 above], therefore the whole sector ABD, is equal to the sector CBD. Q.e.d.

Any two parallel lines AD, BC may cut the ellipse ABC ; and AB, CD may be joined.

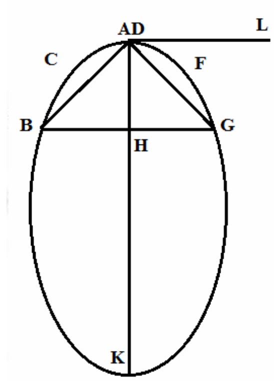
I say the segments AB, CD to be equal and if the segments were equal, I say the lines BC, AD to be parallel.



Demonstration.

AD, BC to be bisected at F and E by the line FE acting through F and E, crossing the ellipse at H, that line will be the diameter, whereby the segments BFH, CFH are equal; again because AE, DE are equal, and the segments AHE, DHE will be equal : therefore with the equal segments BFH, CFH removed the segments AB, FE, DC, FE will remain equal. Then because AE, ED are equal, and the heights of the common parallel lines BC, AD, will be AE, FB, equal to the trapezium DEFC, therefore from the equalities, the remaining segments AB, CD remain equal to each other. Which was the first part.

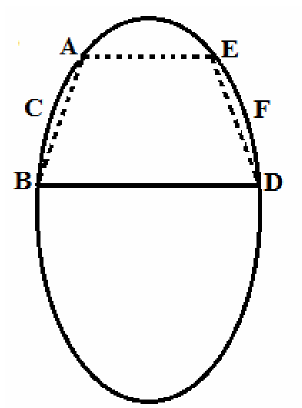
Now the segments AB, CD shall be equal, and BC, AD may be joined ; I say the lines AD, BC to be parallel to each other: truly if not, AG may be drawn parallel to BC itself, and CG may be joined: therefore by the first part of this Prop., the segment AB will be equal to the segment CG: however from the hypothesis the segment CD is equal to the segment AB; therefore the segments CG, CD are equal, which cannot happen, since the point G either falls above or below D, and thus the segment CG shall be greater or less than the segment CD: therefore the line AG shall not be parallel to BC, but to AD only. Q.e.d.



But if the given point D the same as the point A, the tangent AL to the ellipse at the point A, AL shall be drawn tangent to the ellipse at the point A or D, (indeed now from the hypothesis A and D are one and the same point) and with BG drawn parallel to AL, join DG.

I say by this contraction the segment DFG to be equal to the segment ACB.

From the point of contact the diameter AK may be drawn; therefore since BG shall be parallel to the tangent, it is the ordinate put in place for the diameter AK, and thus bisected at H, therefore the triangles BAH, GAH are equal; and truly the segments BCAH, GFAH shall be equal ; therefore the remaining segments BCA, GFA or GFD are equal. Therefore what was sought has been accomplished.



PROPOSITION LII.

Some right line AB may cut the ellipse : taking away the segment ACB, and some point D may be given on the periphery D, it will be required to draw the right line DE from D, which shall remove the segment DEF, equal to the segment ABC.

Construction and demonstration.

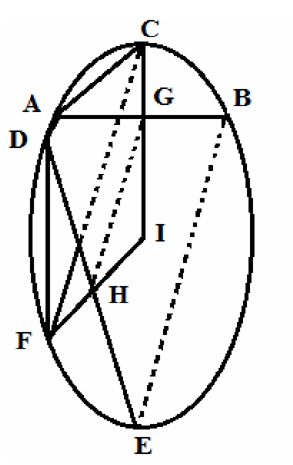
BD shall be joined, and from A there may be put AE parallel to BD, and ED may be joined; it is evident from the preceding that the segment DEF to be equal to the segment ABC. Therefore from the given point, &c. Q.e.f.

PROPOSITION LIII.

Some two chords AB, DE may cut the ellipse ABC, removing equal segments : moreover with the right lines AB, DE bisected at G and H , the diameters IGC, IHF shall be drawn through G and H.

I say these to be divided proportionally at G and H. And if the diameters shall be divided proportionally : I say the segments to be equal.

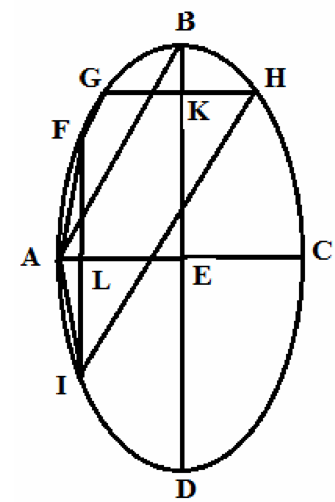
Demonstration.



The lines AD, BE, AC, DF, GH, CB, FE, CF may be joined. Because the segments ACB, DFE may be placed equal, the lines AD, EB are parallel: moreover as AG to GB, thus DH to HE, since the lines AB, DE [§.51. above] shall be bisected at G and H, and therefore the line GH to be parallel to AD, BE. Now truly since AB shall be the ordinate put in place to the diameter IC, the segments AGC, BGC [§.45. above] shall be equal, and thus the segment AGC shall be half the segment ACB; from the same reason the segment DFH is half the segment DFE. Whereby since the whole segments ACB, DFE may be put equal, there will be also the segments AGC, DFH [§.42. above], the halves of which shall be equal to each other. Then the triangles ACB, DFE shall be the greatest of these triangles able to be inscribed, and since AD, BE have been shown to be parallel, also are equal to each other [§.44. Cor.2], and the halves of these, the triangles AGC, DFH: which, if taken from the equal segments AGC, DFH, then the segments AC, DC will remain equal, therefore the line CF [§.52. above] will be parallel to the right line AD, that is GH: whereby as CG to GI, thus FH to HI. Q.e.d. Hence now the truth of the converse shall be evident.

PROPOSITION LIV.

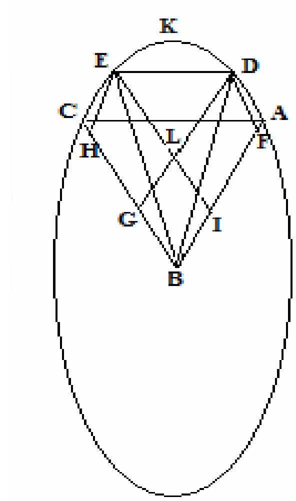
AC, BD shall be some of the diameters conjugate in the ellipse ABC and AB may be joined, some line FG may be drawn parallel to AB; and from F and G, the right lines GH, FI may be placed as ordinates to the diameters BD, AC.
I say the diameters AC, BD to be divided proportionally at K and L.



Demonstration.

The lines GB, BH, FA, AI, HI may be drawn. Because the lines AB, GF are parallel, the segments GB, FA will be equal [§.52. above]. Moreover the segment GB is equal to the segment HB, (for the segment of the line GKB is equal to the segment HKB, and the triangle GBK to the triangle HBK) and, by the same reason, the segment FA is equal to the segment AI. Therefore the segment HB is equal to the segment AI, and HI is parallel to the line AB [§.45. above], that is, FG. Whereby the whole segment GBH is equal to the whole segment FAI: and thus by the preceding, the diameters AC, BD are divided similarly at K and L. Q.e.d.

PROPOSITION LV.



In the given ellipse some of the conjugate diameters AB, CB shall be given, and AC shall be joined, to which some ED shall be given parallel, moreover from the points D and E the ordinate for the diameters DF, EH, DG, El, may be drawn : therefore the figures DFBG, EIBH will be parallelograms. I say these parallelograms FG, HI to be equal.

Demonstration.

Since by the preceding BA, BC are divided proportionally, AB will be to BF, as CB to BH: and on interchanging, as AB to CB, thus BF to BH; similarly by the preceding AB is to BI as CB to BG, and on interchanging, as AB to CB, thus BI to BG; therefore BF is to BH, as BI to BG. Therefore [§.14. ch.6], the parallelograms AG, HI are equal. Q.e.d.

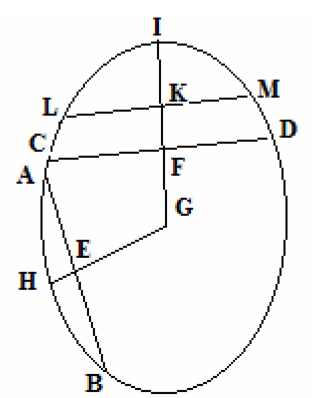
PROPOSITION LVI.

The diameters AB, BC shall be joined; with the points joined AC, ED may be drawn parallel to the right line AC, then from D and E the ordinate right lines EI, DF, DG, EH may be put in place for the diameters AB, CB, and EB, DB may be joined.
I say the sector EBD to be equal to the figure EIFDKE.

Demonstration.

From the halves of the preceding equal parallelograms FG, HI, the triangles DFB, EIB are equal to each other; therefore with the removal of the common triangle LIB, the triangle ELB will be equal to the trapezium DFIL; whereby with the addition of the common figure ELDKE, the sector EBD is equal to the figure EIFDKE. Q.e.d.

PROPOSITION LVII.



Any two lines AB, CD cut the ellipse ABC; it is required to draw some line parallel to the line CD, which will remove a segment equal to the segment AHB.

Construction & demonstration.

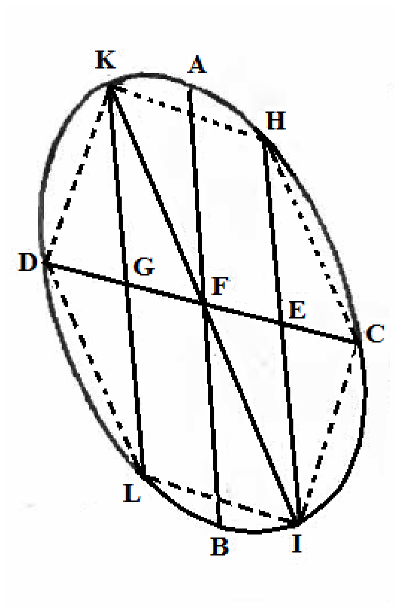
Bisect AB, CD at E and F, the diameters GH, GI shall act from the centre G through E and F; then GI shall be divided at K, just as HG has been divided at E, and the ordinate LM may be placed through K , it is apparent from §53 that LIM, AHB to be equal segments ; but LM is parallel to the given CD, therefore with the segment of the ellipse given, &c. Q.e.f.

PROPOSITION LVIII.

Two conjugate diameters AB, CD shall cut the ellipse ABC ; and of these the other AB divides CD at E and G fourfold, with the right lines HI, KL acting through E and G parallel to the diameter AB, and the points D, K, H, C, I, L, D may be joined.

I say the lines LD, DK, KH, HC, CI, IL to bear equal segments.

Demonstration.



Because KL, HI are parallel to the right line AB, which is the diameter conjugate to DC itself, KL, Hl will be the ordinates put in place for DC Therefore so that the rectangle DGC shall be to the rectangle DEC, thus as the square KG to the square HE, and thus since the rectangles shall be equal, also the squares will be equal, so that the right lines KG, HE also are equal.

Truly GF, EF are equal, and the angles KGF, FEI (since Hl, KL shall be parallel) are equal; therefore the triangle KGF is equal to the triangle FEI, and the angle IFE is equal to the angle GFK. Since GFE is a right line, the angles IFE, KFG are going to constitute equal angles with the vertex: from which the points of KFI lie on a straight line, and thus the sectors KFD, CFI are to be made equal by the vertex.

Moreover, since I have shown the triangle KFG to be equal to the triangle FEI, and by a similar discussion it may be shown also, that the triangle DKG to be equal to the triangle ICE, the whole triangle DKF to be equal to the whole triangle ICF.

In addition, the sectors DFK, IFC are equal with regard to the vertex. Therefore also the segments remaining DK, IC will be equal to each other. In the same manner, the segments DL, HC are shown to be equal to each other.

Again, since the two sides KG, GF shall be equal to the two sides GL, DG, and the angles contained by equal sides to be equal to the angles through the vertex; the triangles GKF, DGL to be equal to each other, and the angle GKF is equal to the other angle GLD.

Thus KFI, DL to be parallel lines; whereby the segments DK, LI are equal to each other [§.51. above]. Now truly it has been shown the segments CI, DK also to be equal to each other, therefore the three segments DK, LI, CI are equal. Further since DC, LI taken together are parallel, and for which GL, EI, themselves also are parallel, from which again the segments CI, DL are to be equal, for the four segments DL, CI, LI, DK to be equal. Again since KH, LI, taken together with KL, HI are equal and themselves parallel : therefore the segments KH, Ll are equal. Therefore the five segments KH, LI, KD, DL, CI are equal; and also the equality of the segments HC, DL has been shown; therefore all six segments are equal. Q.f.d.

Moreover the figure DKHCIL inscribed in the ellipse may be called a regular polygon.

PROPOSITION LIX.

With the same figure proposed remaining, to inscribe a regular hexagon figure in the ellipse.

Construction and demonstration.

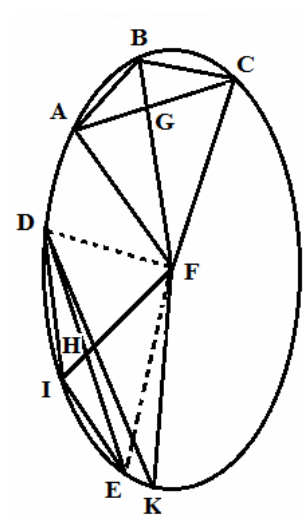
Any two conjugate diameters AB, CD may be assumed and with CD divided fourfold at E and G, the lines HI, KL shall act through the points E and G parallel to AB: and DK, KH, HC, CI, IL, LD may be drawn: it will be evident from the preceding: these right segments to be removed equally, and thus the hexagon figure DK, HC, ILD to be regular; therefore we have inscribed the figure of a regular hexagon. Q.e.f.

PROPOSITION LX.

Any two lines may cut the ellipse ABC bearing the equal segments AC, DE, and the ends of the radii FA, FC, FD, FE may be drawn to these.

I say the sectors AFC, DFE to be equal, and if the sectors were equal, I say the segments to be equal.

Demonstration.



With AC, DE bisected at G and H, the diameters FB, FI shall act through G and H, and AB, BC, DI, EI may be joined; since the segments AC, DE are equal from the hypothesis, therefore the right lines, which may join the points C and E, A and D, may become parallel; therefore the greatest triangles from the inscribed segments AC, DE shall be equal, but ABC, DIE, are the greatest of the prescribed, therefore the triangles ABC, DIE shall be equal : but since the diameters FB, FI at G and H are divided proportionally: therefore as the triangle ABC is to the triangle AFC, thus triangle DIE is to triangle DFE, and on interchanging so that triangle ABC is to triangle DIE, thus triangle AFC is to triangle DFE. Therefore since the triangles ABC, DIE shall be equal, also the triangles AFC, DFE will be equal. Whereby with the equal segments AC, DE added , they will be equal to the sectors FAC, FDE.

Now the sectors AFC, DFE shall be equal, and AC, DE may be joined. I say the segments ABC, DIE also to be equal; truly if not : either may be considered (for example) DIE lesser than the other, and the line DK may be drawn from D, the segment removed is equal to the segment ABC, the sector DFK is equal to the sector AFC, that is equal to the sector DFE, which cannot happen; therefore the segments AC, DE are not unequal, but equal. Q.e.d.

Corollary.

From this, and from §.53 above, it follows in the first place, that if AFC, DFE were some equal sectors, the chords of which were bisected at G and H, then the diameters drawn through G and H, may be divided proportionally by the same points.

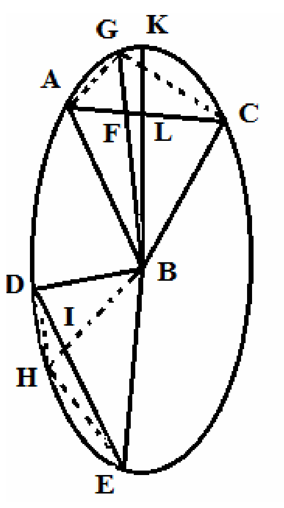
In the second place if the two sectors AFC, DFE were equal, and right angles AC, DE were subtended by these : so that the triangles AFC, DER shall be equal.

In the third place such problems hence are solved, with some sector AFB given, it will be required to draw radii from F which shall constitute a sector equal to the sector AFB: for the construction join AB, some radius FD shall be drawn from F: then from D the right line DI may be drawn taking away a segment DI equal to the segment AB and IF may be joined, it is evident the sector IFD to be equal to the sector AFB.

PROPOSITION LXI.

ABC, DBE shall be equal sectors: AC, DE may be joined and some line BE shall be drawn from B as if cutting the line AC at F: then the sector DBH shall be made equal to the sector ABG, and the line HB may cut the right line ED at I.
I say both the lines AC, DE as well as BG, HB to be divided proportionally at F and I.

Demonstration.



AG, GC, DH, HE shall be joined. Since from the hypothesis both the sectors ABG, DBH, as well as ABC, DBE are equal, and the remaining GBC, HBE shall be equal; whereby the triangle AGB is equal to the triangle DHB, and the triangle CGB is equal to the triangle HEB, whereby in order that the triangle BAG shall be to the triangle GCB, thus as the triangle BDH triangle shall be to the triangle BHE, but (which has been shown easily from ext.6.) the ratios of the right lines AF, FC, and DI, IE, are the same as from the ratios of the triangles BAG, GCB, and BDH, BHE.

Therefore also, AF is to FC, as DI is to IE. Then since the trapeziums GABC, BDHE shall be equal (for indeed the triangle BAG is equal to the triangle BDH, and the triangle BGC is equal to the triangle BHE) moreover the triangle ACB shall be equal to the triangle DEB, the remaining triangle AGC shall be equal to the remaining triangle DHE: therefore as the triangle AGC to the triangle ACE, that is as GF to FB, thus the triangle DHE to the triangle DEB, that is, HI to IB. Q.e.d.

PROPOSITION LXII.

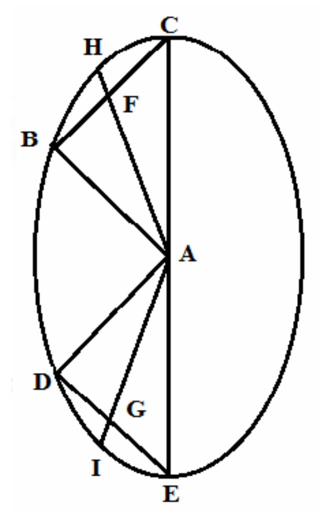
Now the lines AC, DE shall be divided proportionally at F and I : and the radii BG, BH acting through F and I.
I say the sectors ABG, DBH to be equal.

Demonstration.

But if truly one such as ABG shall be smaller than the other : ABK shall be made the sector equal to the sector DBH, and BK shall cut the right line AC at L. Since the sectors ABK, DBH are equal: by the first part of this Prop. there will become AL to LC, as DI to IE. And also AF is to FC, as DI to IE, therefore as AF to FC, thus AL to LC, which cannot happen : since the point L falls either further or nearer than F; whereby the sector ABK is not equal to the sector DBH, nor to any other besides ABG. Q.e.d.

PROPOSITION LXIII.

AB, AC shall be any two diameters and with the ends of these joined, some other two diameters AD, AE may be drawn such that with DE joined, ABC, ADE shall be equal triangles : moreover with BC, DE bisected at F and G, the diameters AH, AI acting through F and G shall be divided proportionally at F and G.
I say the sectors BAC, DAE to be equal.

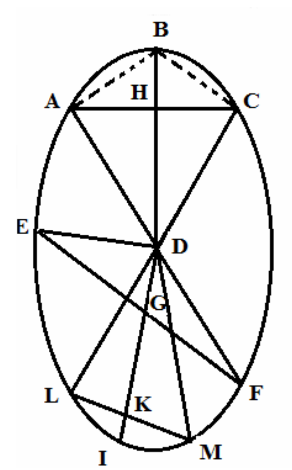


Demonstration.

Because the diameters AH, AI are divided proportionally at F and G, the segments BHC, DAE will be equal: but from the hypothesis the triangles ABC, ADE are equal; therefore the sectors BAC, DAE are equal to each other.
Q.e.d.

PROPOSITION LXIV.

There shall be two sectors DABC, DEIF, and with the right lines drawn AC, LM, and bisected at H and G, the diameters DHB, DGI may be drawn. Moreover the ratio DG to DI shall be smaller than the ratio DH to DB.
I say the sector DEIF to be greater than the sector DABC.

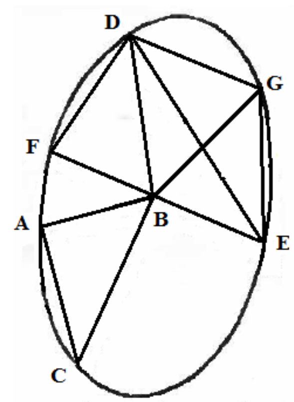


Construction and demonstration.

Because DG to DI is in a smaller proportion than DH to DB. There may become DK to DI, as DH to DB, therefore DK will be greater than DG, and the point K falls between G and I, the ordinate LKM shall be drawn through K ; and DL, DM may be joined; there the segments LIM, ABC are equal. Whereby the sectors DLIM, DABC are equal also, and therefore the sector DEIF is greater than the sector DAVC. Q.e.d.

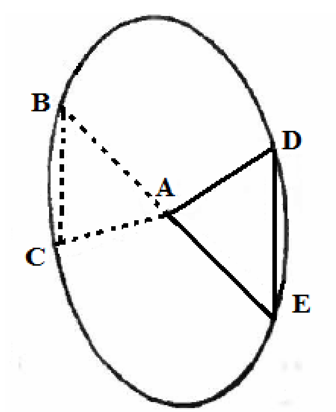
PROPOSITION LXV.

ABC, DBE shall be two unequal sectors, thus so that ABC, DBE shall be equal triangles:
I say the sectors ABC, DBE taken together to equal a semi-ellipse.



Demonstration.

Since, from the hypothesis, the sectors shall be unequal, BDGE shall be the greater sector, as, according to the hypothesis, the triangle DBE is equal to the triangle BAC, therefore the segment DGE to be greater than the segment AC, and thus a segment EG equal to the segment AC itself may be removed from EG, and BG, GD may be joined and EB produced to F, FD shall be drawn : therefore since the segments GE, AC are equal, the sectors also are equal, and indeed the triangle GEB is equal to the triangle ACB, that is, equal to the triangle BDE ; whereby BE, DG are parallel and thus the segment GE, that is the segment AC is equal to the segment DF; and thus the sectors BAC, BFD are equal. And the sectors BFD, BDGE, constitute a semi-ellipse. Q.e.d.



PROPOSITION LXVI.

The two sectors BAC, DAE may be taken, thus so that the triangles ABC, ADE shall be equal: if the sectors of these likewise may be taken, they would be greater or smaller than a semi-ellipse:
I say these sectors to be equal to each other.

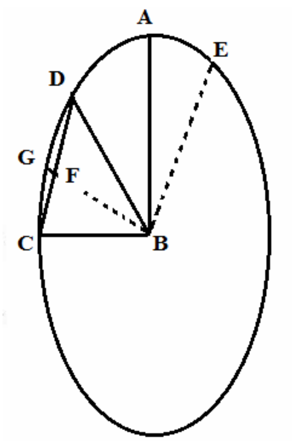
Demonstration.

Indeed if they were not equal, let BAC be smaller than the sector DAE: therefore since the triangles ABC, DAE are equal, the sectors BAC, DAE taken together shall be equal to a semi-ellipse : Which is contrary to the hypothesis.
Therefore the sectors BAC, DAE are not unequal, but equal. Q.e.e.

PROPOSITION LXVII.

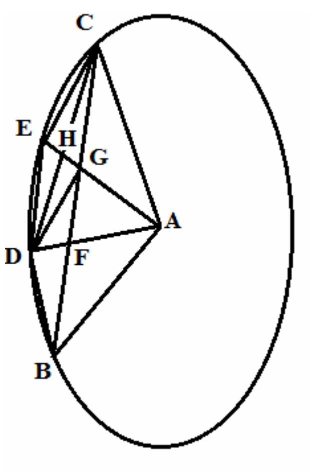
AB, BC shall be conjugate diameters, and some right line CD may be drawn from C, from B the line BE may be drawn parallel to the line CD; and DB shall be joined.

I say the sector DBC to be the double of the sector ABE.



Demonstration.

CD shall be bisected at F, the diameter BG may be drawn, and since BE shall be parallel to the ordinate CD put in place for the diameter BG, so that also is an ordinate indeed through the centre. Therefore BG, BE are conjugate diameters. Moreover, from the hypothesis, CB, BA also are conjugate. Therefore the sector CBA is equal to the sector GBE, and with the common sector GBA removed, the sectors CBG, ABE are equal. And the sector CBD is the double of the sector CBG.
Therefore also CBD is the double of the sector ABE. Q.e.d.



PROPOSITION LXVIII

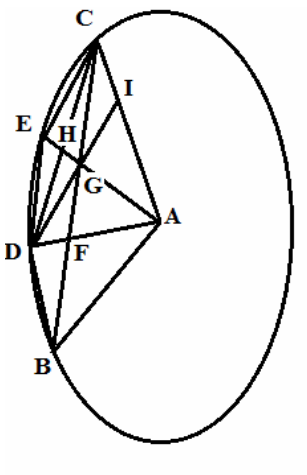
If the sectors ABD, ADE, AEC shall be equal, and the right lines DE and BC may be drawn, cutting the right lines AD, AE in F & G.

I say BF, DE, EC to be equal.

Demonstration.

The right lines BD ,EC, DC, DG may be drawn. Since the sectors ADE, AEC are equal, also the triangles CHA, DHA are equal; therefore the right lines DH, HC are equal. Then since the sectors ADB, ACE are equal, the segments BD, EC also are equal. Therefore CB, ED are parallel. Therefore the angles GCH, EDH, are equal : truly the angles GHC, DHE are equal also, and now the right lines DH, HC are shown to be equal; therefore CG, ED shall be equal. Similarly we may show BF, DE to be equal. Therefore the truth of the proposition may be agreed on.

PROPOSITION LXIX.



With the same in place DG may be produced, then it shall cross the line AC at I.

I say AGI, AGC, AEC to be triangles in continued progression.

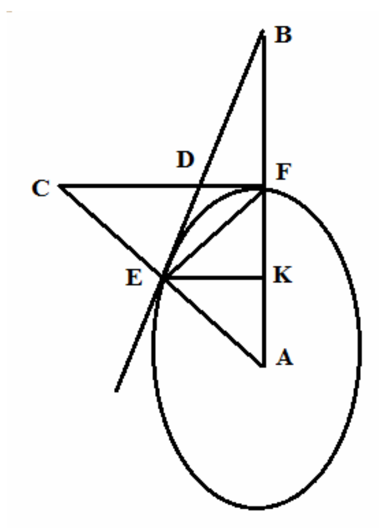
Demonstration.

From the above demonstration, it is evident everything is equal in triangles CHG, EHD, and thus also the sides EH, HG, then only the triangles DHG, BHC may be considered to be equal. In which since the two sides DH, HE, shall be equal to the two sides CH, HE, and the angle DHG equal to the angle CHE, for the base also the angles HGD, CEH will be equal; hence DI, CE shall be parallel and thus as AI to AC, thus AG to AE. But as AI is to AC, thus triangle AGI to triangle AGC: and as AG to AE, thus triangle AGC to triangle AEC. Therefore as triangle AGI is to triangle AGC; thus triangle AGC is to triangle AEC. Q.e.d.

PROPOSITION LXX.

Two diameters AE, AF shall cut the ellipse at the points E and F, from which the tangents to the ellipse FC, EB may be drawn, crossing the diameters at C and B, and EF may be joined.

I say the triangle ACF to be equal to the triangle ABE.



Demonstration.

From the point E, draw the ordinate EK to AF, this will be parallel to the tangent at F, and thus the triangles AEK, ACF are similar, and therefore have the two fold ratio of the ratio AK ad AF : that is because AK, AF, AB are three ratios in continued proportions, they have the ratio as AK ad AB, but it is also as AK ad AB , thus as the triangle AEK to the triangle AEB: therefore the triangle AEK has the same ratio to triangle ACF, as it has to triangle AEB; therefore the triangles ACF, AEB are equal. Q.e.d.

Translated from Latin by Ian Bruce; 24/7/2021.Free download at 17centurymaths.com.

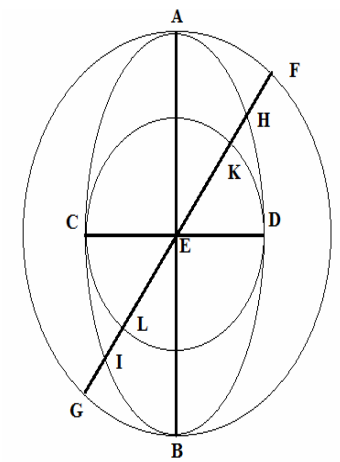
THE ELLIPSE: PART THREE

Part three considers both the similar and dissimilar properties of conjugate axes and diameters of the ellipse.

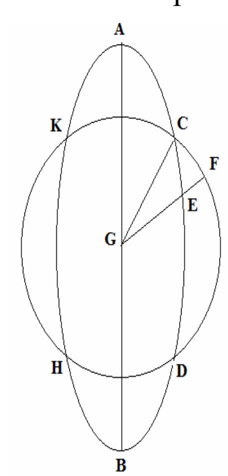
PROPOSITION LXXI.

For an ellipse, the axes are the maximum and minimum of the diameters.

Demonstration.



The axes of the ellipse ABC shall be AB and CD ; and indeed AB shall be the major axis and CD truly the minor axis. I say AB to be the maximum of the diameters, CD truly the minimum, with the centre of the ellipse E: the circle AFG shall be described with the interval EA : it passes through B, and the whole of its remaining part will lie outside the ellipse: then some diameter FG may be drawn through E, intersecting the ellipse at H and I, but the circle at F and G.
Because the whole circle AFG falls outside the ellipse, the line FG will be greater than the line HI : and therefore AB is greater than HI. It is shown likewise for any other diameter; therefore the axis AB is the greatest of the diameters of the ellipse ABC. Which was the first part to be established.
Again, with the centre E and with the interval ED, the circle DKL is described intersecting the line FG at K and L: this circle passes through C and the whole of its remaining part will fall within the section; therefore the line HI is greater than the line KL, that is equal to CD: Whereby since the same may be shown for all the other lines which do not pass through C and D, CD will be the minimum diameter of all the lines which are able to be drawn through the ellipse ADB. Q.e.d.

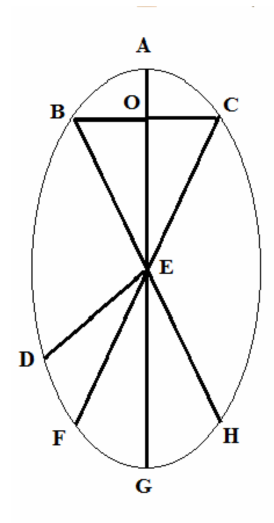


First Corollary.

AB shall be the major axis of the ellipse ABC, centre G; a diameter which is closer to that major axis is greater; and that which is more removed is smaller: and the diameter GC may be put closer to the axis than GE. I say GC to be greater compared with that GE, indeed with centre G and with the interval GC, a circle may be described which will cross that ellipse only at the four points CKHD, whereby

GE does not extend to the periphery, since it is smaller than GC. Q.e.d.

Second Corollary.



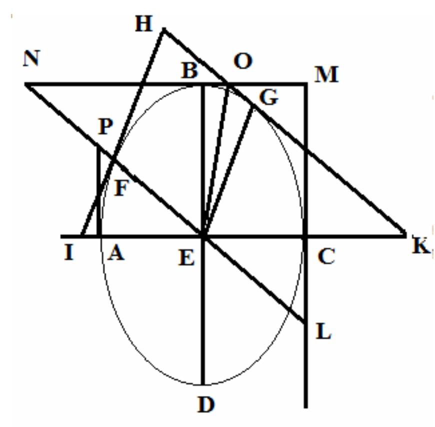
Again, I show as follows alternately from the first corollary: a diameter is closer to the axis, which makes a smaller angle or sector with the axis. The major axis of the ellipse shall be AG and the diameter BE shall make the smaller sector BEA with the axis, than the diameter DE shall make the sector DEG with the axis.

Therefore since the sector DEG is greater than the sector BEA, the sector FEG shall be equal to the sector BEA, and FE shall cross the ellipse at C, and the line BOC may be drawn; the sector BEA is equal to the sector FEG ; from the construction this is the sector to the vertex AEC. Therefore BC is bisected at O by the axis AG. Therefore the angles at O are right. Therefore it is evident the angle BEA to be equal to the angle AEC, that is, to the angle FEG, that is smaller than the angle DEG; therefore it is clear from the first diameter BE that the sector made to be smaller closer to the axis than DE, which shall be greater.

PROPOSITION LXXII.

The rectangle formed under half the axes is equal to the parallelogram formed by the conjugate radii.

Demonstration.



AC, BD shall be the axes of the ellipse ABC with centre E, and EF, EG shall be any radii taken together as it pleases. And with the tangents acting through F and G which shall meet at H, which shall meet the axis AC at I and K: also the tangent lines are drawn through C and B: which moreover shall meet at M: and which tangent lines shall cut [the other tangent line] HK, and EF in O, L, N: then with the points E O joined the tangent acts through A, cutting the line EN at P. Since both the lines NO, KE as well as the lines OK, NE are themselves mutually equidistant; NO, KE will be a parallelogram, with half the diameter OE: moreover the triangles EOB, EOC are equal, and therefore the remaining triangles EBN, EGK are equal to each other. Again since the lines AP, CL shall be equidistant, and the lines AE, CE shall be equal, the triangle ECL shall be equal to the triangle EAP, that is to the triangle EIF. Therefore since the triangle EGK is equal to the triangle EBN, and the triangle IFE to the triangle ECL, these four triangles will be in proportion; truly also since they shall be similar to each other, especially EGK to IFE, and EBN to ECL, therefore the right lines KE, EI, NE, EL, are proportionals, whereby since the above with the proportionalities requiring to be put in place in a straight line, the triangles IFE, EGK, IHK are similar to each other, and the triangles CLF, EBN, LMN also are similar between each other, so that the two triangles IFE, EGK are equal to the two triangles IFE, EGK thus as shown above, to the two triangles LCE, EBN. Therefore also the triangle IHK is equal to the triangle LMN, and hence with the equal parallelogram GEPH from the conjugate radii taken together, shall be equal to the rectangle BECM contained by the half axes. Q.e.d.

First Corollary.

Hence it follows, if in the ellipse any two of the conjugate diameters AE, EB, EF, EG the triangles upon EA, EB, EF, EG with the angles AEB, FEG, to be equal to each other: indeed are equal to half of the equal parallelograms.

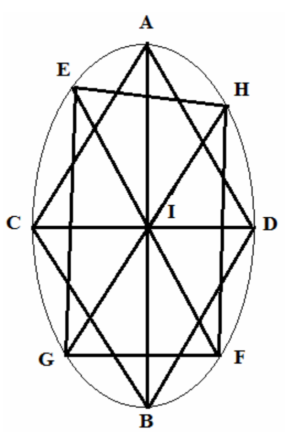
Second Corollary.

It follows in the second place, the parallelogram taken jointly within the whole diameters, to be equal to each other: since the squares of these have been shown to be equal by this proposition.

PROPOSITIO LXXIII.

The parallelogram, which arises from the outermost lines of the conjugate axes in an ellipse, is equal to the parallelogram contained by some of the conjugate diameters, on being compared.

Demonstration.



The axes of the ellipse ABC shall be AB, CD, and some other conjugate axes of the diameters shall be EF, GH. And both the extremes both of the axes, as well as of the diameters: I say the parallelogram CA,DB to be equal to the parallelogram EHFG. The rectangle formed from AB and DI is the double of the triangle ADB: and the rectangle from AB and CI, is the double of the triangle ACB. Therefore the rectangle formed from AB, CD, is the double of the parallelogram ACBD, or the parallelogram ACBD, is half of the rectangle on AB, CD. Similarly I shall show the parallelogram EGFH to be half of the parallelogram on EF, GH within the angle EIH: but the parallelogram on EF, GH is equal to the parallelogram on

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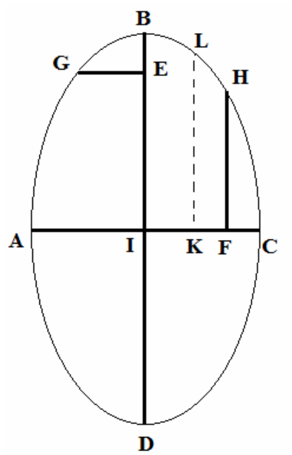
ABCD; and therefore the parallelogram ACBD is equal to the parallelogram EGFH. Q.F.D.

PROPOSITION LXXIV.

The axes or conjugate diameters of the ellipse ABC shall be AC, BD. And with BD divided in some manner at E, AC shall be divided in the same proportion at F, and the ordinate lines EG, FH may be drawn through E and F.

I say the rectangle AFC to be equal to the square GE, and the rectangle BED to be taken equal to the square HF, and if the square GE shall be equal to the rectangle AFC. I say BD, AC to be divided in the same proportion at E and F.



Demonstration.

Since from the hypothesis DE shall be to EB as AF to FC, there will become on interchanging, DE to AF, as BE to FC. And whereby the whole length DB, will be to the whole length AC, as DE to AF and as EB to FC. Therefore the ratios BE to FC, and DE to AF taken likewise, are the squares of the ratio DB to AC. And the ratio of the rectangle BED to the rectangle AFC, is composed from the ratios BE to FC, and DE to AF. Therefore the ratio of the rectangle BED to the rectangle AFC is the square of the ratio BD to AC, that is, of the ratio BI ad AI. Therefore the rectangle BED is to the rectangle AFC, as the square BI to the square AI; but likewise also the rectangle BED is to the square GE, as the rectangle BID; that is, the square BI to the square AI :

$$i.e.\left[\frac{DE}{EB}=\frac{AF}{FC};\frac{DE}{AF}=\frac{EB}{FC}\therefore\frac{DB}{EB}=\frac{AC}{FC}\ \&\ \frac{DB}{AC}=\frac{EB}{FC}=\frac{DE}{AF}.\right]$$

$$\&\left[\therefore\left(\frac{DB}{AC}\right)^2=\frac{EB}{FC}.\frac{DE}{AF}=\frac{BED}{AFC}=\frac{BI^2}{AI^2}.\right]$$

;

therefore the square GE is equal to the rectangle AFC. Similarly we may show the rectangle BED and the square HF to be equal.

Now the square GE and the rectangle AFC shall be taken equal: I say the sectors BD, AC to be in proportion: For if the ratio were not be as BE to ED, then neither thus will AF to FC be as BE to ED, nor thus AK to KC. Therefore the square GE will not be equal to the rectangle AKC by the first part of this above: Which cannot happen, since by the hypothesis the square GE shall be equal to the rectangle AFC. Therefore AC cannot be cut in proportion at any point K other than the point F. Q.E.D.

PROPOSITION LXXV.

If the axes or conjugate diameters may be cut in proportion at E and F, and the ordinates EG, FH may be drawn.
I say the square FH to be to the square EG, as the square BD to the square AC.

Demonstration.

The square FH is equal to the rectangle BED, moreover the rectangle BED [§.74 above] is equal to the square EG as the rectangle BID, that is the square BI, is to the square IA. Therefore also the square FH is to the square EG, as the square BI to the square IA, that is, as the square BD is to the square AC. Q.e.d.

So that if the ordinates EG, FH shall be put in place for the conjugate diameters, and so that the square BD to the square AC, thus shall be as the square FH to the square EG : I say BD, AC to be proportional to the sections at E and F. Indeed if AF to FC may be negative, so that DE to EB, there shall become AK to KC, as DE to EB, and the ordinate shall be KL. Therefore so that as the square BD shall be to the square AC, the square KL shall be to the square EG: which cannot happen, since from the hypothesis the square FH shall be to the square EG, as the square BD to the square AC. Therefore so that there shall not be the ratio the square BD to the square AC, thus the square KL, or any other besides the square FH, to the square EG. Q.e.d.

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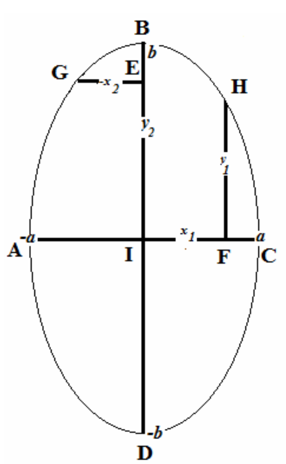


Let us apply the above formulae to obtain the formula for the standard ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, for these conjugate points, taken arbitrarily.

In the first place, we label the x and y intercept lengths $a,-a\,;b,-b$ in the usual manner with the origin I, and consider at first the formula in §74: $FH^2 = BED = BE.ED$, applied to the points E (x_2, y_2) and F (x_1, y_1) as shown on the diagram, where $BE = b - y_2$ and $ED = b + y_2$; while $FC = a - x_1$, and $AF = a + x_1$ hence $FH^2 = BED = BE.ED$ becomes

$FH^2 = y_1^{\,2} = BE.ED = (b - y_2)(b + y_2) = b^2 - y_2^{\,2}$, or

$y_1^2 + y_2^2 = b^2$

Likewise, from the above formula $GE^2 = AF.FC$, we have :
$GE = x_2; AF = a + x_1; FC = a - x_1;$ and $GE^2 = AF.FC$ becomes :
$GE^2 = x_2^2 = AF.FC = (a + x_1)(a - x_1) = a^2 - x_1^2$, or $x_1^2 + x_2^2 = a^2$.
Hence, we may write:

$$\frac{x_1^2}{a^2} + \frac{x_2^2}{a^2} = 1 \; and \; \frac{y_1^2}{b^2} + \frac{y_2^2}{b^2} = 1 \; \therefore \frac{x_1^2}{a^2} = 1 - \frac{x_2^2}{a^2} \; and \; \frac{y_1^2}{b^2} = 1 - \frac{y_2^2}{b^2}$$

$$\therefore \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1 = 1 - \frac{x_2^2}{a^2} + 1 - \frac{y_2^2}{b^2} - 1 = 1 - \left(\frac{x_2^2}{a^2} + \frac{y_2^2}{b^2}\right).$$

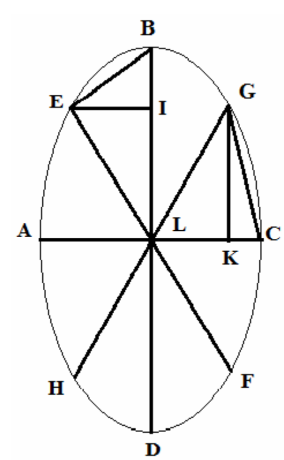
For this equation : $l.h.s. \leq 0; r.h.s. \geq 0$. Hence both sides of the equation are equal to zero, and the standard formula for the ellipse is established for these points.

****

PROPOSITION LXXVI.

In the ellipse ABC there shall be any two conjugate pairs of diameters AC, BD; EF, GH; and from E and G the ordinate lines EI, GK may be drawn to the diameters BD, AC.
I say the square EI to be equal to the rectangle AKC, and the rectangle BID shall be equal to the square GK.

Demonstration.

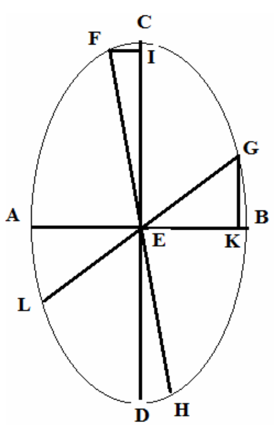


EB, GC may be put in place. Therefore since both the diameters AC, BD, as well as EF, GH are conjugate, the sector BLC will be equal to the sector ELG [Cor.§46], and with the common sector BLG removed, the sector ELB shall be equal to the sector CLG. And indeed the triangle LEB to be equal to the triangle LGC [§.45 Cor.]: and since BD is conjugate to AC itself, BD is parallel to KG, which is put to be the ordinate for AC. Therefore the angle GKL is equal to the angle BLA. Similarly since AC is conjugate to BD, AC is parallel to the ordinate EI put in place for BD; therefore the angle EIB is equal to the angle BLA, that is to the angle GKL; therefore since the triangles shall be equal, there will become (as shown below): as the base LB to the base LC, thus KG to EI, and thus as the square BL shall be to the square LC, that is, as the square BD to the square AC, thus the square KG to the square EI. From which the lines BD, AC are divided in proportion at I and K. Whereby the square EI is equal to the rectangle AKC, likewise the rectangle BID is equal to the square GK. Q.f.d.

PROPOSITION LXXVII.

The squares of the axes taken together are equal to the squares of any conjugation diameters taken together.

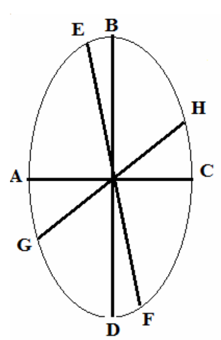
Demonstration.



AB; CD shall be the axes of the ellipse ABC, and FH, GL some other conjugation of the diameters. I say the squares AB, CD taken together to equal the squares of FH, GL taken together. The ordinate lines FI, GK may be drawn, which will be perpendiculars since they are drawn to the axis; moreover E may be put to be the centre of the section. The square EC is equal to the square EI together with the rectangle CID, that is to the square GK: moreover, the square EB is equal to the square EK together with the rectangle AKB, that is to the square FI : from which the two squares EB, EC taken together are equal to the squares FI, IE, EK, GK taken together. But the squares EF, EG are equal to the same squares; therefore the squares EB, EC are equal to the squares SE, EG. Whereby with the squares AB, CD taken together, the quadruple of the squares shall be EB, EC and FH, GL the square of the quadruple of the squares EF, EG; it is apparent the squares AB, CD taken together to be equal to the squares FH, GL taken together. Q.e.d.

PROPOSITION LXXVIII.

The axes of the ellipse taken together shall be the minimum of all the conjugate diameters taken together.



Demonstration.

AC, BD shall be the axes and EF, GH some other conjugate of diameters : I say the axes taken together to be the smaller of the conjugate diameters taken together. Because the squares AC, BD taken together are equal to the squares EF, GH taken together: moreover BD shall be the maximum of the diameters, and AC truly the minimum, AC, BD taken together will be smaller than the right lines EF, GH; therefore etc. Q.f.d.

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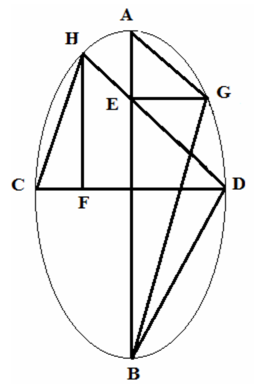
PROPOSITION LXXIX.



AB, CD shall be the axes of the ellipse ABC, divided proportionally at E and F: and with the ordinate lines EG, FH drawn (which here are perpendiculars): AG, GB, CH, HD may be joined: I say the four squares AG, GB, CH, HD, taken together, to be equal to the squares of the two axes.

Demonstration.

The square AB is equal to the squares of AE and EB together, with the rectangle AEB equal to the square HF [§76 above]; in the same manner, the square CD is equal to the square of CF and FD together, with the rectangle CFD equal to the square EG; [consequently on combining, and via Pythagoras,] from that the same squares are equal to the squares AG, GB, CH, HD. Therefore the squares of the axes taken together are equal to the AG, GB, CH, HD. Q.e.d.

[*i.e.* $AG^2+GB^2+CH^2+HD^2=\left(AE^2+EG^2\right)+\left(EB^2+EG^2\right)+\left(CF^2+FH^2\right)$

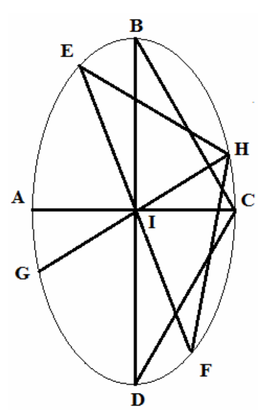
$$=\left(AE^2+CF.FD\right)+\left(EB^2+CF.FD\right)$$
$$+\left(CF^2+AE.EB\right)+\left(FD^2+AE.EB\right)$$
$$=2CF.FD+CF^2+FD^2+2AE.EB+AE^2+EB^2$$
$$=\left(CF^2+FD^2\right)+\left(AE+EB\right)^2=CD^2+AB^2.]$$

PROPOSITION LXXX.

The squares of the whole lines of the axes taken together are equal to the squares of the whole of each of the conjugates taken together.

Demonstration.

AC, BD shall be the axes of the ellipse ABC, and EF, GH shall be some other conjugate diameters. And BC, CD, EH, FH shall be joined. I say the squares BC,CD taken together to be equal to the

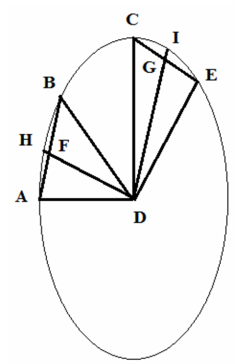


squares EH, FH taken together. The squares BC, CD taken together are equal to the squares BI, IC taken together twice; but the squares EH, HF are equal to the squares EI, IH taken together twice; therefore the squares BC, CD taken together, are equal to the to the squares EH, HF taken together. Q.e.d.

PROPOSITION LXXXI.

Two conjugations of the diameters AD, DC; BD, DE of which the centre is D, cut the ellipse ABC, and with the points AB, CE joined, the lines AB, CE are bisected at F and G, and DF, DG may be drawn, which produced cross the ellipse at H and I.
I say HD, ID to be conjugate diameters.

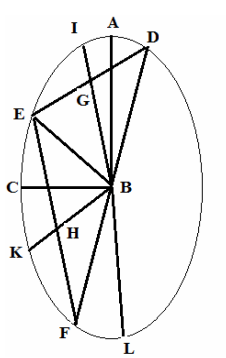
Demonstration.



Since AD, DC; BD, DE are conjugate diameters, ADC, BDE are equal sectors: therefore with the common sector BDC removed, the remaining sectors ADB, CDE are equal : again since the line AB shall be cut at F, the diameter DH bisecting into two equal parts, both the sectors ADH, BDH shall be equal, as well as the triangles AFD, BFD. In the same way the sector EDI is equal to the sector CDI; therefore the sectors ADH, EDI are equal to each other : Therefore with the addition of the common sector HDI, the sector ADI to equal the sector HDE, therefore DH, DI are conjugate.

PROPOSITION LXXXII.

The axes of the ellipse ABC shall be AB, BC; but the conjugate of some other diameter DF shall be EB; DE, FE may be joined; indeed DE which cuts the major axis, EF truly the minor : then DE, FE themselves are bisected by the diameters BGI, BHK.
I say IB to be the greater diameter than the diameter KB.

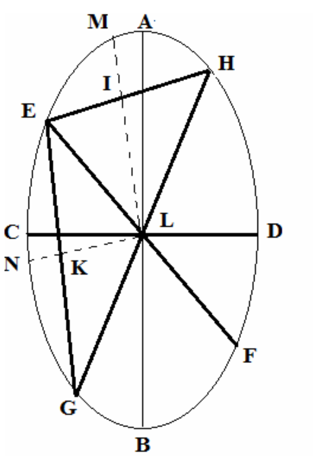


Demonstration.

Because the diameters BI, BK bisect the right lines ED, EF both the sectors DBE, EBF are bisected. Whereby since they themselves shall be equal, also the half sectors of these IBD, KBF will be equal; therefore the sector IBA is smaller than the sector KBL. Therefore IB is closer to the axis than BK, and therefore is greater than BK. Q.e.d.

PROPOSITION LXXXIII.

AB, CD shall be the axes of the ellipse ABC, and EF, GH some other pair of the conjugate diameters, and EH, EG may be joined.
I say the line EH which cuts the major axis, to be smaller than the line EG, which cuts the minor axis.
Demonstration.



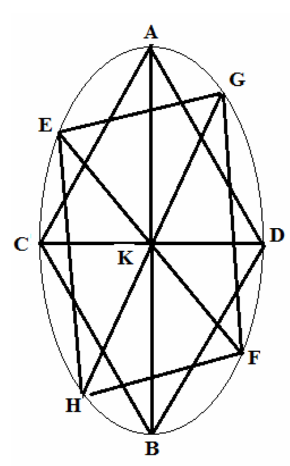
EH, EG are bisected by the diameters LM, LN, at I and K. Because GH, EF are conjugate; the sectors GLE, ELH are equal, and therefore the segments GNE, EMH are equal. Whereby LM, LN bisecting the chords EH, EG, are divided proportionally. Therefore MI shall be to IL as NK to KL; and on placing together and interchanging, so that as LM to LN, thus LI ad LK. But LM is greater than LN; therefore LI also is greater than LK. Now truly since LN, LM also shall be conjugates, and EG from the construction shall be put the ordinate to LN, LM will be parallel to EK. For the same reason, LN, EI will be parallel ; therefore EI, LK is a parallelogram, and thus LI, KE; LK, EI are equal. Therefore since LI shall be shown to be greater than LK, and KE greater than LK, that is than EI. Whereby the double of this, EG, is greater than the double of EH. Q.e.d.

PROPOSITIO LXXXIV.

The largest values of the axes taken together are the maxima of all the diameters of any conjugations taken together to the limit.

Demonstration.

AB, CD shall be the axes and EF, GH shall be some other conjugation of the diameters : and both the greatest of the diameters as well as of the axes may be joined together. I say the greatest values of the lines CA, AD, DB, BC taken together, to be greater than EG, GF, FH, HE taken together. Because GK is equal to HK itself, truly with EK common, and the right line EH greater than EG [§.83], the angle EKH is greater than the angle



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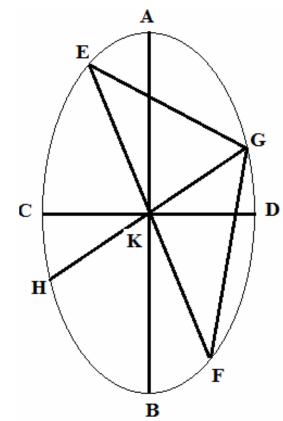
EKG: and therefore the angle EKH is greater than the right angle AKC. But the triangles AKC, EKH are equal[§.73], whereby EH is greater than AC [Elem.]: in the same manner it is shown the line AC to be greater than the line EG. Therefore EG is the minimum, and EH the maximum of the lines EH, AC, AD, EG. Therefore since the squares EH, EG taken together shall be equal to the squares AC, AD taken together, the lines EH, EG taken together shall be less than the lines AC, CD taken together. In the same manner it will be shown the lines GF, FH to be less than the lines CB, BD. Therefore, etc. Q.e.d.

PROPOSITION LXXXV.

In no ellipse are conjugate diameters to be found which cut each other at right angles, besides the axes.

Demonstration.



AB shall be the major axis, CD the minor axis, and some other conjugate diameters EF, GH: moreover, K shall be the centre of the ellipse. I say neither of the angles EKG, GKF to be right: for the points EG, GF may be joined : Since EK, KG are equal to the two right lines FK, KG, and EG smaller than FG [§.83], therefore the angle EKG shall be smaller than the angle GKF [Elem.]: Whereby since the sum shall be equal to two right angles, neither of these is right: the same is shown for the rest. Therefore no ellipse is to be found, etc. Q.e.d.

PROPOSITION LXXXVI.

AC, BD shall be the axes of the ellipse ABC and EF, GH the conjugate pair of some other diameters: and with the points AB, BC joined, the right lines EH, EG are drawn.
I say the angle ABC which exists about the smaller axes, to be greater than the angle GEH, and hence to be the maximum of all the angles which will held by the greatest of the conjugate diameters connected together.

Demonstration.

Since the line GH shall be smaller than the axes AC, it may be produced equally at each end to K and L so that LK shall be equal to the axis AC; and the points EK, EL may be joined, and the line EM may be sent from E normal to LK: therefore the triangle KEL shall be greater than the triangle GEH, that is to the triangle ABC ; whereby since the bases AC, LK of the unequal triangles shall be equal, the altitude EM of the triangle KEL, shall be greater than the height IB of the triangle ABC: and indeed the angle KEL shall be greater than the angle ABC: therefore the angle GEH is much smaller than the angle ABC. Q.e.d.

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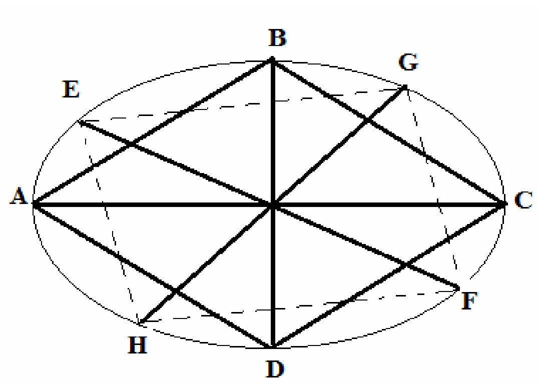
PROPOSITION LXXXVII.

AC, BD shall be the axes of the ellipse ABC : and the ends of these AB, BC, CD, DA are joined: moreover some other of the conjugate diameters shall be EF, GH, the ends of which also are connected.
I say the angles ABC, EGF, HFG, BCD to be in arithmetical proportion.

Demonstration.



Since both AC as well as EF is a parallelogram, both the angles ABC, BCD, as well as the angle EGF, GFH, are equal to two right angles: and whereby the angles ABC, BCD taken together are equal to the angles EGF, GFH taken together: but we have shown the angle ABC thus to be greater than the angle EGF, and therefore GFH is greater than the angle BCD: and because, as I have shown now, the angles B and C taken together are equal to the angles G and F taken together, by which excess the angle ABC exceeds the angle EGF, likewise it is necessary that the angle GFH will exceed the angle BCD. Q.e.d.

Corollary.

Hence it is evident BCD, which is the angle present about the major axes, to be the smallest of all the angles which are constructed from the diameters taken conjointly.

****

Let A, B, C, D & E, F, G, H *represent the angles of the parallelograms* ABCD & EFGH:
Since $B+C=G+F$; *we have shown* B>C, G>F & B>G ∴ F>C.
Hence, *for the equality to hold* : B>G>F>C.

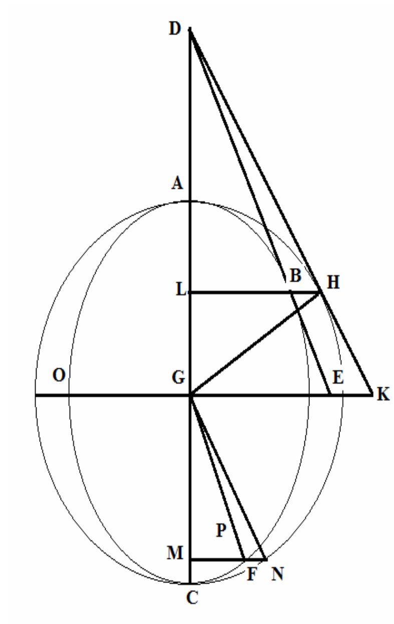
****

PROPOSITION LXXXVIII.

A certain line DE is a tangent to the ellipse ABC at B, of which the axes are AC, EG, meeting each axis at D & E: Truly a right line GF may be sent from the centre G parallel to the line DE.

I say the lines DB, GF, BE to be in continued proportion.

Demonstration.



The circle AHC is described with centre G and radius AG, : and from D the tangent DK is sent to the circle at H crossing the axis to the ellipse EG at K, and from H the line HL normal to the axis, which by the Cor. to §.33, also passes through B; another line FMN normal to the axis, passes through the point F on the ellipse, and the points NG, HG may be joined. Since the lines LH, MN are equidistant, the angles LDB, MGF will be equal: but the angles BLD, FMG are right by the construction; and therefore the remaining angle LBD is equal to the remaining angle MFG. Whereby the angles DBH, GFN are equal to each other. Moreover, since the triangles DLB, GMF are similar, so that there shall be LB to MF, thus as DB to GF. But from the demonstration in the Scholium of §.4 above, as LB to MF, thus BH to FN. Therefore DB to GF, as BH to FN. Whereby since the angle DBH, GFN now shall be shown to be equal, the triangles DBH, GFN will be equal. Therefore HD is to BD, that is HK is to BE, as GN to GF, and on interchanging, as HK to GN, thus BE to GF. Then since in triangle DGK, the angle at G shall be right and GH drawn from the centre to the point of contact, shall be normal to DK, HK will be to GH, as GH to HD. But GN, GH are equal, therefore, as KH to GN, that is just as shown before, to be as BE to GF, thus GN to DH. But on account of the similitude of the triangles as GN to DH, thus GF to DB. Therefore, as BE to GF, thus GF to DB. Q.e.d.

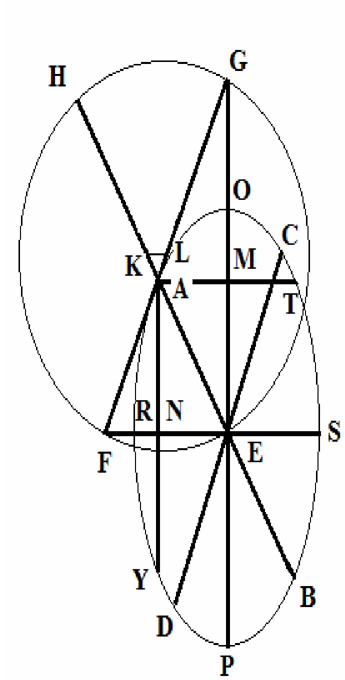
PROPOSITION LXXXIX.

With the same in place, if GF shall be the mean proportional between DB, BE.
I say the point F to be found from the ellipse curve.

Demonstration.

If the point F did not pertain to the ellipse, then the right line GF will pass through the ellipse at P, either above or below F. Therefore from the preceding, DB, GP, BE are continued proportionals; which cannot happen, since DB, GF, BE shall be the continued proportionals in place. Therefore no other point of the right line GF lies on the ellipse, apart from F. Q.e.d.

PROPOSITION XC.



With the conjugate of some diameters given, to find the axes of the ellipse.

Construction & demonstration.

The conjugate diameters AB, CD shall be given, bisecting each other at E; and the line FG acting through A which shall be parallel to CD, so that as AE to ED, thus there become ED to AH; then EH shall be bisected at K, the normal KL erected from K crossing FG at L, then with the centre L and with the radius LE, the circle EFG is described, this will pass through H and will cut the line FG at some points F and G: finally the points EF, EG are joined. I say the lines EF and EG found to be satisfactory. Indeed since the right lines AE, ED, AH from the construction are in continued proportion, for the square ED will be equal to the rectangle EAH, that is to the rectangle FAG. And thus the lines GA, DE, FA also are in continued proportion, and hence the point D lies on the ellipse, the axes of which lie on the lines EF, EG. But A also lies on the same ellipse, and therefore the points B, C lie on the same ellipse.

Again the ends of the axes will be found thus: with AM drawn, with AN drawn normal to the line EG, the mean EO may be found between EM, EG, and the mean ER between inter EN, EF: and the ellipse may be described through the points D, A, C ; since CD is conjugate to AB itself, and thus for the same ordinate put in place, and the line FG parallel to CD itself, FG will be a tangent to the ellipse ABC; but AM, AN are normal to the lines in which the axes for the section are present; and both EM, EO, EG, as well as

EN, ER, EF to be in continued proportion, therefore the ellipse ABC passes through the points R and O: whereby R and O are the ends of the axes, which it was required to show.

Scholium.

This proposition is taken from the problem of Pappus, Book 8, Mathematical Collection, Prop.14, and indeed the truth of this construction evidently will follow both from this source, as well as from that which we now present, but have not yet demonstrated, Frederico Commandino had endeavored to supply a demonstration, thus writing:
AM may be produced as far as T thus so that TM shall be equal to MA itself: also AN may be produced as far as to Y, so that YN shall be equal to NA : the points T and Y lie on the ellipse, from these matters which have been demonstrated by Apollonius in Prop. 47, Book 2 on Conic Sections. But RS is parallel to AT itself, indeed there is a right angle in a semicircle, and whereby OP itself will be parallel to AY. Therefore since the ordinate to AB is the applied line drawn through A parallel to DC, evidently FG will touch the section at the point A, and since FG touching the section shall cross the diameter at G and AM shall be the applied ordinate, by §.37 of the first book of Conics of Apollonius, the rectangle GEM is equal to the square from EO or EP. Also, by the same reasoning, the ordinate AN may be applied to the rectangle FEN equal to the square from ER or ES; therefore OP, RS will be the conjugate axes of the ellipse.

This explanation by which Commandino has shown the right lines OP and RS to be the conjugate axes of the ellipse, which advances through the points A T Y, and is a tangent at the point by the line FG: truly this proposition cannot be true. For according to the conjugate axis requiring to be found there is no need for the circle FEG to be described, to make the rectangle EAH equal to the square ED, and to bisect EH at K and thence to erect the normal, which meeting FG at L, provides the centre L of the circle FEG, if indeed with the centre taken somewhere on the line FG, if the circle may be drawn around E, which will cut the line FG: now indeed the right angle will be contained not by the points F and G but by others which are erected from the right line through E, and not cut by EF and EG, and whereby if from A some normals may be drawn to these latter lines themselves, whereby there will become AM & AN, which may be doubled, so that AT and AY will provide the points which in turn allow the entrance of the points T and Y on the ellipse, and which are destined to become infinitely close, the tangent of the line FG, which passes through A. But it is observed this ellipse (which was required to be shown) to pass through the end points C & D, therefore so that the centre of the circle shall be assumed to be from some other point L, neither shall the square ED be equal to the rectangle under EA, nor be equal to any other line contained by AH. And thus so that it may be shown OP and RS to be the conjugate axes of the ellipse, which pass through the ends of the conjugate diameters AB and CD, another account is required to be entered into, which we have now proposed in our demonstration.
This so far has been concerned with the Commandino demonstration.
Thus indeed , the text remaining of Pappus may be had, for which I do not know what unfortunate circumstances may have passed over time [for the indistinct parts lost in the

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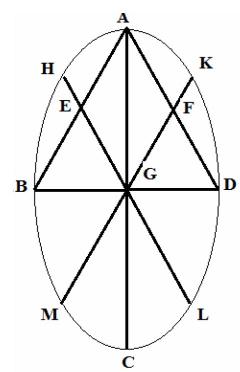
text used by Gregorius]: But it is easy to find the axis related to these, for any conjugate diameters of the ellipse. Which indeed will be done by this method. Words which have no proper sense do not find a place, since these are not related to the construction. But everything shall be geometric, so that it becomes apparent on being read: whereby I consider the omission of geometric words, and thus these being required to be read: But for an ellipse found with any conjugations of the diameters, its related axes to be found easily, which indeed will be done in this account geometrically. Then these words are added into the same construction itself: since DE shall be greater than EA: for since the construction shall be general, DE shall be lesser or greater, or may be put equal to EA itself, as can be deduced from our demonstration, Pappus also is in no way certain about each side, frustrated he assumed DE greater than EA. Hence it is a wonder that Frederick Commandino did not turn his attention to this error, especially since with that assumption in his demonstration, as we have given above, he made no use: but he could have brought it into the general demonstration: and since from which the demonstration itself was abandoned, to which there is no doubt Pappus would have added. It is clear enough from the error of these matters that were touched upon, from the hand of whom this proposition arrives (which scarcely was considered from the whole, as I would judge), which now clearly changed and imperfect, have frustrated attempts to restore the original.

PROPOSITION XCI.

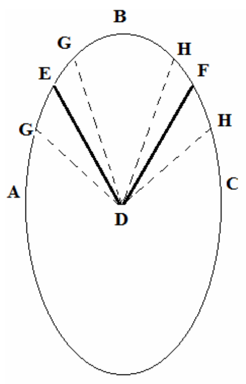
To show equal conjugate diameters from the given axes of the ellipse.



Construction and demonstration.

AC, BD shall be the axes of the ellipse ABC : moreover it shall be required to show the conjugate diameters to be equal to the joined points A B, A D; the right lines AB, AC shall be bisected at E and F; and from the centre G the right lines GH, GK shall be drawn through E and F, meeting the ellipse at the points H, K, L, M. I say these points satisfy the proposition. For since the two right lines EB, BG, are equal to the two right lines FD, DG (moreover are equal to each other, with equal sides and contained angles) , also the angles to the base EGB, FGD and thus the remaining angles AGE, AGF are equal to each other. Again since the angle AGB shall be right and the base AB bisected at E, if a circle with centre E and with the radius EA may be described, it will pass through B also, and thus EA, EG are equal lines. Whereby the angle EAG to be equal to the angle EGA, that is, AGF. Therefore AB, KM shall be parallel : in the same way the lines AD, HL are shown to be parallel: so that since the diameters HL, KM will bisect the mutually parallel lines, they will be conjugate. Truly since the angle HGA has been shown to be equal to the angle AGK, also the line HG is equal to the line GK, as is evident from §18 above; therefore the diameters HL, KM are conjugate and equal. Therefore we have shown, etc. Q.e.d.

PROPOSITION XCII.



In an individual ellipse only two conjugate equal diameters are to be found.

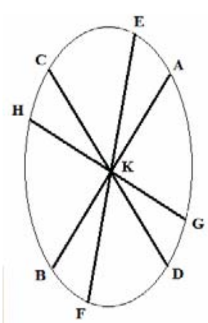
Demonstration.

D shall be the centre of the ellipse ABC and in that the equal conjugate diameters ED, FD : I say no other equal and conjugate diameters can be shown in that: for if it can be done, besides the diameters ED, FD: if there shall be the other equal and conjugate diameters GD, HD: therefore the sector GDH will be equal to the sector EDF. Which cannot happen, for if the diameters GD, HD shall be equal since it is by necessity these must be greater or smaller with the diameters ED; FD, and thus, both taken together to fall either greater or smaller than the diameters ED, FD: Therefore, besides the equal conjugate diameters ED, FD, no others can be shown equal in the ellipse. Q.e.d.

PROPOSITION XCIII.

In an ellipse, the equal conjugate diameters taken together, are the greatest of all the conjugate diameters taken together.

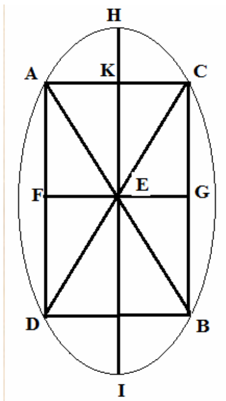
Demonstration.



AB, CD shall be the equal conjugate diameters, but if there shall be some other of the diameters EF, GH conjugate : I say the diameters AB, CD likewise taken to be greater than the diameters EF, GH taken together: for since the sectors AKC, GKE shall be equal to each other ; it is necessary one of the unequal (it shall be EF) to be closer to the axis than whichever of the equal axis AB, CD: truly than the other HG to be more distant, so that from the four diameters EF is the maximum and GH the minimum. But the squares EF, GH taken together are equal to the squares AB, CD taken together ; therefore the lines AB, CD taken together are greater than the lines EF, GH also taken together: Q.e.d.

PROPOSITION XCIV.

The lines which are conjugate to the greatest of the conjugate diameters, are bisected by the axes.
Demonstration.

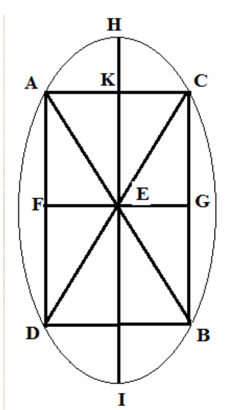


AB, CD shall be the equal conjugate diameters, and the ends of these joined together AD, AC, CB, DB. I say these to be bisected by the axes, with AD bisected at F, FEG acting through the centre E crossing the right line CB at G. On account of which therefore AB, CD are put equal, the halves of these AE, DE, also are equal. Moreover from the construction , similarly AF, DF are equal. And thus in the triangles AEF, DEF since FE shall be common, all the sides themselves are equal in turn. Therefore the angles at F are equal, and thus also FEA, FED are equal with right angles. Whereby also the angles GEC, GEB vertically opposite to the first are equal. Truly again the sides CE, EB are equal and with EG common to each of the triangles GEC, GEB. Therefore CG, BG are equal, and thus the angles at G to be right. Therefore the right lines AD, CB (which by §19 are parallel) are bisected at right angles by the axis FG. Therefore with the right lines AD, CB bisected by the axis , the ends of the equal conjugates AB and CD are connected. We may show in the same manner the two remaining lines AC, BD to be bisected by the axis HI. Therefore the truth of the proposition is agreed on.

PROPOSITION XCV.

If the lines which join the ends of the conjugates may be bisected by the axes:
I say these diameters to be equal to each other.

Demonstration.



The same figure as before may be put in place, and the lines AD, CB, AC, the extremities of the conjugates bisected at F, G, and K and divided at right angles by the axes Hl, FG. I say the conjugate diameters AB, CD to be equal to each other: indeed since AD, CB per §19 shall be parallel and from the hypothesis shall be bisected by the axis at F and G, the angles at F are right, and the two sides AF, FE are equal to the two sides DF, FE; therefore the remaining sides AE, ED also are equal to each other. Similarly I may show CE, EB to be equal. From which the whole diameters AB, CD are equal. Q.f.d.

Corollary.

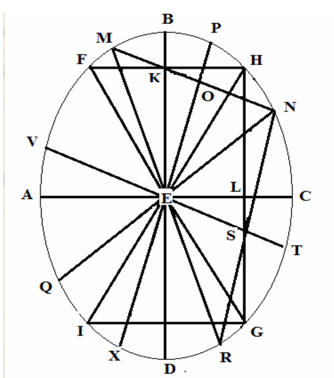
Hence it is evident the lines, which connect the ends of unequal conjugates diameters, at no time to be bisected by the axes or by any other diameter, or to be cut at right angles.

PROPOSITION XCVI.

Of the lines which join the ends of any conjugate diameters, that one is the greatest which joins equal conjugates which shall cut the minor axis; the smallest the one which shall cut the major axis.

Demonstration.

AC, BD shall be the axes of the ellipse ABC, truly the equal conjugates FG, HI ; and the join FH meets the major axis at K: and HG the minor axes at L. I say HG to be the maximum line of these which join the ends of any conjugations whatsoever, and FH the



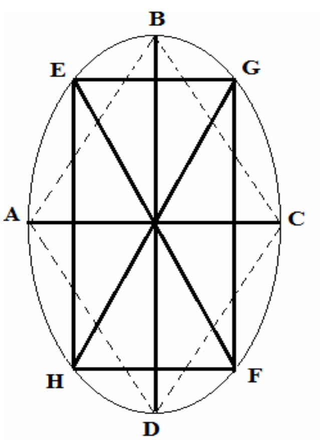
minimum. Indeed some other conjugations of the axes may become MR, NQ. The ends of which M N, N R may be joined, the lines XOP, VST may be drawn by which these conjugate lines are bisected at O and S through their centres. Therefore since FG, HI are conjugates, the sectors EFBH, EHCG are equal, (§46). Therefore the segments FBH, HCG are equal, (§60). Whereby, since the axes BD, AC also bisect the right lines FH, HG, (§94), which join equal conjugates, the axes themselves to be cut proportionally at K and L (§53). Therefore the rectangle BKD is equal to the square LH (§74). Clearly by a similar discourse we will show the rectangle POX to be equal to the square NS. Then (§46, Cor.) since the sectors MEN, FEH are equal (§60), and thus the segments MPN, FBH, and both FH, MN bisected at K and O, DB, XP (§53) are cut in proportion at K and O; but the axis DB is greater than the axis XP: therefore the rectangle BKD, that is, as shown before, the square HL, is greater than the rectangle POX, that is to the square NS. Therefore the right line HL is greater than the right line NS, but HG is the double of HL, as shown before, and NR (§53) from the construction is the double of NS. Therefore HG is greater than NR: but since NR is greater than MN, HG also will be greater than MN. We show in the same manner HG to be greater than the connected ends of any other conjugates. Therefore HG is the greatest of all. Which was the first part of the proposition.

Clearly we will demonstrate by the opposite means that FH may be the smallest of all the connectors. For the construction may be repeated for some other conjugation of the diameters MR, NQ taken, than has been used above. We may show in the same way the square FK to be smaller than the square MO, and the line FK to be smaller than the line MO, and hence FH, to be smaller than MN; but RN is greater than MN. Therefore FH also is smaller than NR. And thus we will show FH to be smaller than any of the ends of the conjugate diameters joined together. Therefore it is the smallest of all. Which was required to be shown in the second place.

PROPOSITION XCVII.

The ends of the equal conjugate diameters joined together are the smallest of all the conjugate diameters of any kind joined together.

Demonstration.



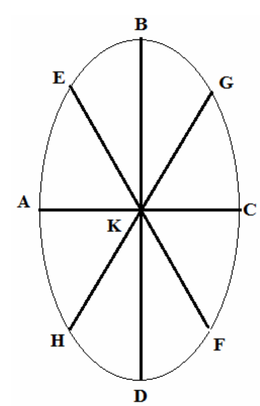
EF, GH shall be the equal conjugate diameters in the ellipse ABC. Moreover some other conjugate of the diameters EF, GH may be put in place. I say the lines which join the ends of these equal conjugate diameters taken together to be lesser than the lines which connect the ends of the other conjugate diameters (§80). Indeed the squares EG, GF are equal to the squares AB, BC, and in addition EG is the shortest line of the connection, and FG the greatest by the preceding; therefore the lines EG, GF are shorter than the lines AB, BC; in the same manner the lines EH, HF are shown to be lesser than the lines AD, DC : Therefore the lines , etc. Q.f.d.

PROPOSITION XCVIII.

AC, BD shall be the axes of the ellipse ABC and EF one of the equal conjugate diameters.
I say the squares AK, BK taken together to be the double of the square EK.

Demonstration.

GH shall be drawn, the other of the equal conjugate diameters. Since the squares AC, BD taken together are equal to the squares EF, GH taken together (§77), and the squares under half the axes will be AK, BK, equal to the squares EK, GK under half the equal diameters; but the squares EK, GK are equal to each other, therefore the squares AK, BK taken together are equal to the square EK. Q.e.d.



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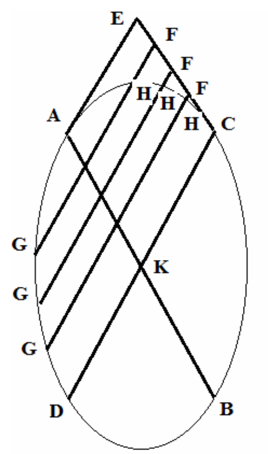
PROPOSITION XCIX.

Apollonius has a proposition of this kind in book.3.Conics.Prop.16 : if the tangents to an ellipse meeting at E shall be AE, CE, and on taking some point G in the section, GHF may be drawn parallel to one of the tangents AE, the rectangle GFH shall be to the square BC, as the square AE to the square CF.

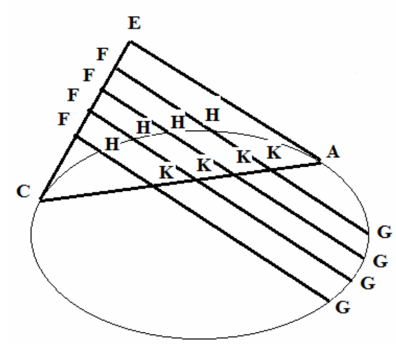
Truly not only the same ratio shall be found, but also the areas of equal conjugate diameters shall be found.



The equal conjugate diameters of the ellipse shall be AB, CD, at the ends A, C of which the two tangent right lines shall meet at E, if several more may be drawn parallel to GH, the rectangles GFH will be equal to the squares FC.

Demonstration.

Since CD is the conjugate diameter of the diameter AB, for that the ordinate itself will be put in place: therefore CD is parallel to the tangent AE. In the same way AK is parallel to the tangent EC. Therefore the figure KAEC is a parallelogram. Whereby since AK, KC shall be equal from the hypothesis, also AE, CE are equal: therefore the squares AE, EC are equal. And as the square AE to the square EC, thus the rectangle GFH to the square FC, therefore the rectangle GFH is equal to the square FC. Q.e.d.



And since we now have that theorem of Apollonius at hand, I add three things to this that likewise Apollonius does not seem to have observed: without doubt if with the tangents drawn AE, CE, the points of contact A, C may be joined together, with the rectangles GFH to be equal to the squares KF.

Since FK, AE are parallel, the triangles AEC, KFC are similar. Therefore AE to EC, thus as KF to FC. There as the square AE to the square EC, thus the square KF to the square FC. But also as the square AC to the square EC, thus the rectangle GFH to the square KF. Therefore the square KF and the rectangle GFH have the same ration to the square EC; therefore the equations are equal. Q.e.d.

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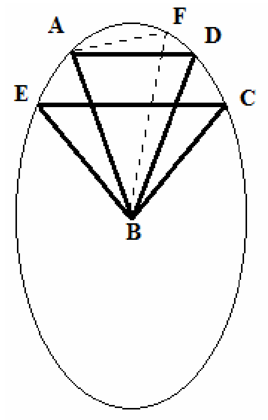
PROPOSITIO C.

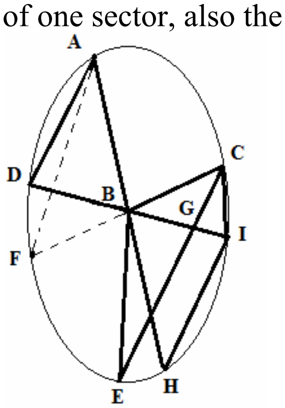


AB, BC shall be unequal conjugate diameters, and from A some right line AD may be drawn cutting the ellipse at D; to which the a line CE may be drawn parallel from C, and EB, DB may be joined.

I say EB, DB to be the opposite conjugate diameters.

Demonstration.

In the first case they lie parallel on the same part of the ellipse. The right lines EA, DC are drawn. Since the lines AD, EC are mutually equidistant from each other, the segments EA, DE are equal to each other (§51), and thus the sectors ABE, DBC are equal (§59). Therefore with the common angle ABD added, the sectors EBD, ABC will be equal to each other. Whereby the diameters BA, BC shall be conjugate with the sides of one sector, also the other sides EB, BD are conjugate.



In the second case the lines AD, CE lie parallel on opposite parts of the ellipse : the radii AB, DB may be produced to H and I, and the points H and I may be joined. Since AB, BC are conjugate the sector ABC shall be the fourth part of the ellipse, but the diameter AH bisects the ellipse, and thus the portion ACH is half of the ellipse. Therefore the sector ABC is half of the semi-ellipse ACH. And hence half of the sector CBH. But since IH by §19 is parallel to DA, to which from the hypothesis CE also is parallel, IH and CE shall be parallel to each other. Therefore the segments CI, EH and thus the sectors CBI, HBE are equal ; therefore with the common sector IBH added, the sector IBE is equal to the sector CBH, that is, as now shown before, equal to the sector ABC. Whereby since the sector ABC shall be a quarter of the ellipse, or half of the half ellipse, also the sector IBE will be half of the half ellipse, that is, of the portion IED, which is apparent to be half of the semi ellipse from the Coroll. to §45. Therefore the sector IBE, that is the sector ABC, is equal to the sector EBD. Whereby since AB, BC shall be conjugate, also DB, EB will be conjugate.

Now AB, CB, likewise EB, DB shall be conjugate diameters and AD, EC may be joined together. I say the lines AD, EC to be parallel. Truly if not, AF may be drawn parallel to EC itself from A , and FB joined: therefore FB will be the conjugate diameter to EB itself through the second part of this: but DB by the construction is the conjugate to the diameter EB, therefore several diameters are conjugate to that same EB. Which cannot be the case. Therefore AF cannot be parallel to EC. The same is shown for any other. Therefore AD alone is parallel to the right line BC. Q.e.d.

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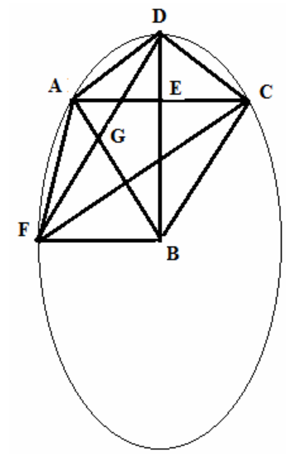
PROPOSITION CI.

AB, BC shall be some conjugate of the diameters of the ellipse ADC, the centre of which shall be B: and with the points A C joined, the diameter BD may cut the line AC at some point E, of which the conjugate BF may be drawn, and with the line FD drawn, it shall cut the diameter AB at some point G.



I say the lines AC, FD, as well as the lines BD, AB to be divided in proportion at E and G.

Demonstration.

Since both the diameters AB, BC as well as the diameters DB, FB are conjugate, the sectors ABC, FBD will be equal (§46.Cor.): therefore with the common sector ABD removed, the equal sectors DBC, ABF remain. So that the lines BD, AB (§61), likewise the lines AC, FD are divided proportionally at E and G.

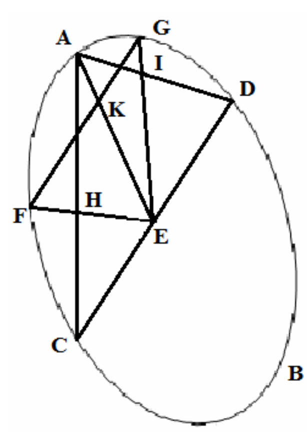
PROPOSITION CII.

With the same in place:
I say the joined lines AD, FC to be parallel.

Demonstration.

By the preceding the sectors DBC, ABF have been shown to be equal; therefore the segments DE, AF also are equal to each other (§60). Therefore the lines AD, FC are parallel (§61). Q.e.d.

PROPOSITION CIII.



The conjugate of some diameter AE shall cut the ellipse ABC at CD: and moreover there shall be the conjugate of other diameters, FE, GE which cut the joined lines AC, AD at H and I:

I say that as AH to HC, thus there shall be DI to IA.

Demonstration.

FG shall be drawn which will cut AE at K so that AH to HC, thus as FK to KG: but as FK is to KG, thus DI is to AI (§101), therefore as AH to HC thus DI to AI. Q.e.d.

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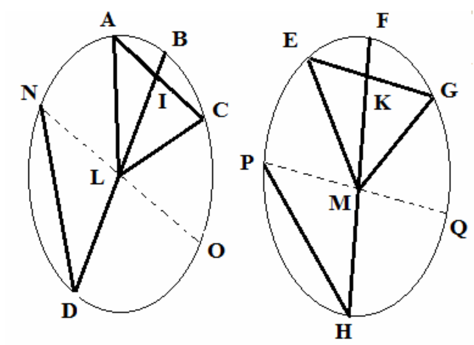
PROPOSITION CIV.

Some diameter BD shall cut the ellipse ABC: moreover EFG shall be an ellipse similar and equal to ABC : which some other diameter FH may cut : henceforth with the diameters BD, FH divided in proportion at I and K, the ordinate lines AC, EG shall act through the points I and K : the radii of the lines with the same ends AL, CL, EM, GM are drawn.

I say ALC, EMG to be equal triangles.



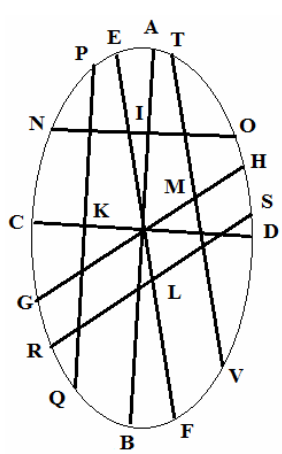
Demonstration.

The conjugate NO is drawn to the diameter BD, and PQ to the diameter FH: ND and PH shall be drawn. Since the diameters BD, FH are divided proportionally both by IK as well as LM, so that the rectangle BID will be to the rectangle BLD, thus as the rectangle FKH to the rectangle FMH: whereby, as the square AI to the square NL and thus as the square EK to the square PM : and as the line AI to the line NL, thus EK to PM: but also there is by the construction, as IL to BL, that is LD, thus KM to FM, that is, MH; therefore so that as the triangle NLD to the triangle AIL, thus the triangle PMH to the triangle EKM (since the ratio is composed from the same sides of these:) and by interchanging, as the triangle NLD to the triangle PMH, thus the triangle AIL to the triangle EKM. But the triangles NLD, PMH are equal, therefore the triangles AIL, EKM also, and thus the whole triangles ACL, EGM are equal. Q.e.d.

PROPOSITION CV.

Two conjugations of diameters AB, CD, EF, GH shall cut the ellipse ABC and all four diameters shall be divided proportionally at the points I, K, M, L, through which the ordinate lines are drawn NO, PQ, RS, TV.

I say the squares NO, PQ taken together to be equal to the squares RS,TV taken together.

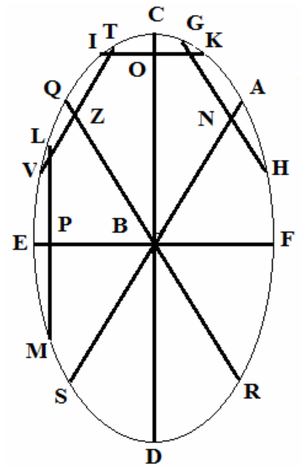


Demonstration.

Since both AB, EF, as well as CD, HG are divided proportionally, so that the square AB shall be to the rectangle AIB, thus as the square EF to the rectangle ELF, and the square CD to the rectangle CKD, and the square HG to the rectangle HMG. Indeed from these same ratios, the proportions of the individual squares to the rectangles are composed. Therefore so that the squares AB, CD taken together with the rectangles AIB, CKD taken together, shall be the squares EF, GH taken together, are as the rectangles ELF, HMG ; that is the squares

PK, NI to the squares SL, TM (§74) : but the squares AB, CD taken together are equal to the squares EF, GH taken together (§77); and therefore for the squares SL, TM to be equal to the squares NI, PK : therefore the squares NO, PQ taken together are equal to the squares SR, TV taken together. Q.e.d.

PROPOSITIO CVI.



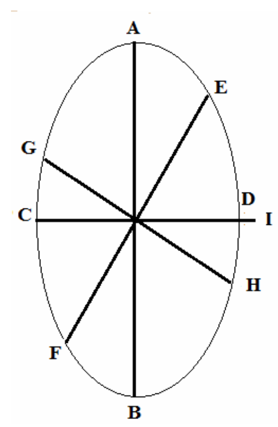
Some conjugates may cut the diameters CD, EF of the ellipse ABC, by which the diameters are divided proportionally at O and P : together with the conjugate drawn from one of the two equal conjugate diameter AS, which may be divided at N, in the same proportions as CD is divided at O. The ordinates GH, IK, LM are drawn through N, O, and P. I say the squares IK, LM taken together to be the double of the square GH.

Demonstration.

The other conjugate QR of the two equal conjugates may be drawn, which is to be divided similarly at Z, as SA is divided at N, and CD, EF, at O and P, and the ordinate VT may be placed through the point Z : since QR, AS therefore are cut similarly, the rectangle QZR is equal to the rectangle ANS (§74). But the rectangles QZR, ANS are equal to the squares GN, TZ. Therefore the squares GN, TZ and the squares GH, TV thus are equal. But the squares ML, IK are equal to the squares VT, GH. Therefore the squares ML, IK (§105) together are twice the square GH. Q.e.d.

PROPOSITION CVII.

Two pairs of conjugate diameters AB, CD; EF, GH shall cut the ellipse ABC : and AB shall be the largest in magnitude and EF the second largest.
I say the ratio AB to EF to be smaller than the ratio GH to CD.



Demonstration.

Since the squares AB, CD are taken equal to the squares EF, GH taken together: the ratios cannot be as AB to EF, thus GH to CD, for then AB, CD shall be the maxima of the squares and the minima of the squares EF, GH. Therefore there shall become, as AB to EF, thus GH ad CI: and AB, CI shall be the greater square of the squares EF, GH; that is of the squares AB, CD. Whereby the line CI is greater than the line CD, and the ratio GH to CD; that is from the construction, the ratio AB to EF is smaller than the ratio GH to CD. Q.e.d.

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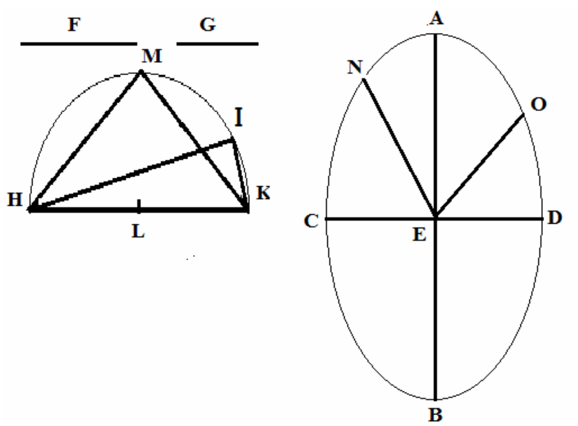
PROPOSITION CVIII.

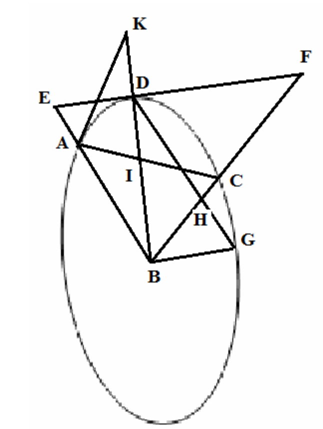
In a given ellipse, to show the conjugate diameters to be in a given ratio.

Construction and demonstration.



AB, CD shall be the axes of the ellipse ABC with centre E : truly as the given ratio of the greater to the lesser axis shall be as F to G, will be required for that not to be greater than the ratio of the axes AE, EC. If the ratio given is equal to the ratio of the axes, the situation is clear: if the ratio is less, the lines HI, IK may be taken, equal to the right lines AE, EC, and HIK may be put equal to a right angle: and with HK drawn, and bisected at L, with centre L the circle HIK is described with radius LH, which will pass through the points I and K, then in the segment HIK (§49, Circle) the two lines HIK shall be inclined in the given ratio F to G. Therefore since HM is to MK as F to G, and the ratio F to G is less than the ratio of the axes, that is, from the construction in the ratio HI to IK, it is clear the point M to fall between I and H, and thus the right line HM to be smaller than HI, that is, smaller than EA; truly the line MK to be greater than IK, that is than EC. Therefore with centre E and with radius HM, the described circle shall cut the ellipse at some point N between A and C, therefore the diameter NE may be drawn to that; with EO joined. I say NE, OE (§77) are required to satisfy the diameters. Since these squares NE, OE are to be equal to the squares AE, EC ; that is HI, IK to the squares HM, MK: but the line NE is equal to the right line HM. Therefore the line OE is equal to the line MK. Therefore as HM to MK, that is as F to G, thus NE to OE. Therefore we have shown, etc. Q.e.f.

PROPOSITION CIX.

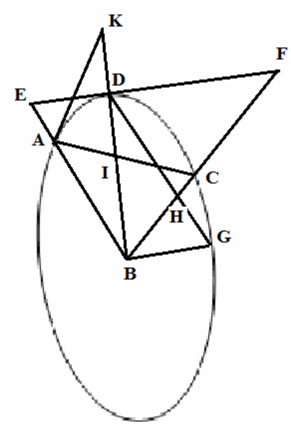


Some conjugate of the diameters AB, BC shall cut the ellipse ACD, and with some point D assumed on the periphery between A and C, the tangent EF is acting through D, which since it shall cross the lines AB, BC at E and F; the line BG is drawn parallel to that tangent through the centre.
I say the lines DF, BG, DE to be in continued proportion.

Demonstration.

Since from the hypothesis DF, BG are parallel, the triangles DFH, BHG will be similar, as evident from the Elements, and hence DF is to BG, as DH to HG; that is, as the triangle DHB is similar to the triangle

BHG. Thence, since BG drawn through the centre is parallel to the tangent DF, it is clear the conjugate to be the diameter DB itself: truly from the hypothesis also AB, BF to be

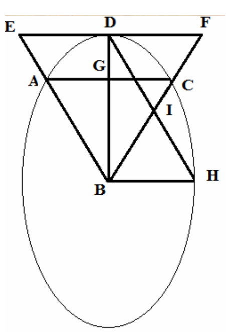


conjugates. Therefore the sectors BDCG, BADC (§46,Cor.), and thus also the triangles BDG, BAC are equal: of which the bases DG, AC, since they shall be divided proportionally by H and I (§60, Cor.), so that DH shall be to HG, thus as CI is to IA, it may be agreed from the Elements the triangles BHD, BIC and BHG, BIA to be equal. Therefore since before I shall have shown DF to be to BC as the triangle BHD to the triangle BHG, there will be also DF to BG, as [the area of] triangle BHD to triangle BHG. Also there will become DF to BG, as triangle BIC to triangle BIA. In addition since triangle BHG shall be to triangle BHD, as GH to HD, that is, as AI to IC, that is as KI to IB (since indeed CB shall be the conjugate of AB, and AK, the tangent from the construction, it is apparent AK, CB to be parallel) that is as triangle AIK to triangle AIB : therefore by adding triangle BDG to triangle BHD, to become as triangle AKB to triangle AIB; and by interchanging, triangle BDG to triangle AKB, as triangle BHD, just as shown before, as triangle BIC to triangle AIB. But triangle AKB is to triangle EDB, that is triangle BDG is to triangle EDB, that is, since ED, BG are parallel, BG is to ED, as triangle BIC to triangle AIB, that is just as shown above, as DF to BG. Therefore they are in the continued ratio DF, BG, ED. Q.e.d.

PROPOSITION CX.

AB, BC shall be some conjugate diameters, and on assuming some point D on the periphery between A and C, the tangent shall be acting through D, crossing the diameters AB, BC at E and F, and with AC joined it shall cross the diameter DB at G.
I say the right line DF to DE, to be in the square ratio of that which CG has to GA.

Demonstration.



The line BH may be drawn from B parallel to EF, and from D the right line DH crossing FB in I. Therefore, from the preceding, the lines FD, BH, ED will be in continued proportion. And therefore the ratio FD to ED, will be in the square ratio of FD to BH, that is DI to IH, since DF, BH will be parallel, from the construction: again since the right line HB shall be parallel to the tangent DF, the diameters DB, BH will be conjugate; moreover from the construction AB, BC also shall be conjugate; therefore as DI to IH, thus CG ad GA (§101): whereby the ratio FD to DE, is the square of the ratio CG ad GA. Q.e.d.

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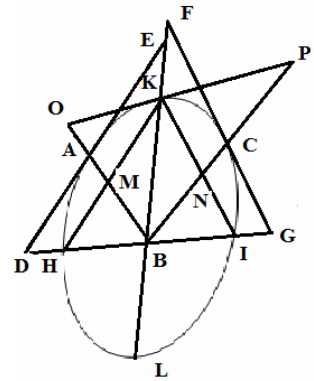
PROPOSITION CXI.



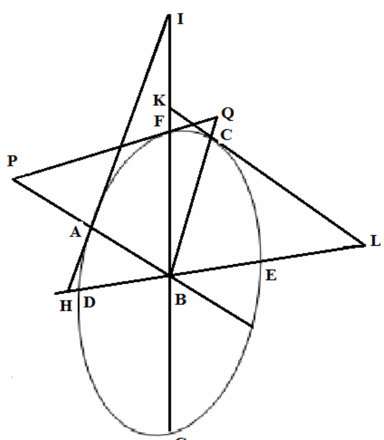
AB, BC shall be conjugate diameters and the tangents DE, FG may be drawn through A and C. Moreover there shall be some other conjugates of the diameters, HI, KI, which produced will intersect the tangents DE, FG at E, F, D, and G. I say the lines DE, FG to be divided proportionally at A and C, without doubt to be EA to AD, as GC to CF.

Demonstration.

The lines HK, KI may be drawn which will cut the right lines AB, BC at M and N. The ratio EA to AD is the square of the ratio KM ad MH (§110), and GC to CF is the square of the ratio IN to NK. And the ratio KM to MH is equal to the ratio IN to NK. Therefore the ratios EA to AD and GC to CF of the squares of equal ratios, are equal; therefore DE, GF are cut proportionally at the points A, C. Q.e.d..

Corollary.

So that if a third tangent may be drawn through K, agreeing with the conjunctions BA, BC at O and P, I say there becomes OK to KP, as EA to AD. Which will be shown with the right line AC drawn in the same manner as we have used.
And thus the three tangents DE, OP, FG thus are divided similarly so that EA shall be to AD, just as OK to KP, and GC to CF.



PROPOSITION CXII.

AB, BC, FG, DE shall be two sets of conjugate diameters, with the tangents HI, KL acting through A and C, which intersect the diameters FG, DE at the points H, I, K, L.
I say BC shall be to BA thus as HI to KL.

Demonstration.

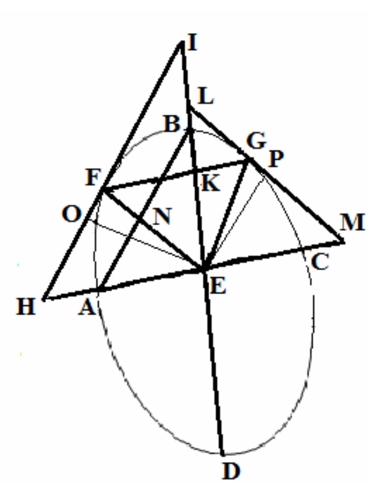
Since the right line BC shall be parallel to the right line HI, and KL parallel to AB, both HA, BC, AI as well as KC, AB, CL are lines in continued proportion. Therefore since just as for HA to AI, first to third, thus also there will be KC to CL, first to third; there will be also HA to BC, first to second, as well as KC to AB; I have shown before HA to be to AI, as KC to CL, and thus on inverting the compounded ratios, and on interchanging, there shall be as HA to KC, thus HI to KL, so that there shall be BC to BA, thus as HI to KL.

$$\left[\begin{array}{l} \text{HA, BC, AI \& KC, AB, CL in continued proportion provide} \\ BC^2 = HA\cdot AI \;\&\; AB^2 = KC\cdot CL \\ \dfrac{BC^2}{AB^2} = \dfrac{HA\cdot AI}{KC\cdot CL} = \dfrac{HI^2}{KL^2}. \end{array}\right]$$

Corollary.

It is shown in the same manner, if a tangent may be acting through F which crosses AB, BC at P and Q , that BC shall be to BD, thus as HI to PQ.

PROPOSITION CXIII.



AC, BD, EF, EG shall be the two pairs of conjugate diameters: and with the tangents acting through F and G which cross the diameters AC, BD at H, I, L, M, the right line FG may be drawn cutting the diameter BE at K.

I say the lines LK, KE, KI to be in continued proportion.

Demonstration.

Since FE, as the conjugate of EG, shall be parallel to the tangent LG , so that LK to KE, thus GK will be to KF; Similarly since EG, as the conjugate of EF, shall be parallel to the tangent FI; therefore as LK to KE, thus KE is to KI . Q.e.d.

PROPOSITION CXIV.

With the same in place:

I say the triangles IHE, LME to be similar.

Demonstration.

The right line AB shall be drawn which shall cut the line FE at N : and from E the right lines EO, EP shall be drawn normal to the lines HI, LM. Since the line LM shall be parallel to FE (for LM is the tangent, and FE conjugate to EG), the angle FEG shall be equal to the angle EGP. In the same manner the angle OFE will be equal to the angle FEG, where the angles OFE, EGP are equal to each other: moreover the angles EPG, EOF are right; Therefore the triangles EGP, EFO are similar. Whereby as EG to EF, thus EP to EO. But as EG is to EF, thus HI to LM. Therefore as EP to OE, thus inversely HI to LM. Therefore the triangles IHE, LME are similar. Q.e.d.

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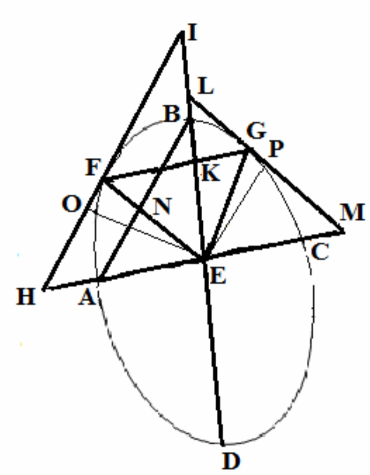
PROPOSITION CXV.



With the same triangle EGM in place equal to the triangle EFI, then triangle EGL is equal to triangle EFH.

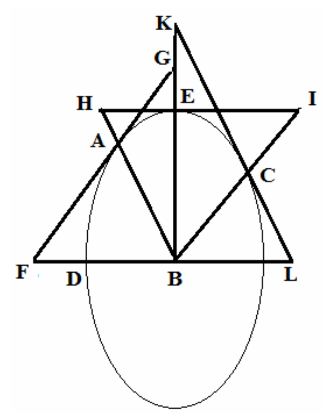
Demonstration.

Indeed as HF is to FI, thus LG to GM, and on adding together, as Hl to FI, thus LM to GM: but as HI to FI, thus HIE triangle is to triangle FIE, and as LM ad GM, thus triangle ELM to triangle EGM, therefore as triangle HIE to triangle FIE, thus triangle ELM is to triangle EGM. Whereby the triangles FIE, EGM are equal. In the same manner the remaining triangles EGL, HFE are shown to be equal.

PROPOSITION CXVI.

If the ellipse ADC may cut the two sets of conjugate diameters AB, BC; EB, BD in such a way that by acting through A, E, C, the tangents FG, HI, KL shall indeed cross the diameters EB, BD at G, K, F, L. Truly with the diameters AB, BC, at H and I.

I say the triangles FGB, HBI, KLB to be equal to each other.



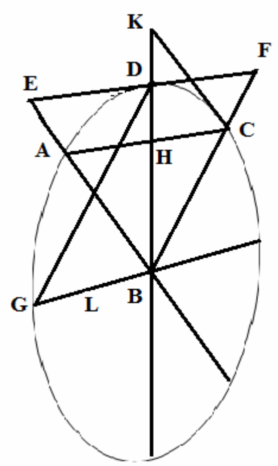
Demonstration.

For triangle BIE is equal to triangle BKC (§.70), that is, by §.115, with triangle BFA and triangle BHE, shall be equal to triangle GAB, therefore the whole triangle BIH is equal to the whole triangle BFG. Whereby since also FGB, KBL shall be equal; it is evident the three triangles to be equal.

PROPOSITION CXVII.

AB, BC shall be two conjugate diameters of the ellipse ABC, and from some point assumed between A and C, evidently D, the tangent DE may be drawn crossing over the conjugates AB, BC produced at E and F: then BG may be drawn from the centre B, parallel to EF itself, and GB shall be the mean between ED, DF.
I say the point G to be in the periphery of the ellipse, of which the conjugate diameters are AB, BC, and the tangent ED.

Demonstration.



BD is drawn from the centre to the tangent point, and since by the hypothesis GB is parallel to the tangent ED, GB will be the ordinate in place for BD, and indeed for the centre. From which thus BD, GB are conjugate diameters. Then AHC and DG may be joined, and from C, CK may be drawn parallel to AB, crossing BD produced at K : which is a tangent to the section at C, and there will become as AH to HC, thus BH to HK: but as AH to HC, thus the triangle ABH is to the triangle HBC; and thus as BH ad HK, thus triangle HBC is to triangle HCK: therefore as triangle ABH to triangle HBC itself, thus triangle HBC is to triangle HCK: therefore on adding, and interchanging triangle ABC with triangle BCK , thus there is triangle BHC to triangle HCK, that is now as shown, so that triangle ABH is to triangle HBC, that is, as the line AH to the line HC: but the triangles BCK, BDF are equal (§.70); therefore also the triangle ABC to the triangle BDF, shall be as AH to HC. Further since DF, GB are parallel, there will be triangle GDB to triangle DBF, as GB to DF: but, since from the hypothesis ED, GB, DF are in continued proportion , the ratio GB to DF, is half of the ratio ED to DF [i.e. the geometric mean of the ratio]. Therefore the ratio of triangle GDB to triangle DBF is half of the ratio ED to DF. And the ratio AH to HC, that is as shown above, the ratio of triangle ABC to triangle DBF, also is half of the ratio ED ad DP. Therefore triangle GDB is to triangle DBF, as triangle ABC to the same triangle DBF. Therefore the triangles GDB, ABC are equal. Therefore the parallelogram held by the radii, as shown above, with the conjugates GB, BD in the angle GBD is equal to the parallelogram held under the conjugate radii AB, BC. Therefore the point G lies on the ellipse (§.72). Q.e.d.

PROPOSITION CXVIII.

Again there shall be the two conjugate diameters BA, BC. And with the point D taken on the perimeter of the ellipse between A and C, it shall be the tangent EF of the ellipse at D, crossing with the diameters E and F. Then from the centre B, BG is drawn to the perimeter parallel to the tangent.
I say ED, GB, DF to be in continued proportion.

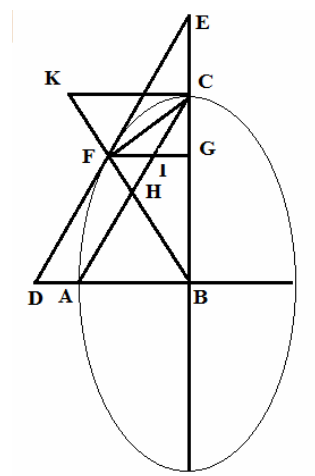
Demonstration.

Thus if not, there shall be some other mean value LB between ED, DF greater or smaller than GB. Therefore by the preceding the point L lies on the ellipse, which cannot happen, since from the hypothesis the point G shall be present on the ellipse. Therefore no other besides GB, is the mean between ED, DF. Therefore ED, GB, DF are continued proportionals. Q.e.d.

PROPOSITION CXIX.

AB, BC shall be conjugate diameters for the ellipse and the ends of these may be joined ; the line DE parallel to this line AC shall be a tangent at F, and crossing the conjugate diameters extended at D and E.
I say the triangle EDB to be the double of triangle CAB.

Demonstration.



The diameter BHF shall be drawn from the centre B through the point of contact F, to which it shall meet at K the right line tangent to the ellipse at C, then from the point of contact F the ordinate FIG may be put in place. Since AC is parallel to the tangent DE from the hypothesis, it will be the ordinate to the diameter BF (§.13). Therefore KB, FB, HB are continued proportionals (§.32). Therefore as KB is to FB, thus KF is to FH. But KB is to FB as the triangle KCB is to the triangle FCB, that is, since the triangles KCB, EFB are equal (§.70), so that as triangle EFB to the same triangle CFB, that is, as EB to CB. Therefore as EB to CB, thus KF to FH. Then since from the construction KC to be a tangent, and FG is put the ordinate to BC, the right lines EB, CB, GB are in continued proportion. Therefore as EB to CB, thus EC to CG. But also as I have now shown, thus as EB to CB, thus KF to FH. Therefore as KF to FH, thus EC to CG. Therefore as triangle KCF to triangle FCH: thus triangle EFC to triangle CFG. And since the whole triangles KCB, EFB shall be equal, with the common triangle FBC removed, KCF, EFC are equal. Therefore FCH and CFG are equal; therefore with the common triangle FIC removed, FIH, CIG remain equal, from which if with the common area BHIG added, FGB will be equal to CHB. Now truly since the ordinate AC to BF put in place as shown above, is AC bisected at H, and thus the triangle CAB is twice the triangle CHB, and DE parallel to AC also shall be bisected at F: truly is FG, just as with the ordinate drawn to CB, parallel to CB, the conjugate diameter to CB. Therefore BE shall be bisected at G. And thence EFB is twice GFB. And CHB, FGB have been shown to be equal. Therefore the double of these CAB, EFB are equal. But triangle DEB doubled is triangle EFB, is indeed DE bisected at F. Therefore triangle DEB doubled is also the double of triangle CAB. Q.e.d.

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PART FOUR : THE ELLIPSE

The poles [or foci] of a section [of the cone]: that shall designate the shortest line from a given point on the axis to a point on the periphery, [and back again to the other pole or focus.]

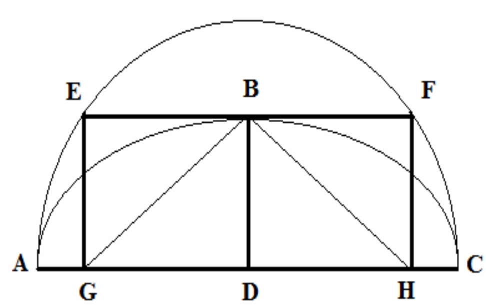
This part, which I am about to undertake, is concerned with the poles, to which we will advance several items to that investigation of the poles, demonstrated by Apollonios, in Propositions 41& 45 of Book Three ; and indeed it has been necessary to be introduced by this work; since besides the logical development of these which Apollonius has brought together, so that everything is understood clearly, thus more items shall be introduced that I consider necessary. Therefore Apollonius shall show the poles on the axis of the ellipse, use is made of the construction from propositions 45 &46. It is understood to be equal to the fourth part of the figure prepared from each part; that is, the axis of the section AC may be cut thus by the two points G & H, so that both the rectangle AGC as well as the rectangle CHA [i.e. AG.GC & CH.HA] shall be equal to a fourth part of the figure: with which in place further it will be shown G and H to be the poles of the section: which points he calls forth from the construction made; clearly from the comparison of the rectangles under the segments from the fourth part of the figure. Again Apollonius here calls the figure of the ellipse rectangular since it shall be with a right side under the major axis with a right side under the major, and with the axis itself: and that with its fourth part may serve to be of exceptional use; clearly before the discovery of the individual poles etc., the figure with the remaining right angles he called by the ancient name rectangle: but the fourth part of this is equal to the square of the minor semi-axis: which Pergaeus demonstrated splendidly in Book 3, Prop. 42, and thus we will indicate by a single word.

Lemma.

From §11, a rectangular figure as if under the major axis and with the right side of this, is equal to the square of the minor axis ; but the fourth part of the square of the minor axis is the square of half the minor axis; therefore the square of half the minor axis is equal to the fourth part of the figure. From which I will call this the fourth part to be used, by which I mean to be understood the square of the semi-minor axis.

In this part some propositions arise concerning focal points, the same as those which Apollonius demonstrated: which I have made here with further deliberations, lest the more studious reader may wish for more on this subject matter.

PROPOSITION CXX.



The axes of the ellipse ABC shall be AC, BD, and with the tangent EF acting at B: with centre D the circle AEFC is described with radius DA, which it

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may meet the circle touching at E and F. Then the normals EG, FH may be dropped from E and F to the axes AC.

I say both the rectangle AGC as well as AHC to be equal to the fourth part of the figure.

Demonstration.

Since both the line EB shall be parallel to the line AD as well as EG parallel to BD, the lines EG, BD will be equal : but the rectangle AGC is equal to the square EG, since the right line EG shall be drawn normal to the diameter of the circle ; and therefore the square BD is equal to the rectangle AGC. In the same manner FH, that is the square BD is equal to the rectangle AHC, but the square BD is equal to the fourth part of the figure, therefore both the rectangle AGC as well as the rectangle AHC is equal to the fourth part of the figure. Q.e.d.

Corollary.

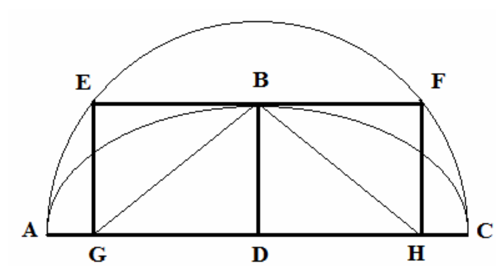
Hence it is apparent the lines AG, HC to be equal and GH to be bisected at D.

PROPOSITION CXXI.

With the same in place the right lines BG, BH may be drawn.

I say the lines BG, BH taken together to be equal to the axis AC.

Demonstration.

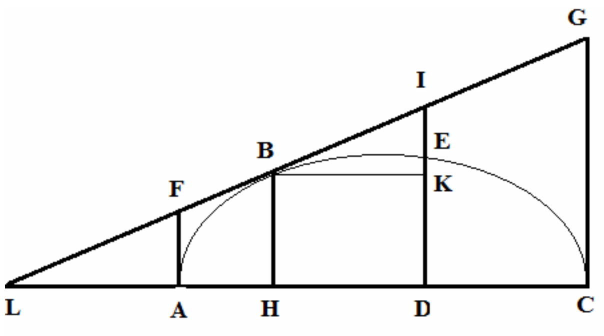


Because AG, HC are equal (§120.Cor.), the square AG is equal to the rectangle from AG and HC. Again since GD, DH are equal, the rectangle AGD taken twice will be equal to the rectangle AGH. Whereby since the square AD shall be equal to the squares DG, AG and with the rectangle AGD taken twice; the same square AD will be equal to the square DG, and with the rectangle from AG, HC together with the rectangle AGH. And the rectangles AG, HC, and AGH are equal to the rectangle AGC. Therefore the square AD is equal to the square DG together with the rectangle AGC; that is to the squares DG, GE, that is to the squares DG, DB. But from the same, the squares GB are equal; therefore the squares AD, GB are equal, and therefore the right lines AD, GB are equal. It will be shown in the same manner the right lines CD, HB to be equal. Therefore both GB, BH taken together are equal to the axis. Q.e.d.

Corollary.

Hence it follows: if the axes of the ellipse ABC were AC, BD, and from the vertex B of the minor axis the right lines BG, BH were dropped equal to the lines AD, DC, cutting the axes AC at G and H . So that both the rectangles AGC as well as AHC shall be equal to the fourth part of the figure, thus G and H are poles of the section.

PROPOSITIO CXXII.



The ellipse ABC, of which the conjugate diameters AC, DE have tangents at A and C, and at some other point B the three lines AF, CG, FG are tangents at A, B, and some other point B: and indeed FG shall cross the right lines AF, CG at F and G, and on being produced, ED will cross the line FG at I, and the right ordinate line BH dropped from B to the diameter AC:

I say the rectangle on the lines AF, CG to be equal to the rectangle on BHID.

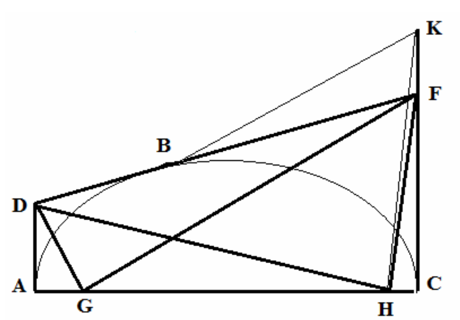
Demonstration.

The line FG may be produced until it reaches the axis at L. Since DH, DA, DL are lines in continued proportion [§32], there will become: LD to AD, that is as DC thus LA to AH, and on inverting and adding, so that as CL to DL, thus HL to AL, but as CL to DL, thus there is CG to DI, and as HL to AL, thus HB to AF, therefore as CG to DI, thus HB to FA: and thus the rectangle on the lines AF, CG to be equal to the rectangle BH, ID. Q.e.d.

Corollary.

Hence it follows the rectangle AF, CG or HB, ID, to be equal to the fourth part of the figure: for BK may be drawn parallel to the axis AC: the rectangle DK, DI shall be equal to the square ED; and therefore the rectangle AF, CG is equal to the square ED, that is to the fourth part of the figure.

PROPOSITION CXXIII.



The lines AD, CF, and DF shall be tangents to the ellipse ABC, of which the axis is AC, at A and C and at some other point B, indeed it may be agreed with the lines AD, GF at D and F, but with the line AC may be divided at G and H, so that AGC, AHC to be rectangles equal to the fourth part of the figure; and lines DG, GF, DH, HF shall be drawn.

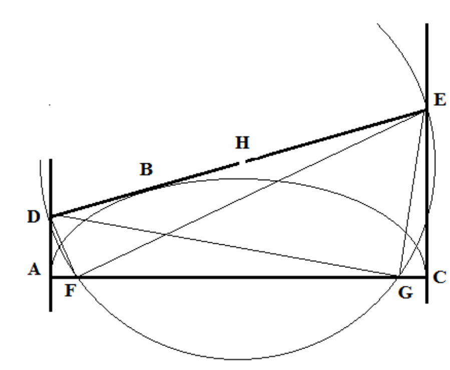
I say the angle DGF, DHF to be right; and if they shall be right: I say DF to be a tangent line of the ellipse.

Demonstration.

From the preceding Cor., the rectangle DACF is equal to the fourth part of the figure, that is, to the rectangle AGC. Therefore as AG to AD thus FC to CG , but the angles DAG, FCG are right; therefore DAG, FCG are similar triangles : and the angle ADG is equal to the angle CGF, but the angle ADG, together with the angle AGD is equal to a right angle, since the angle DAG in triangle ADG shall be right : and therefore the angle CGF, together with the angle AGD are equal to one right angle: therefore the remaining angle DGF is right: it may be shown in the same manner the angle DHF is right. Q.e.d.

PROPOSITION CXXIV.

The lines AD, CE, DE shall be tangents to the ellipse ABC at A, C, B, the axis of which is AC, and indeed DE shall meet the lines AD , CE at the points D and E. Moreover they shall become the fourth part of the figure, equal to the rectangles AFC, AGC, and ED shall be bisected at H:
I say the circle described with centre H and with the radius DH and EH, to pass through F and G.



Demonstration.

The points DF, FE, DG, GE may be joined. Since both the angles DFE, as well as DGE, is right, and DE some line bisected at H, it is evident the circle described with centre H and with the radius HD to pass through F and G. Q.e.d.

Corollary.

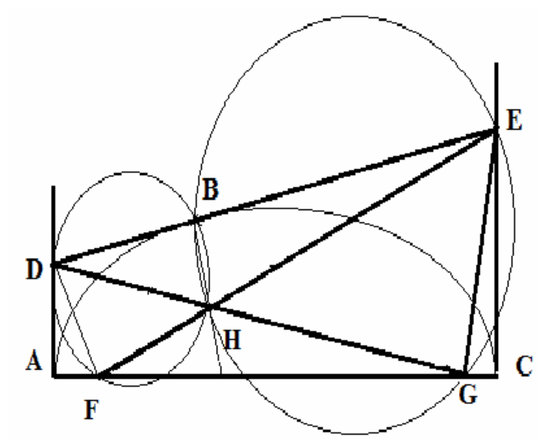
Hence it follows the angles EDG, FDA to be equal to each other, for the angle ADF is shown in the preceding demonstration to be equal to the angle GFE: but the angle EDG is equal to the angle GFE since it stands on the same arc EG, therefore the angles EDG, FDA are equal to each other.

PROPOSITION CXXV.

The lines AD, CE, DE shall be tangents of the ellipse ABC, of which AC is the axis, at A, C, B : and DE indeed shall meet the lines AD, CE at D and E. Moreover AFG, AGC shall become rectangles equal to the fourth part of the figure: and with the lines FE, GD drawn which intersect each other at H, from the point H to the contact point B, the right line HB may be drawn.
I say HB to be normal to the tangent DE.

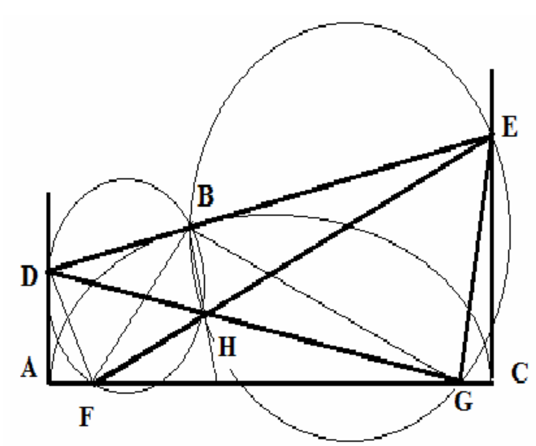
Demonstration.

The right lines ED, CE are drawn. The angles DFE, EGD are right. Now with the lines HD, HE as the diameters , the circles DBH, EBH are described. Since the lines DH, HE are not parallel, it is evident the circles DBH, EBH in turn cut each other at some point B. Therefore with the points H and B joined; the right lines DB, EB may be drawn: the angles DBH, EBH are right, and thus the lines DB, EB are collinear, and HB shall be normal to the line DE. But as shown before in §123, the angles DFE, EGD are right; therefore the line DE is a tangent.



Whereby the right line BH is normal to the tangent ED. Q.e.d.

PROPOSITION CXXVI.

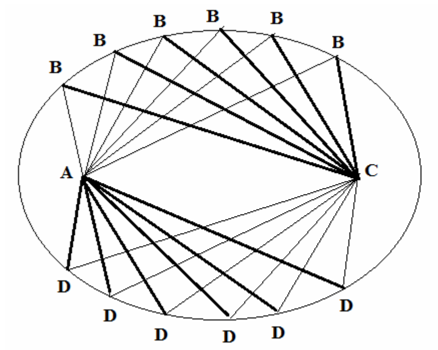


With the same figure remaining: FB, BG are drawn.
I say the angles DBF, EBG at the tangential point B to be equal.

Demonstration.

Since the angles DFH, EGA are right, the circles DFH, EGH will pass through F and G, but each also passes through B: since the angles EBH, DBH are right: therefore both the angles DBF, DHF as well as the angles EBG, EHG are equal to each other: but the angle

DHF is equal to the angle EHG: and therefore DBF is equal to the angle EBG [i.e. the point B on the ellipse can be considered as an elemental mirror, to which the usual law of reflection can be applied]. Q.e.d.



Scholium

Since the point B shall be assumed to lie on the periphery, it follows all the lines drawn from F to whatever point on the periphery are required to be reflected through G. Whereby the points F and G are to be called by no other name than the foci or poles: which are called from the comparison made with Apollonius, again these have extraordinary properties for ellipses :between those it has pleased to add the following here.

A, C shall be the foci of the ellipse, the distance between which shall be the separation of the eyes, and the left eye may be placed at A, and the right eye at C. I say that everything reflected by the whole mirror to appear to the right eye placed C: and in turn the right eye placed at C to see everything seen by the left eye placed at C: for the kinds of objects A reflected by the whole mirror, are reflected at C, and the kinds of objects seen at C by the whole mirror are reflected at A. Whereby an object placed at A reflected by the whole mirror to be used by the eye at C, and an object at C, by the eye at A. Hence it follows that a small visible object placed at C will appear large to the eye placed at A: since it will be reflected by the whole surface of the mirror.

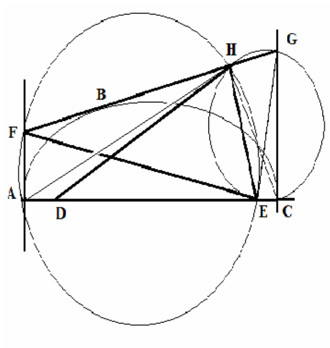
[There is a certain naivety in this description; for it is an inverted real image of the object at one focus that can be viewed at the other focus, subsequently the rays will return to the initial focal point on being inverted again; with the eye placed elsewhere conveniently; the eye itself should never placed at the focal point of a large mirror: if it were, damage to the retina could occur, and in any case, none of the light would be reflected, as it would be absorbed mostly by the eye. However, it is a pleasing demo., using sound waves to have two large elliptical or spherical mirrors set up confocally as parts of the ellipse considered above, and to have someone whisper at one focus, to be heard by someone with an ear placed at the other focus.]

PROPOSITION CXXVII.

The ellipse ABC, of which the axis is AC and poles D E, has the right line tangents AF, CG, FG at the points A, C, B ; and indeed FG shall meet the lines AF, CG at F and G. The line EH, erected from E, shall be put in place normal to the tangent FG, and the points AH, CH joined.

I say the angle AHC to be right.

Demonstration.



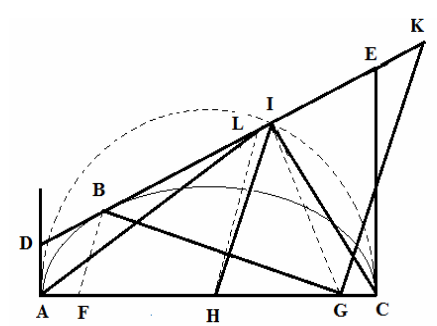
With the lines FE, EG drawn, the circles FHE, HGC are described with the diameters FE , EG : and indeed the circle FHE, with the right angles EHF, EAF, will pass through the points H, F, A ; truly the circle HGC: also shall be with the right angles DHE, ECG, shall pass through H, C. Therefore both the angles AHF, AEF as well as the angles EHC, EGC, shall be equal angles: but the angle GCE by the demonstration shown above in §121, is equal to the angle FEA, therefore the angle FHE is equal to the right angle AHC, and whereby to be right itself. Q.e.d

PROPOSITION CXXVIII.

The lines AD, CE, DE shall be tangents to the ellipse ABC at A, C, B, the axis of which is AC, and indeed DE shall cross the lines AD, CE at D and E : moreover the poles shall be F, G, the centre H, and the right line FB drawn from F to the point of contact, and from H the line HI shall be drawn parallel to the line FB, crossing the line ED at I.

I say the line HI to be equal to the line HC, and if HI crosses the line ED, it shall be equal to HC. I say the line HI to be parallel to FB.

Demonstration.



BI shall be made equal to IK: and BG, GK may be joined, and the right lines AI, IC may be drawn: Since IB, IK are equal, BI will be to IK, as FH to HG: and thus the lines BF, KG are parallel, and the angle BKG equal to the angle DBF, that is IBG, from §115 : whereby the lines BG, GK are equal: moreover the two remaining sides BI, IG are equal to the sides KI, IG. Therefore the angle BIG, is equal to the angle KIG: and thus GI is normal to the tangent DE, and the angle DIG is right.

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Whereby the circle with centre H and described with radius HC will pass through I, and the line HI will be equal to the line HC. Which was to be shown first.

For the rest remaining, now HI shall be a line which crosses the tangential line ED at I, equal to the line HC. I say the right line HI to be parallel to the line BF: truly on the other hand, if a line HL may be drawn from H, parallel to the right line FB crossing the tangential line ED at L; hence the line HL would be equal to the line HC, that is, to HI. Whereby a circle with centre H described with the radius HC would pass through the points I and L. Which cannot happen ; therefore HL is not parallel to FB: nor any other line apart from the line HI. Q.e.d.

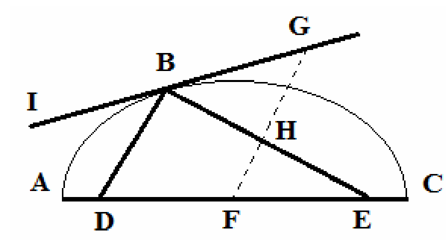
PROPOSITION CXXIX.

ABC shall be an ellipse with axis AC, moreover the poles shall be D, E ; from D and E the lines DB, EB meeting at some point B of the periphery will be reflected.

I say the lines DB, EB taken together shall be equal to the axis AC.

Demonstration.

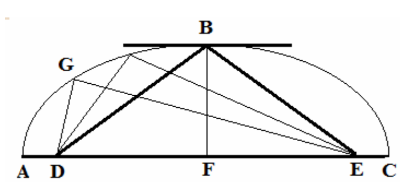
F shall be the centre of the ellipse; and with the tangent BG acting through: the right line FG shall be drawn parallel to the line DB cutting the line EB at H. Since BD, FG are parallel lines, therefore the angle FGB is equal to the angle DBI that is, to the angle EBG, and thus the lines HB, HG are equal : again, since DE shall be to FE, thus as BE to HE, and DE shall be twice as much as FE, and EB shall be twice as much as right line BH, that is HG: thus also BD is FH doubled, thus since DE shall be to FH, thus as DB to FH; therefore the lines EB, BD taken together are twice as much as FG, that is, FC, [from the previous prop.]: and whereby equal to the axis AC. Q.e.d.



PROPOSITION CXXX.

The maximum of the isoperimetric triangles is isosceles.

Demonstration.



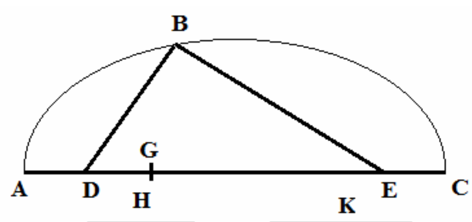
Some ellipse ABC may be described, of which the axes shall be AC, FB; the poles D, E; and the points D B, B E may be joined while above the base ED some triangles may be put in place and the points may be joined then some triangles DGE may be constituted upon the base of the triangle, the vertices of which G shall be on the periphery. Since both the lines DB, BE as well as DG, GE taken together are equal to the axis AC: it is apparent the triangles DBE, DGE to be isoperimetric: moreover I say the maximum triangle of these to be the triangle DBE acting through the tangent B: which since it shall occur only at the one point B of the ellipse, and the rest of that kind to fall outside the ellipse, it is clear the triangles DGE which are terminated on the ellipse to have a lesser height than the triangle DBE, and thus these to have a lesser perimeter: moreover the triangle DBE is isosceles, because the equal sides DF, FB are with the sides EF, FB, and with these to contain right angles; therefore of the triangles the isosceles triangle is of the maximum isoperimetric form. Q.e.d.

PROPOSITION CXXXI.

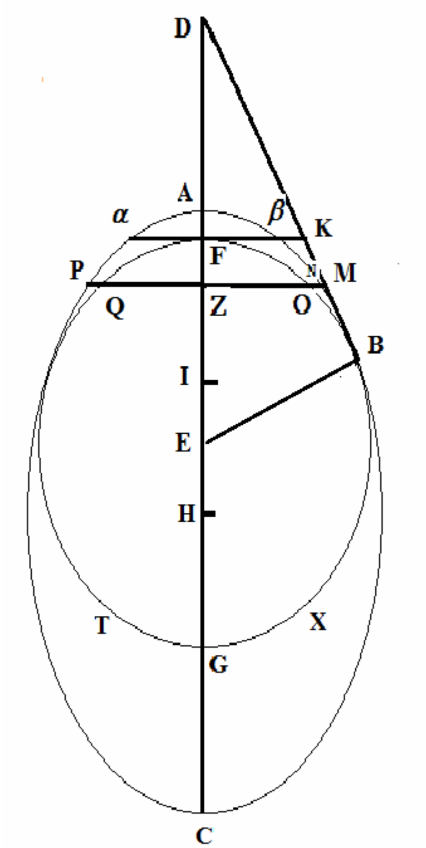
It shall be required to incline two lines from the foci D E of the ellipse to the same point of the perimeter which may be held in the given H to K.

Moreover, the ratio must be greater than the ratio AD to DC, yet smaller than the ratio AE to EC.

Construction & demonstration.



The axis AC shall be cut at G, following the given ratio H to K, which since it shall be put greater than the ratio AD to DC, and lesser than the ratio AE ad EC, evidently the line AG to be greater than the line AD: truly smaller than AE, and hence the point G to fall between the poles D, E and therefore the right line DB may be put in place from D to the perimeter equal to the right line AG. And the points B E shall be joined. I say what is required to be accomplished. For since the two lines DB, BE taken together shall be equal to the axis AC, moreover by the construction the line DB shall be equal to the line AG, BE will be equal to the remaining line GC: therefore DB is to BE, as AG to GC, that is as H to K. Therefore we have inclined the lines, etc. Q.e.f.



PROPOSITION CXXXII.

The line BD shall be a tangent to the ellipse ABC at B meeting the major axis CA at D; moreover, from the point of contact B, the normal

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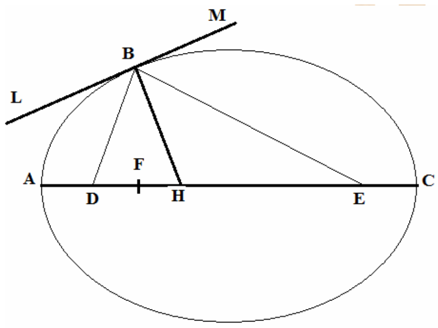
BE to the tangent may be put in place, meeting the axis at E.
I say EB to be the shortest line of these which are able to be drawn from the point E to the periphery of the ellipse.

Demonstration.

The circle FBG is described with centre E and radius EB crossing the axis at F and G; the centre of the ellipse shall be H. Since the ellipse tangent line DB meets the major axis at D, and the angle DBE is right: the line BE does not pass through the centre of the ellipse H: for if E were the centre, the right line EB will be placed normally to the conjugate axis AC, (since all the tangent lines DB shall be parallel and divided into two right lines) and thus DB would be parallel to the axis AC: therefore neither is E the centre of the ellipse, nor is EB the diameter : truly since DB meets the axis in the region of A, the line EB is smaller than the radius parallel to itself: and thus is smaller than the semi-axis HC, and much smaller than the right line EC, whereby the circle described with radius EB, meets the axis at G within the ellipse, therefore the point G is above C.
Again since HC, that is AH, shall be greater than EB shown, that is EG, with the common term EH removed, AE will remain greater than HG, but on putting EI equal to EH, the right line FI is equal to HG, therefore AE also is greater than FI: therefore with the removal of the common term IE from FE and AE, IA will remain greater than IF: and from which the point F falls within the ellipse below A. Further the tangent FK may be put through F, to which PQNM will be parallel, therefore so that the square MB shall be to the square BK thus as the rectangle QMO to the square FK; but so that the square MB shall be to the square BK thus as the rectangle PMN to the rectangle αKβ ; therefore so that the rectangle QMO is to the square FK thus as the rectangle PMN is to the rectangle αKβ: and on interchanging and inverting, so that the square FK shall be to the rectangle αKβ, thus as the rectangle QMO is to the rectangle PMN: but the square FK is greater than the rectangle αKβ, and therefore the rectangle QMO is greater than the rectangle PMN : again the rectangle QMO together with the square ZO is equal to the square ZM, and the rectangle PMN together with the square ZN, is equal to the same square ZM ; therefore the rectangle QMO, together with the square ZO, is equal to the rectangle PMN, together with the square ZN; from which if unequally the rectangles QMO, PMN may be taken away, the unequal squares remain ZO and ZN: and since the rectangle QMO is greater than the rectangle PMN, the square ZO is smaller than the square ZN: and the square ZQ less than the square PZ; therefore the points O and Q are within the ellipse: the points X T may be shown similarly, and any other points of the perimeter of the circle FHG to be within the ellipse ; therefore the whole circle FBG falls within the ellipse: from which since all the right lines drawn from the centre of the circle E to the periphery of the ellipse, first shall meet the circle then the ellipse : and thus the radii of the same shall be greater than EB, the shortest of all of these is that which is terminated at the common point B of the ellipse and circle to be drawn from the point E to the periphery of the circle. Q.e.d.

PROPOSITION CXXXIII.

To draw the shortest line from the point (H) on the axes of the ellipse to the perimeter.



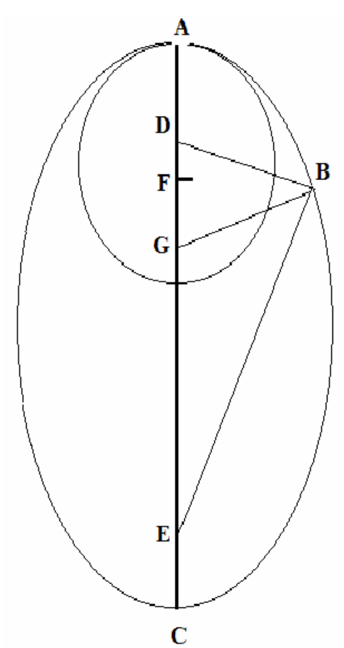
Construction & demonstration.

D and E shall be the foci of the ellipse. Cut the axis AC at F, thus so that AF shall be to FC, just as DH is to HE. Then from the pole to the perimeter the distance DB may be prepared equal to AF itself, and HB may be joined. I say HB to be the shortest distance.

For the right line EB may be drawn from the pole E to B, and LM the tangent to the ellipse at B; DB, and BE are equal in length to the axis. But DB is equal to AF. Therefore BE is equal to FC. Therefore DB is to BE, as AF to FC, that is , from the construction, as DH to HE. Therefore the angles DBH, EBH are equal, but also the angles are equal to the angles at the tangent DBL, EBM; therefore the sum of the angles HBL, HBM are equal ; therefore HB in normal to the tangent, therefore from the preceding the shortest of all the lines which can be drawn from the point H can be drawn to the perimeter. Therefore what was desired has been done.

If the point F may fall on the pole D, or between A and D; then the shortest distance from the given point H to the perimeter will be part of the axis, as is apparent from the first construction and demonstration being considered.

PROPOSITION CXXXIV.



In the given ellipse to describe the maximum circle of these which are tangential at the end of the axis and which are held within the ellipse.

Construction & demonstration.

The poles of the ellipse shall be D and E. The ratio shall become so that CD to DA; thus EF to FD. I say the circle described with centre F and with the radius AF to be that which is desired. Indeed since from the construction, there shall be CD to DA, thus as EF to FD. It is apparent from the preceding, FA to be the shortest of all the lines which can be drawn from the point F to the perimeter ; therefore the circle described with centre F is tangential to the ellipse at A, which was the first part:

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Moreover I may show thus that it shall be the maximum of all the tangents within the ellipse. For with some other point G taken for the centre of a greater circle, since therefore EG is to GD in a smaller ratio than EF to FD, that is, than CD ad DA, for the sake of an example there shall become EG to GD, thus CF ad FA: and by necessity FA shall be greater than DA: and thus the point F will fall beyond the pole D towards E: therefore if from the pole D to the perimeter DB may be put in place equal to GA, and GB may be joined, it will be clear from the preceding that GB to become the minimum of all the lines which may be drawn from G to the perimeter. Whereby GA is greater than GB, therefore the circle described through A with centre G falls outside the ellipse. Similarly we may show any other greater circle that that described before with the radius FA, to fall outside the ellipse : therefore that is the maximum of all the circles touching within the ellipse. Therefore in the given ellipse, etc. Q.e.f.

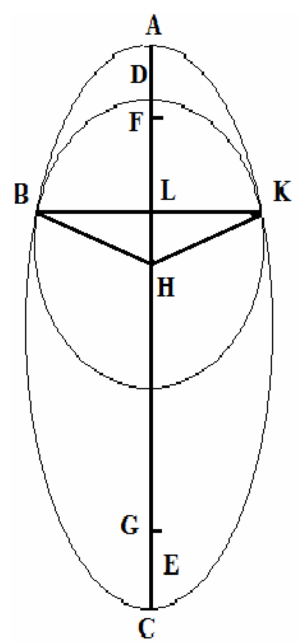
Corollary.

Clearly it is agreed from the discussion of this proposition, the circles described for all the radii, that the smallest to be tangential at the point A is for the radius FA. If the vertical axis, with the centre put in place between F and A, may pertain as far as A. These circles also will be tangential with that circle which is described with the radius FA, and these will be smaller, whereby also will be tangential within the ellipse, the axis of which is AC.

PROPOSITION CXXXV.

The major axis of the ellipse ABC shall be AC and in that the poles D, E, shall become as CD to DA, thus EF to FD, and DG to GE.

I say circles can be described from any point of the right line FG which touch the ellipse inside at two points: truly the centres of these to stand between the two excluding ends F and G.



Demonstration.

For some point H may be taken on the right line FS, and from H the line HB may be drawn, the shortest of these which will be able to be drawn from H to the periphery ; then from B the ordinate BLK to the axis may be drawn, and with HK, HB, joined, it is clear from the Elements that HK to be equal to HB, and thus the circle with centre H and with the element HB described to pass over through K and B: and since the lines HK, HB shall be the shortest by the construction, it is evident the whole circle BDK to fall within the ellipse, and on that account for the points B and K on that to be tangential. But since the centres of the circles tangential to the ellipse at two points, shall stand between the two points F and G: from that it is evident that the shortest lines FA, GC shall be those which are able to be drawn from F and G to the perimeter, and thus to be the circles with centres F or G, and for any radius which shall be described greater than FA or GC, the circle will cut the ellipse: truly the greatest will be described with the radius FA or GC, of these which lie inside the ellipse being tangential at only one point.

PROPOSITION CXXXVI.

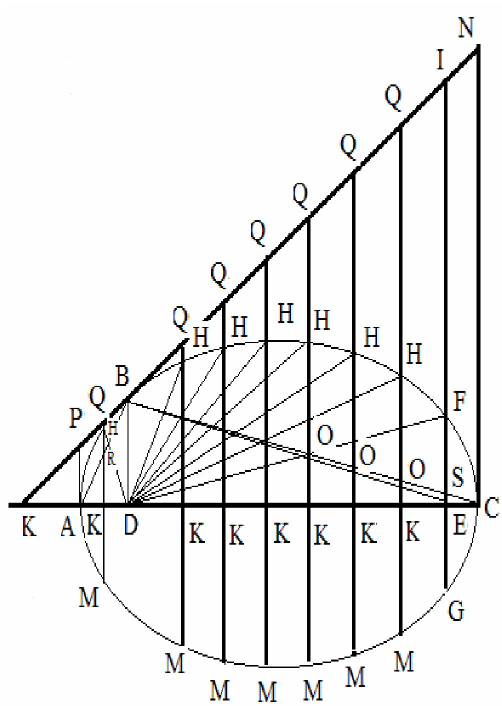
With the same figure remaining: it shall be proposed to designate a point on the axis from which centre a circle may be described, which shall be tangential at the given point.

Construction & demonstration.

The point B shall be given on the perimeter, through which if it is understood to be the touching point, the line BH shall be sent from B, normal to the tangent, crossing the axis at H. Clearly H is the point requiring to be satisfied; for since HB shall be the shortest of these which can be drawn from H to the periphery, the circle described with centre H and with radius HB shall be tangential to the ellipse at the point B; therefore, &c. Q.e.f.

PROPOSITION CXXXVII.

The axis of the ellipse shall be A, C, the poles D, E, from which the normals DB, EFG shall be drawn to the axis ; moreover KBI shall be a tangent to the ellipse at B, crossing the axis at K, truly the line GF at I, and DF may be joined.
I say the rectangle FIG to be equal to the square DE.



Demonstration.

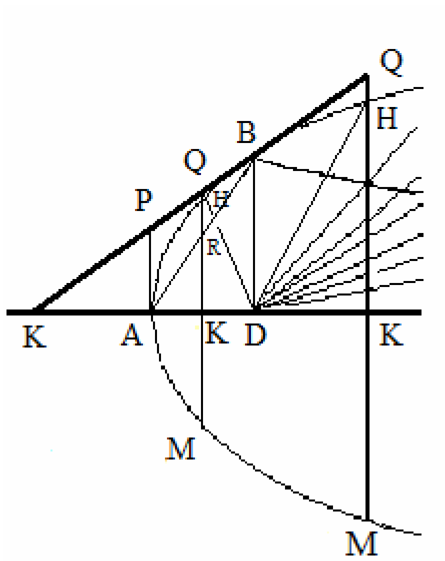
Since DB, EB are drawn from the poles to the point of contact B, the angles KBD, IBE are equal, but, since DB, EI are parallel, the angle KBD is equal to the angle EIB, therefore the angles IBE, EIB, and therefore BE, IE are equal. Then since the rectangles CEA, CDA are equal, also the squares EF, DB are equal, and thus the right lines EF, DB are equal. Whereby since [the sums of:] DB & BC are equal to EF& FD, see §121, which also will be equal to BE & DF. And BE has been shown equal to EI. And therefore DF is equal to EI, and hence the squares DF, EI are equal. But the squares DE, EF are equal to the square DF; and since GF is bisected at E, and FI may be added to that, the rectangle GIF with the square EF is equal to the square EI. Therefore the squares DE, EF are equal to the rectangle GIF with the square EF. Therefore with the common square EF removed, the rectangle GIF and the square DE will remain equal. Q.e.d.

PROPOSITION CXXXVIII.

With the same in place some other Q, H, K, M, normal to the axis may be put in place. I say the rectangles HQM, to be equal to the squares DK.

Demonstration.

CN shall be a tangent to the ellipse at C, crossing the tangent line BN at N, and BC may be drawn cutting QM at O, and IG at S ; since NC, IG, QM are normal to the axis, they are parallel ; therefore by that which we have shown in §99, the rectangle FIG shall be equal to the square SI, and the rectangle HQN is equal to the square QO; whereby the square IS to the square QO, that is as the square SB to the square OB, that is as the square ED to the square KD, thus the rectangle FIG to the rectangle GQM; and on interchanging so that as the rectangle FIG to the square ED, thus the rectangle HQM to the square KD, but by the preceding, the rectangle FIG is equal to the square ED. Therefore also the rectangle HQM is equal to the square KD.



Similarly we will demonstrate for another position of the pole D, the rectangle HQM is equal to the square KD. Indeed AP is a tangent to the ellipse at A, crossing the other tangent line at P, and AB cutting QM at R. In the triangle BCN [in the larger diagram], some other QO shall be drawn, parallel to NC normal to the axes, thus itself having the ratio QO to QB, as QR is to QB, therefore on interchanging there will become thus, QB to BQ as KD ad DK, thus as QO to QR. Therefore so that the rectangle HQM to the square DK, thus the square QO to the square QR, that is the rectangle HQM to the square DK; thus so that on interchanging, as the rectangle HQM to the square KD, thus the rectangle HQM to the square DK. But it has been shown above, the rectangle HQM (evidently that which is towards C) to be equal to the square KD, therefore the rectangle HKM also is towards A, is equal to the square DK. Therefore all the rectangles HKM, &c. Q.e.d.

Corollary.

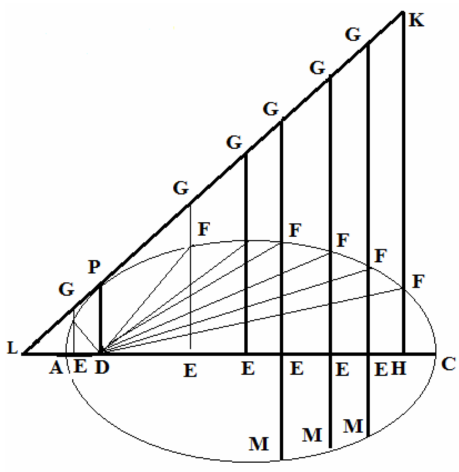
From the discussion of the demonstration now brought forth, it is allowed to deduce the squares of the tangents CN, AP, to be equal to the squares CD, DA.

PROPOSITION CXXXIX.

The ellipse shall be given the axis of which shall be AC, the poles D, H; from the pole D to the perimeter DP shall be drawn normal to the axis, and the line GPG shall be a tangent to the ellipse at P. Now some normals GFE may be drawn to the axis, and DF, DF may be joined.

I say all the lines DF, to be equal to all the lines GE.

Demonstration.



One of the right lines GE may be produced to M: from the preceding, the rectangle FGM is equal to the square DE; therefore by adding the common square EF, the squares DE, EF, are equal to the square DF, which is equal to the rectangle FGM with the square EF, that is, to the [next] square GE. Therefore since the square DF is equal to the square GE, also the right line DF is equal to the right line GE. By the same discussion all the remaining DF, are equal to the remaining GE. Q.e.d.

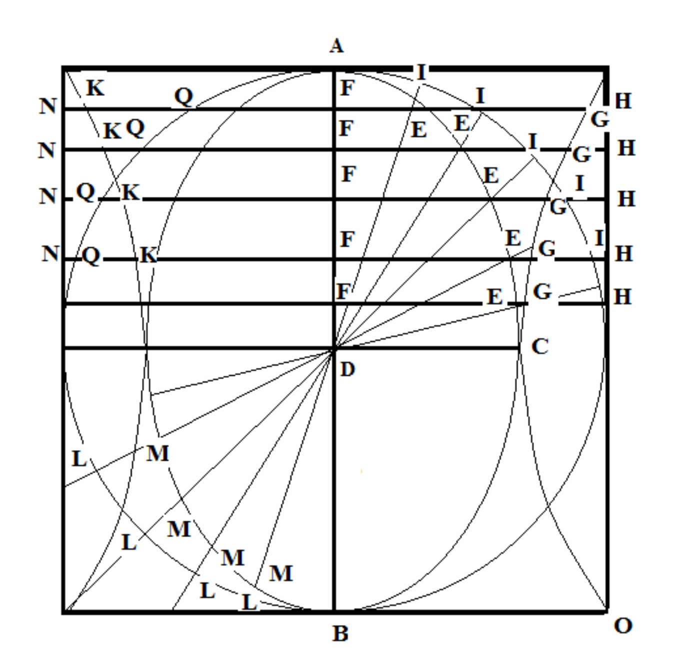
The three following theorems shall be allowed to be demonstrated in the book on the hyperbola, which shall depend on the properties of hyperbolas, yet on account of the favorable disposition of the properties of ellipses related to hyperbolas, this is not seen to be the place to propose these other properties.

PROPOSITION CXL.

With the same figure remaining, if the lines EFG normal to the axis AC may be equal to the right line DF drawn from the pole. I say the line drawn through the point G shall be a tangent to the ellipse at the point P.

Evidently the demonstration is from the preceding proposition, with the tangent at C. See the demonstration in the book on the hyperbola.

PROPOSITION CXLI.



The ellipse shall be given having the axis AB, the centre point D may be taken on the axis which in the first place shall be the centre of the ellipse, from that DC is drawn to the perimeter normal to the axis, and then some other lines DE, DE : from which the lines FEG shall be made normal to the axis.

I say the line described through the points G to be a hyperbola which shall have the same centre D as the ellipse, and shall be a tangent to the same at C.

See the demonstration in the book on the hyperbola.

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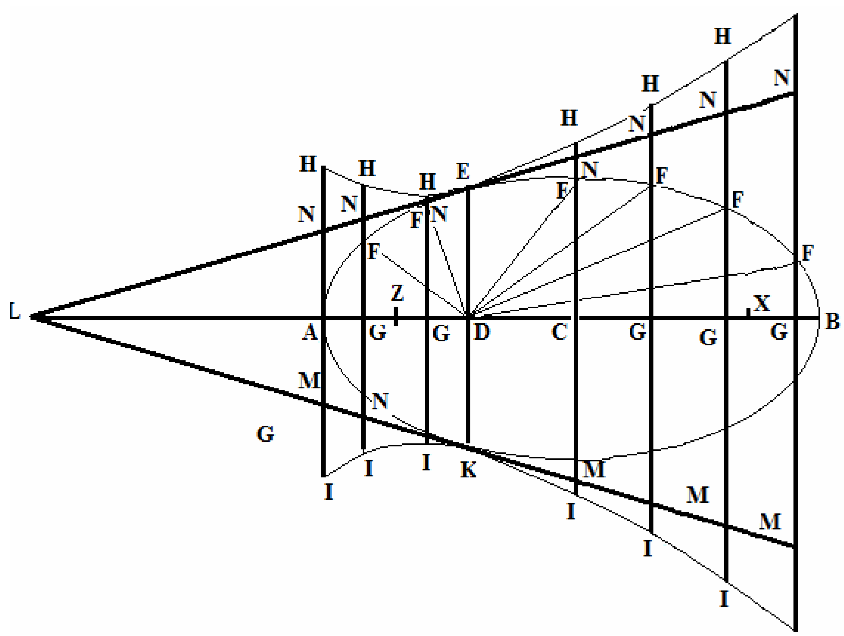
PROPOSITION CXLII.



An ellipse shall be given having the axis AB, centre C, the poles X, Z ; take some other point D on the axis between the centre C and the pole X, from which DE is drawn to the periphery normal to the axis, and then some others DF, DF are drawn; from which equal normals GFH may be drawn to the axis.

I say the line described through the points H, H to be a hyperbola, which shall be a tangent at F.

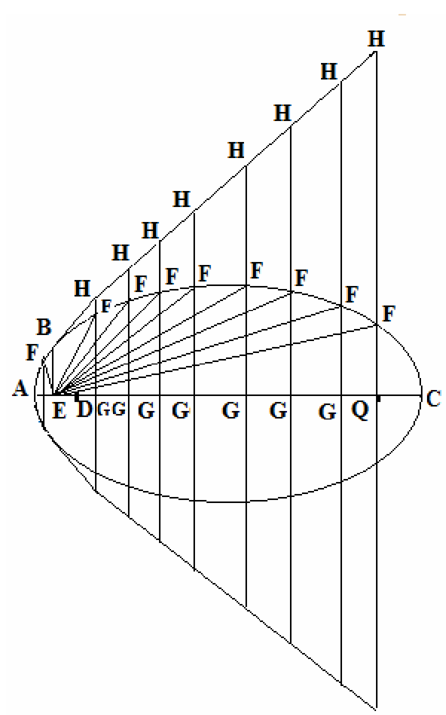
This will be demonstrated in the book concerning the hyperbola.

PROPOSITION CXLIII.

Again the ellipse shall be given, having the axis AC, the poles D, Q, the point E may be taken on the axis between the pole D and the vertex A, from which the normal EB may be drawn from the axis to the perimeter : and some other normals EF, by which GFH become normal to the axis.

I say the line described by the point H on the hyperbola may be embraced by the ellipse and shall be a tangent at the point B.

We will give the demonstration in the book on the hyperbola.

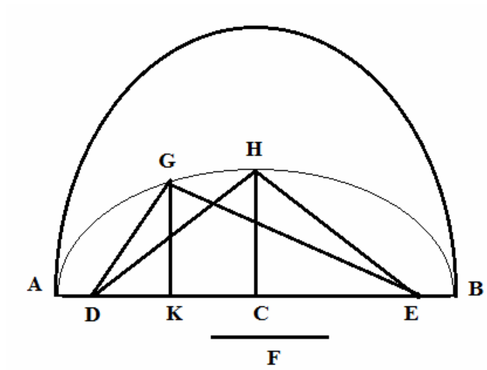


PROPOSITION CXLIV.

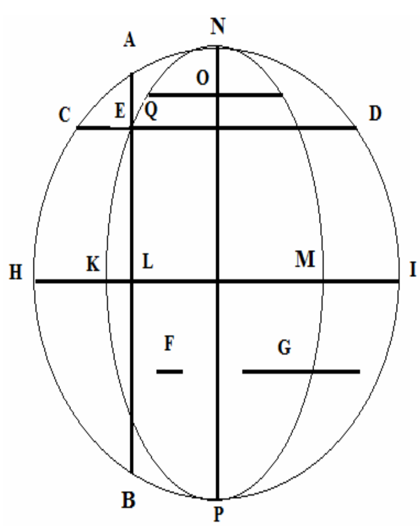
To show the triangle for a given sum of the sides, and for a given height and base.

Construction & demonstration.

AB may be put equal to the given sum of the sides of the triangle, DE shall be made equal to the base of the triangle which shall be bisected at C, thus so that at each end the equal lengths AD, BE may be left off, moreover F shall be put equal to the height. Triangle DHE may be composed from the sides AC, CB, DE (for AC, CB, taken together are greater than DE,) and therefore DHE will be isosceles. Thence it shall happen that the square HC shall be to the square F, thus as the rectangle ACB to the rectangle AKB, and KG may be erected equal to F and parallel to HC, and DG, GE may be joined. I say DGE to be the triangle sought, since the rectangle ACB is



to the rectangle AKB, as the square HC to the square F, that is to the square GK, therefore A,G,H,B will be points on the same ellipse, of which AB is the axis: and since AD shall be equal to EB itself, and likewise DH together with HE, shall be equal to AB itself, DE will be the points made from the comparison, or from the foci of the ellipse, whereby DGE are equal to the sides of the axis AB, that is the sum of the sides is from the given base DE, and from the height F that is GK. Therefore we have shown the triangle which was sought.



PROPOSITION CXLV.

The right line AB, subtended by some arc of the circle ABC, to cut another CD, the same at right angles, so that CE to ED, may maintain the ratio F to G.

Construction & demonstration.

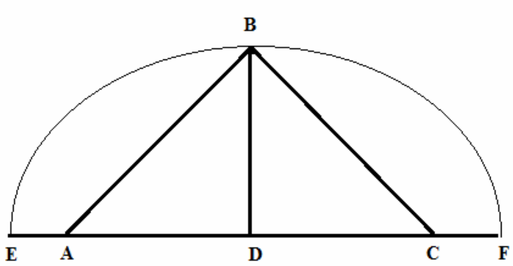
We have proposed this problem in the book concerned with the properties of the circle : but since its demonstration depends on a property of the ellipse, therefore we have delayed that demonstration to this place: truly the construction is as follows.

The diameter HI shall be drawn normal to AB cutting AB at L. And there shall become as F to G, thus HK to KI: then with IM taken equal to HK itself, it may divide the diameter HP, at the point O so that KM shall be divided at L: then the rectangle NOP may become equal to the square LE; finally the right line CED may be drawn through E. I say CED to be divided at E, following the ratio F to G. Since the lines NP, KM are bisected at right angles ; therefore the ellipse NKP may described around these described put as the axis of the ellipse, and thus OQ, LE will be put in place in order for the individual axes; and because the axes are similarly divided at O and L, the rectangle NOP is equal to the square LE, it is evident the point E to belong to the ellipse described through the points N, K, P, M. Therefore as HK is to KI, that is, as F to G, there shall be CE to ED, it is apparent the applied right line at D to be normally to the circle for CD, so that CE to ED may obtain F to G in the given ratio. Q.f.d.

PROPOSITION CXLVI.

With the right line AC and the height BD given, to describe the ellipse the poles of which shall be A and C.

Construction & demonstration.



The isosceles triangle ABC shall be established on the line AC with the height BD , then the line AC may be produced equally in each side to E and F: so that the total length EF shall be equal to twice AB, BC, since the ellipse may be described through the points E, B, F. I say that to be what is desired. Since the line EF is bisected at D and not to be bisected at A: the rectangle EAF together with the square AD, to be equal to the square ED, that is equal to the square AB by the construction ; but also the squares AD, BD are equal to the square AB; therefore with the common square AD removed, the rectangle EAF remains equal to the square BD, that is, to the fourth part of the figure. It is shown in the same way for the square BD to be equal to the rectangle FCE : whereby A and C, are the foci of the ellipse EBF described. Therefore, given the line and the height, etc. Q.e.f.

Corollary.

Hence it follows for some given isosceles triangle ABC with some angle held to the vertex, an ellipse can be described of which the foci shall be the ends of the given base of the triangle ABC. The demonstration to be apparent from the proposition.

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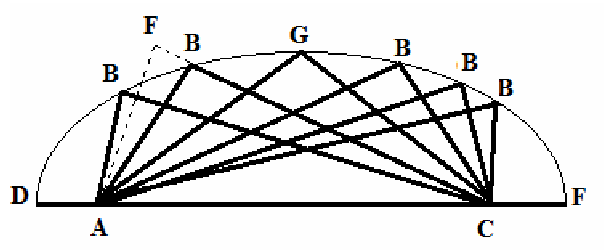
PROPOSITION CXLVII.

Some isoperimetric triangle ABC, AGC shall be described on the line AC.
I say the points G, B, G to be on the same ellipse, the poles of which shall be A and C.

Demonstration.



AC shall be produced equally on each side to D and F, so that the whole length DF shall be equal to double the lengths AB, BC, then the ellipse may be described through the points D, B, F. I say that truly it be allowed to pass through the remaining points B; it may pass either above or below B, and first above to pass through the point F, CB produced until it may cross the periphery at F, AF may be joined. Therefore since the points G and F are points on the ellipse, of which the poles are A and C, AGC, AFC shall be isoperimetric triangles ; but the triangle AGC by the construction is isoperimetric to the triangle ABC; therefore AFC, ABC are isoperimetric triangles, which cannot happen; whereby the ellipse DGF does not pass through the above point B, but it can be shown in the same way, neither may it fall below B. Therefore the ellipse passes through the points B, B; therefore the points G B B belong to the ellipse of which the poles are A, C. Q.e.d.

[There are numerous errors in labeling the last diagram especially, in the original text.]

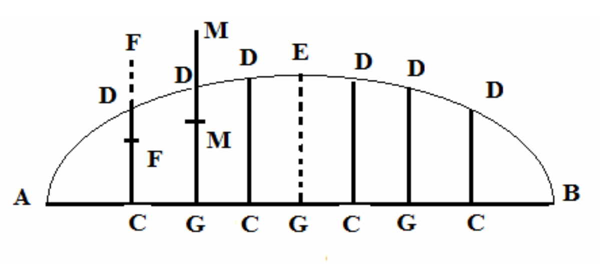
PART FIVE OF THE ELLIPSE

Showing the production of various kinds of ellipses.

PROPOSITION CXLVIII.

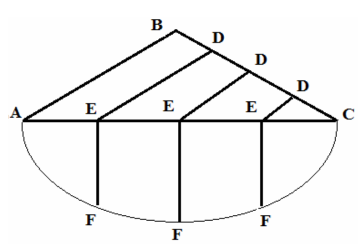
AB shall be a line divided in some manner at C, and from C some number of parallel lines CD shall be erected, so that the rectangle ACB shall become to the rectangle ACB thus as the square DC to the square DC.
I say the points ADB to be the same for the ellipse or the circle.

Demonstration.



With AB bisected at G, GE shall be erected parallel to CD, so that the rectangle ACB shall become to the rectangle AGB, thus as the square CD to the square GE, thence it is understood the ellipses to be described the conjugate diameters of which shall be AG, GE : therefore if the ellipse does not pass through the point D, it may pass through the right line CD either above or below D at F so that AG, EG shall be conjugate diameters ; the lines DC will be put parallel to the ordinate EG for the diameter AB. Whereby since the ellipse may be said to pass through the point F, the square FC shall be to the square EG as the rectangle ACB to the rectangle AGB, that is, from the construction, to be as the square DC to the square EG: Which is absurd. Therefore there is no other point F either above or below D, which lies on the ellipse, so that the point D itself lies on the ellipse. Now some other point D may be taken D, for the sake of an example the following closer point ; again if the ellipse does not pass through D, it may pass through the right line CD at M, either above or below D. Because from the hypothesis, the square DE is to the square DE, thus as the rectangle ACB to the rectangle ACB, and from the construction, so that the square DE is to the square EG, thus the rectangle ACB to the rectangle ACB, from the equality it would arise that not only the square DE to the square EG, thus the rectangle ACB to the rectangle AGB: but also the square MC to the square EG, as the rectangle ACB to the rectangle AGB since the point M shall be placed on the ellipse. Therefore the squares DC, MC have the same ratio to the square EG. Which is absurd: Therefore no point M nor any other point besides D lies on the ellipse; we will be able to show by a similar discussion the remaining points D to lie on the ellipse. From which the truth of the theorem may be agreed on.

PROPOSITION CXLIX.



ABC shall be some triangle, and with the side BC divided in some manner at the points DD: from the points D the right lines DE may be drawn parallel to AB, and from A the lines EF may be dropped so that the squares EF shall be equal to the rectangles BDE.

I say the points A, F, C to lie on the same ellipse.

Demonstration.

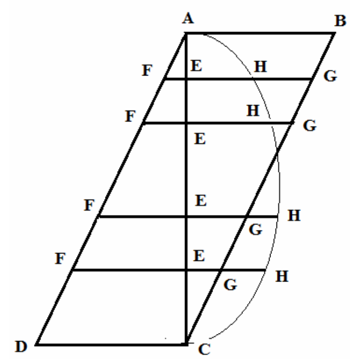
As the rectangle BDE to the rectangle BDE, thus the rectangle BDC is to the rectangle BDC; that is, the rectangle AEC to the rectangle AEC: but as the rectangle BDE is to the rectangle BDE, thus the square EF is to the square EF; therefore as the rectangle AEC is to the rectangle AEC, thus the square EF is to the square EF. Therefore the points AFC lie on an ellipse.

PROPOSITION CL

AC shall be the diameter of the parallelogram AB, CD; lines of some size FG may be drawn parallel to the side AB, cutting the line AC at E : then the [geometric] means EH between FE, EG may be put in place.

I say the points A, C, and all the points H to lie on the same ellipse, unless they shall lie on a circle.

Demonstration.



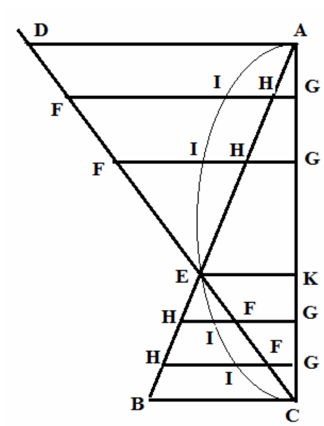
The ratio of the rectangle FEG to the rectangle FEG is composed from the ratio FE to FE, that is AE to AE; and from the ratio EG to EG, that is EC to EC: but from the same the ratio is composed of the rectangle AEC to the rectangle AEC, therefore as the rectangle FEG to the rectangle FEG; that is as the square EH to the square EH, thus the rectangle AEC to the rectangle AEC. Whereby the points AH, HC lie on an ellipse. Q.e.d.

Corollary.

If the right line AB shall be normal to AC, and were equal to that, the points AHH lie on the same circle; for the rectangle AEC will be equal to the rectangle FEG, that is for the square EH; and thus the points HH lie on a circle.

PROPOSITION CLI.

Any two lines AB, CD shall cut each other at E which shall connect the two parallel lines AD, BC; likewise the points AC may be joined together : then the right lines FG may be drawn parallel to the lines AD, BC, meeting the line AB at HH, and the rectangles HGF may be made equal to the squares GI.
I say the points I, I, I to lie on an ellipse.



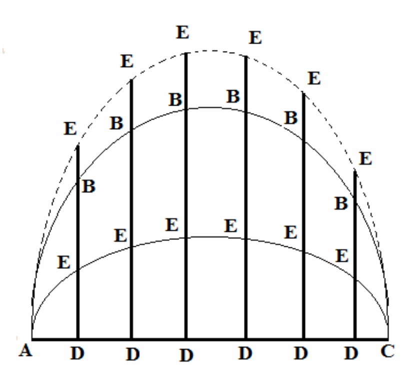
Demonstration.

The ratio of the rectangle HGF to the rectangle HGF is composed from the ratio HG ad HG, that is AG to AG, and from the ratio FG to FG, that is GC to GC : but the ratio of the rectangles AGC to AGC is composed from the same, therefore as the rectangle HGF is to the rectangle HGF, that is the square IG to the square IG, thus the rectangle AGC to the rectangle AGC: whereby the points I, I lie on an ellipse. Q.e.d.

Corollary.

The right line EK may be drawn parallel to the line AD from the point of intersection E : if AK, EK, KC will have been continued, and the line EK normal to the right line AC, I say the points I, I, to lie on a circle where indeed it shall be so that as the rectangle AKC to the rectangle AGC, thus the square EK to the square IG: (that indeed we have proved by the same discussion, where we have shown the rectangle AGK to be to the rectangle AGK as the square GI to the square GI); on interchanging will become: as the rectangle AKC to the square EK, thus the rectangle AGC to the square GI. And thus the square IG is equal to the rectangle AGC. Therefore I, I lie on a circle.

[It is of course a trivial exercise to prove this theorem for two points on the standard ellipse using coordinate geometry.]



PROPOSITION CLII.

AC shall be the diameter of the semicircle ABC, divided in some manner at D D, and from D the normals DE may be erected, and they shall be produced so that as BD to BD, there shall be ED to ED. I say the points E, E to lie on the same ellipse.

Demonstration.

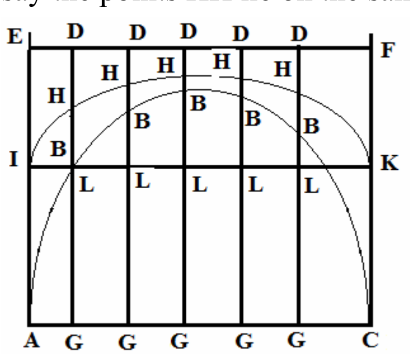
As the square DB is to the square DB, thus as the square ED is to the square ED: but as the square BD is to the square BD thus the rectangle ADC is to the rectangle ADC: therefore as the square ED is to the square ED, thus the rectangle ADC is to the rectangle ADC.

Whereby the points E, E lie on an ellipse. Q.e.d.

PROPOSITIO CLIII.

A rectangle AF shall be described on the semicircle ABC with the diameter AC : and the lines DG may drawn parallel to the side AE which cross the circle at BB, some line IK shall be drawn parallel to the right line ED crossing the lines DG at LL; and there shall become, as AI to IE thus BH to HD.

I say the points HH lie on the same ellipse.



Demonstration.

As AE to AI, that is GD to LD, thus BD is to DH, therefore by interchanging and by dividing, and by interchanging again so that as GB to LH, thus BD to

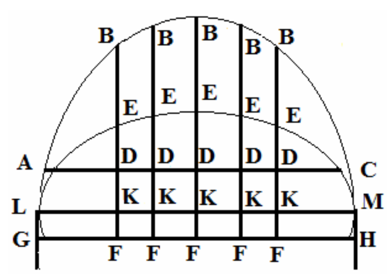
DH, and thus BD is to HD, as HD to HD, for both the ratios BD to HD, are of the same ratio AE to IE; whereby as GB to LH, thus GB to LH: and by interchanging as GB to GB, thus LH to LH: and as the square GB to the square GB, thus the square LH to the square LH: but as the square GB to the square GB thus the rectangle AGC to the rectangle AGC, that is, the rectangle IIK to the rectangle IIK; therefore as the square LH is to the square LH, thus the rectangle IIK is to the rectangle IIK : whereby the points H, H lie on an ellipse. Q.e.d.

PROPOSITION CLIV.

ABC shall be a segment of some circle, subtended below from which from AC, divided in some manner at DD, the normals DB may be erected from D, and it shall become so that as BD to BD, thus ED to ED.

I say the points E, E to lie on an ellipse.

Demonstration.

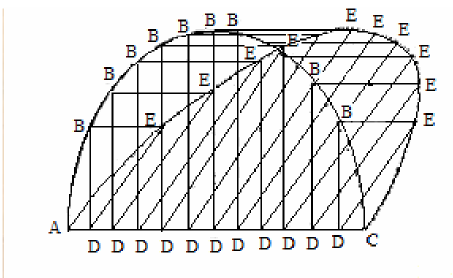


With the semicircle ABC completed : the diameter GH of the circle GBH way be drawn parallel to the line AC, which lines BD produced may cut GH at FF: and it shall be made so that as BD to DF, thus ED ad DK: then from G and H the right lines GL, HM may be erected parallel to the lines BF cutting the line KK at L and M. Since as BD is to DF, thus DE is to DK, on interchanging there will become, as BD to ED, thus DF to DK. And thus BD is to DE, as BD to DE. Therefore as DF to DK, thus DF to DK : whereby the points K, K lie on the same line, and indeed are parallel to the line GH. Again since BD shall be to DK, therefore as ED to DK, on adding together and on interchanging, BF shall be to EK, as DF to DK. Therefore as BF to EK, and again as interchanging, as BF to BF, thus EK to EK, and as the square BF to the square BF, the square EK shall be to the square EK : but as the square BF shall be to the square BF, the rectangle HFG is to the rectangle HFG, that is the rectangle MKL to the rectangle MKL, therefore as the rectangle MKL shall be to the rectangle MKL, the square EK shall be to the square EK. Whereby the points E, E lie on an ellipse. Q.e.d.

PROPOSITIO CLV.

Let AD be the diameter of the semicircle ABC, which however many normals may cut BC at D, then with the right lines BD may become equal to BE parallel to the diameter AC.
I say the points E, E to belong to the ellipse, the diameter of which AC.

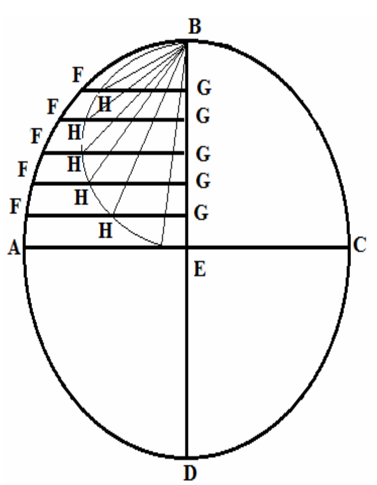
Demonstration.



The right lines DE shall be drawn: because the angles BDC are right, and the lines BE are parallel, the angles DBE are right also; therefore the squares DE are equal to the squares BD, BE, that is, since BD, BE are equal, they are twice the squares BD. Therefore as the square BD to the square BD, that is, as the rectangle ADC to the rectangle ADC, thus the square DE to the square DE. Truly the right lines DE are parallel to each other : since indeed the angles DBE shall be right; and the sides BD, BE equal ; BDE will be semi-right angles. Therefore since also BDC shall be right, the remaining angles EDC are semi-right, and thus equal: hence the lines DE are parallel. Therefore the points E, E lie on an ellipse. Moreover since AC shall be the diameter, it will be readily apparent by the same construction with a complete circle a complete ellipse for the other part as well shall be produced, then indeed all the parallel lines DE will be bisected by the right line AC.

PROPOSITION CLVI.

The diameters AC, BD shall cut the circle ABC at right angles, and with the right lines FG which shall be parallel to the diameter AC, the right lines may be dropped from B equal to the right lines BH cutting the lines FG in HH.
I say the points B, H, E to lie on the same ellipse.



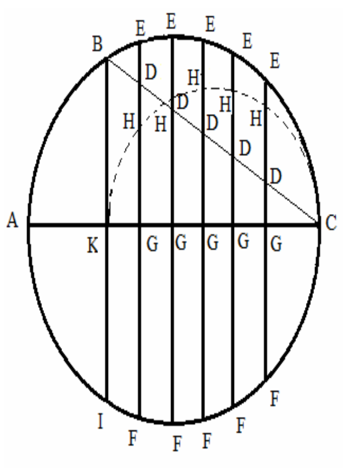
Demonstration.

Since the square FG is equal to the rectangle BGD, that [in turn] is equal to the rectangle BGE taken twice, together with the square BG; but the square HB is equal to the squares HG, BG: therefore with the common square BG taken away the square HG remains, equal to the rectangle BGE taken twice; similarly the remaining squares HG are equal to double of the remaining rectangles BGE; therefore so that as the square HG shall be to the square HG: the rectangle BGE is to the rectangle BGE; whereby the points B,E, and all the points H, lie on the same ellipse. Q.e.d.

PROPOSITIO CLVII.

Let AC be the diameter of the circle ABC and from C with some lines CB drawn crossing the perimeter of the circle at B, then with the right line BI dropped from B which will cut the diameter AC at right angles at K. Some number of lines EF may be drawn parallel to the line BI crossing the diameter AC at G, and the line BC in D: and the rectangles EDF shall be made equal to the squares GH.

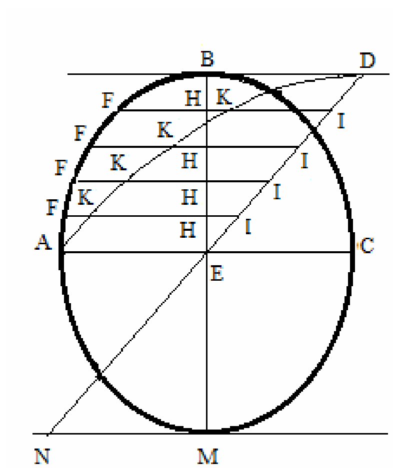
I say the points KHC to belong to the same ellipse.



Demonstration.

As the rectangle EDF is to the rectangle EDF, thus the rectangle BDC is to the rectangle BDC, that is, the rectangle KGC to the rectangle KGC: but (just as becomes apparent from the hypothesis in turn), as the rectangle EDF to the rectangle EDF, thus the square HG is to the square HG: therefore as the rectangle KGC is to the rectangle KGC, thus the square HG to the square HG; whereby the points KHC pertain to an ellipse. Q.e.d.

PROPOSITION CLVIII.



The diameters AC, BE shall cut the circle ABC orthogonally, and with the tangent BD acting at the point B some right line ED shall be drawn through the centre E meeting the tangent BD at some point D. Then the right lines FHI may be drawn, parallel to the tangent BD, meeting the diameter EB at HH, and the line ED at I, I : and there becomes FH to FH, thus as IK to IK.

I say the points AKD to be for the same ellipse.

Demonstration.

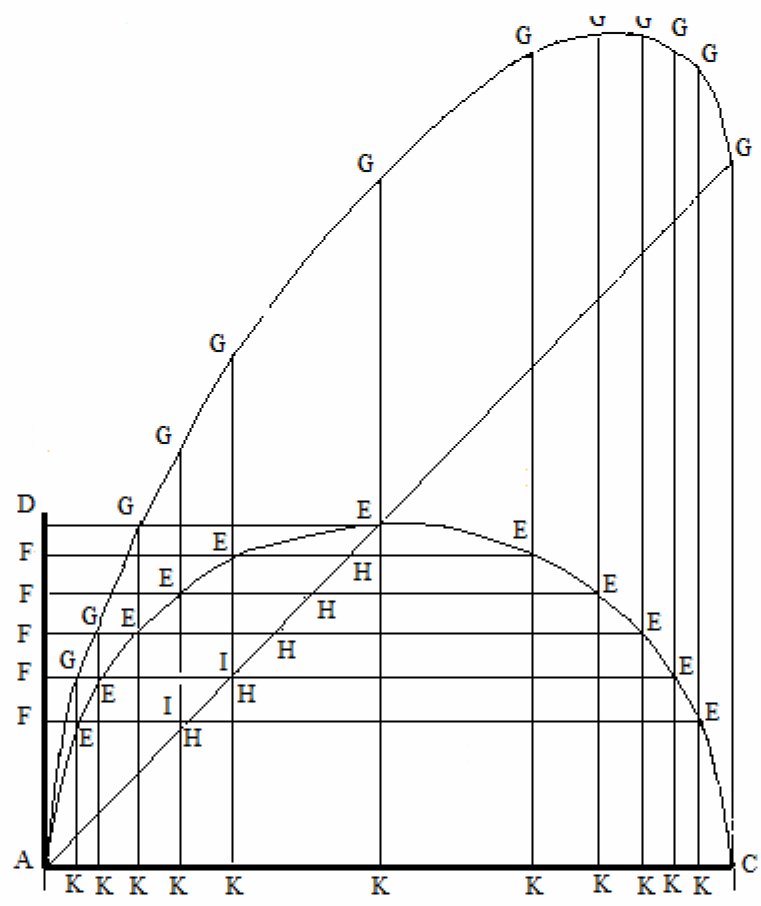
The diameter BE shall be produced to M; and the line DE shall be produced until it may meet the tangent acting through M at N. Since BD, NM, HI shall be parallel lines, so that the rectangle BHM shall be to the rectangle BHM, thus as the rectangle DIN shall be to the rectangle DIN ; but as the rectangle BHM shall be to the rectangle BHM, thus the square FH shall be to the square FH, that is the square IK to the square IK: therefore as the rectangle DIN shall be to the rectangle DIN, thus the square IK shall be to the square IK. Whereby the points AKD are for an ellipse. Q.e.d.

PROPOSITION CLIX.

The two lines AD, BD intersecting each other at right angles at D, shall be tangents to the circle ABC, the diameter of which is AC: and with the points AB joined, and with the tangent CL acting through C: and with the points AB joined, the tangent CL shall be acting through C, crossing the line AB at L; then some lines FE may be drawn parallel to the line DB crossing the line AB at HH, and the circle at EE; then the right lines GK shall be drawn through E parallel to the line AD crossing the diameter AC at K and the line AL at I I. And all the lines FE equal to EG.

I say the points AGL to be for an ellipse.

Demonstration.



As AD is to DB, thus AF is to FH: but AD, DB are equal lines, and therefore AF, FH shall be equal lines ; and whereby EK, FH shall be equal lines. Again, since AF shall be to FH, thus as EI to EH, the lines EI, EH are equal to each other: moreover from the construction the lines FE are equal to the lines EG: therefore the whole length IG, is equal to the whole length FH, that is FA is equal to EK; whereby as the square EK to the square EK, thus the square IG is to the square IG: but as the square EK is to the square EK, thus the rectangle ACK is to the rectangle ACK, that is the rectangle AIL to the rectangle AIL; therefore as the square IG is to the square IG, thus the rectangle AIL is to the rectangle AIL. Whereby the points AGL are for an ellipse. Q.e.d.

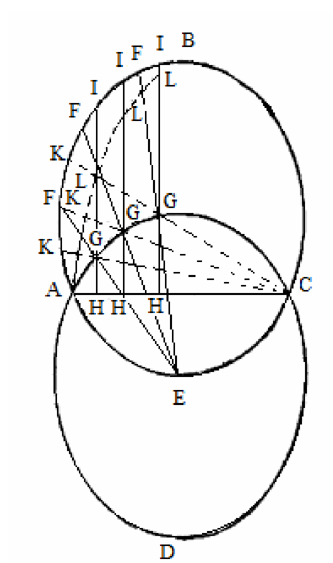
So that if the same construction may be continued to the other part, this part of the ellipse also will have been completed, which will fall within the circle. Whereby it is noteworthy to be observed from the hypothesis, that the circle and the ellipse themselves in turn may be allowed to be cut by the same lines GK, still the same right line DA to be a tangent at the same mutual point A for the circle and the same also shall be the tangent for the ellipse, thence it becomes evident that all the points of the perimeter of the ellipse shall be on the lines GK which are drawn parallel to DA between the points C and A.

PROPOSITION CLX.

The two circles ABC, ADC shall cut each other so that the former ABC shall pass through the centre E of the latter ABD, and with these joined together by the points AC, some line EF shall be drawn from the point E meeting the circle ABC at the points F and with the circle ADC at the points G: then through G the right lines HI are acting normal to the line AC, meeting the line AC at H H and the circle ABC in I I: and the right lines GF shall be made equal to the lines GL.

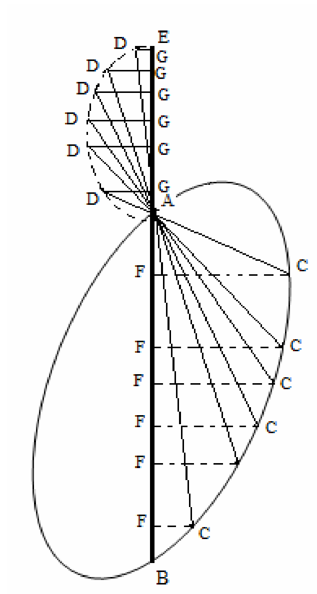
I say the points ALL to lie on an ellipse.

Demonstration.



The lines CGK shall be drawn from C through G, so that as the rectangle KGC shall be to the rectangle KGC, thus the rectangle FGE shall be to the rectangle FGE; but as the rectangle KGC shall be to the rectangle KGC, the line GH shall be to the line GH (§84 Cor., Circle Book III), and as the rectangle FGE to the rectangle FGE, thus the line FG to the line FG, therefore so that as GH to GH, thus FG to FG, that is, LG to LG, and on adding together and interchanging as LH to LH, thus GH to GH. Whereby the points ALL
or an ellipse.

PROPOSITION CLXI.



AB shall be a diameter of some ellipse ABC, and with the line CD acting through the point A, which crosses over the ellipse at CC, there shall become as AC to AC, thus AD to AD, and as AC to AD, thus AB to AE.

I say the points A, D, E to lie on the same ellipse.

Demonstration.

Since DG, FG are parallel, and hence the triangles FCA, DGA are similar, so that the individual lines CA will be to the individual lines AD one to one in turn, thus so that the individual FC will be to the individual GD. And thus the individual CA to the individual AD are as BA to AE. Therefore the individual FC are to the individual DG, as BA to AE. Whereby since the ratio has one particular FC to one particular DG, the remaining individual FG have the same ratio to the individual remaining DG. Therefore on interchanging so that as the FC are to the FC, thus the GD are to the GD, and thus as the squares FC are to the squares FC, thus the squares GD are to the squares GD; similarly we will demonstrate, as the AF are to the AF, there shall be AG to AG. From which as the remaining FB are to the remaining FB, thus the remaining GE are to the remaining GE. Whereby since the rectangles AFB shall have the same ratio between themselves, in turn to be composed from the ratios AF to AF, and FB to FB, which are shown to be in the same ratios AG to AG, and GE to GE, from which the ratio of the rectangles AGE is composed ; so that the rectangles AFB to the rectangle AFB, shall be as the rectangles AGE to the rectangles AGE. And the rectangles AFB are to the rectangles AFB, as the square FC to the square FC; this has been shown above, to be as the square GD to the square GD; therefore the rectangles AGE are to the rectangles AGE, as the squares GD to the squares GD. Therefore the points D A, A D E belong to an ellipse. Q.E.D.

PROPOSITION CLXII.

The diameter of some ellipse ABC shall be AB, divided in some manner at D and from D the right lines DC, DF shall be drawn to the periphery, and which lines shall be divided in some manner at E and G: then AD shall be divided at L, and DB at M, so that DC, DF are divided at E and G.

I say the points L E G M to belong to an ellipse.

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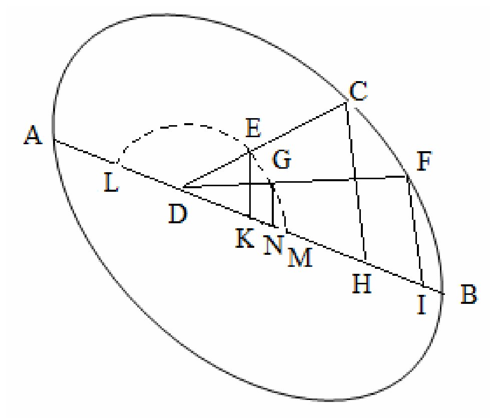
Demonstration.

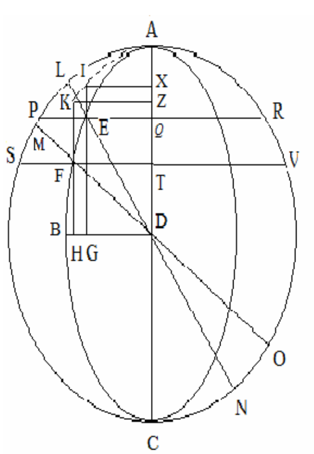
The ordinate lines CH, FI may be drawn from C and F to the diameter AB, from which, from E and G the parallel lines EK, GN may be drawn so that as DC to DE, thus DF to GD, therefore so that as CH to EK, thus FI to GN, and EK to GN, as CH to FI: and therefore the square EK to the square GN, as the square CH to the square FI. Then, since DI is to DN, as DF to DG, and DA to DL, as DF to DG, as there will become DI to DN, there shall be DA to DL, therefore as the preceding one DI to the following one DN, (i.e. , as DF to DG, i.e. as DC to DE, i.e. as DH to DK) thus both the antecedents, i.e. the whole AI to both the following, i.e. to the whole LN, similarly we may infer AH to be to IK, as DH to DK. From which AI is to LN, as AH to DK, and on interchanging AI is to AH, as LN to IK. Besides, since as DF is to DG (i.e. as the whole DB is to the whole DM) there will be with DH removed to DN removed, the remainder IB to the remainder NM, as the whole DB to the whole DM. Similarly we may deduce HB to KM, to be as DB to DM. Therefore on interchanging IB to NM, and HB as to KM, and by inverting, HB to IB to be as KM ad NM. Therefore since we shall have shown the ratios AH to AI, and AK to AN, likewise the ratios HB to IB, and KM to NM to be the same, also the ratios of the rectangle AHB to the rectangle, and of the rectangle AIB to the rectangle AKB, composed from these equal ratios, will be the same; but the rectangle AHB is to the rectangle AIB, as the square CH to the square FI; that is by the above demonstration as the square EK to the square GN. Therefore the rectangle IKM is to the rectangle LNM, as the square EK to the square GN. Therefore the points L, E, G, M lie on an ellipse. Q.e.d.



PROPOSITION CLXIII.

The axes of the ellipse ABC shall be AC, BD: and with several radii DE, DF drawn from D, the lines IG, KH may act through E and F equal to ED, FD themselves, truly parallel to the axis AC, crossing the axis BD at G and H.

I say the points AIK to be for the ellipse of this axis.



Demonstration.

The circle ALC shall be described on AC as diameter, and the lines DE, DF may be produced in each way until the cross the perimeter of the circle at L, M, N, O: and with the lines PER, SFV acting through E and F, which cross the circle at R, V and cross the diameter AC at Q and T, and will be parallel to the axis BD, the ordinate lines IX, KZ may be drawn for the axis AC. Since the lines PER, SFV are divided in proportion at E

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and F, the ratio of the rectangle PER to the rectangle SFV is twice the ratio of PE to SF, and thus the rectangle PER to the rectangle SFV shall be as the square PE to the square SF, that is, as the square EQ to the square FT, that is as the square IX to the square KZ. Whereby since the rectangles LEN, PFO shall be equal to the rectangles PER, SFV, also the rectangle LEN is to the rectangle MFO, as the square IX to the square KZ: then since IG, that is, XD is equal to ED, and DC is equal to DN, XC will be equal to EN; and truly the whole AC is equal to the whole LN. Therefore the remainder AX is equal to the remainder LE : and thus the rectangle AXC equal to the rectangle LEN: it may be shown in the same manner that the rectangle AZC [§148] to be equal to the rectangle MFO, therefore there will be as the rectangle AXC to the rectangle AZC, thus the square IX to the square KZ. Whereby AIKC shall be points on an ellipse. Q.e.d.

THE ELLIPSE

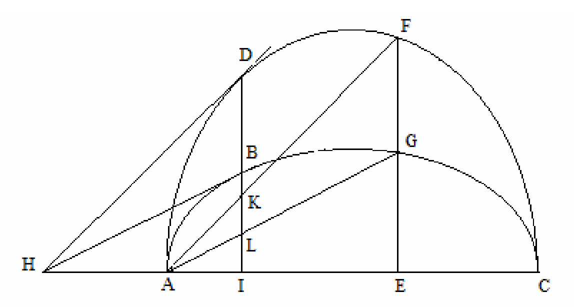
PART SIX

The circle to be compared with the ellipse.

PROPOSITION CLXVIII.

The ellipse ABC and the circle ADC shall have the common axis AC, and with the ordinate EF drawn, it shall cross the circle at F and the ellipse at G: AF and AG may be joined: then with DH drawn parallel to AF, which shall be a tangent to the circle at D; and it shall cross the axis at H; from D the ordinate of the line DI may be dropped to the diameter cutting the ellipse at B and AF, AG at K & L: HB may be joined.
I say the line AG to be parallel to the line HB.

Demonstration.



As EG is to EF, thus IL is to IK, but as EG to EF, thus IB is to ID, [Schol. Prop. 4] therefore as IL to IK, thus IB to ID, and on interchanging, so that as IK to ID, thus IL to IB, but as IK to ID, thus IA to IH (because AF, HD are parallel by the hypothesis), therefore as IL to IB, thus IA to IH. Whereby the lines AG, HB parallel. Q.e.d.

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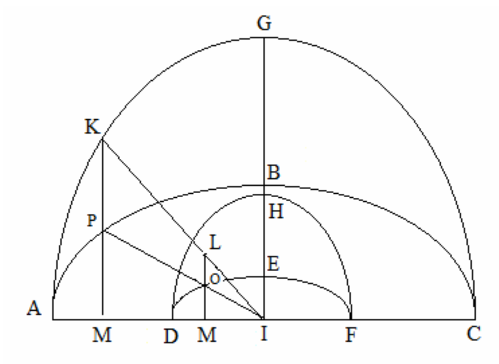
PROPOSITION CLXIX.

ABC, DEF shall be similar ellipses, and likewise with the same centre I put in place : and the circles AGC, DHF shall be described on the diameters AC, DF: moreover from the centre a certain line IK may be put in place crossing the circles at the points L & K, from which the normals dropped LM, KN, shall cut the ellipses at O & P: and the lines IO, OP shall be drawn.
I say that as IL shall be to LK, thus IO shall be to OP.

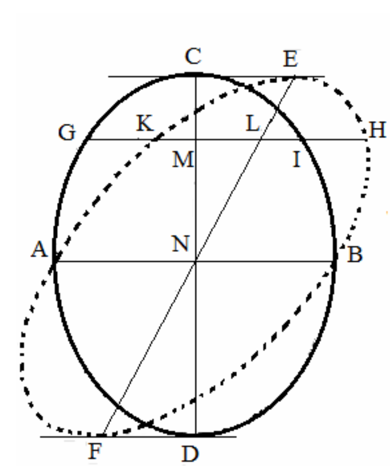
Demonstration.



The normal IG shall be erected from the centre I crossing the ellipses at E, B, truly the circles at H & G. Since both the circles AGC, DBF as well as the ellipses ABC, DEF are similar according to the same centre put in place, so that IG to IB, thus as IH is to IE, but as GI to BI, there shall be KN to PN; and as HI to EI, thus LM to OM: therefore as KN to PN, thus LM is to OM, and on interchanging so that as KN to LM, this becomes as IN to IM, thus PN to OM, I, O, P are collinear. Whereby since KN, LM shall be perpendicular to AC, and hence parallel to each other, so that as IL to LK, thus there shall be IO to OP. Q.e.d.

PROPOSITION CLXX.

The two diameters AB, CD shall intersect the circle ABC, crossing each other at right angles, and the tangent lines acting through C & D also shall be tangents to some ellipse AEB, the diameter of which shall be AB; then some line GH may be drawn parallel to CE, crossing the circle at G & I, truly the ellipse at K & H, & the line CD at M.
I say the lines GI, HK to be equal.
Demonstration.



EN may be put in place from the point of contact E to the centre N, the diameter for which line produced will occur at the point of contact F as shown elsewhere in this book [§15], the ordinates put in place will be KH, AB, from which the rectangle ELF is to the rectangle ENF, as the square LH to the square NB; but the rectangle ELF is to the rectangle ENF, as the rectangle CMD to the rectangle CND, (since indeed from the hypothesis CE shall be parallel to DF & KH, the ratios of these rectangles shall be composed from the same ratios) ; and the rectangle CMD is to the rectangle CND, as the square

MI, to the square NB, and therefore the square MI is to the square NB, as the square IH to the square NB; therefore the squares MI, LH are equal, and thus the right lines MI, LH and the doubles of these GI, KH are equal. Q.e.d.

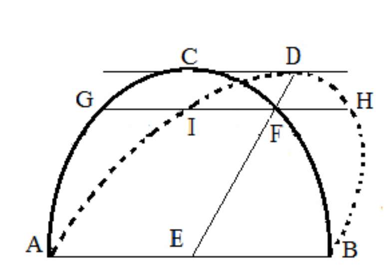
PROPOSITION CLXXI.

The right line CD shall be a tangent to the semicircle ABC of which the diameter is AB and the centre E , parallel to AB, and an ellipse shall be described through the points A & B, which shall have the diameter AB, and shall have the tangent line CD at some point D; truly DE may be drawn from D crossing the periphery of the circle at F, and the parallel line GH shall be acting through F, intersecting the ellipse at H & I, and indeed the circle at G & F.

I say the line GH to be trisected by the points I & F.

Demonstration.

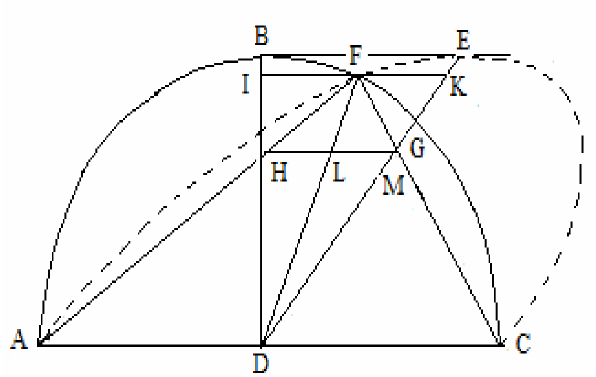
Since from the preceding, HI is equal to GF, with the common IF removed, FH remains equal to GI; but FH itself is equal to IF (since HI is placed as the ordinate line to the diameter DE) : therefore the lines GI; IF; FH are equal. Q.e.d.



PROPOSITION CLXXII.

The line BE shall be a tangent to the circle ABC, of which the diameters AC, BD shall be crossed at right angles at D, then with the ellipse AEC drawn through the points A & C, BE being a tangent at E, the ellipse shall cross over the circle at F, and the right line ED shall be drawn from the point of contact to the centre D, and the right lines AF, CF shall be drawn : & indeed AF cuts the diameter BD of the circle at H, truly the diameter CF of the ellipse ED at G.

I say the line joined GH to be parallel to BE.



Demonstration.

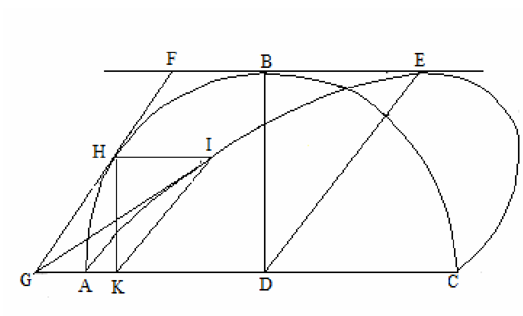
The line IK shall be put in place through F, parallel to the line EB, and with the points FD joined, from H the right line HG may be drawn, parallel to IK, crossing the lines FD in L & ED in G. Since IK is parallel to the tangent

EB, the lines IF, FK are equal to each other [§ 170]; and whereby the line HG parallel to the line IK, also is bisected at L; now if the point G may not be common to the lines FC, ED, HG, it may meet the line HG itself at M: Therefore since HM shall be parallel to EB and thus to AC, as AD to DE, thus as HL to LM, whereby MH is bisected at L: however as HG is bisected at L ; therefore the points G & M, are one and the same: from which G is common to the lines FC, ED; therefore HG, BE to be parallel. Q.e.d.

PROPOSITION CLXXIII.

The right line BE shall be a tangent to the circle ABC at B, the diameters of which AC, BD cross each other at right angles, and with the right line ED drawn from E, the ellipse AEC is described, with the tangent line FE at E, also at E some tangential line FG may be drawn to the circle at H, crossing the diameter AC at G: then HI may be drawn from H, parallel to BE crossing the ellipse at I, and IG may be drawn.
I say that that the line IG to be a tangent to the ellipse at I.

Demonstration.

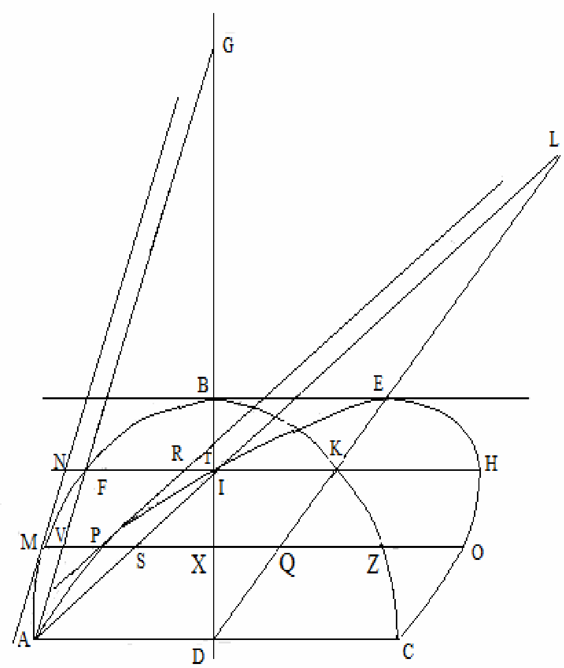


IK shall be put parallel to ED, there with the ordinate put in place for the diameter AC, since AC, DE shall be conjugate diameters: indeed from K the line KH is drawn, parallel to BD, crossing the line HI at H. Since both HK, BD, as well as IK, ED are parallel, so that there: the square ED will be to the square IK, thus as the square BD to the square HK: but as the square ED to the square IK, thus the rectangle ADC to the rectangle AKC, therefore as the rectangle ADC to the rectangle AKC, thus the square BD to the square HK. Whereby the point H lies on the periphery of the circle : therefore since HK shall be normal and HG the tangent, as GC to CK, thus CK to AK: but IK is the ordinate put in place for the diameter AC ; therefore GI is the tangent line to the ellipse [§30]. Q.e.d.

PROPOSITION CLXXIV.

The line BE shall be a tangent at B to the circle ABC, the diameters of which cross each other at right angles at D : then an ellipse may be described through the points A & C touching the line BE at E, of which one of the diameters shall be AC, and with ED joined, cutting AF drawn from A at G, and FH is drawn acting through F, parallel to BE, crossing the ellipse at I. Then the right line AI is drawn through A & I, but the right line MN shall be a tangent to the circle at M, & from M the line MO is drawn, parallel to the tangent BE, crossing the ellipse at P; and the right line BE may be drawn through P, parallel to the right line PR, cutting the line FH at R.
I say the line PR to be a tangent to the ellipse at P.
[Reductio ad absurdum proof to follow.]

Demonstration.



The lines cut by the circle AF to DB, and by the ellipse AI to DE, may meet at G & L. Then FH shall meet BG itself at T, and the tangent at N, & the right line DE at K. Similarly MO shall cross AF at V, AI at S, DE at Q, & GD at X. Since the triangles AGD, ALD, have the same base AD, and FT, KI shall be equal, the triangles AGD, ALD lie between the same parallel lines. Again since the line MO shall be parallel to FK, VX & SQ shall be equal lines; and indeed again, MX shall be equal to PQ; and therefore the remainder MV, shall be equal to the remainder PS: but the right line MV is equal to the line NF, & PS is equal to RI, and therefore NF, RI shall be lines equal to each other : and indeed FT shall be equal to KI, & therefore NT, KR shall be equal.

But, since the tangent MN falls completely outside the circle, NT therefore is greater than FT, that is, greater than KI; and therefore KR is greater than KI; therefore the point R falls outside the ellipse; in the same manner if some parallel lines may be drawn both above as well as below MO, all the point of the right line PR will be shown to fall outside the ellipse as well as the point P; therefore the right line PR is a tangent to the ellipse. Q.e.d.

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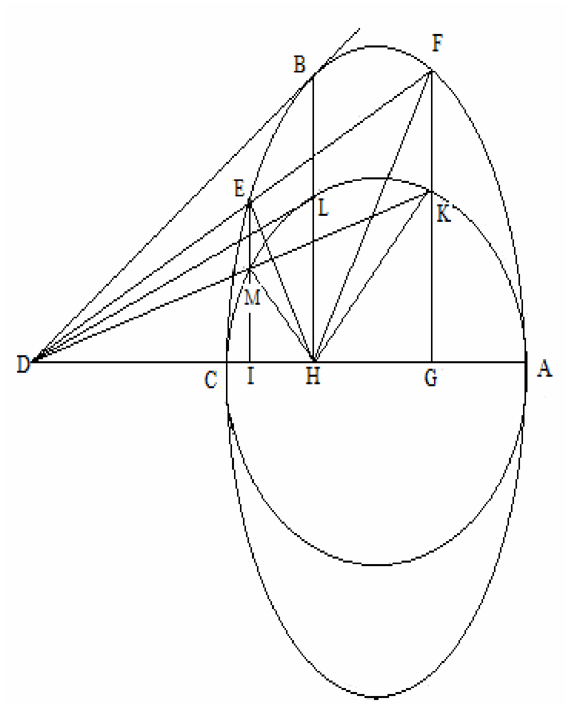
PROPOSITION CLXXV.

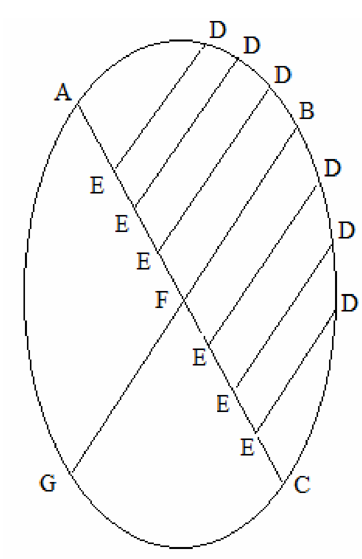


Let AC be the axis of some ellipse ABC, produced to D in some manner; and with the line DB drawn from D, which shall be a tangent to the ellipse at B; and another line DF shall be put in place, crossing the ellipse at E & F: then with the normals FG, BH, El dropped from the points F, B, E on the arc EF to the axis AC; FH, EH may be joined. I say the triangles FGH, EIH to be similar.

Demonstration.

The circle AKC shall be described on the diameter AC, crossing over the right lines FG, BH, EI at K, L, & M, moreover the lines MH, KH, EH, FH may be drawn : therefore since FG shall be to EI, (that is GD to ID), thus KG to MI, it is evident MK produced meets the other lines at D : moreover DL is the tangent, therefore HKG, HMI are similar triangles. Whereby as KG to MI thus HG to HI: but as KG to MI, thus FG to EI, therefore as FG to EI, thus HG is to HI ; but the angles contained by the proportional sides to be right, therefore the triangles FGH, EIH are similar. Q.e.d.

PROPOSITION CLXXVI.



The diameter AC shall be equal to one of conjugates in the ellipse ABC : to which some number of the ordinates DE may be put in place.

I say the rectangles AEC to be equal to the squares DE.

Demonstration.

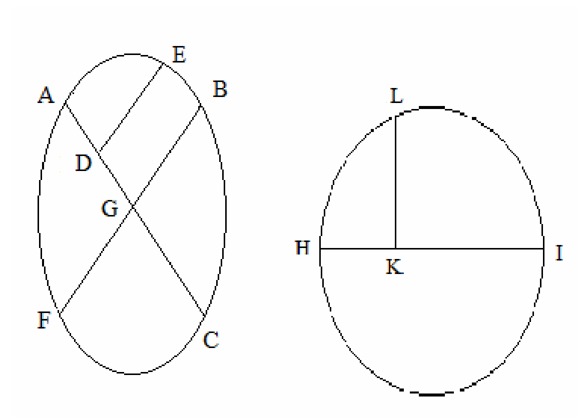
One of the equal conjugate diameters BG shall be put in place: moreover, the centre of the section [i.e. of the cone] shall be F: therefore a rectangle AEC shall be to a square ED, as the rectangle AFC to the square FB: but the rectangle AFC, that is equal to the square AF, is equal to the square FB, since the diameters shall be equal, therefore the rectangle AEC is equal to the square DE: Q.e.d.

PROPOSITION CLXXVII.

One of the conjugate diameters AC may cut the ellipse ABC, so that the ordinate line ED may cut the conjugate diameter AC at D, and on assuming the line HI which shall be equal to AC, and with the circle HLI described on HI, HI shall be divided at K, as AC is divided at D, and KL is erected normal at K.

I say the squares ED, KL to be equal to each other.

Demonstration.



The other of the equal conjugates FB may be drawn: moreover the centre of the section shall be G, the rectangle ADC is to the square ED, as this rectangle AGC is to the square AG : as the squares AG, GB are equal. Therefore the rectangle ADC is equal to the square ED: again since the lines HI, AC shall be put equal and divided proportionally at D and K , the rectangle ADC that is equal to the square ED, is equal to the rectangle HKI, that in turn is equal to the square LK. Q.e.d.

PROPOSITION CLXXVIII.

One of the equal conjugate diameters AB may cut the ellipse ABC, to which the ordinate CD may be put in place, and AC, AD may be joined ; since in the above proposition EF shall be equal to the right line AB, and the circle EFK described to be itself divided proportionally: and with the normal IK acting through H, EI, EK may be joined.

I say the squares AC, AD taken together, to be equal to the squares IE, EK taken together.

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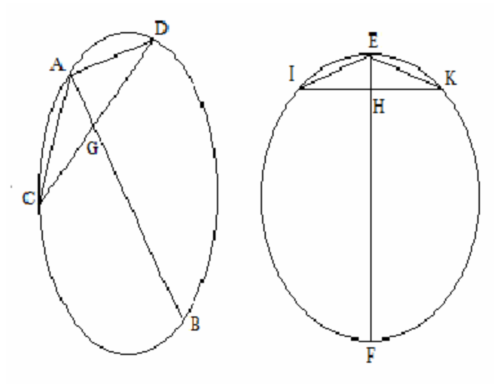


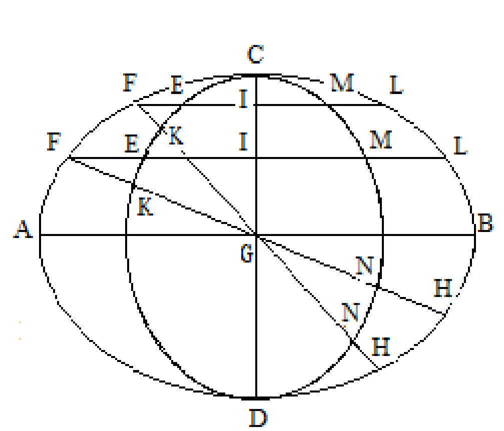
Demonstration.

The squares AC, AD taken together are equal to the squares CG, AG taken twice, & the squares IE, EK are equal to the squares IH, HE taken twice; but by the preceding, the squares CG, IH are equal, and likewise the squares AG, EH are equal to each other, because from the construction the right lines AG, EH shall be equal; therefore the squares CG, AB taken twice are equal to the squares GH, EH taken twice ; and therefore the two squares CA, AD taken together are equal to the squares IE, EK taken together. Q.e.d.

PROPOSITION CLXXIX.

AB, CD shall be the axes of the ellipse ABC and the circle CED is described on the minor axis CD, with the ordinates FL for the axis CD, which cross the circle at E and M, but the axis of the circle at I, and of the ellipse at L; we shall draw FH from F through the centre G, crossing the circle at K and N, and the ellipse at H.

I say to be the case that the square FI to be to the square FI, thus as the rectangle FKH to the rectangle FKH.



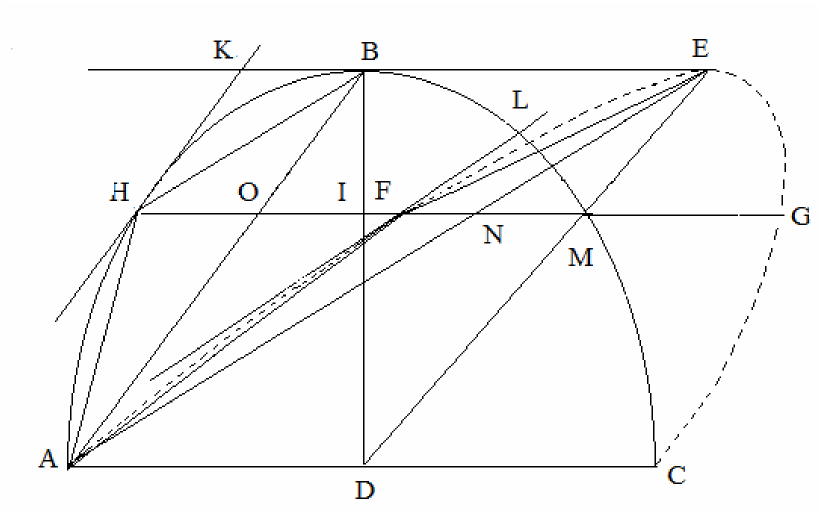
Demonstration.

It is clear from the fourth scholium of this book these two proportions FE to FE, and EL to EL to be the same as the ratio EI to EI whereby since the ratio of the rectangle FEL to the rectangle FEL is composed from the ratios FE ad FE, and EL to EL, that will be the ratio of the rectangle FEL to the rectangle FEL, with the ratio EI to EI squared, and hence the same as with the square EI to El: but the rectangles FEL are the rectangles MFE, that is the rectangles NFK, that shall be FKH. Therefore the rectangles FKH themselves are in turn as the squares EI , that is as the squares FI. Q.e.d.

PROPOSITION CLXXX.

The line BE described shall be a tangent to the circle ABC at B, the diameters of which AC, BD cross each other at right angles, then BE shall be a tangent at E to the ellipse described, passing through the points A, C, and the points AE, AB may be joined.
I say the segments AFE, AHB to be equal.

Demonstration.



With HK drawn parallel to AB which shall be a tangent to the circle at H ; the line HG shall be drawn to the tangent BE, crossing the ellipse at F, and crossing the right lines drawn AB, AE, ED, BD, at M, N, I, O, and the points BH, AH, EF, FA may be joined; then through F the line FL shall be drawn parallel to AE: Since both the tangent HK shall be parallel to the right line AB [§'s 13, 42, 174], moreover FL will be parallel to AE, since FL shall be the tangent to the curve AFE itself; AHB, AFE will be the largest of the triangles which can be inscribed in the segments AHB, AFE ; and hence more than half of these segments [§ 43]. Again since the triangles ABD, AED shall be on the same base and put in place parallel to each other, and the line HM parallel to the base AD, OI & NM shall be equal lines ; but the whole lengths HI, FM are equal ;and therefore the remaining lengths HO, FN are equal to each other : Whereby both the triangles HOB , NEF, as well as the triangles HAO, NFA, and thus the whole triangles BHA, EFA are equal. In the same manner if triangles may be inscribed in the remaining segments, we will show triangles inscribed in the remainder of the circle to be equal to the triangles inscribed for the remainder of the ellipse, and for each to be the greater parts of their segments. Whereby since the said inscription of the triangles, it shall be possible on both sides always for the equality and more halving of the segments to be continued without end; therefore the segments AHB, AFE are equal. Q.e.d.

Corollary.

With the same in place, it follows the semicircle ABD to be equal to the semi ellipse AEC, for the segment AHB has been shown to be equal to the segment AFE, moreover the triangles ABD, AED on the same base and set up between the same parallel lines are equal to each other, therefore the quadrant of the circle ABC is equal to the quadrant of the ellipse AED. Therefore the semicircle ABC is equal to the semi ellipse AEC. Q.e.d.

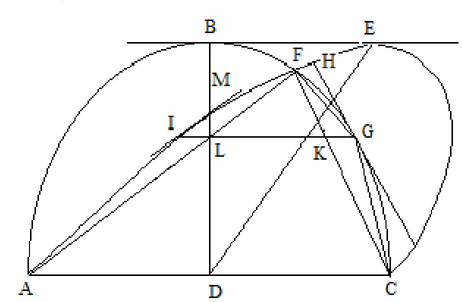
[Thus, an attempt is made geometrically to compare the area of a semi ellipse with that of the associated semicircle, by adding together indefinitely more and more smaller slices of equal areas. This can be seen as an early form of the limiting process leading towards integration.]

PROPOSITION CLXXXI

The semicircle ABC and the semi ellipse AEC shall have the common diameter AC and the tangent BE, parallel to the diameter AC: moreover the ellipse AEC shall cut the circle ABC at F, and the lines AF, CF may be drawn.
I say the segments AIF, CGF to be equal.

Demonstration.

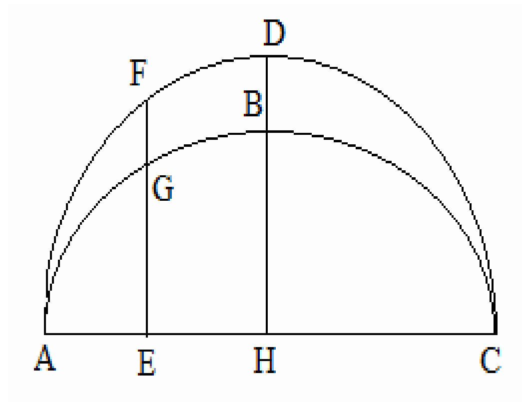
With the line GH drawn parallel to the right line CF, which shall be a tangent to the semicircle at G, from G a line may be drawn parallel to the right line AC, crossing the lines CF, AF at K & L, and indeed the ellipse at I: and the line IM shall be drawn through I, which shall be parallel to the line AF: the points AI, FI, CG, FG shall be drawn : Since the tangent line IM is parallel to the section AF, IM, and thus the maximum of these triangles AIF can be inscribed in the segment : but since the maximum of these triangles inscribed in the circle evidently shall be CGF : each triangle is more than half of its segment. Then, since IL, KG are equal, as may be deduced easily from §174, and IG, AC are parallel, the triangles IAL, GCK are equal: truly for the same reason the triangles IFL, GFK are equal. Therefore AIF, GCF are equal. We can show it possible for the remaining elliptical and circular segments to be inscribed similarly, with the triangle boundaries greater than half of the segments and equal to each other ; therefore the segments AIF, CGF are equal. Q.e.d.



PROPOSITION CLXXXII.

The ellipse ABC and the circle ADC shall have the same axis AC, and some right line EF may be drawn to the axis AC, crossing the ellipse at G.
I say to be the case that as EG to EF, the ellipse ABC shall be to the circle ADE.

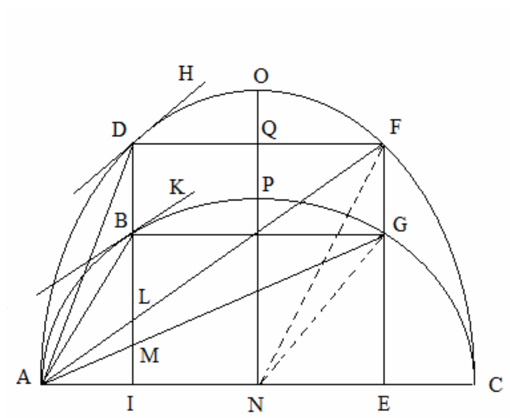
Demonstration.



The normal HBD shall be erected from the centre H, crossing the ellipse at B and the circle at D, so that as HB to HD, thus EG is to EF: but as HB to HD, thus the ellipse ABC is to the circle ADE: therefore, as EG to EF, thus the ellipse ABC is to the circle ADE. Q.e.d.

PROPOSITION CLXXXIII.

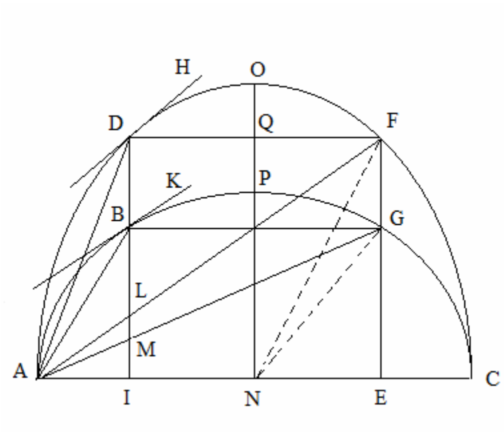
The ellipse ABC and the circle ADC shall have the same axis AC for which the right line EF shall be the ordinate put in place crossing the ellipse at G and the circle at F.
I say the segment ADFE to be to the segment ABGE as the circle ADC is to the ellipse ABC.
Demonstration.



AF, AG may be joined, and the right line DH may be drawn parallel to AF, tangent to the circle at the point D: from which the right ordinate DI may be dropped to the axis, crossing the ellipse at B and the lines AF, AG at L and M, and the right line BK shall be acting through B parallel to AG, which by §154, shall be a tangent to the ellipse. Then, the lines AD, DF, AB, BG shall be drawn, so that EF shall be to EG, thus IL is to IM, but as EF is to EG, thus ID is to IB. Therefore as ID to IB, thus IL to IM; and therefore the remaining DL to the remaining BM, as the whole length ID to the whole length IB; moreover as DL is to BM, thus the triangle DAL to the triangle BAM, and (because DI, FE are parallel), triangle DFL to triangle BGM, therefore triangle DAL is to triangle BAM, as ID to IB, and triangle DFL is to triangle BGL, as ID to IB. Therefore the whole triangle ADF is to the whole triangle ABG, as ID to IB . Moreover, since DH, BK are tangents, ADF and ABG are the greatest triangles of these which can be inscribed in the segments [§13 & §42], and hence greater than the two half segments remaining. Similarly we will show the triangles able to be described both of the circle and of the ellipse, which shall be greater than the two half segments remaining, and which shall have the ratio as ID to IB. Whereby since this procedure may be able to be continued without end, the segment ADF is to the segment ABG, as ID to IB: and indeed the triangle AFE is to the triangle AGE, as EF to EG, that is as ID to IB. Therefore the whole segment ADFE is to the whole segment ABGE, as ID to IB, that is, as the circle to the ellipse. Q.e.d.

PROPOSITION CLXXXIV.

With the same figure remaining, some diameter NF may be drawn from the centre N diameter NF, and with the normal FE dropped from F which shall cut the ellipse at G, NG may be joined :
I say the sector ANF shall be to the sector ANG, as the circle ADE to the ellipse ABC.



Demonstration.

The segment ADFE is to the segment ABGE, as FE ad GE : & triangle FNE is to triangle GNE, as FE to GE. And therefore truly the remaining sector ANF is to the remaining sector ANG, as FE to GE, that is as the circle to the ellipse. Q.e.d.

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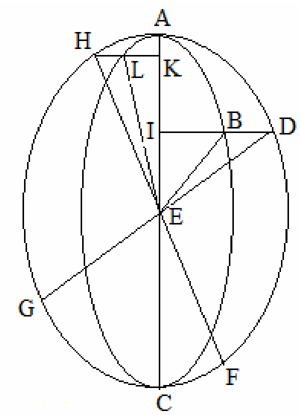
PROPOSITION CLXXXV.

The ellipse shall become ABC, and the circle ADC, having the common axes AC: and through the common centre E two diameters of the circle may themselves be drawn intersecting at right angles, and crossing over the circle at the points D, F, G, H: then from D and H, the lines DI, HK supply the ordinate to AC crossing the ellipse at B and L: EB and EL may be joined.

I say LEB to be a quadrant of the ellipse, just as HED is a quadrant of the circle.

Demonstration.



The segment ABI is to the segment ADI as IB to ID, and as IB ad ID, thus the triangle IEB to be to triangle IED. Therefore as IB to ID, thus the sector AEB is to the sector AED. In a similar manner we may show the sector LEA to the sector HEA, to be as KL to KH, that is, as IB to ID: therefore as IB to ID, thus the whole sector BEL is to the whole sector DEH; but as IB to ID, that is the minor axis of the ellipse to the diameter of the circle, by Book 5 of Archimedes On the Sphere, thus as the ellipse ABC is to the circle ADC; therefore as the sector BEL is to the sector DEH, thus the ellipse to the circle, and on inverting and interchanging, as the sector DEH to the circle ADC, therefore the sector BEL to the ellipse ABC; but the sector DEH is the quadrant of the circle, therefore BEL will be the quadrant of the ellipse, and thus BE, EL will be conjugate diameters, therefore, &c. Q.e.d.

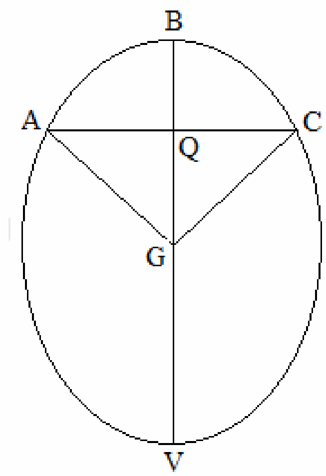
Note the same to be shown if ADC shall be an ellipse of which the axis is AC, and HF, DC shall be its conjugate diameters; truly as DEH shall not be the quadrant of the ellipse as is the sector DEH to the circle, thus it will be the sector BEL to the ellipse, as follows from the demonstration.

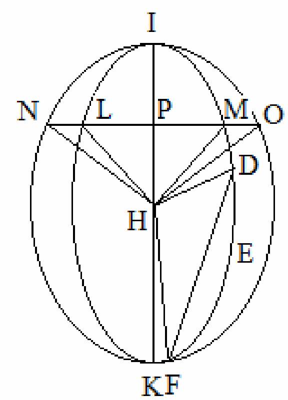
PROPOSITION CLXXXVI.

Let AGC be some sector of the circle ABC, and DHF the sector of the ellipse DEF : moreover the sector to the sector shall be as the circle to the ellipse: and the right lines AC, DF may be drawn.

I say the segment ABC to be to the segment DEF, as the circle ABC is to the ellipse DEF, and vice versa.

Demonstration.



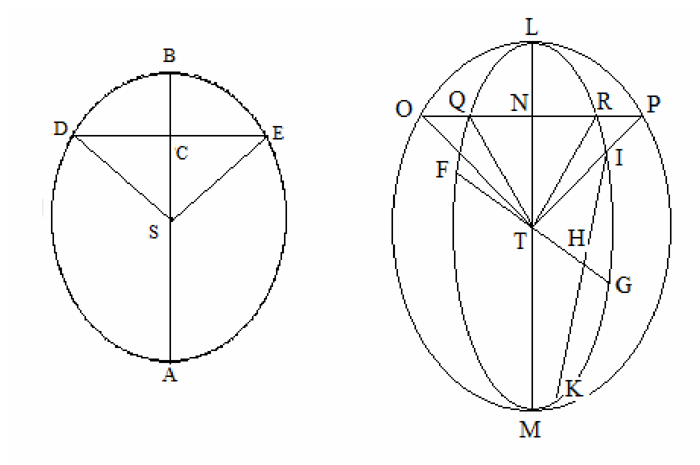


With the greater axis IK of the ellipse found, and with the circle drawn on the diameter IK ; for the ordinate axis LM, which bears the segment LIM equal to the segment DEF, shall cross the circle at N and O, and the right lines shall be drawn LH, MH, NH, OH. Since the segment MIL is equal to the segment DEF, the sector LHM shall be equal to the sector DHF. Again since as the ellipse shall be to the circle NOK, thus the sector LHI shall be to the sector NHI, and thus the sector LHM to the sector NHO, and the triangle LHM shall be to the triangle NHO, as LM ad NO, that is so that as the ellipse to the circle NOK, thus the segment LIM to the segment IO, and on interchanging, the circle NOK to the segment NIO, as the ellipse to the segment LIM. Now truly the circle ACV is to the sector AGC, from the hypothesis, as the ellipse to the sector HDF, that is, (as shown above) as the ellipse to the sector LHM, that is as the circle NOK to the sector NHO. Therefore also the circle ACV to the segment ABC, as the circle NOK to the segment NIO, that is, as shown before, so that the ellipse to the segment LIM, that is (since from the construction the segments LIM, DEF are equal) as the ellipse to the segment DEF. Therefore on interchanging, as the circle ACV to the ellipse, thus the segment ABC to the segment DEF. Q.e.d.

Now truly if the segment ABC were to the segment DEF, as the circle ABC is to the ellipse IDF: I say the sector AGC to be to the sector DHF, as the circle ABC to the ellipse IDF: indeed with the axis IK of the ellipse DEF found as before, the circle NOK may be described with the axis IK : and with the ordinate LM drawn which may be taken equal to the segment DEF, the rest will become as at first. Therefore the segment NIP is to the segment LIP, as the circle NOK to the ellipse IDF. And thus the segment NIO to the segment LIM, as the circle NOK to the ellipse IDF : and on interchanging, the circle NOK to the segment NIO, as the ellipse to the segment LIM, that is from the hypothesis, as the circle ACV to the segment ABC. Therefore as the circle NOK shall be to the segment NIO, thus the circle ACV to the segment ABC, now will be as the circle NIK to the sector NHO, thus the circle ACV to the sector AGC. And as the circle NOK to the sector NHO, thus the ellipse to the sector LHM, that is, since the segments LIM, DEF are equal to the sector DHF. Therefore as the circle ACV to the sector ABC, thus the ellipse to the sector DHF. Q.e.d.

PROPOSITION CLXXXVII.

Let the diameter AB of the circle ADB be divided at some point C, and through C the normal DE may be put in place, and moreover FG shall be some diameter of the ellipse divided at H, so that AB is at C, and with the ordinate IK drawn through H.
I say the segment DBE to be to the segment IGK as the circle ADB is to the ellipse FLG, and vice versa.
Demonstration.



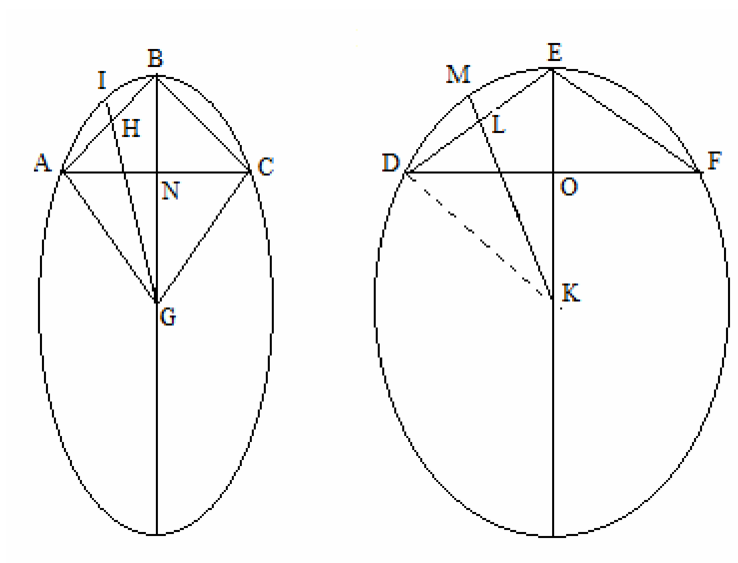
With the major axis of the ellipse LM found, the circle LOM may be described on the line LM, and with LM divided at N, so that AB is divided at C, it shall be acted on by the normal OP through N, cutting the ellipse at Q and R: and the radii OT, QT, RT, PT shall be drawn. The rectangle BCA to the square DE, shall be as the rectangle LNM to the square ON, and on interchanging, the rectangle BCA is to the rectangle LNM, as the square DC to the square ON, but the ratio of the rectangle BCA to the rectangle LNM, is composed from the ratios BC to LN, and CA to NM. Therefore the ratios BC to LN, and CA to NM taken together shall be equal to the ratio of the squares DC, ON, that is of the ratio DE, ON taken twice. And the ratios BC to LN, and CA to NM, are the same or equal, since from the hypothesis BA, LM shall be divided proportionally; therefore one of these BC ad LN, is the same as the ratio DE to ON :but , since BC shall be to LN, thus as CA to NM, also there will be, as BC to LN, thus BA to LM. Whereby BA to LM, that as SD to OT, thus DC is to ON: but the angles DCS, ONT are right; therefore the triangles DCS, ONT are similar, and the angles DSC, OTN equal: whereby the doubles of these angles DSE, OTP are equal, and the sectors DSE, OTP are similar, and thus the segments DBE, OLP are similar ; therefore as the segment OLP to the circle LOM, thus the segment DBE to the circle DBA: but also as the segment OLP to the circle LOM, thus the segment OLR to the ellipse. Whereby as the segment DBE to the circle ADB, thus the segment QLR shall be to the ellipse FLG, that is from the construction, the segment IGK to the ellipse FLG, and on interchanging, the segment DBE is to the segment IGK, as the circle ADB to the ellipse FLG. Which was the first part.

Now the segment DBE shall be to the segment IGK, as the circle ADB to the ellipse FLR: I say the lines BS,GT to be divided proportionally at C and H, the same may be put everywhere ad before: Since the segment DBE is to the segment IGK, that is QLR, as the circle ADB to the ellipse FLR, and on interchanging, the segment DBE to the circle ADB, as the segment QLR to the ellipse FLR, and moreover there shall be OLP to the circle OLM, as the segment QLR to the ellipse FLR, there will become, as the segment OLP to the circle LOM, thus the segment DBE to the circle ADB. Whereby the sectors DSE, OTP are similar and the angles DSE, OTP and thus the halves of these DSC, OTN are equal: but the angles DCS, ONT are right; therefore the triangles DCS, ONT are similar and as DS to OT, that is BS to LT, thus SC to NT. Whereby BS, LT shall be divided proportionally into C and N ; but since by the construction the segments QLR, IGK shall be equal, thus as LT divided into N, GT will be divided into H. Therefore as BC to CS, thus there will be GH to HT. Q.e.d.

PROPOSITION CLXXXVIII.

ABC shall be some segment of the ellipse ABC; moreover the segment DEF may be taken in the circle DEF, so that thus it may itself be had to its own circle, as the segment ABC to its own ellipse : then the greatest triangles ABC, DEF may be inscribed for the segments, and with the lines AB, DE divided in two at H and L, the diameters GI, KM shall be acting through H and L.
I say these cuts at H and L to be proportional.

Demonstration.



The diameters BG, EK may be drawn from B and E, and since DK, KF, GA, GC are the diameters [i.e. radii] drawn from the vertices of the greatest triangles, certainly in the circle it is evident DF to be bisected at C, moreover also to be bisected in the ellipse; you may deduce AC from §42 above. Whereby both in the ellipse as well as in the circle the sectors AGC, DKF shall be bisected. Therefore the sector AGB is to the sector DKE, as the sector AGC to the sector DKF. Now truly on interchanging the hypothesis, the segment ABC shall be to the segment DEF, as the ellipse to the circle, also the sector AGC will be to the sector DKF, that is (as now shown) the sector AGB, to the sector DKE, as the ellipse to the circle. And because the sector AGB to the sector DKE, as the ellipse to the circle, there will be also the segment AIB to the segment AME, as the ellipse to the circle. Therefore the diameters IG, MK are divided proportionally in H and L. Q.e.d.

PROPOSITIO CLXXXIX.

With the same figures in place: if the segment ABC were to the segment DEF, as the ellipse ABC to the circle DEF.

I say the maximum triangle ABC is the maximum triangle DEF as the ellipse ABC is to the circle DEF.

Demonstration.

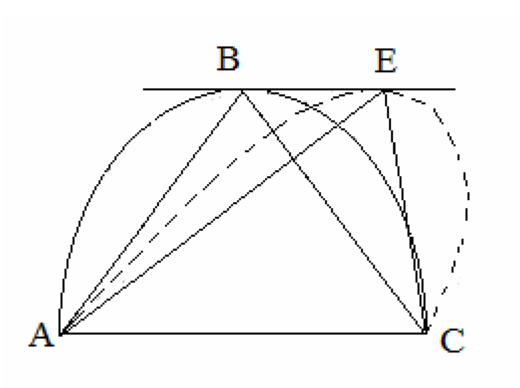
As the ellipse ABC is to the circle DEF, we have show thus in the previous section, AIB to be to the segment DME, as BC to EF. Whereby since also from the hypothesis there shall be, as the ellipse to the circle, thus the whole segment ABC to the whole segment DEF, and therefore the remaining triangle ABN is to the remaining triangle DEO as the ellipse ABC, to the circle DEF. Q.e.d.

PROPOSITION CXC.

ABC shall be the maximum triangle inscribed in the semicircle ABC ; and moreover AEC shall be the maximum triangle inscribed AEC inscribed which shall have the common diameter AC for the ellipse AEC: if ABC, AEC were equal triangles:

I say the semicircle ABC to be equal to the semi ellipse AEC.

Demonstration.



The points B and E may be joined. Since the triangles ABC, AEC described on the same base are equal by the hypothesis are equal, the joined line BE shall be parallel to the right line AC; and thus since both ABC as well as AEC, shall be the greatest triangles, the right line BE shall be a tangent both to the circle and the ellipse : therefore the semicircle ABC shall be equal to the semi ellipse AEC. Q.e.d.

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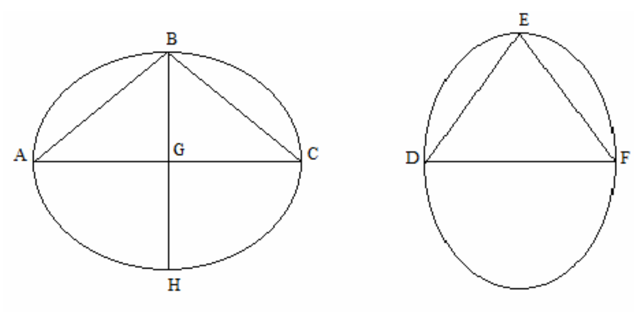
PROPOSITION CXCI.

The ellipse is to the circle or the ellipse is to the maximum triangle inscribed in the semi ellipse as the maximum triangle inscribed to the semicircle or to the semi ellipse.

Demonstration.



Indeed since the semi ellipse shall be to the semicircle or vice versa, thus the whole ellipse shall be to the whole circle or vice versa; so that the maximum triangle AEC inscribed to the semi ellipse to the maximum triangle DEF inscribed to the semicircle or vice versa, thus, thus the ellipse to the circle or vice versa. Q.e.d.

PROPOSITION CXCII.

For a given circle or ellipse, to show an equal ellipse DEF. And to show an equal circle for the ellipse.

Construction & demonstration.

The maximum triangle DEF may be inscribed in the semicircle or semi ellipse DEF, to which some other triangle ABC may become equal, and AC bisected at G, BG shall be drawn, and BG shall be produced to H, so that BG, GH shall be equal, and if AC, BH shall intersect each other at right angles, the ellipse ABCH shall be described, the axis of which shall be ACBH, but if they do not intersect each other at right angles, the axis of the given conjugate diameters ACBH may be found around which the ellipse ABCH may be described. I say the ellipse ABCH to be equal to the circle or to the ellipse DEF : indeed the triangle ABC is the maximum of these which can be inscribed in the semi ellipse, that are placed conjugate to the diameters ACBH. Whereby since the inscribed maximum triangle of the semi ellipse shall be equal to the maximum triangle inscribed for the circle or the ellipse, it will be equal to the ellipse for the circle ABC or for the ellipse DEF, therefore what was demanded.

From these the construction and demonstration of the second part is evident.

Corollary.

Hence it is evident infinitely many ellipses to be given equal to the circle or the ellipse ABC, since from the triangle ABC an infinitude of equal triangles are given.

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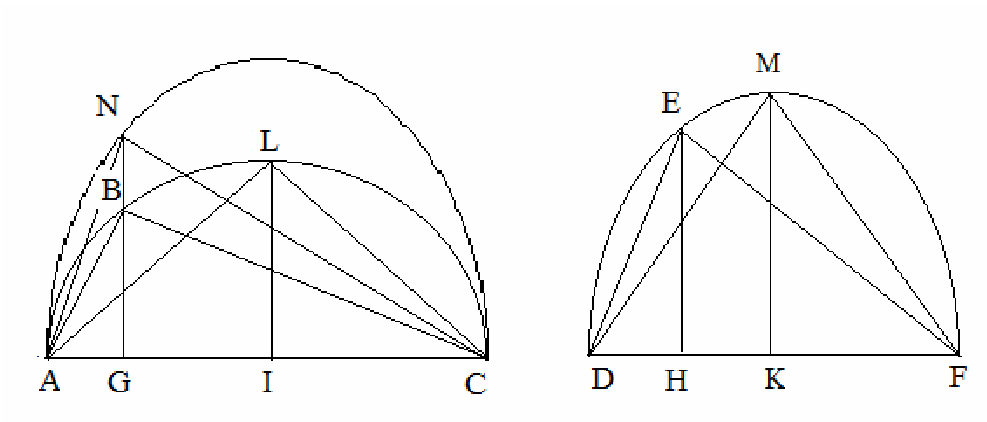
PROPOSITION CXCIII.

AC shall be the axis of the semi ellipse ABC, moreover the semicircle DEF shall be put equal to the semi ellipse ABC; then the triangles ABC, DEF, equal to each other, shall be inscribed to the semi ellipse and semicircle. and from B and E, the normals BG and EH may be dropped to the base.
I say the lines AC, DF to be divided similarly at G and H.

Demonstration.



From the centres I and K, the normals IL, KM are erected to the diameters AC, DF ; and AL, LC, & DM, MF may be joined : since the semi ellipse ABC is equal to the semicircle DMF, and indeed the greatest triangles ALC, DMF are equal to each other ; therefore as triangle ALC is to triangle ABC, that is as LI is to BG, thus triangle DMF is to triangle DEF, that is thus the line MK to EH, and thus as the square LI to the square BG, thus the square MK will be to the square EH: but as the square LI is to the square BG, thus the rectangle AIC is to the rectangle AGC: and as the square MK to the square EH, that is the rectangle DKF is to the rectangle DHF, therefore as the square AI is to the rectangle
AGC, thus the square DK is to the rectangle DHF: and on interchanging, as the square AI to the square DK, or as the square AC to the square DF, thus the rectangle AGC is to the rectangle DHF, therefore it is in agreement with Serenus I.1. Prop. 12, the lines AC, DF to be divided proportionally at G and H. Q.e.d.

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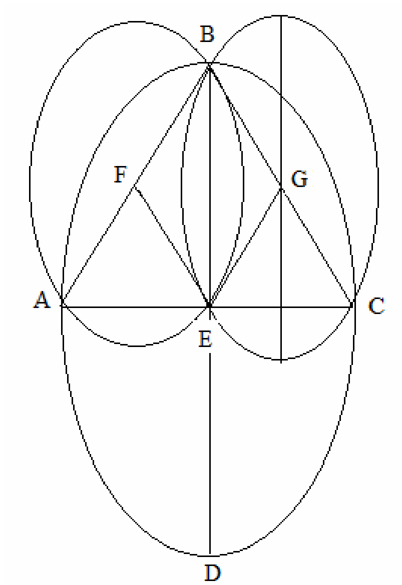
PROPOSITION CXCIV.



Some conjugate diameters AC, BD will cut the ellipse ABC and with the points AB, CB joined the lines AB, CB shall be bisected at F and G; from the centre of the lines E, the lines EF, EG may be drawn: then from the points the ellipses AEB as well as CEB will be described, of which the conjugate diameters shall be AB, EF, CB, EG :

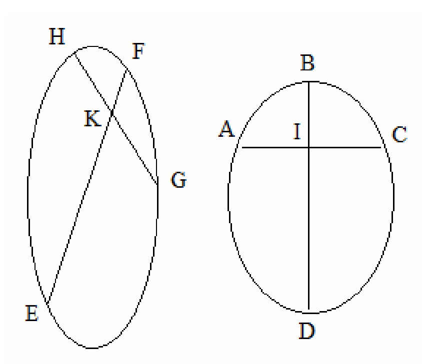
I say the ellipse ABC to be equal to the ellipses AEB, CEB taken together.

Demonstration.

Because ABEF, CBEG, ACBE are conjugate diameters, certainly the triangles ABC, AEB, CEB to be the greatest which can be inscribed by the semi ellipses themselves : but triangle ABC is the double of triangle CEB, and therefore the ellipse ABC is twice as great as the ellipse CEB; similarly I may show the ellipse ABC to be twice as great as the ellipse BEA ; therefore the ellipse ABC is twice as large as each of the individual ellipses BEA, CEB and therefore the ellipse ABC is equal the two ellipses taken together. Q.e.d.

PROPOSITION CXCV.

Some line AC shall cut the circle ABC, removing the segment ABC: it is required in the given ellipse EFG, to deduce the ordinate line GH for a given diameter EF, which may remove the segment HFG, which may have the same ratio to the ellipse as the segment ABC has to the circle ABC.



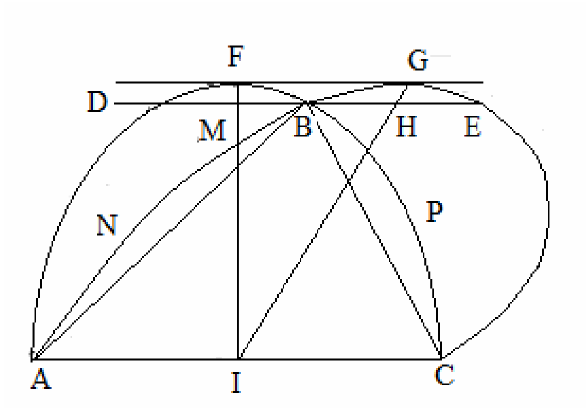
Construction & demonstration.

The right line AC in the circle ABC shall be bisected at I, the diameter BD is acting normally through I : then in the ellipse EFG, the diameter EF shall be divided at K, as BD has been divided at I, and the ordinate line HG shall be acting through K, it is apparent the segment HFG to be to the ellipse EFG, as the segment ABC is to the circle ABC. Therefore with the given segment from the circle, etc. Q.e.d.

PROPOSITION CXCVI.

Let some triangle ABC be inscribed in the semicircle ABC, it shall be required to describe an elliptic segment on the line BC, equal to the circular segment ADB.

Construction & demonstration.



The right line FG shall be drawn parallel to the diameter AC tangent to the circle at F, then the right line DE may be drawn through B acting parallel to the diameter AC, and BE shall be made equal to DB itself, with which bisected at H, the right line IG shall be drawn through H from the centre of the circle I : then the ellipse may be described through the points A, G, C : of which the conjugate diameters shall be AC, IG, and since FG itself is through the extremity of the diameter IG, AC itself shall be parallel to the tangent to the ellipse at G. Whereby the ellipse is equal to the circle [Cor. §180]. Then since FG, MH are parallel, we may show easily from The Elements, the rectangle under GH, and the remaining part of the diameter, to be to the rectangle under GI, and with the remaining part of the diameter to be as the rectangle under FM, and the remaining part of the diameter, as the rectangle under FI and the remaining part of the diameter as the square DM to the square AI; that is, as the square BH to the square AI; therefore that is as the rectangle under GH and the remaining part of the diameter is to the rectangle GI, and the remaining part of the diameter, as the square BH to the square AI. Therefore the ellipse passes through the point B. Now since the ellipse is equal to the circle: as shown above, with the common curvilinear segment ANBPC removed, the equal segments BPCE, ANBD will remain, which if the segments BPC, ANB may be added, which are equal, the elliptical segment BCE having BC will be equal to the segment of the circle ADB, the base having the right line AB. What was sought has been done.

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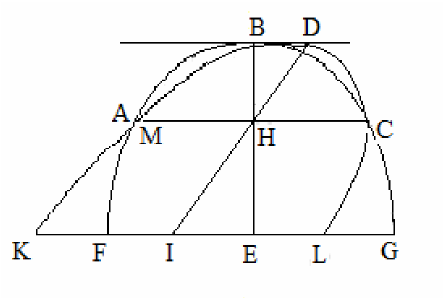
PROPOSITIO CXCVII.

ABC shall be some circular segment, it is required to subtend an elliptical segment on AC, equal to the given segment ABC, of which one shall be given from the conjugate diameters FG of the circle ABC.

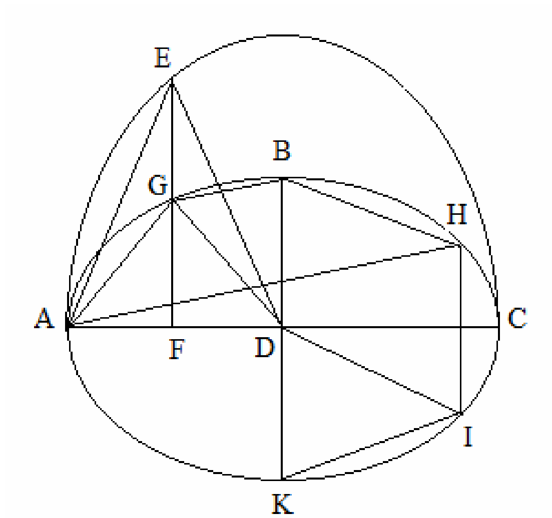
Construction & demonstration.



From the centre E found of the circle ABC, the diameter FG may be drawn through E parallel to the right line AC and with the normal EB erected from E, which shall bisect the line AC at H, the tangent BD shall act through B, then the right line ID of the given equal axis shall be drawn through H, equal to the given tangent at D crossing the tangent line at D and the diameter FG at I: and IK shall be made equal to the right line FE, and IL equals EG, then the points of the ellipse may be drawn through KDL, crossing the line AC in some manner at M. I say to be made what was sought. Indeed since LK, DI shall be conjugate diameters, and BD parallel to LK itself, it is evident BD to be a tangent to the ellipse KDL, and thus the right line MH equal to AH itself: therefore the ellipse will pass through the point A. In the same manner it is shown to pass through the point C: therefore the elliptic segment ADC is equal to the segment of the circle ABC; therefore on the given line AC, &c. Q.e.f.

PROPOSITION CXCVIII.

From a given point on the periphery, to divide an ellipse into a given number of equal sectors.



Construction & demonstration.

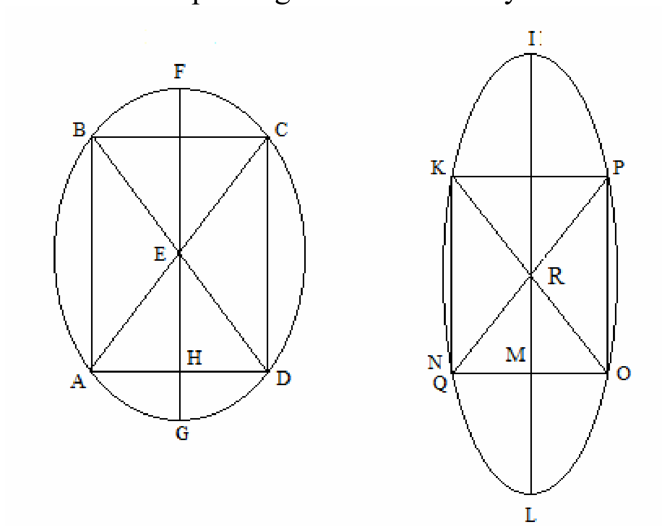
The major axis AC of the ellipse ABC shall be given, the point B given on the periphery, it shall be required to cut this ellipse into as many equal sectors as you wish, for example into six.

Draw the semicircle itself with the diameter AC, inscribe AE, the side of the polygon having as many sides as the sectors sought. But since the sectors sought shall be equal to six, AE will be the side of a hexagon. Therefore EFG shall be put normal to AC, and ED, GD, BD, GB may be joined, and GB shall be parallel to AH: I say what was required to be produced: so that the circle thus is to the ellipse: the segment EFA to the segment GFA ; that is as FG to FE and likewise to be as FE to FG, thus as triangle EDF to triangle GDF: therefore as the circle to the ellipse, thus the sector EDA is to the sector GDA: but the sector EDA is the third part of the semicircle, since the line AE shall be the side of the [regular] hexagon; and therefore the sector GDA will be the third part of the semi ellipse. But since GB shall be parallel to AH, the segments AG, BH: and thus the sectors GDA, BDH are equal to each other : thus the sixth part of the whole ellipse ; just as the segment BH shall become equal to the segments HI, IK and the sectors DH, DI, DK may be joined ; therefore DBH, DHI, DIK are equal and thus are the sixth parts of the ellipse. And taken together they establish the semi ellipse BCK, therefore the remaining semi ellipse BAK cut as BCK has been divided, and the whole ellipse divided into six equal parts. As was required to be made.

PROPOSITION CXCIX.

Let some regular polygon ABCD be inscribed in the circle ABC, the centre of which is E, and the diameter FG drawn shall bisect some side AD at H: moreover IL shall be some diameter of the ellipse IKL drawn by which the side FG has been divided at M, just as FG has been divided at H, and the ordinate line NO to the diameter IL is acting through M.

I say the right line NO, to be one of so many sides of the regular polygon requiring to be inscribed in the ellipse, as the number of sides of the polygon inscribed in the circle ABC. Moreover I call the elliptic polygon regular, the individual sides of which are the same as the elliptic segments taken away.



Demonstration.

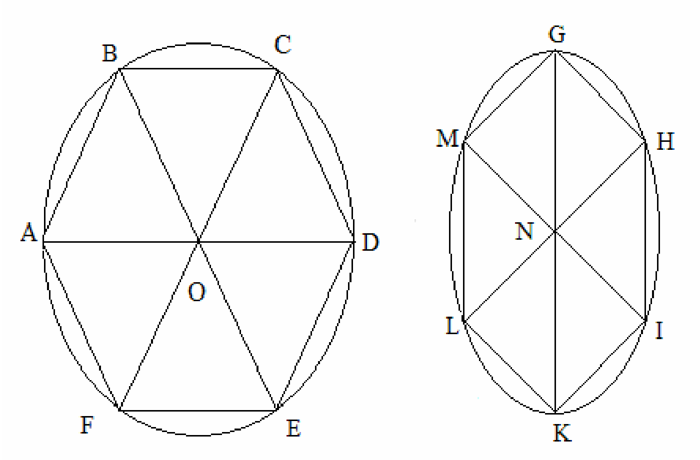
From O the line OP is drawn taking away a segment equal to the segment NLO, and from P the right line PK which takes away a segment equal to the segment NLO, then from K the line KQ takes away a segment equal to the segment NLO, Q will be the same as the point N. Indeed with the points EA, EB, FC, ED joined in the circle ABC, with the points drawn in the ellipse the radii become RK, RN, RO, RP. Since the diameters FG, IL are divided proportionally in H and M, and AD, NO the lines through H and M, the ordinates acting for the diameters FG, IL; as the sector AED to the circle ABC, thus the sector NRO to the ellipse IKL: but just as the sectors AFD, DEC, CEB, BEA, as well as the sectors NRO, ORP, PRK, KRQ are equal to each other, (since the segments NO, OP, PK, KQ are equal) therefore as the four circular sectors to their circle, thus the four elliptic sectors to their ellipse: but the circular sectors are equal to the whole circle, and therefore the elliptic sectors are equal to the ellipse. Whereby the point Q is the same as the point N, and the polygon KNOP is regular just as many sides inscribed as is in the polygon for the circle. Q.e.d.

PROPOSITION CC.

Let some regular polygon A, B, C, D, E, F be inscribed in the circle ABC : and moreover the regular polygon G, H, I, K, L, M with just as many sides shall be inscribed in the ellipse GHI, as the polygon inscribed in the circle.

I say a circular segment taken from some side of the polygon, to be to the elliptical segment, taken from some side of the elliptic polygon, to be as the circle ABC to the ellipse GHI.

Demonstration.



O shall be the centre of the circle and N the centre of the ellipse: and from O and N the radii for the angles of the polygon may be drawn: Since the segments GH, HT, IK, &c. in the ellipse are equal by the construction, and the sectors GNH, HNI, INK, &c. are equal: and moreover the sectors of the circle AOB, BOC, COD, &c. are equal and equal to the number for the ellipse, therefore the sector AOB is to the sector GNH, as all the circular sectors, that is the as the circle ABC to all the sectors of the ellipse, that is to the ellipse GHI, and whereby the segment AB is to the segment GH, as the circle to the ellipse. Q.e.d.

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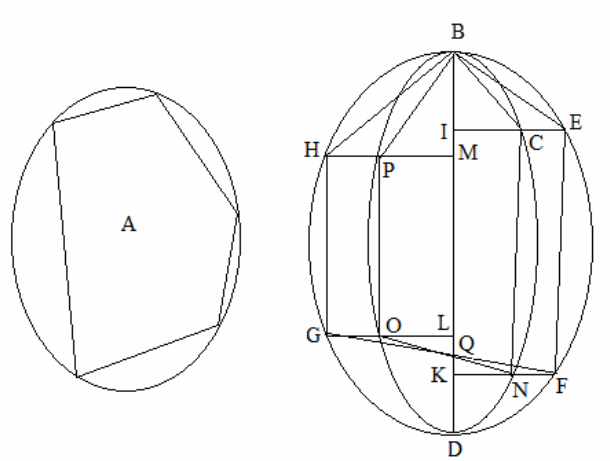
PROPOSITIO CCI.

Some kind of polygon shall be inscribed in the circle A, it will be required to inscribe a polygon with an equal number of proportional sides in the given ellipse BCD, that is, so that it may have the same proportion as the circle to another circle; and the individual sides of which bear segments which shall have such a ratio to the ellipse as they have to the circular segments, which the individual sides of the other circle bear to its original circle.



Construction & demonstration.

With the greater axis BD of the ellipse found, the circle BEF is described with the diameter BD, to which the polygon BEFGH may be inscribed, similar to that, as A is the polygon in the circle : and the side FG shall cut the axes BD at Q; then with the normals EI, FK, GL, HM drawn from E, F, G, H to the axis BD which will cut the ellipse in C, N, O, P. The right lines shall be drawn BC, CN, NQ, OQ, OP, PB. I say to be done what was sought. Since GL, FK are parallel, the triangles GLQ, KFQ will be similar ; and thus as GL to KF, that is as OL to KN, thus LQ to QK, and truly the angles OLQ, FKQ are right; therefore the triangles OLQ, KFQ are similar. Therefore OQ, QF are on the same line; therefore the figure BC, NO, PB is the polygon inscribed in the ellipse. As the line IE id to the line IC, thus the triangle IBE is to the triangle IBC, but as IE is to IC, thus the trapezium IEFQ is to the trapezium ICNQ; therefore the whole figure BEFQ is to the figure BCNQ, as the line IE is to the line IC, that is the segment IBE is to the segment IBC, that is as the circle BEP is to the ellipse BCD. In the same manner the figure BHGQ is shown to be to the figure BPOQ, as the circle BEC to the ellipse BCD. Whereby the total polygon BEFGH will be to the polygon HCNOP, as the circle BEF to the ellipse BCD, and on interchanging, as the polygon BEFGH to the circle BEF, that is from the construction, as the polygon to its inscribed circle A, thus the polygon BCNOP will be to the ellipse BCD: Which was the first part.

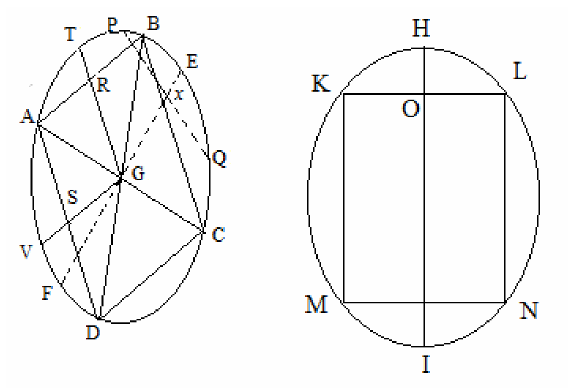
Again just as since the segment IBE shall be to the segment IBC, then as well the triangle IBE shall be to the triangle IBC; as the circle BED will be to the ellipse BCD, and the segment BE to the segment BC, as the circle BED shall be to the ellipse BCD : and on interchanging, so that as the segment BE to the circle BED, thus the segment BC is to the ellipse BCD: likewise similarly concerning the rest of the segments to be shown: therefore we may inscribe the polygon in the given ellipse BCD, &c. Q.e.f.

PROPOSITION CCII.

ABCD shall be some regular quadrilateral inscribed in the ellipse ABC, and one from the conjugate diameters EF drawn through the centre G, may be taken equal to the right line HI : with which diameter the circle HIK may be describe, to which the square KLMN may be inscribed.

I say the squares KL, LN, NM, MK taken together to equal the squares AB, BC, CD, DA taken together.

Demonstration.



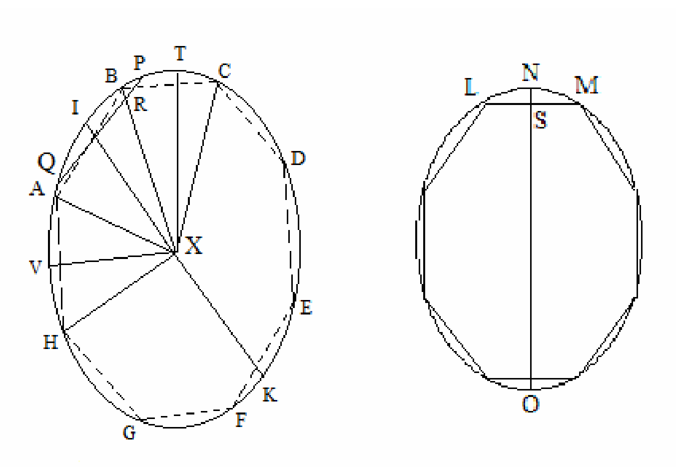
KL bisected at O, the diameter HI shall be drawn through O : then the ordinate line PQ may be adjoined to the diameter FE, bearing a segment equal to the segment AB: and with AB, AD bisected at R and S, and the radii GRT, GSV shall be joined, and the points AG, BG, CG, DG. Because the segments AB, BC, CD, DA are equal by the construction, and the sectors AGB, BGC, CGD, AGD are equal, and therefore AG, BG shall be conjugate diameters. Besides from the construction GT, GV from the centre will bisect the right lines AB, AD; the sectors AGT, AGV are half the part of the sectors AGB, AGD; that is half of the ellipse; therefore the sector TGV is the quarter part of the ellipse; therefore GT, GV are conjugate and AB, AD the ordinate lines put in place for these: therefore since by the construction the segment PEQ shall be equal to the segment ATB, or AFD will be the square PQ taken twice, equal to the squares AB, AD likewise taken, and therefore the square PQ taken four times is equal to the squares AB, BC, CD, DA. Again since the segment PQ shall be to the segment KL, as the ellipse ABC to the circle KLM, the diameters EF, HI are divided proportionally at X and O, and thus the line PQ is equal to the line KL. Therefore the square KL taken four times is equal to the squares AB, BC, CD, AD. Q.e.d.

PROPOSITION CCIII.

Some regular polygon ABCD, EFGH shall be inscribed in the ellipse ABC : and with one diameter IK drawn from the equal conjugate diameters; the circle LMN shall be described having a diameter equal to the diameter IK: then a regular polygon is inscribed in a circle with just as many sides as the polygon inscribed in the ellipse.

I say the squares of all the sides of the inscribed polygon of the ellipse taken together, likewise to equal the squares of the sides of the polygon of the circle taken together.

Demonstration.



We may establish, for example, regular octagons to be inscribed in the ellipse and circle, certainly the same demonstration will be agreed on, in the circle LMN the diameter NO may be drawn cutting LM, bisecting the line at S; truly the diameter IK shall cut the ordinate line PQ bearing a segment equal to the segment AB, then the radii may be drawn HX, AX, BX, CX, likewise TX, VX, which bisect the lines AH, CB , shall bisect the lines AXB, BXC, &c.: but these taken together are equal to the total ellipse, therefore the two sectors AXB, BXC that is the fourth part of the sector, will be the quadrant of the ellipse ABC. Now since XT, XV drawn from the centre of the ellipse will bisect BC, AH, the sectors CXT, XAV will be half of the sectors CXT, XAV half of the equal sectors BXC, AXH , and hence equal to each other; therefore with the common sector added AXT, the whole sector VXT, shall be equal to the whole sector AXC, whereby since AXC shall be the quadrant of the ellipse, and will equal VXT. Therefore VX, TX are conjugate diameters, for which AHCB are the ordinate put in place bearing the equal segments; therefore the square PQ taken twice is equal to the squares AH, CB taken at the same time : it is shown likewise in the same way the square PQ taken twice to be equal to the squares AB, GH taken at the same time: and thus the square PQ taken four times is equal to the squares CB, BA, AH, HG, that is to the squares GF, FE, ED, DC. Whereby the square PQ taken eight times as many as the sides of the polygon; is equal to the squares of the total number of sides of the polygon inscribed in the ellipse. Again since as the ellipse ABC shall be to the circle LMN, thus the segment AB that is PQ to the segment LM, the diameters IK, NO will be divided proportionally to the diameters at R & S, and the square PQ equal to the square LM. Therefore the square PQ taken eight times shall be equal to the squares of the sides of the circular polygon. But the square PQ taken eight times is equal also, as shown above, to the squares of the sides of the elliptic polygon;

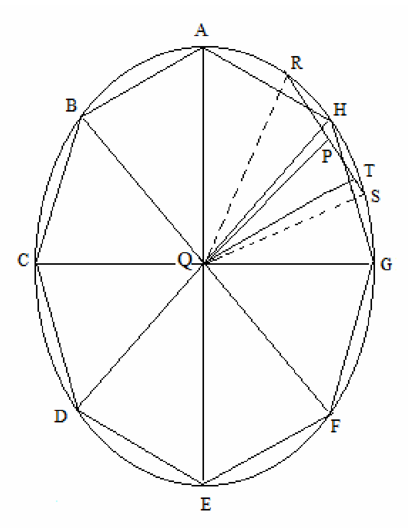
therefore the squares of the sides of the elliptic polygon taken together are equal to the squares of the sides of the circular polygon taken together. Q.e.d.

Corollary.

Hence it is apparent: if the same two ellipses may be inscribed with polygons with equal number of sides, the squares of the sides of one polygon taken together, to be equal to the squares of the sides of the other polygon taken together.

PROPOSITIO CCIV.

A regular polygon shall be inscribed in an ellipse, moreover QP shall be one of the equal conjugates, for which the ordinate shall be RS.
I say the squares of the sides of the polygon added together to be to the whole polygon as the line RS to half of the line TQ.
Demonstration.



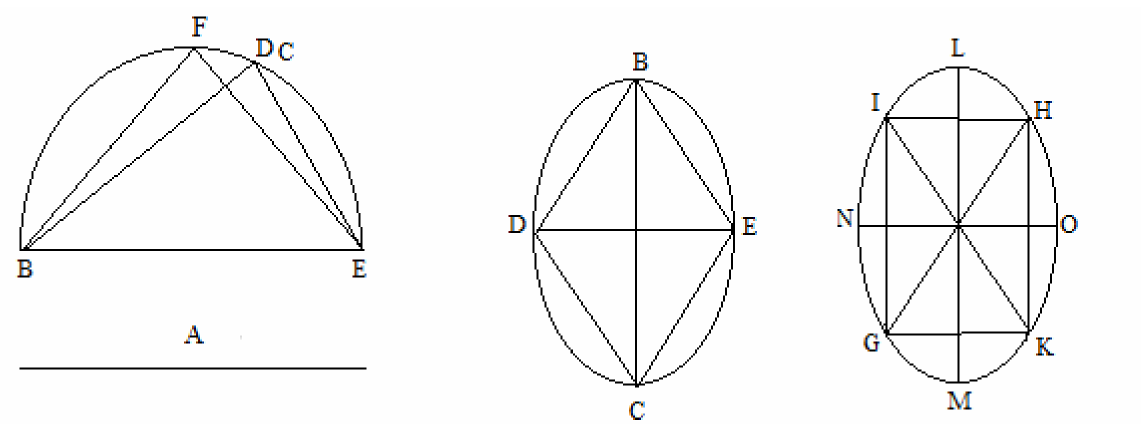
The radii are drawn from the points A, B, C, D, E, F, G, H, R, S. Since by the construction the segment RS is equal to the segment AB, and the triangle RQS is equal to the triangle AQB: I may show similarly the triangles BQC, CQD, &c. to be equal to the triangle RQS: and thus the triangle RQS taken eight times to be equal to the whole polygon : moreover the square RS taken eight times to be equal to the squares of all the sided of the polygon, therefore as the square RS taken eight times is to the triangle RQS taken eight times, that is as the square RS taken once to the triangle R, Q, S, thus all the squares of the sides of the polygon is to the whole polygon. But since the square RS shall be to the rectangle on RS,TQ, [TQ is the normal to RS] as the line RS to the line TQ, the square RS to the triangle RQS, therefore will be as half the rectangle RS,TQ, as the line RS to half the right line TQ, and all the squares of the sides of the polygon are to the whole polygon as the line RS to half the line TQ. Q.e.d.

PROPOSITION CCV.

From the given axes and diameter of the ellipse to find its conjugate, and from the position of the given axes, to establish the ellipse.

Construction & demonstration.

A shall be the given diameter, and BC, DE the given axes. It is required to find the conjugate diameter of A, since it is required to deduce the ellipse with the same given axes; the axes ED, BC may be put in place with the right angle ECD, and BE may be described on that semicircle ECF, in which A is put equal to EF, and FB shall be drawn: therefore since the squares of the axes ED, CB are equal to the square of any conjugate on the ellipse, and the same square, and the same squares of the axes shall be equal to the squares EF, FB, and EF shall be equal to A from one of the diameters, the right line FB is conjugate to the diameter FE: therefore we shall have shown the conjugate to the diameter A, which was the first part requiring to be found.



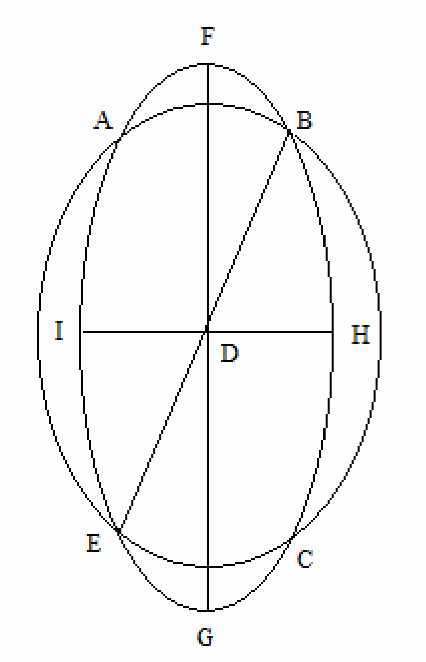
Thence the ends of the axes DBEC may be joined which shall show the parallelogram DBEC: to which the parallelogram IHKG shall become equal, so that the diameters IK, HG may have the equal right lines EF, FB. Therefore with the given conjugate of the diameters IK, HG , and thus we may show the axes with the position LM, NO for the ellipse LMN. That ellipse will be equal to the ellipse BEC of which the axes BC , ED have been given; since only a single ellipse may pass through the position of the given ends of the conjugate axis, and with IK in place, GH shall be the conjugate in the ellipse LMN ; and the same shall pertain for the ellipse BEC, the ellipses LMN, BEC and thus the axes are equal: therefore LMN is that ellipse, in which the conjugates are EF, FB; that is IK, GH in place with the given axes BC, DB that is: it was required to deduce LM, NO.

PROPOSITION CCVI.

To show geometrically those common points of intersection for a given ellipse and circle.
Moreover it will be required for the circle and the ellipse to have the same centre.

Construction & demonstration.

The ellipse ABC of which the centre is D, shall intersect the circle AB, it is required to show the point of intersection CE ; therefore since the circle and the ellipse shall have a common centre D, if from that to some point of intersection it is understood some right line to be drawn, that will be a radius of the circle and of the ellipse, therefore the circle and the ellipse shall have some common diameter, that shall be AC; then since the ellipse shall be given , also some axes FG, HI shall be given and the diameter AC. Therefore the given conjugate of this may be found, and from the preceding, that conjugate with the given axes in the same position, the position of the ellipse may be deduced, and from these alone, the points of intersection may be deduced. Which was required to be shown.



End of book four.

Translated from Latin by Ian Bruce; 24/7/2021.Free download at 17centurymaths.com.

QUADRATURAE CIRCULI

LIBER QUARTUS : DE ELLIPSI.

ARGUMENTUM

Ellipsis proprietates illiusque naturam methodicè propostituri, rem totam in sex partes dividire placuit. Ac prima quidem è cono sectionem educit, affectionesque illius essentiales, dein accidentales reliquis necessarius & fundamentales.

Secunda ellipsim dividit illiusque sectores & segmenta comparat.

Tertia, axium ac diametrorum coniugatarum tam aequalium quam inaequalium ampliorem continet considerationem. Ac illarum primo quidem contemplatur potentiam:inde lineas, quae extrema diametrorum coniungunt.

Quarta sectionis polos eorumque passiones ac lineam breuissmam à puncto in axe dato ad peripheriam designat.

Quinta varias ellipsis geneses quae tum ex lineis, tum è circula, tum ex ipsa ellipsi oriuntur continet.

Sexta ellipsim cum circulo comparat, in qua hic etiam ordo tenetur, ut primo linearum proportiones ac potentiae, secundo segmenta & ipsae sectiones, dein figurae utriq; inscriptae inter se conferantur.

Caeterùm propositiones nonnullae huius libri ac sequentium duorum sunt Apollonii, sed via longè aliâ à me demonstratae, paucis exceptis, quas nihilominus caeteris apponere visum fuit, ne quid hoc in opere quod ad conicam doctrinam pertineat, studiosus Geometriae lector desideraret. Caetera omnia, quae longe maximam atque praecipuam operis partem constituunt, a nobis & inventa sunt & demonstrata. Quare si quis in recentium quorundam Geometratum libris theoremata quaedam reperiat quae cum nostris conveniant, is velim intelligat, ea ab annis iam plurimis ac multo ante fuisse a me reperta, quam authorum illorum libri in lucem prodierint. Quae paucis lectorem meum docere volui, non ut cuiusquam inventis detraham, sed ut plagii suspicionem a me remoneam.

ELLIPISIS.

DEFINITIONES.

I.

Diameter ellipseos est, recta linea intra ellipsim ducta, quae omnes lineas, rectae cuidam aequidistantes bifariam dividit & si quidem ad rectos illas secet angulos, axis dicetur: in quaevis autem ellipsi binos esse axes, & quidem coniugatos (qui extremae dicuntur diametri) hoc est qui mutuas parallelas bisecent ad angulos rectos, suo loco patebit.

II.

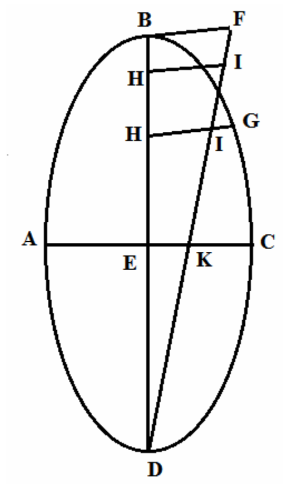
Ordinatim ad diametrum applicari dicitur unaquaeque linearum aequidistantium, ac bifariam divisarum.

III.

Centrum ellipseos est punctum quod diametrum bifariam dividit.
Quod autem lineae in ellipsi per centrum ductae bifariam secentur, propos. septima huius libri demonstrabimus.

IV.

Diametri coniugatae dicuntur quae mutuas parallelas bifariam secant.

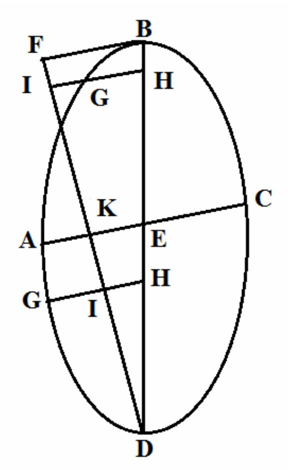


V.

Latus rectum voco lineam, iuxta quam possunt ordinatim ad diametrum applicatae sive, latus rectum est mensura iuxta quam comparantur potentiae linearum ordinatim ad diametrum positarum.

Res in exemplo erit clarior: sit ABC ellipseos diameter BD, illiusque latus rectum repraesentet FB: iunctisque FD, sumantur in diametro puncta quaevis H, ponanturque HG normales diametro BD, occurrences FD in I : singula igitur quadrata ordinatim positarum aequalia erunt singulis rectangulis BHI, (ut propositione undecimi huius demonstrabimus) quae deficiant a rectangulis FBH, rectangulo simili, ipsi FBD.

Atque quidem latus rectum tum Apollonius, tum caeteri illum hactenus secuti exposuere. Verum mihi minime videtur necessarium ut latus rectum diametro ad rectos applicetur angulos : & quadratorum ac rectangulorum loco possunt Rhombi ac Rhomboides inter se compariri. Itaque ad veterem lateris recti acceptionem, novam aliam adiicio eiusmodi ad ellipseos diametrum sint ordinatim positae quotvis rectae GH : & quaedam BF latus rectum, aequidistans ponatur ordinatim applicatis : singuli ordinatim positarum Rhombi HG in angulis IHB aequales erunt singulis IHB, Rhomboidibus in iisdem angulis, qui deficiunt a Rhomboidibus FBH per Rhomboides similes Rhomboidi FBD: demonstratiomen huius vide propos.12 huius libri.



Porro latus rectus eo ab antiquis consilio inventum est, ut certi aliquid & noti haberent, per quod reliquas sectionem proprietates intelligere ac notas sibi reddere facilius possent: ut in singulis coni sectionibus illae plane diversae sunt, ita & latera recta diversas in singulis obtinent passiones ; & rectangula lateribus rectis ac diametrorum partibus inter verticem earundem & puncta quibus ab ordinatim positis secantur interceptis contenta longe diversam in singulis, ad quadrata ordinatim positarum habent proportionem; in ellipsi quidem quadrata illa deficiunt figura simili illi quae latere recto & transverso continetur a rectangulis praedictis ; in parabola iisdem aequantur; in hyperbola vero excedunt figura simili illi, &c. Unde & nomenclaturam singulae suam sortitae sunt.

Caeterum uti potentiae ordinatim positarum ad diversas diametros , diversae quoque sunt, ita & diametris singulis, proprium & unicum latus rectum assignatur: quae omnia, uti & lateris recti inventionem, suis locis demonstrata invenies

VI.

Figura est rectangulum quod latere recto & transverso (id est diametro, nam illa quoque transversa vocari solet) continetur.

VII.

Poli seu foci ellipseos, puncta sunt (quae ex comparatione facta vocat Apollonius) in quibus axis divisus rectangulum exhibet sub segmentis contentum aequale quartae parti_ figurae: de quo suo loco agendum.

VIII.

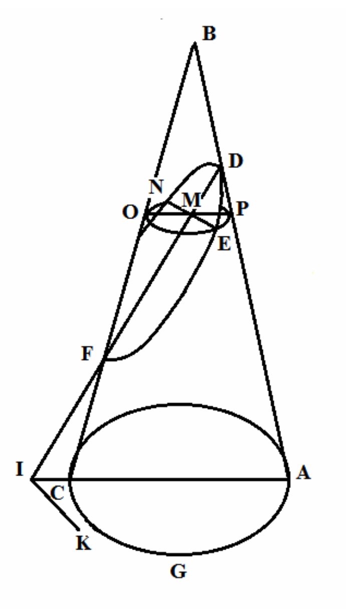
Sectio subcontraria est quando conus plano per axem sectus triangulum producente, alio rursum secatur plano, quod abscindat (triangulo producto) triangulum simile quidem, sed ita positum ut anguli qui in utroque triangulo sunt aequales ad diversa sint latera.

ELLIPSIS

PARS PRIMA

Sectionem e cono educit, primasque essentiales eiusdem exhibet proprietates.

PROPOSITIO PRIMA.



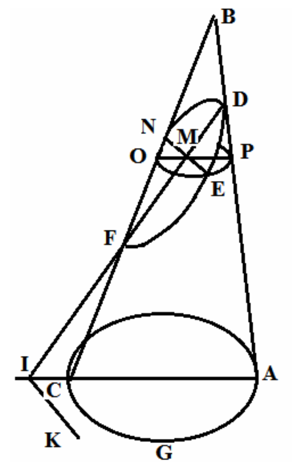
Conus rectus AGCB sectus sit plano per axem faciente triangulum ABC. Secetur alio deinde plano basi coni AGC non parallelo, cum utroque trianguli latere conveniente in D & F: ex qua (sectione producta) sit in cono figura DEFN , communis autem sectio illius plani secantis cum triangulo ABC sit DFI; eiusdem vero sectio communis cum plano in quo est coni basis AGC, sit recta IK, quam perpendicularem esse oportet ad AC diametrum basis coni, vel ad rectam quae diametro AC in directum constituitur.

Dico figuram DEFN circulum non esse.

Demonstratio.

Per punctum aliquod M rectae DF ducatur NE parallela ad IK, in plano figurae DEFN: & per idem illud punctum M ducatur in plano trianguli ABC recta OP parallela ad ACI, per lineas autem NE, OP agatur planum. Erit hoc a parallelum basi AGC [15.undecimi], ac proinde producet circulum OEPN [16.prolegone], cuius diameter erit OP.

Quoniam igitur OP est parallela ad AC, triangula BOP , BCA similia sunt, sed BCA isosceles est, ergo & BOP isosceles est. Ergo [34.de lineis] rectangulum FMD maius est rectangulo OMP ; sed rectangulum OMP [35.tertii] aequale est rectangulo NME. Ergo rectangulum FMD maius est rectangulo NME patet igitur ex 35.tertii figuram DEFN circulum non esse. Quod erat demonstrandum.



PROPOSITIO II.

Datus iam sit conus scalenus ABC, & planum secans quod producit in cono figuram DEFN, neque sit parallelum basi coni AGC, neque subcontrarie positum. Caetera vero omnia ponantur & fiant eadem quae propositione prima.

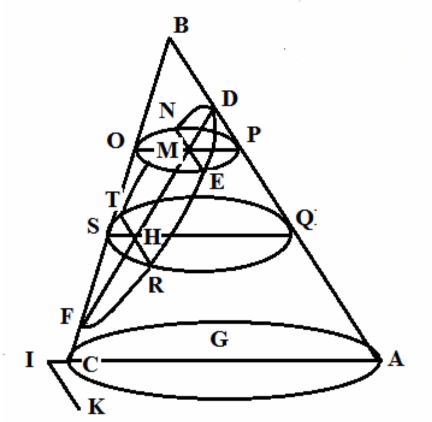
Dico rursum figuram DEFN circulum non esse.

Demonstratio.

Quoniam OP est parallela ad AC eamque secat FD in M non subcontrarie, hoc est angulum BFD non constituens aequalem angulo BAC, patet ex 36. libri nostri primi rectangulum FMD inaequale esse rectangulo OMP; sed rectangulum OMP aequatur rectangulo NME. Ergo rectangulum FMD rectangulo etiam NME inaequale est, liquet igitur ex 35.tertii figuram DEFN non esse circulum. Quod erat demonstrandum.

PROPOSITIO III.

Datus sit conus quicunque sive rectus sive scalenus, & caetera ponantur & fiant eadem quae supra:



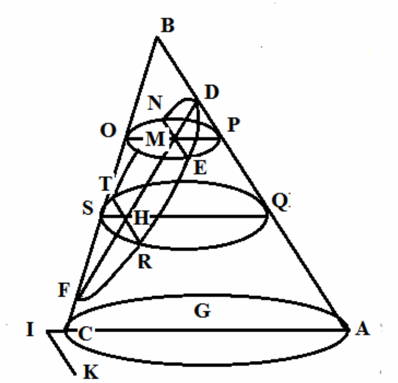
Dico rectam NE a recta DF secari bifariam in M.

Demonstratio.

Recta PM ex hypothesi est parallela ad rectam AC, & ME parallela ad IK. Quare PM, EM angulos comprehendunt aequales; atqui angulus AIK ex hypothesi rectus est communis enim sectio IK posita fuit perpendicularis ad ACI, propositione primae ergo etiam PME rectus est. Itaque cum sectioONPE sit circulus, eiusque diameter OP, manifestum est EMN, a diametro circuli OP, ad quam normalis est , bisecati in M. Sed ex hypothesi punctum M tribus rectis OP, NE, DF communis est. Ergo NE ad DF, bisecatur in M. Quod erat demonstrandum.

Corollarium.

Hinc patet, si ducantur quotcunque rectae ad IK sive NE parallelae, omnes a DF bifariam dividi eadem enim est in omnibus demonstratio. Ex quo ulterius sit manifestum sectionis DEFN ; (quam ellipsim deiceps nominabimus) diametrum esse lineam DF [Def.1], rectam vero NE [Def.2] caeterasque huic parallelas ordinatum esse ad diametrum DF applicatas.



PROPOSITIO IV.

Iisdem positis, ducatur ellipsi DEFN linea quaevis RT, parallela ad EN sive IK, secans diametrum DF in puncto H.

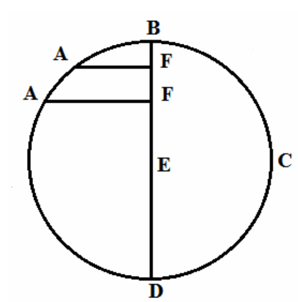
Dico rectangulum DMF esse ad rectangulum DHF ut quadratum EM ad quadratum RH.

Demonstratio.

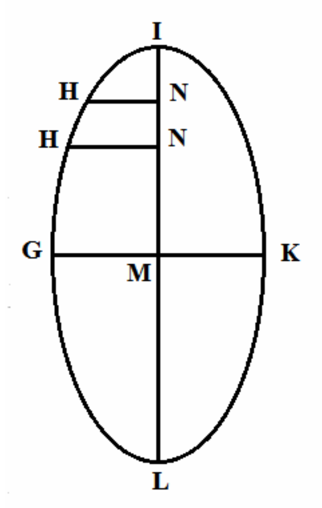
Per punctum H ducatur recta QHS parallela recta OP occurrens lateribus triangluli ABC in Q & S. Tum per rectas QS, TR agatur planum, erit hoc parallelum basi AGC ac proinde sectionem producet circulum QRS, iam vero ratio rectanguli. DMF ad rectangulum DHF componitur ex ratio DM ad DH, (hoc est, quia PM, QH sunt parallelae ex construct. ex ratione PM ad QH) & ex ratione MF ad HF (hoc est, quia MO, HS ex const. sunt parallelae, ex ratio MO ad HS.) Atqui ratio rectanguli PMO ad rectangulum QHS componitur etiam ex rationibus PM ad QH, & MO ad HS. ergo rectangulum DMF est ad rectangulum DHF ut rectangulum PMO ad rectangulum QHS; hoc est quoniam sectiones PEO, ORS circuli sunt, ut rectangulum EMN ad rectangulum DHF ut rectangulum RHT, hoc est, quia EN, RT bisectae sunt a diametro DF in M & H, ut quadratum EM ad quadratum RH. Quod erat demonstrandum.

Scholion.

Exhibuimus propositione hac proportionem rectangulorum quae a segmentis diametri ellipseos constituuntur ad quadrata, ordinatim ad eandem diametrum applicatarum: quae quidem proprietas ellipseos est primaria, & essentialis: verum quia haec ita ellipsi inest, ut etiam in circulo suo modo reperiatur, operae pretium me existamavi, si differentiam, illum inter & ellipsim, breviter in scemate apposito ostendam.

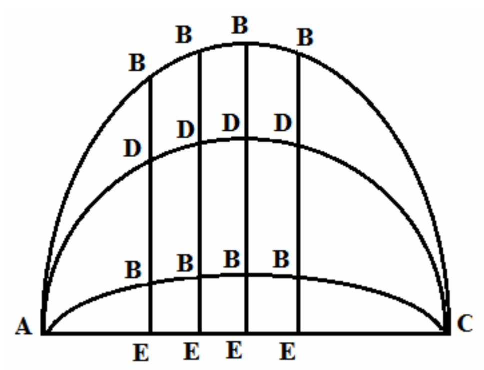


Esto circuli ABC diameter BD centrum E: & normales to diametrum AF: sit autem & ellipsis HIK diameter quaecunque IL, quam ordinatim secent HN: centrum vero sectionis M. Quoniam igitur in circulo, AF, rectae sunt normales ad diametrum BD, erunt rectangula BFD aequalia quadratis AF, proindeque, AF quadratum, est ad quadratum AF ut BFD rectangulum : eodem modo cum HN rectae in ellipsi ordimatim positae sint ad diametrum IL, erit HN quadratum ad quadratum HN, ut INL rectangulum ad rectangulum INL: illud igitur utrique sectioni convenit; eam esse proportionem inter quadrata ordinatim positarum, quae rectangulorum est, sub segmentis diametri ad quam ordinatim sunt positae: in hoc vero differunt, quod in circulo proportio rectangulorum sub segmentis diametri, ad quadrata ordinatim positarum sit aequalitatis; in ellipsi vero (si casum diametrorum coniugatarum aequalium excipias, de quo plura suo loco) inaequalitatis. quod prima & secunda huius planum fecimus.



Ex quo sequitur primo; in ellipsi axem unum altero maiorem esse, quod sic ostendo: sin in HIK ellipsi axis aliquis IL quem ordinatim secent HN: agatur per M centrum recta GK aquidistans ipsi HN: dico illos axes esse inaequales: est enim ut INL rectangulum ad rectangulum IML, sic quadratum HN ad quadratum GM, &permutando ut INL rectangulum ad quadratum HN sic IML rectangulum ad quadratum GM; sed rectangulum INL quadrato HN, est inequale ; ergo & rectangulum IML, (hoc est quadratum IM) quadrato GM inaequale est: ergo recto IM recta GM est inaequalis, ergo totas IL, nempe axis, toti GK hoc est axi alteri, inaequalis est. Quod erat propositum.

Deinde si axes in ellipsi aequales essent, iam non differret ellipsis a circulo: eo quod rectangula sub segmentis axeos aequalis essent quadratis ordinatim positarum.

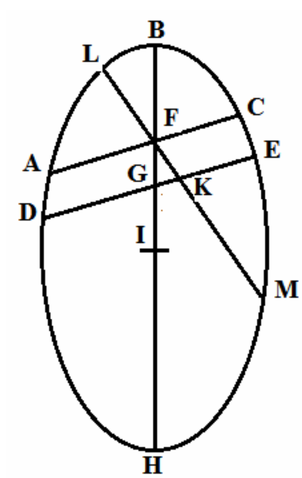


Sequitur secundo, si super axe AC ellipseos ABC, describatur semicirculus ADC, ducanturque ordinatim lineae BE occurrentes semicirculo in D , quod BE sit ad BE, ut DE est ad DE; est enim tam in ellipse quam in semicirculo, ut AEC rectangulum ad rectangulum AEC, sic BC quadratum ad quadratum BE, & DE quadratum ad quadratum DE; unde quoque est quadratum BE ad quadratum BE, sit DE quadratum ad quadratum DE.

PROPOSITIO V.

In data ellipsi diametrum invenire.

Constructio & demonstratio.



Intra ellipsim ducantur parallelae AC, DE, quas bifariam seca in punctis F, G; & per F ac G, ducatur recta BH.

Dico hanc esse diametrum.

Demonstratio est manifesta, si enim BH non est diameter, sit LFM, secans DE in K. Quoniam igitur LM ponitur esse diameter, & bisecat e parallelis unam AC, in F, bisecat alteram quoque DE in K. Quod fieri non potest cum ex construct, DE bisecta sit G.Non igitur LM aut alia quaevis ducta per F est diameter praeter eam , quae etiam transit per G, hoc est praeter ipsam BH. In data igitur ellipsi invenimus diametrum. Quod erat faciendum.

PROPOSITIO VI.

Datae ellipseos centrum reperire.

Constructio & demonstratio.

Per praecedentem quaere diametrum ellipseos BH, quam seca bifariam in L. Ex definitione tertia patet ellipseos centrum esse I.

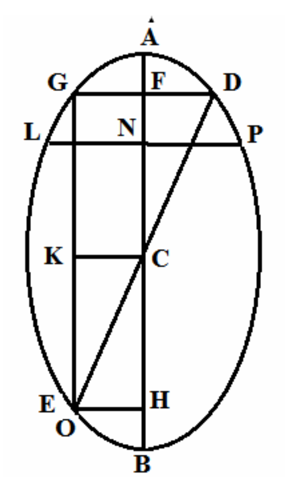
Corollarium.

Patet ex hac propositione omnem diametrum transire per centrum. Ex quo & conversam facile deduces, omnes nimirum lineas per centrum transeuntes esse diametros.

PROPOSITIO VII.

Data sit ellipsis ADB, cuius diameter AB, recta vero LP, una sit earum, quas propositione tertia huius demonstravimus a diametro secari bifariam: centrum ellipseos sit C.

Dico omnes lineas per centrum ductas in centro dividi bifariam.



Demonstratio.

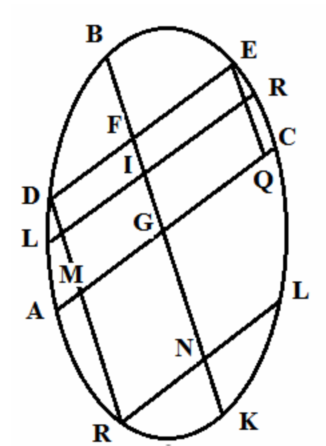
Ducta sit enim quaecunque recta DO, per centrum C; ex D ducantur DFG parallela ad LP, GE parallela ad AB, & EH, CK parallelae ad GD, sive LP. Quoniam igitur FGEH parallelogramum est, erunt GF, EH aequales : unde & quadrata GF, EH aequales sunt. Atqui ut quadratum GF est ad quadratum EH, ita rectangulum AFB est ad rectangulum AHB, aequantur igitur rectangula AFB, AHB, ergo ut AF ad AH , sic BH ad BF : ergo dividendo ut AF ad FH, sic BH ad HF aequantur ; igitur AF, BH. Quare cum tota quoque diameter AB bisecta sit in C, ut paret ex definitione centri, reliqua etiam FH, bisecta est in C.

Quoniam igitur KC ipsis GD, EH est parallela, recta quoque GE bisecatur in K, est vero & DG bisecta in F, utpote ipsi LP parallela. Ergo est ut DG ad GF, hoc est ut DG ad CK, sic GE ad KE. Ergo puncta DCE sunt in directum ; sed etiam puncta DCO sunt in directum, cum ex hypothesi DCO sit linea recta. Una igitur eademque recta sunt DCE, & DCO. Atqui DCE bisecta est in C, (cum enim ex constr: GE, FC sint parallelae, erit ut DF ad FG, sic DC ad CE.) Ergo etiam DCO bisecta est in C. Quod erat demonstrandum.

PROPOSITIO VIII.

Data sit ellipsis ABCH, cuius diameter sit BH; ordinatim vero; ad diametrum applicata LPR: centrum ellipsos G. Ducta autem sit per centrum G, recta AGC ordinatim applicatae parallela.

Dico BH, AC diametros esse coniugatas.



Demonstratio.

Sumatur in AG quodvis punctum M, per quod ducatur KD diametro BH parallela, occurrens ellipsi in punctis D & K ; ex quibus ducantur DFE, KNL; ipsi LR parallel. Quoniam igitur DK, NF parallelogramum est, rectae DF, KN, adeoque & quadrata DF, KN aequantur. Quare cum rectangulum BFH sit ad rectangulum BNH ut quadratum DF ad quadratum KN, rectangula BFH, BNH etiam sunt aequalia, ac proinde, ut ostensum in praecedenti, BF & NH aequantur. Sunt vero & BG, HG aequales. Ergo & reliquae FG, N G aequales sunt, sive FN bisecta est in G. Ergo & KD parallela diametro BH bisecatur in M ab AC. Similiter ostendam quasvis alias diametro BH parallelas bisecati ab AC. Quare cum etiam AB bisecet DE, LR ceterasque omnes quae sunt ordinatim positae ad BH, & parallelae ex hypothesi ipsi AC; patet ex definitione
BH, AC diametros esse coniugatas.

Corollarium.

Quae per centrum ad ellipseos axem datum perpendicularis ducitur, est axis dato axi coniugatus.

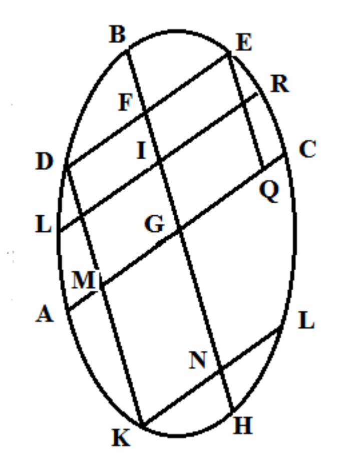
Ex discursu iam allato facile sibi lector demonstration huius rei eliciet.

PROPOSITIO IX.

Data sit ellipsis eiusque diameter BH.
Oporteat diametro BH coniugitam diametrum exhibere.

Constructio et demonstratio.

Ducatur recta aliqua KD parallela ad BH, & utraque D K, BH divisa bifariam M & G, per M & G, ducatur AC.



Dico AC, BH coniugatas esse diametros.

Ac primo quidem rectam AC esse diametrum patet ex 5.huius & BH diameter est ex hypothesi; ambae igitur sunt diametri. Quod autem sint coniugatae sic ostendo. Quoniam AC diameter est & bisecat KD, erit KD ad AC ordinatim applicatae; ergo & reliquae ipsi KD parallelae, erunt ad AC, ordinatim applicatae, hoc est a diametro AC bifariam secabuntur. Sed DK ex constructione cum sibi parallelis, parallela est ad diametrum BH. Ergo diameter AC bisecat diametro BH parallelas. Ducantur deinde DE, parallela diametro AC & EQ, paralla rectae DK parallelogrammum igitur est MDEQ, in quo quia FG ipsis DM, EQ ducta est parallela, erit DF ad FE, ut MG ad GQ; latera quoque parallelogrammi DM, EQ adeoque & quadrata DM, EQ aequalia erunt. Iam vero quia DM, EQ, sunt ad diametrum AC, ordinatim positae, ratio inter rectangula AMC, AQC erit eadem quae inter quadrata DM; EQ, hoc est aequalitatis : ac proinde ut patet ex demostratis in septima huius, aequales erunt AM, QC. Quare cum & totae AG,CG aequales sint (est enim B centrum ellipseos ; quia bisecat diametrum BH) etiam reliquae MG, QC, aequales sunt; quoniam igitur est ut MG ad GQ, sic DF ad FE, etiam DF, FE aequantur, hoc est DE bisecta est in F. Ergo ex definit, DE est ad diametrum BH ordinatim posita, ergo & reliquae ipsi parallelae sunt ad BH ordinatim positae, hoc est bisecantur a BH. Atqui DE ex constructione, cum sibi parallelis, est alteri diametro AC parallela. Ergo diameter BH bisecat parallelas diametro AC. Quare cum etiam prius ostenderim AC bisecare parallelas ad BH, erunt BH, AC diametri coniugatae. Factum igitur est quod petebatur.

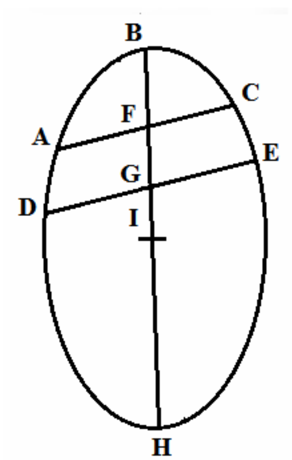
Corollarium primum.

Dato ellipseos axi; axem coniugatum invenies, si per centrum ellipseos duxeris rectam lineam dato axi perpendicularem, res patet ex corollario octavae.

Corollarium secundum.

Ex hoc problemate sit manifestum qua ratione ex dato in ellipsi puncto D, ad diametram BH, recta linea ordinatim debeat applicari. lnveniatur enim AC diameter coniugata diametro BH; & ex dato puncto D ducatur DFE ipsi AC parallela.

Dico DFE ordinatim esse positam ad diametrum BH. Demonstratio patet ex propositione.



PROPOSITIO X.

Ordinatim positarum (AC, DE, &c.) illa maior est quae centro (I) vicinior.

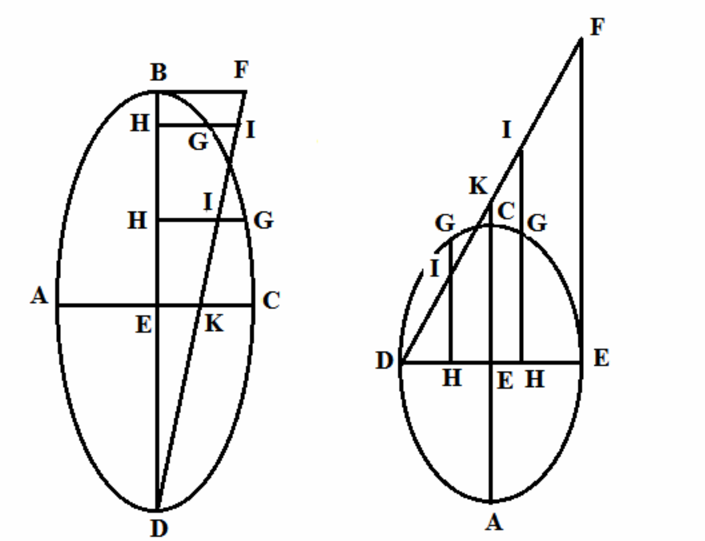
Demonstratio.

Rectangulum BGH maius est rectangulo BFH ut patet ex quinta secundi. Atqui quadratum EG est ad quadratum CF ut rectangulum BGH ad rectangulum BFH. Ergo quadratum EG maius est quadrato CF. Ergo & ordinatim posita EG maior est ordinatim posita CF. Quod erat demonstrandum.

PROPOSITIO XI.

Esto ABC ellipsis axis BD; oportet illius latus rectum exhibere.

Constructio & demonstratio.



Axi BD per E centrum ducatur coniugatus AC, fiantq; continuae BD, AC, BF : dico FB esse latus rectum aequidistet enim BF ipsi AC: ducanturque ordinatim lineae GH quae iunctae FD occurrant in I ipsa vero FD secet AC lineam in K. Quoniam EC, GH ordinatim positae sunt ad axem BD, erit ut quadratum GH ad quadratum EC, sit BHD rectangulum ad rectangulum BED: sed ut BHD rectangulum ad rectangulum BED, sit IHB rectangulum est ad rectangulum KEB (quia ex iisdem rationem habent compositam scilicet ex BH ad BE, & ex HD ad ED, hoc est HI ad EK.) igitur ut quadratum GH ad quadratum CE, sic IHB rectangulum est ad rectangulum KEB, & permutando invertendo ut KEB rectangulum ad quadratum CE, sit IHB rectangulum est ad quadratum GH: sed cum AC quadratum sit aequale rectangulo super FBBD (cum ex construct: BD, AC,BF sint trcs continuae) erit EC quadratum, (nimirum quarta pars quadrati AC, est enim F, AC bisecta in E) aequale rectangulo KEB quartae parti rectanguli sit per FBBD. Igitur & quadratum HG aequale est rectangulo IHB: ergo HG potest spatium quod adiacet ipsi FB latitudinem habens HB, deficiens ab FBH rectangulo, similis figurae rectangulo, BFD quare FB latus rectum est; exhibuimus ergo, &c. Quod erat faciendum.

Corollarium.

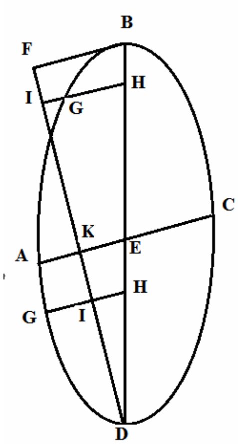
Hinc sequitur primo quatuor lineas, nimirum latus rectum axis minoris, axem maiorem, axem minorem , & latus rectum axis maioris in continua esse analogia.

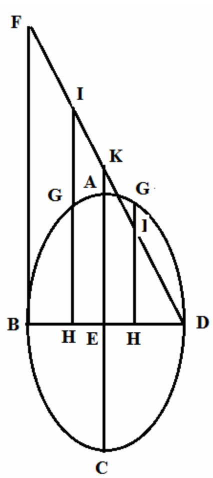
Sequitur secundo qui datis lateribus rectis axium, ellipsiu exhibuerit, quod inter binas datas, duas medias invenerit.

PROPOSITIO XII.

Esto ABC ellipsis diameter quaecunque BD , oportet illius latus rectum exhibere.

Constructio & demonstratio.



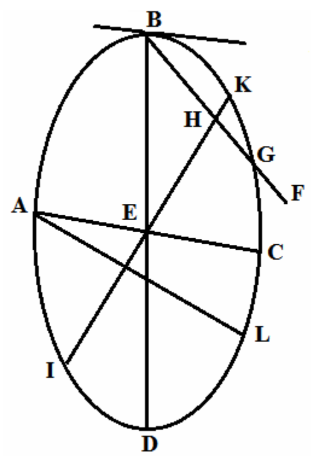


Ducatur per E centrum diameter AC coniugata ipsi BD, fiantque continuae BD, AC, BF, ac BF quidem aequidistet diametro AC : ducaturque linea FD, quae AC rectum secet in K, ducanturque ordinatim lineae GH, quae FD lineae occurrant in E Quoniam BD, AC , FB lineae sunt continuae, erit AC quadrato aequale rectangulum super FB, BD; igitur & quadrato AE (nimirum quartae parti quadrati AC , est enim AC bisecta in E) aequale rectangulum super KE, EB, quarta pars rectanguli FBD: quare, ut in praecedenti ostendimus, ita etiam ostendemus HG quadrato aequale esse rectangulum IHHB, atqui Rhombus IH in angulo IHB est ad Rhombo idem IHB in eodem angulo, ut quadratum IH ad rectangulum IHB, (rationes enim Rhombi ad Rhomboidem & quadrati ad rectangulum ex iisdem rationibus componuntur nempe ex IH, ad IH & IH ad HB,) ergo cum quadratum IH aequale sit rectangulo IHB, etiam Rhombus IH, Rhomboidi IHB aequalis erit; recta igitur IH potest Rhomboidem in angulo ordinatim applicatae IHB, qui (quod demonstratum est facile) a Rhomboide FBH in eodem angulo, deficit Rhomboide simili ei qui in eodem angulo sit a diametro DB & recta BF. Igitur FB est latus rectum. Quod petebatur.

PROPOSITIO XIII.

Omnis recta (BF,) quae per terminum diametri (BD) ducitur ordinatim applicatae (AC) aequidistans, ellipsim contingit.
Et quae tangenti ducitur parallela, est ordinatim ad diametrum applicata.

Demonstratio.

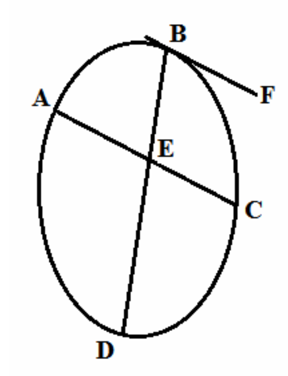


Si enim recta BF non contingat ellipsim, secet illam in G; divisaque BG bifariam in H, agatur per H & E, KI occurrens utrimque peripheriae in I & K. Quoniam recta BG per constructionem aequidistat AC, utramque autem bifariam secet recta IK erit IK diameter & AC, BG lineae ordinatim ad illam positae; secat autem per constructionem recta AC ordinatim quoque diametrum BD, igitur una eademque recta AC ordinatim secat duas diametros BD, IK. Quod fieri non potest; alias enim quae ipsi AC duceretur parallela, etiam utraque diametro BD, KI ac proinde in duobus punctis bisecaretur, igitur patet FB lineam , sectionem contingere : quod erat primum. Quod si tangenti BF parallela ducatur quaevis AC occurrens diametro in E, erit ordinatim ad diametrum posita. Si non ducatur ex A ordinatim AL: erit AL parallela contingenti BF; quare & ipsi AC aequidistat, quod fieri non potest, cum eandem secet in A : igitur AL non est ordinatim posita nec quaevis alia praetcr AC, quod erat alterum. Patet igitur veritas propositionis.

Translated from Latin by Ian Bruce; 24/7/2021.Free download at 17centurymaths.com.

PROPOSITIO XIV.

Per datum in peripheria punctum contingentem ducere.

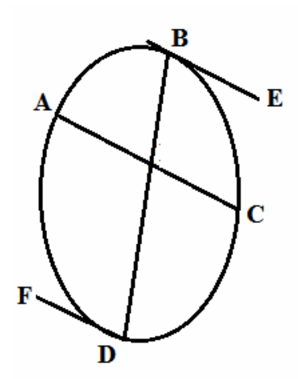


Constructio & demonstratio.

Esto ABC ellipsis & punctum in peripheria datum B, oportet per B rectam ducere quae sectionem contingat in B inveni centrum, & per hoc ex dato puncto B duc diametrum BD, ad quam ponatur ordinatim quaevis linea AC, cui per B agatur parallela BF, manifestum igitur est BF esse tangentem; igitur per datum in peripheria punctum, &c. Quod erat faciendum.

PROPOSITIO XV.

Lineae quae per extremitates diametri ductae, ellipsim contingunt, inter se aequidistant.

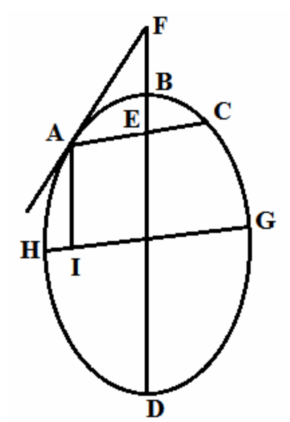


Demonstratio.

Ducatur enim quaevis AC ordinatim ad diametrum manifestum est ex 13.huius tam BC quam DF lineas illa aequidistare, adeoque & inter se. Quod fuit demonstrandum.

PROPOSITIO XVI.

Contingentes ductae per extremitates ordinatim positae, conveniunt cum diametro extra sectionem.

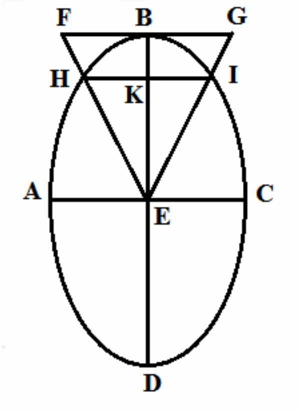


Demonstratio

Sit ABC ellipsis diametcr BD, & ordinatim posita AEC, agaturque per tangens AF; dico illam cum diametro convenire in F. Inventa enim HG diametro coniugata ipsius BD, demittatur ex A linea AI aequidistans BD, quoniam igitur AI, BD aequidistant, & AF occurrat rectae AI, patet productam quoque convenire cum BD. Quod fuit demonstrandum.

PROPOSITIO XVII.

Ellipsim ABC cuius axis BD, contingat in B linea FG, sumptisque; in contingente aequalibus partibus FB, BG, dem ittantur ex F & G diametri duae FE, GE occurrentes ellipsi in H & I. Dico iunctam HI aequidistare ipsi FG



Demonstratio.

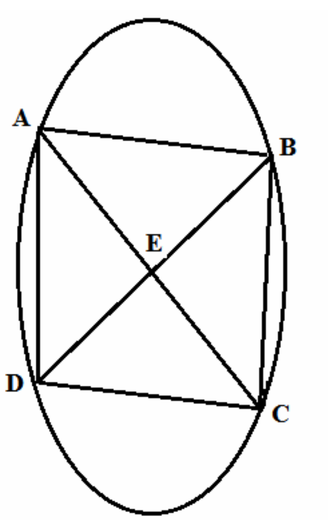
Ponatur HK parallela FG , quae producta occurrat EG in I; erit itaque HK aequalis KI. Ergo cum rectangulum BKD ad BED, eam habeat rationem quam HK quadratum ad quadratum AE, erit quoque BKD, ut IK quadratum ad quadratum EC: unde punctum I est ad ellipsim, & HI linea perimetrum BIC, & EG rectam in eodem puncto intersecat. Quod fuit demonstrandum.

PROPOSITIO XVIII.

Eadem manente figura, sint ABC ellipseos axes AC, BD, & HE diameter quaecunque, oportet ex E versus C diametrum educere, aequalem ipsi HE.

Constructio & demonstratio.

Fiat angulo BEH aequalis angulus BEI, dico rectam EI satisfacere petitioni; productae enim lineae HE, EI, occurrant actae per B contingenti in F & G. Iungantur puncta H, I; quoniam anguli BEH, BEI ponuntur aequales, sunt autem & EBF, EBG anguli recti, & BE linea communis, patet FBE, GBE triangula, adeoque & latera FB, BG inter se aequaliae unde & HI aequidiidistat FG, estque Ut FE ad GE, sit HE ad le, quare HE, IE lineae aequales; igitur ex E diametrum eduximus, &c. Quod erat faciendum.



PROPOSITIO XIX.

Lineae in ellipsi coniungentes extrema quarumcunq; diametrorum inter se sunt aequales & parallelae.

Demonstratio.

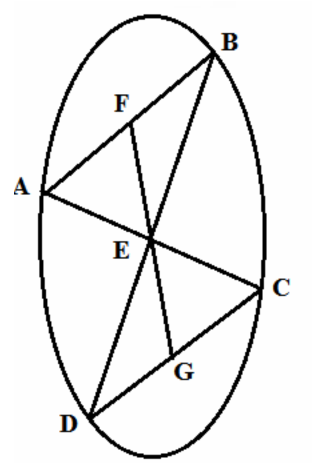
Secent ABC ellipsim diametri duae quaevis AC, BD; dico iunctas AB, CD, item AD, BC, esse interse aequales & parallelas: cum DB, AC bisectae sint in E, erit ut DE ad EB , sic CE ad AE , & permutando ut DE ad CE, sic BE ad AE sunt vero & anguli E aequales, similia igitur sunt triangula DEC, AER; ergo ut DE ad EB, sit DC ad AB; quare cum DE: EB aequentur, etiam AB, DC aequales erunt. Similiter ostentemus AD, BC aequales esse. Quod erat demonstrandum.

PROPOSITIO XX.

Lineae quae ad extremitates diametri, intra sectionem aequidistances ponuntur, aequales quoque erunt inter se,

Demonstratio.

Secet ABC ellipsim diameter quaecunque BD, ducanturque ex B & D, intra sectionem parallelae AB, CD. Dico illas inter se esse aequales. Invento centro E, & AB bisecta in F, iunge FE, & produc in G, & quoniam EF diameter bisecat AB, biseceat etiam DC, ipsi AB parallelam. Deinde quia similia sunt triangula FEB, DEG; erit DE ad DG, ut EB ad BF: & permutando ut DE ad EB sit DG ad BF; sed DE, EB aequantur, ergo & BF, DC, qui sunt, ut iam ostendi, ipsarum AB, DC dimidiae. Ergo & totae AB, DC aequales sunt, Quod erat demonstrandum.

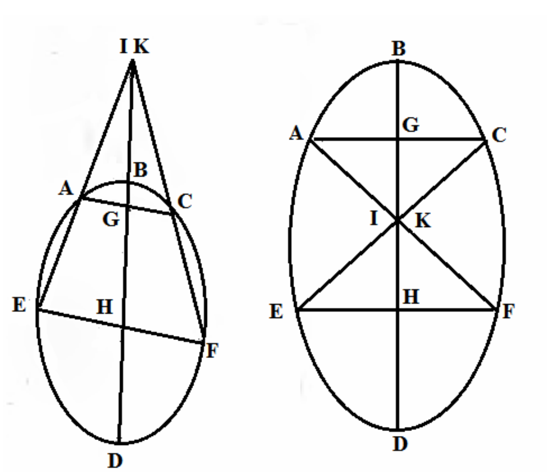


Corollarium.

Hinc sequitur iunctas AE, EC esse in directum: cum enim latera AF, FE aequalia sint duobus lateribus CG, GE & anguli aequalibus lateribus contenti, aequales, patet AFE, CGE triangula esse inter se aequalia, & angulum AEF aequalem angulo CEG, adeoque AE:EG lineas in directum.

PROPOSITIO XXI.

Lineae per extremitates duarum parallelarum inaequalium in ellipsi ductae, conveniunt in eodem puncto cum diametro, ad quam ordinatim positae sunt paralleleae.



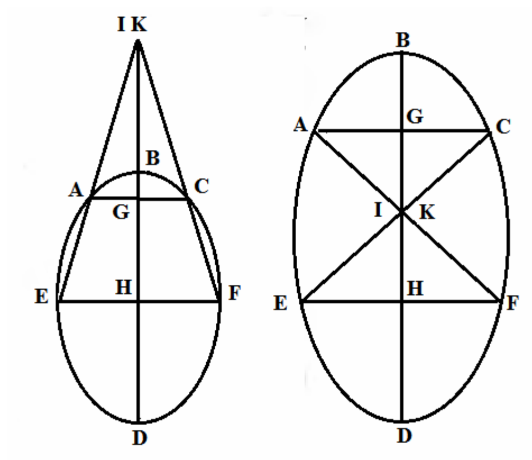
Translated from Latin by Ian Bruce; 24/7/2021.Free download at 17centurymaths.com.

Demonstratio.

Secent ECD ellipsin duae quaevis parallelae inaequales AC, EF, ordinatim positam ad diametrum DB. Dico iunctas EA, FC cum BD diametro quam secant ordinatim in eodem puncto convenite. Quoniam ordinatim ponuntur lineae AC, EF ad diametrum BD, ambae bisecantur in G & H, unde AG ad GC ut EH ad HF, & permutando ut AG ad EH, sic GC ad HR, concurrat iam EA cum diametro in Ii alter vero FC in K, erit ergo ut IG ad IH, sic IA ad IE, sed est, ut ante ostendi, ut AG ad EH, hoc est ut IG ad IH, ergo puncta I & K eadem sunt; ergo punctum I communis est intersectio rectarum EI, FI, HI. Quod erat demonstrandum.

PROPOSITIO XXII.

Sit ABC ellipseos diameter BD ad quam ordinatim posita sit EF, ducanturque ex E & F linea occurrentes diametro in puncto G, ellipsi vero in A &C.
Dico iunctam AC, aequidistare EF.
Demonstratio.



Ponatur AI parallela EF & producta occurrat FG lineae in C, quoniam igitur EH aequalis est HF, erit & AI ipsi IC aequalis; sed quia AI aequidistat EF, erit BID rectangulum ad rectangulum BHD , ut AI quadratum ad quadratum EH. Ergo etiam, ut rectangulum BID ad rectangulum BHD, ita quadratum IC ad quadratum HF; unde punctum C est ad ellipsim & communis intersectio rectarum FG, AI cum perimetro BCF; ac proinde AC iungens puncta A, C, aequidistat EF. Quod erat demonstrandum.

PROPOSITIO XXIII.

In ellipsi ductae sint parallelae AC, EF, per quarum terminos ducantur EA, FC coeuntes in G & per G ducta GIH bisecet parallelam AC.
Dico etiam alteram bisecati.

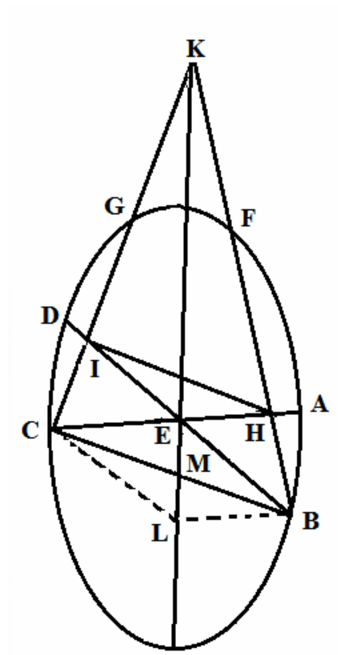
Demonstratio.

Ut HG ad IC, sic EH ad AI, & ut HG ad IG, sic FH ad CI; ergo EH ad AI, ut HF ad IC; ergo permutando EH ad HF, ut AI ad IC; sed AI, IC aequantur, ergo & EH, HF aequantur, adeoque tam EF quam AC sunt bisectae; ergo GIH diameter est. Quod erat demonstrandum.

PROPOSITIO XXIV.

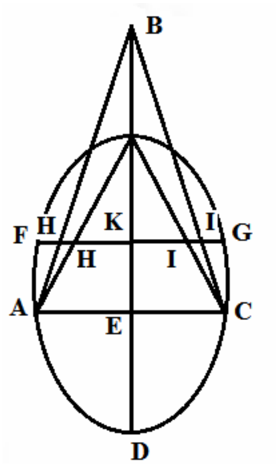
Secent ABC ellipsim diametri duae AC, BD, iunctaque BC, agatur per E centrum diameter KL, secans BC bifariam K secantes secans BC bifariam in M, & ex B & C rectae ducantur BF, CG, ad idem diametri punctum K secantes AC, BD lineas in H & I.
Dico rectangulum AHC esse ad rectangulum DIB ut quadratum AC ad quadratum DB.

Demonstratio.



Ponatur ex C linea CL parallela BD , occurrens diametro KL in L, & iunge BL, quoniam CL aequidistat DE, erunt EMB, CML triangula inter se similia : quia vero CM, MB aequales sunt, aequalia quoque erunt triangula CML, EMB & lateri EM, aequale latus LM; igitur in triangulis BML, CME, duae latera CM, ME,aequalia sunt duobus lateribus BM, ML, sed & anguli iis contenti BML, EMC aequantur. Ergo ad bases anguli LBM, ECB aequantur ; ergo BL, CEA sunt parallelae. Ergo BH ad HK , ut IE ad EK, hoc est (quoniam ex constructione BI, CL sunt parallelae ut CI ad IK. Ergo IH aequidistat CB, & est ut HE ad EC, sic IE ad ER, & componendo ac permutando ut EC ad EB, sic HC ad BI, sed ut CE ad BE, sic AC est ad BD, cum utraque in centro divisa sit bifariam; igitur ut AC ad BD, sic HC ad BI. Ergo etiam ut AC ad DB, sic AH ad dL. Quare cum rectangulum AHC ad rectangulum DIB rationem habeat compositam ex laterum rationibus AH ad DI, & HC ad BI, quae ambae ostensae sunt eadem esse cum ratione AC ad BD, erit rectangulorum ratio duplicata ratione AC ad BD, hoc est eadem quae quadratorum AC, BD. Quod erat demonstrandum.

PROPOSITIO XXV.



Duae lineae CG,BF intra ellipsim ductae occurrant diametro ellipseos MK in eodem puncto K. Ductae sint deinde binae aliae diametri BD, CA quae ita secentur a rectis CG, BF ut rectangula BID, CHA quadratis BD, AC proportionalis sint.

Dico iunctas IH, CB esse parallelas.

Demonstratio.

Quoniam est ut quadratum BD ad quadratum CA, hoc est ut quadratum ED ad quadratic EA, sic rectangulum BID ad rectangulum CHA ; erit permutando ut quadratum ED, (hoc est rectangulum BID cum quadrato EI ad rectangulum BID est ad quadratum EI) ad rectangulum BID, ut quadratum EA, (hoc est rectangulum CHA cum quadrato EH) ad rectangulum CHA ad quadratum EH. Permutando igitur rectangulum BID est ad rectangulum CHA ut quadratum EI ad quadratum EH sed etiam est rectangulum BID ad retangulum CHA ut quadratum BD ad quadratum CA; hoc est ut quadratum ED ad quadratum EA. Itaque quadratum EI est ad quadratum EH ut quadratum ED ad quadratum EA: adeoque recta EI ad rectam ED, hoc est EB, ut recta EH ad rectam EA, hoc est EC, parallelae sunt igitur IH, CB. Quod erat demonstrandum.

PROPOSITIO XXVI.

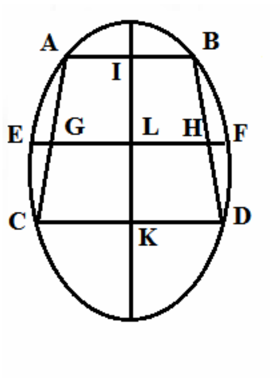
Esto ABC ellipsos diameter BD, ad quam ordinati, posita sit recta AC : ductisque ex A & C lineis quae diametrum in eodem puncto B secent, ducatur FG parallela AG, occurrens AB, CB in H & I, diametro vera BD in K.

Dico FH, GI lineas esse aequales.

Demonstratio.

Quoniam FG aequidistat AC ordinatim posite ad BD, erit & FG, quoque ordinatim posita ad diametrum BD, adeoque in K bifariam divisa ; sed & HI in K divisa est bifariam, uti AC in E, demptis igitur aequalibus HK, IK, reliquae FH, IG aequales sunt. Quod erat demonstrandum.

PROPOSITIO XXVII.



Secent ABC ellipsim ducae quaevos parallelae AB, CD, iunctisque AC, BD, ducatur EF parallela AB, secans AC, BD lineas in G & H.
Dico EG, FG rectas esse aequales.

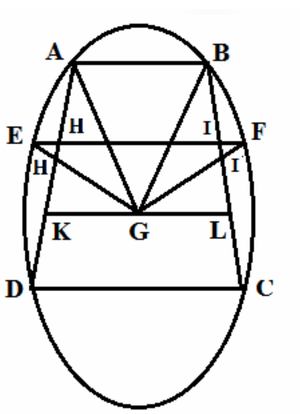
Demonstratio.

Divisis AB, CD bifariam in I & K, agatur per I & K, linea IK; erit illa diameter, & EF lineam, rectae AB parallelam secabit bifariam in L; sed & HG in L secta est bifariam ut CD in K; vel AB in I , ablatis igitur aequalibus GL, LH, manent EG, FH, reliqua aequales.Quod erat demonstrandum.

PROPOSITIO XXVIII.

Secent ABC ellipsim duae quaevis parallelae AB, CD, iunctisque AD, BC ducatur ENMF parallela AB, & ex E & F, semidiametri ponantur EG, FG, quae AD, BC lineas secent in H & I.
Dico EG, FG in H & I proportionaliter esse divisas.

Demonstratio.



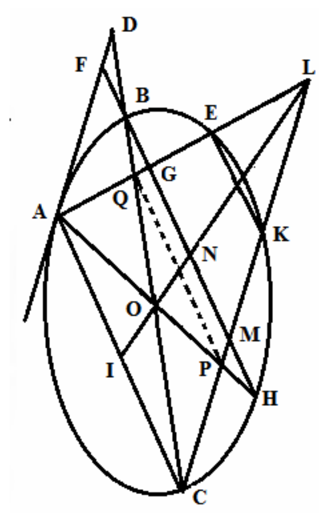
Ducatur per G, KL aequidistans AB , occurrens AD, BC, in K & L. Quoniam EF, KL aequidistant, erit ut EN ad KG, sic EH ad HG; & FI ad IG, ut FM ad LG; sed ut EN ad KG, sic FM est ad LG, (cum EN, FM item KG, LG, aequales sint,) igitur ut EH ad HG, sit FI ad IG. Quod fuit demonstrandum.

Corollarium.

Hinc patet iunctam HI aequidistate DC, adeoqae lineas AD, BC, in H & I proportionaliter esse divisas.

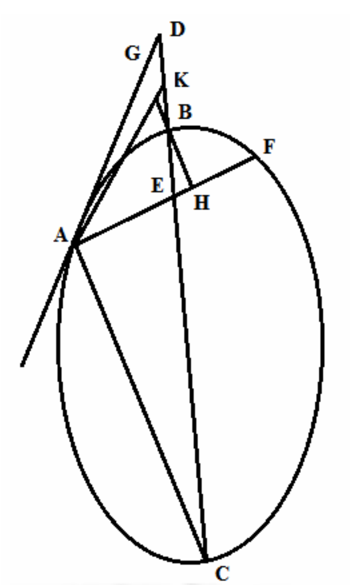
PROPOSITIO XXIX.

Ellipsim, cuius diameter BC centrum O, contingat AD occurrens diametro in D, ductaque ex puncto A ordinatim AQE, & iuncta AC, per B ponatur recta FBG parallela rectae AC.
Dico FB, BG aequales esse.
Demonstratio.



FG occurrat ellipsi in H, iunganturque HO, AO quae erunt in directum. Tum AC bisecta in I, ducatur per I diameter IOL occurrens rectae AE in L & iungantur puncta LC, per rectam LC, occurrentem ellipsi in K, & rectae FG in M; rectae vero HO in P.
Quoniam AC ex constructione ordinatim posita est ad diametrum IL, rectaeque per A & C duplae occurrunt diametro in eodem puncto L, erit EK parallela AC. Est vero & BH parallela ipsi AC ex hypothesi : & semidiametra OB, OH secant AE, CK in Q & P, ergo QP aequidistat rectae BH, tres igitur AC, QP, EK sunt parallelae. Quare cum ex hypothesi AE bisecta sit in Q, erit & CK bisecta in P, ac proinde ordinatim posita ad diametrum AH. Itaque CK aequidistat tangenti AD. Est autem & FM ex hypothesi parallela ad AC, ergo FM aequalis est AC , sed etiam BH aequalis est AC. Igitur FM, BH aequales sunt; quare communi dempta BM aequantur FB, HM. Atqui etiam GB, HM aequales sunt. itaque FB, GB aequales sunt. Quod erat demonstrandum.

PROPOSITIO XXX



Ellipsim ABC cuius diameter BC contingat in A recta AD conveniens cum diametro in D : ductaque ex A sit linea AF ordinatim ad diametrum BD.
Dico rectam DC in B & E divisam esse extrema & media ratione proportionali, hoc est, ut CD est ad BD, sic CH est ad HB: &: si divisa fuerit in B & E extrema & media ratione proportionali agaturque per E ordinatim linea AF ad BC: dico iunctam AD, sectionem contingere.
Demonstratio.

Iuncta AC, agatur per B linea GH parallela recta AC ocurrens AF lineae in H & AD tangenti in G. Quoniam AC, BH lineae aequidistant, erit ut AC ad BH, sic CE ad EH, sic CE ad EB: sed ut AC ad BH, sic AC est ad GB, (quia GB, EH sunt aequales) igitur ut AC ad GB, sic CE est ad BE: est autem ut AC ad GB, sic CD ad DB (quia GB, AC aequidistant) igitur ut CD ad DB, sic CE est ad BE. Quod erat primum iam ut CD ad BD, sic CH ad HB, si per E ordinatim agatur AF: dico iunctam AD sectionem contingere in A. si enim AD non tangit, ponatur per A tangens quae BD diametro occurrat in K, erit igitur ut CE ad EB, sic CK ad KB, sed est ut CE ad EB, sic CD ad DB, igitur CK ad KB, quod fieri non potest, cum punctum K supra vel infra D cadat. Igitur AK non est tangens nec quaevis alia praeter AD. Quod fuit demonstrandum,

Corollarium.

Propositiones 29 & 30 etiam in circulo sunt verae, quamvis autem saepius contingat ut quae hoc libro de ellipsi demonstramus locum etiam habeant in circlo, circuli tamen mentionem non facio nisi ad sequences demonstrationes assumi debeat.

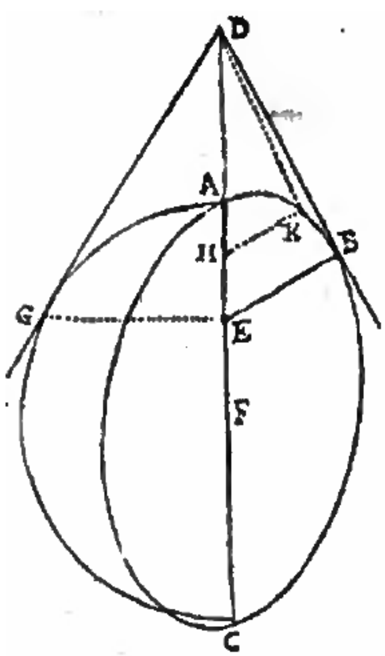
PROPOSITIO XXXI

Eadem manente figura propositum sit a dato extra sectionem puncto D, tangentem ducere.
Constructio & Demonstratio.

Ducatur ex D diameter DBC, fiatque ut CD ad DB, sic CB ad EB, & per E ad BC, ordinatim ponatur AF, iungturque AD , patet per praecedentem AD lineam sectionem in A contingere; igitur a, dato extra ellipsim puncto; &c. Quod erat faciendum,

PROPOSITIO XXXII

Ellipsim ABC cuius diameter AC contingat recta BD in B, conveniens cum diametro in D : & ex B ducatur BE ordinatim ad diametrum AC: centrum autem sectionis sit F.

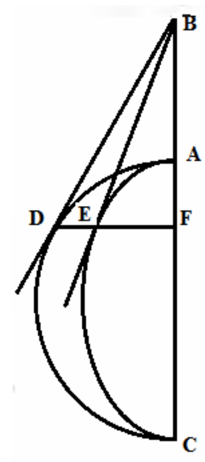


Dico FE, FA, FD lineas esse in continua ratione & si FE, FA, FD fuerint continuae proportionaLEs, & per E ordinatim recta agatur EB, dico iunctam BD sectionem contingere, Est Apollonii.
Demonstratio.

Centro F intervallo FA circulus describatur AGC, tum ex E puncto normalis educatur ad diametrum AC occurrens circulo in G: ducaturque recta GD, quoniam EB recta ponitur ordinatim ad diametrum AC & per B acta tangens convenit cum eadem diametro in D, erit ut CD ad DA sic CE ad EA : est autem in circulo, recto EG normalis ad diametrum

EG, igitur &. recta GD circulum contingit in G; quare in circulo erunt FE, FA, FD lineae continuae proportionaLEs : sunt autem eaedem lineae communes ellipsi, igitur & in ellipsi erunt FE, FA, FD in continua analogia. Quod si FE, FA, FD continuae proportionaLEs sint, & per E ducatur ordinatim EB, dico iunctam BD ellipsim contingere in B; sin vero: ducatur ex D recta DK contingens ellipsim in K, &: ex K ordinatim ponatur KH; igitur per primum partem huius FH ad FA, ut FA ad FD; sed etiam ex hypothesi est, FE est ad FA, ut FA ad FD. Ergo FE est ad FA, ut FH est ad FA, quod fleri non potest, cum FG sit maior aut minor quam FE. unde DK non est contingeris, sed DB. Quod fuit demonstrandum.

PROPOSITIO XXXIII



Esto ABC ellipseos axis AC, super quo ut diametro semicirculus describatur ADC, assumptoque in axe puncto F quod non sit centrum, erigatur ex F orchogona FD occurrens ellipsi in B.
Dico contingentes per B & D actas, axi AC in uno eodemque puncto occurrere.

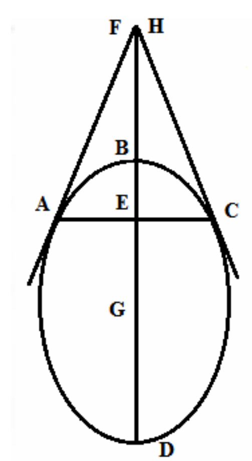
Demonstratio.

Agatur per B contingens BE, conveniens cum axe in E, iunganturque ED; quoniam FB ordinatim posita est ad axem & BE sectionem contingit, erit CF ad FA, ut CE ad EA: unde & iuncta ED circulum contingit; igitur contingentes per B & D actae, conveniunt cum axe in uno eodemque puncto. Quod fuit demonstrandum.

Corollarium.

Hinc facile etiam demonstrabimus si duae tangentes in eodem puncto diametro occurrant normalem FD, quae si per unum contactum D transeat, transire etiam per alterum.

PROPOSITIO XXXIV



Esto ABC ellipsis diameter BD, ad quam ordinatim ponatur AC aganturque; per A & C contingences.
Dico illas diametro in uno eodemque puncto occurrere.

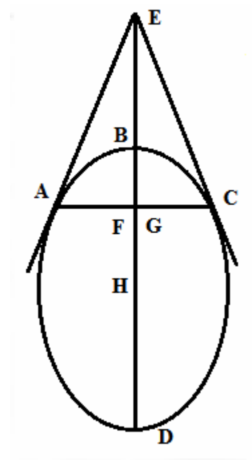
Demonstratio.

Per 16.huius patet singulas contingences per A & C ductas cum diametro convenire si igitur non conveniant in eodem puncto, occurrat AF contingens diametro in F, & CH in H: Quoniam tangens AF concurrit cum diametro in F, erit ut DE ad EB ; sic DF ad FB; sic

DH ad BH, quod fieri not potest; quare tangentes non occurrunt diametro in diversis punctis: ergo in eodem. Quod erat demonstrandum.

PROPOSITIO XXXV

Esto ABC ellipsis, diameter BD producta utcunque in E, & ex E demissae EA, EC sectionem contingant in A & C .
Dico iunctam AC, ordinatim esse positam ad diametrum BD.

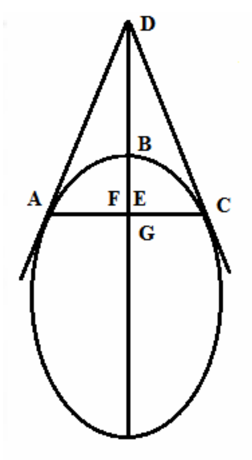


Demonstratio.

Ponatur AF ordinatim ad BD sitque H centrum ellipseos; erit igitur linea EH divisa
in B & F in tres continuae proportionales, demittatur quoque CG ordinatim ad BI erit
denuo EH divisa in B, & G, in tres lineas in analogia continuae igitur F & G, puncta sunt
eadem; quare recta AFC, est ordinatim posita ad diametum BI. Quod fuit demonstrandum.

Corollarium.

Hinc sequitur si ellipsim ABC contingant in A & C, rectae duae AD, CD convenientes in D; iunctaque AC, bifariam secetur in E; rectam DE transire per centrum sive iunctam DE esse diametrum sectionis, si enim ED non sit diameter, ducatur ex D diameter DF; occurrens AC lineae in F; erit igitur per praecedentem AC linea in F; divisa bifariam, adeoque punctum F, idem cum E, unde DF recta eadem cum linea DE: quod est contra suppositum; quare DE sectionis est diameter. Quod erat demonstrandum.



PROPOSITIO XXXVI.

Si ellipsim tangant binae rectae coeuntes in D, & ex centro ducantur GA, GC, GD.
Dico triangula GCD, GAD esse aequalia.

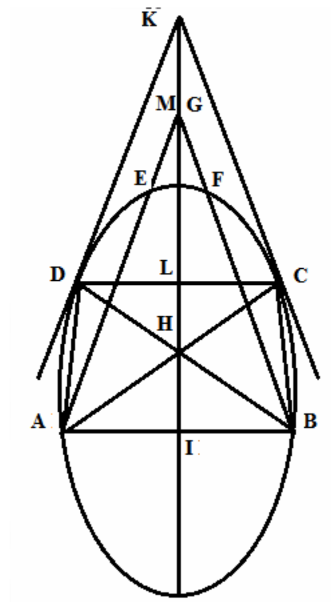
Demonstratio.

Puncta contactuum iungantur recta AC, quoniam AC bisecta est in E, triangula

GAE, GEC, item DEC, DEA aequalia erunt : duo itaque triangula DEC, DEA, hoc est totum DCG, aequabuntur duobus triangulis DEA, EAG, hoc est toti GAD. Quod erat demonstrandum.

PROPOSITIO XXXVII.

Ellipsim ABC secent AC, DB: diametri quaevis agantur per C & D tangentes, quae per 34. huius conveniunt cum diametro HK in eodem puncto.



Dico lineas AG, BG ex A & B ductas, ipsis DK, CK aequidistantes, diametrum HK, in uno eodemque puncto intersecare.

Demonstratio.

Ponatur AE occurrere diametro in G & BF in M; iunganturque DC, AB, DA, CB. Quoniam DC iungit tangentes DK, CK, bisecatur a diametro HK in L, est vero AB parallela ad DC; ergo & haec a diametro bisecatur in I; quare cum totae DC, AB aequales ; cum igitur etiam DL, AI, sint parallelae, quae eas iungunt DA, IL parallelae, ergo figura AGDK parallelogrammum; ergo figure AGKD parallelogrammum est, proindeque DA aequales est GK; simili modo ostendimus BC aequalem esse MK. Quare cum DA, BC,sint aequales, etiam KG, KM aequales erunt, unum igitur idemque punctum sunt G & M in quo parallelae tangentibus DC, CK, ductae e punctis A, B, occurrunt diametro. Quad fuit demonstrandum.

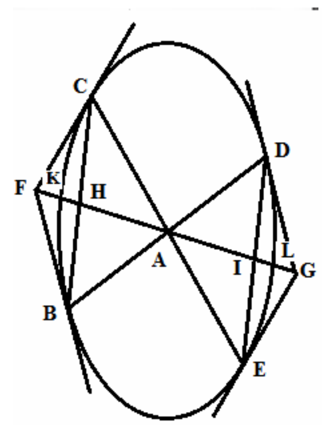
PROPOSITIO XXXVIII.

Ellipsim cuius centrum A secent quaevis duae diametri BD, CE, iunctisque BC, DE ponatur FG diameter, quae BC bifariam dividat in H, ipsi autem ED occurrat in I, tum per C & B, item D & E, contingentes agantur,que FG diametro occurrent in ijsdem punctis F & G.

Dico aequalia esse inter se; primo triangula ACF, ABF, secundo triangula ACF, ADG, tertio triangula CBF, DEG.

Demonstratio.

Occurrat FG diameter ellipsi in K & L. Quoniam igitur BC ex hypothesi in H divisa est bifariam, erit tam ACH triangulum aequale triangulo AHB, quam HCF aequale triangulo HFB, unde totum triangulum ACF, aequale est toti triangulo AFB, quod erat primum. Rursum cum BC, DE



aequidistent & BC in H divisa sit bifariam a diametro FG, erit & ED in I, bifariam divisa. quare cum totae CB, DE sint aequales, erunt & harum dimidiae HC, DI aequales, quare cui, etiam sint parallelae rectae, CD, HI,quae illas iungut, sunt parallelae ; triangula igitur ACH, ADI sunt inter easdem parallelas. Sunt autem & bases AH, AI aequales, est enim AK aequalis, AL, & bases KH, AI aequales , (est enim AK aequalis IL) ergo triangulam ACH aequatur triangulo ADI. Iam vero AH, AK, AF, idemque AI, AL, AG sunt continue ; quare ratio AH ad AF, duplicata est rationis AH ad AK; & ratio AI ad AG duplicata est rationis AI ad AL Cum igitur rationes AH ad AK, AI ad AL, eadem sint (AH enim ipsi AI & AK, ipsi AL aequalis est) erunt & rationes AH ad AF, AI ad AG earundem rationum duplicatae; eaedem inter se : ac proinde triangulum quoque; AHC est ad triangulum AEC, ut triangulum AID ad triangulum AGD. Quare cum triangula ACH, ADI ostensa sint aequalia, etiam AFC, AGD aequalia erunt, quod erat alternum. Ex quo iam patet quibus si addas FCH, GDI, aequalia esse, quibus si addas FBH,GEI, quae eodem plane discursu ostendimus aequalia, erunt tota triangula CBF, DEG, aequalia. Quod erat tertio loco demonstrandum.

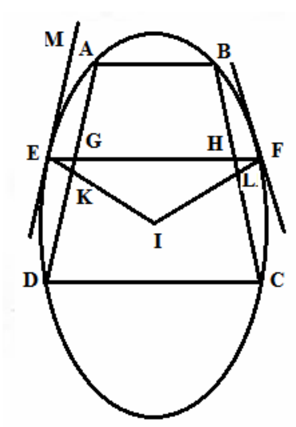
Corollarium.

Hinc patet quadrilatera CFBA, DGEA, aequalia esse, eodem enim discursu probabimus aequalia esse triangula ABF, AEG, quo probavimus aequari ACF, ADG.

PROPOSITIO XXXIX.

Secent ABC ellipsim duae quaevis parallelae AB, CD, iunctisque AD, CB, recta EM parallela ipsi AD, contingat sectionem in E, & ex E ducatur EF, aequidistans AB, secans AD, CB lineas in G & H.

Dico contingentem per F ductam aequidistare ipsi BC.



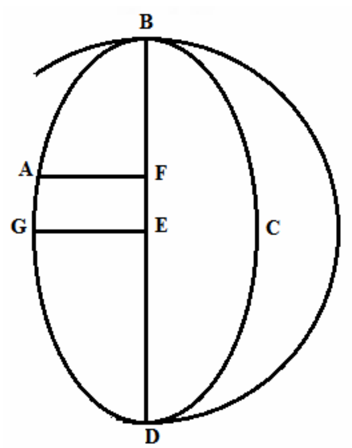
Demonstratio.

Ducantur ex I centro lineae IE, IF occurrentes AD, CB in K & L : quoniam igitur EM contingens, aequidistat AD & IF diameter ad contingentem ducta, secet AD lineam in K, erit AD in K divisa bifariam; est autem recta BC in L divisa sicut AD in K, recta enim KL iungens puncta K, L parallela ellipsis AB, DC. Igitur & BC in L secta est bifariam a diametro IF; unde & tangenti per F ductae aequidistat. Quod fuit demonstrandum.

PROPOSITIO XL.

Circulus super axe maiore ut diametro descriptus ellipsi exterius in duobus tantum punctis occurrit.

Demonstratio.



Sit ABC ellipseos axis maior BD, centroque illius E intervallo EB circulus describatur, dico illum ellipsi in duobus tantum punctis B & D occurrere. Occurrat enim si fieri possit insuper in puncto A & per A ordinatim ad axem agatur AF, ducaturque axis minor GE, erit igitur ut BFD rectangulum ad quadratum FA sic BED rectangulum ad quadratum EG: sed BFD rectangulum in circulo est aequale quadrato FA; igitur & rectangulum BED, id est quadratum BE aequale est quadrato GE, quod fieri non potest cum BE linea maior sit quam GE, igitur circulus ellipsi non occurrit in A: nec in alio quavis puncto, praeter B & D. Quod fuit demonstrandum.

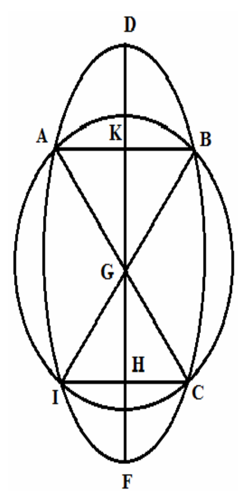
Corollarium.

Simili discursu demonstrabimus circulum circa minorem ellipseos axem descriptum in duobus tantum punctis extremis axeos ellipsi occurrere, & totum intra ellipim existere.

PROPOSITIO XLI.

Circulus centro ellipseos descriptus, si ellipsi in secat, in quatuor punctis secabit.

Demonstratio.



Sit enim ellipseos centro G descriptus circulus secas ellipsim in B, ducatur axis FD, & recta BGI; tum ordinarim applicetur BKA occurrens ellipsim A, ducantur item recta AGC, IC; in triangulis BKG, AKG, BK, AK aequantur, & KG est communis, angulique ad K recti; ergo GB, GA aequales; quare cum punctum B sit ad circulum, erit & punctum A est autem idem punctum etiam ad ellipsim, ergo circulus ellipsim secat in A. Deinde AB, IC sunt parallelae, adeoque cum angulus AKH rectus sit, erit etiam rectus IHK; ac proinde IC ordinatim est posita ad axe in DF, ac bisecta in H sunt autem totea AB, IC aequales, ergo AK, IH earum dimidiae etiam sunt aequales. In triangulis igitur GKA, GHI, AK ipsi IH, & KG ipsi HG est aequalis, anguli vero AKG, IHG etiam aequales sunt; ergo GA, GI aequantur; quare cum punctum A sit ad circulum, erit & punctum I, atqui etiam punctum I est ad ellipsim. Ergo circulus ellipsim secat in I. similiter oftendemus circulum ellipsi occurrere in C. In quatuor igitur punctis secat. Quod erat demonstrandum.

Corollarium.

Quod autem non secet ellipsim circulus in pluribus punctis quam quatuor, facile colligetur ex demonstrationc iam posita.

ELLIPSIS

PARS SECUNDA

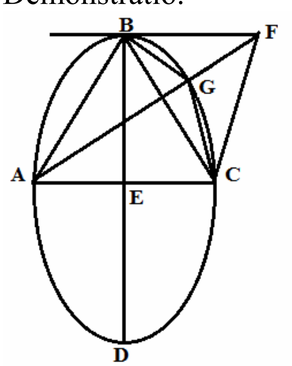
De sectoribus & segmentis Ellipseos.

PROPOSITION XLII.

Sit ABC ellipseos diameter BD, ad quam ordinatim ponatur AEC; iunganturque ABC.

Dico ABC triangulum maximum esse illorum quae segmento ABC inscribi possunt.

Demonstratio.



Acta per B contingence BF, ex A recta ducatur quaevis AF, occurrens ellipsi in G & contingenti in F iunganturque GC, FC : Quoniam FG contingens cadit supra G, igitur triangulum AFC maius est triangulo AGC; sed AFC triangulo aequale est triangulum ABC ob AC, BF aequidistantes; igitur & ABC triangulum maius est triangulo AGC: unde cum idem de aliis omnibus triangulis ostendatur; patet ABC triangulum, maximum eorum esse quae segmento ABC inscribi possunt. Quod erat demonstrandum.

Corollary.

Hinc facilis praxis elicitur ad inscribendum cuivis segmento triangulum maximum: erigendo nimirum diametrum BD, iungendoque AB, BC puncta. Demonstratio patet ex priori.

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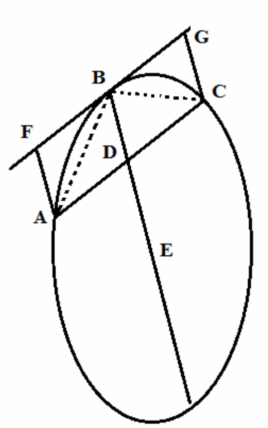
PROPOSITION XLIII.

Triangulum maximum segmento cuivis non maiori semiellipsi inscriptum, maius est dimidio eiusdem segmenti.

Demonstratio.

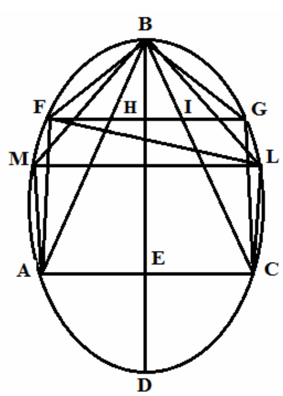


Esto ABC segmento, non maiori semiellipisi inscriptum triangulum maximum ABC. Dico illud maius esse dimidio segmenti ABC: ducta enim diametro BE, quae AC subtensam diuidat bifariam in D, erigature ex A & C lineae AF, CG parallela diametro BE quae FG contingenti per B actae occurrant in F; iunganturque AB, CB: triangulum ABC [§.41, Ch.1] dimidium est parallelogrammi AG. Atqui AG parallelogrammum maius est segmento ABC, cum AF, CG, FG lineae cadant extra ellipsim; igitur & triangulum ABC maius est dimidio cuiisdem segmenti. Quod fuit demomstramdum.

PROPOSITION XLIV.

Ellipsim ABC secet diameter BD, ad quam ordinatim posita sit AEC: iunctis AB, CB inscribatur segmento AFB triangulum maximum AFB, & ex F ponatur FG parallela AC, iunganturque BG, GC.
Dico BGC triangulum maximum esse eorum quae segmento ABC inscribi possunt, & si triangula fuerint maxima, dico FG esse parallelam ad AC.

Demonstratio.



Quoniam FH, GI lineae sunt aequales, triangula FBH, GBL eandem habentia altitudinem, aequalia erunt, similiter triangula FAH, GIC inter parallelas EG, AC, constituta erunt aequalia ac proinde aequabuntur tota triangula BFA, BGC : si igitur BGC non sit maximum, ponatur aliud BLC, maius triangulo BGC, & ex L ducatur LM aequidistans AC; erit igitur ut prius triangulum BLC, aequale triangulo AMB, adeoque & AMB triangulum, maius triangulo BGC id est AFB, quod est contra suppositum, cum BFA maximum ponatur ; igitur BGC triangulum maximum est eorum quae segmento BGC inscribi possunt, quod erat primum; sint deinde triangula BFA, BGC maxima, demonstrabimus iunctam FG parallelam esse AC, si enim non est parallela , sit alia supra vel infra ipsam FG parallela ad AC, nimirum recta FL; iunganturque BL, CL; ergo per primam partem huius triangulum BLC erit maximum, quod fieri non potest cum FGC ex hypothesi sit maximum. Non igitur FL, aut alia ulla praeter FG est parallela ad AC. Quod erat demonstrandum.

Corollary primum.

Hinc sequitur si triangula BFA, BGC maxima sint eorum quae segmentis inscribi possunt, esse aequalia. Nam per secundam partem huius FG est parallela ad AC. Unde FH, GI aequales sunt, ac proinde triangula FBH, BIG, & FAK, GCI ; adeoque & tota BFA, BGC aequalia sunt.

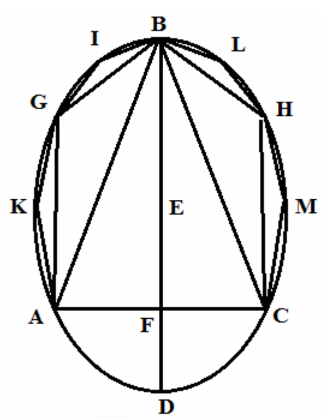
Corollary secundum.

Quod si fuerint binae AC, FG ad diametrum ordinatim positae, iunganturque FA, CG, triangula quae segmentis AF, CG inscribuntur maxima, inter se quoque aequalia esse, eodem plane discursu demonstrabimus, quo usi sumus in propositione & corollario primo, nullo alio immutato, quam quod loco 26 huius assumenda sit vigesima septima.

PROPOSITION XLV.

Sit ABC ellipsis diameter quaecunque BD ad quam ordinatim ponatur AFC.
Dico AG BF segmentum aequari segmento CHBF.

Demonstratio.



Iunctis AB, CB,inscribantur [§.42, Ch.1] segmentis reliquis triangula maxima AGB, CHB erunt illa per primum Coroll. praecedentis propositionis inter se aequalia & [§.43, Ch.1] maiora dimidio segmentorum ABG, CBH: dein & residuis utrimque segmentis triangula inscribantur maxima AKG, GIB, CMH, BLH erunt ut prius triangula AKG, GIB partim per corollar. primum, partim per secundum aequalia triangulis CMH, BLH, & maiora dimidiis segmentorum: igitur cum ea inscriptio semper possit continuari in utroque segmento AGB, BHC, & utrimque partes ablatae sint inter se aequales, & maiores dimidio, segmentorum a quibus auferuntur, constat [§.216, Progress.] AGB segmentum aequale esse segmento CHB. Quare additis aequalibus triangulis ABF, CBF erunt tota segmenta AGBF, CHBF aequalia. Quod erat demonstrandum.

Corollary.

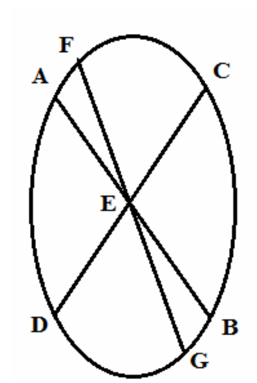
Hinc sequitur a quavis diametro ellipsim bifariam secari: sit enim diameter quaevis BD, & ducatur per quodvis illius punctum ordinatim AFC, per propositionem iam demonstratam segmentum ABF, segmento BCF, aequatur rursum per eandem proposition e segmentum ADF, segmento CDF aequale est; ergo segmenta ABF, ADF, hoc est totum segmentum DAB, aequantur segmentis BCF, CDF, hoc est toti segmento DCB; bifariam igitur divisa est ellipsis a diametro BD.

PROPOSITION XVI.

Diametri duae coniugatae ellipsim quadrifariam dividunt, & diametri ellipsim quadrifariam diuidences, sunt inter se coniugatae.

Demonstratio.

Sint in ABC ellipsi diametri duae coniugatae AB, CD; dico illas ellipsim quadrifariam dividere; & si AB, CD diametri ellipsim quadrifariam dividant, dico illas esse conugatas. Quoniam AB, CD diametri sunt coniugatae ; erit AB ordinatim posita ad diametrum DC, unde iam AEC, CEB quam AED, BED sectores sunt aequales ; sunt autem & AEC, AED sectores ob eandem rationem aequales; sectores igitur quatuor AEC, CEB, BED, DEA, sunt inter se aequales, & AB, CD lineae quadrifariam dividunt ellipsim: Quod erat primum.



Sit iam ellipsis quadrifariam divisa, dico AB, CD diametreos esse coniugatas; sin vero ducatur ipsi CD, coniugata FG, igitur FEC, sectorquadrans ellipseos est per priorem partem huius. Atque sector AEC ex hypothesi etiam quarta ellipseos pars est, ergo sectores FEC, AEC aeqales sunt, pars & totum, quod fieri nequit ; igitur FG diameter non est conjugata ipsius CD, nec quaevis alia praeter AB. Quod erat demonstrandum.

Corollary.

Hinc patet sectores quarumcumque coniugatarum, aequales esse sectoribus cuiuscunque alterius coniugationis singulas singulis : singul enim quadrantes sunt ellipseos, & si sectores sint aequales ac latera unius sint coniugatae, alterius etiam latera esse coniugatas.

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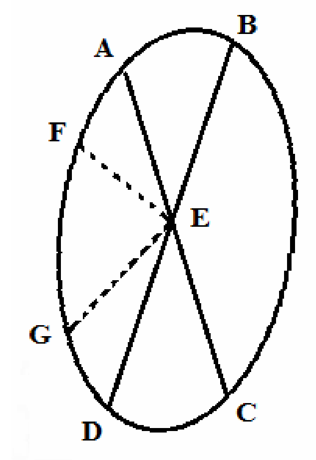
PROPOSITION XLVII.

Sectores ad verticem oppositi sunt inter se aequales.

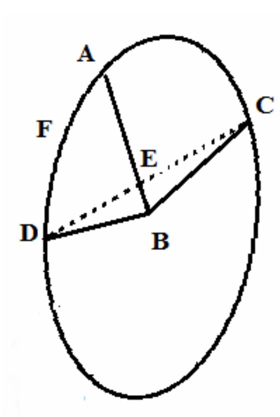
Demonstratio.

Secent ABC ellipsim diametri quaecunque AC, BD. dico sectores ad verticem oppositos esse inter se aequales. Ducantur enim ex E centro diametri duae EF, EG & EF quidem coniugata ipsi: EG vero coniugata ipsi AE. Quoniam igitur sectores BEF, AEG, [§.46, Ch.1] aequales sunt, dempto communi AEF, erit sector AEB, aequalis sectori FEG: rursum cum sectores SED, GEC sint aequales, dempto communi DEG, erit sector DEC aequalis sectori FEG, id est AEB ad verticem opposito. Eodem modo ostenduntur AED, BEC sectores aequales. Igitur, &c. Quod fuit demonstrandum.



PROPOSITION XLVIII.

Sit in ADC ellipsi sector quicunque ABC, oportet ex B rectam ad peripheriam ducere, quae cum AB linea sectorem constituat dato ABC aequalem.



Constructio & demonstratio.

Ducatur ex C ordinatim ad diametrum AB, recta CED, iungaturque BD; dico factum esse quod petitur est enim segmentum AED [§.45, Ch.1] aequale segmento AEC, & cum aequales sint CE, ED; triangulum DE aequale triangulo CEB, igitur & sector ABD aequalis sectori ABC. Eduximus igitur, &c. Quod fuit faciendum.

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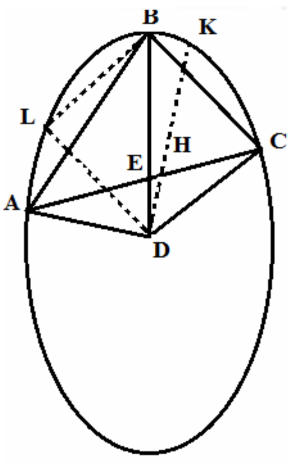
PROPOSITION XLIX.

Habeant ADB, CDB sectores aequales commune latus BD, iunganturque AB, CB. Dico segmenta lineis AB, CB ablata esse inter se aequalia, & si segmenta fuerint aequalia, dico & sectores aequari.

Demonstratio.

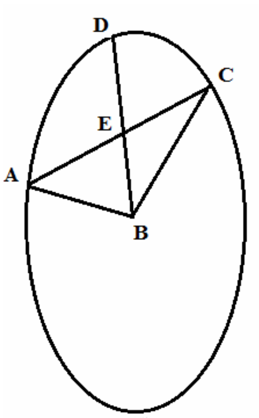


Iungantur A,C occurratque AC, linea diametro BD in E, tum si AC non sit diuisa bifariam in E: dividatur bifariam in H, agaturque per H diameter DK. Quoniam AC, linea ordinatim ducta est ad diametrum DK, erunt AHK, CHK segmenta aequalia ; sunt autem & AHD, CHD triangula aequalia, sectores igitur ADK, CDK inter se aequales sunt, igitur sector CDK aequalis est sectori CDB pars toti. Quod absurdum; quare AH linea non dividitur in H bifariam; nec in alio puncto qua in E: adeoque igitur AC, lineae posita est ordinatim ad diametrum BD. Unde AEB segmentum est aequale segmento CEB; sunt autem AEB, CEB triangula aequalia, ergo reliquum segmentum AB, aequale est segmento CB; quod erat primum.

Sint iam AB, CB segmenta aequalia & ex A, B, C punctis diametri ponatur AD, BD, CD, dico sectores ADB, CDB, esse inter se aequales; sin vero: fiat CDB sectori aequalis sector BDL, iunganturque puncta LB, erit igitur LB segmentum aequale segmento CB, hoc est AB per hypothesin, adeoque pars aequalis toti.
Quod fieri non potest; igitur sector BDL [§.48 above] non est aequalis sectori CDB: nec alius quisquam preater ADB sectorem. Quod erat demonstrandum.

PROPOSITION L.

Sit ABC sector quicunque.



Dico lineam ex centro ductam, quae AC subtensam dividit bifariam, sectorem quoque bifariam secare.

Demonstratio.

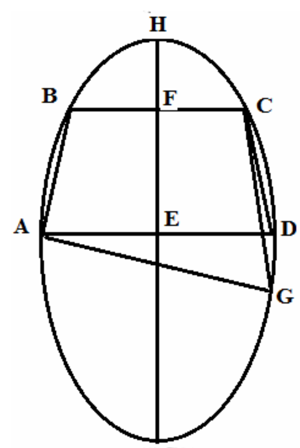
Ducatur ex B centro diameter BD; secans bifariam AC lineam in E: dico ABD, CBD sectores esse aequales: Cum enim AC in E divisa sit bifraiam, erunt ABE, CBE triangula aequalia, sed, quia AC est ordinatim posita ad diametrum BD, etiam segmenta AED, CED sunt aequalia [§.45 above] , igitur totus sector ABD, sectori CBD aequale est. Quod erat demonstrandum.

PROPOSITION LI.

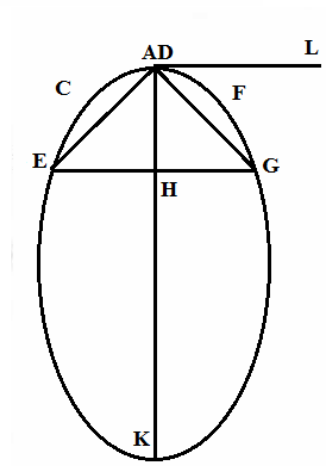
Ellipsim ABC secent duae quaevis parallelae AD, BC; iunganturque ; AB, CD.
Dico AB, CG segmenta esse aequalia & si segmenta fuerint aequalia, dico BC, AD lineas aequidistare.

Demonstratio.

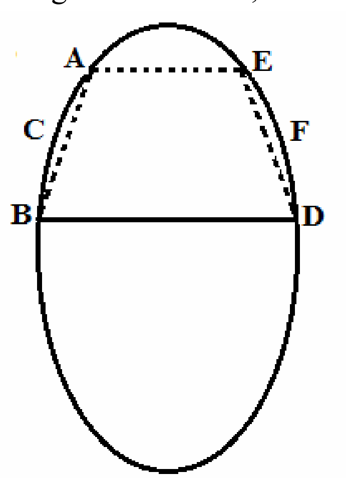
Divisis AD, BC bifariam in F & E agatur per F & E linea FE, occurrens ellipsi in H, erit illa diameter, quare BFH, CFH segmentia sunt aequalia; rursum quoniam AE, DE aequales sunt, & segmenta AHE, DHE aequalia erunt : ablatis igitur aequalibus segmentis BFH, CFH remanent segmenta AB, FE, DC, FE aequalia. Deinde quoniam AE, ED sunt aequales, & altitudo communis parallelarum BC, AD, erunt AE, FB, DEFC trapezia aequalis, igitur ab aequalibus, manent AB, CD reliqua segmenta inter se aequalia. Quod erat primum.



Sint iam AB, CD segmenta aequalia, iunganturque BC, AD; dico AD, BC lineas aequidistare: sin vero, ducatur ipsi BC parallela AG, iunganturque CG : erit igitur per primam partem huius segmentum AB aequale segmento CG: sed & CD segmentum ex hypothesi aequale est segmento AB; segmenta igitur CD, CG sunt aequalia, quod fieri non potest, cum punctum G cadat supra vel infra D, adeoque CG segmentum maius vel minus sit segmento CD: igitur AG linea non aequidistat ipsi BC, sed sola AD. Quod erat demonstrandum.



Quod si punctum datum D idem sit cum puncto A, ducatur AL tangens ellipsim in puncto A, ducatur AL tangens ellipsim in puncto A sive D, (sunt enim A & D iam ex hypothesi unum idemque punctum) ductaque BG parallela ad AL iunge DG.
Dico hanc abscindere segmentum DFG aequale segmento ACB.
Ex contactu ducatur diameter AK; igitur BG quia tangenti aequidistat, est ordinatim posita ad diametrum AK, adeoque bisecta in H, triangula igitur BAH, GAH aequantur; aequantur vero & segmenta BCAH, GFAH; ergo reliqua etiam segmenta BCA, GFA siveGFG aequantur. Factum igitur est quod petebatur.



PROPOSITION LII.

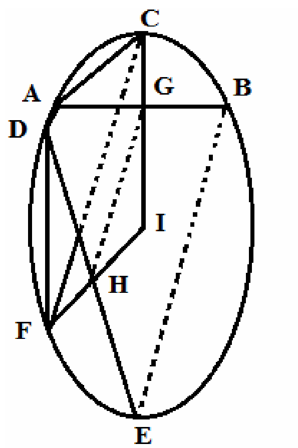
Secet ellipsim recta quaevis AB : auferens ACB segmentum, & detur in peripheria punctum quodvis D, oportet ex D rectam ducere DE, quae auferat segmentum DEF, aequale segmento ABC.

Constructio & demonstratio.

Iungantur BD, & ex A ponatur AE aequidistans BD, iunganturque ED; patet per praecedentem DEF segmentum aequale esse segmento ABC. Igitur ex puncto dato, &c. Quod erat faciendum.

PROPOSITION LIII.

Secent ABC ellipsim duae quaevis lineae AB, DE segmentas auferentes aequalia: divisis autem AB,



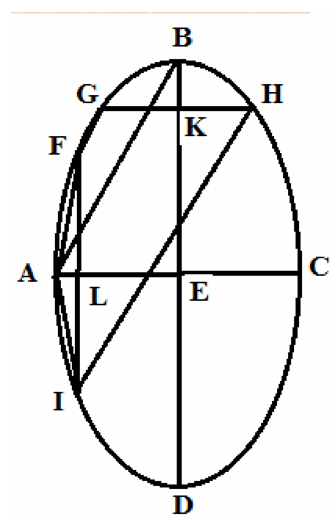
DE segmenta auferentes aequalia: diuisis autem AB , DE rectis bifariam in G & H , ducantur per G & H diametri IGC, IHF.
Dico illas in G & H proportionaliter esse divisas. Et si diametri sint proportionaliter divisae: dico segmenta esse aequalia.

Demonstratio.

Iungantur AD, BE, AC, DF, GH, CB, FE, CF. Quoniam ACB, DFE segmenta ponuntur aequalia, AD, EB lineae parallelae sunt: est autem ut AG ad GB, sic DH ad HE, cum AB, DE [§.51. above] lineae in G & H diuisae sint bifariam, igitur & GH linea aequidistat AD, BE. Iam vero cum AB, sit ad diametrum IC ordinatim posita, erunt segmenta AGC, BGC [§.45. above] aequalia, adeoque segmentum AGC dimidium segmenti ACB; simili de causa segmentum DFH dimidium est segmenti DFE. Quare cum tota segmenta ACB, DFE ponantur aequalia, erunt etiam [§.42. above] segmenta AGC, DFH, eorum dimidia inter se aequalia. Deinde triangula ACB, DFE maxima sunt eorumquae segmentis inscribi possunt, & quoniam AD, BE ostensae sunt parallelae, etiam inter se aequalia erunt [§.44. Cor.2] ; aequabuntur igitur & eorum dimidia triangula AGC, DFH: quae si auferas a segmentis aequalibus AGC, DFH, remanent segmenta aequalia AC, DC, ergo CF lineae [§.52. above] aequidistat rectae AD hoc est GH: quare ut CG ad GI, sic FH ad HI. Quod erat demonstrandum. Hinc iam veritas conversae fit manifesta.

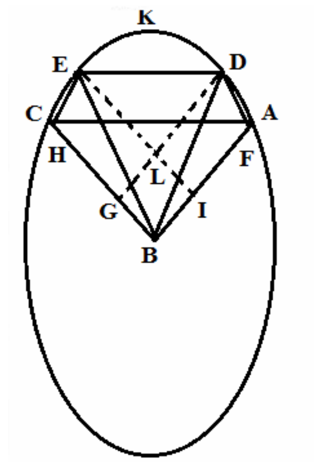
PROPOSITION LIV.

Sit in ABC ellipsi quaevis diametrorum coniugatio AC, BD, iunctaque AB, ducatur quaeuis FG parallela AB; & ex F & G, rectae ponantur GH, FI ordinatim ad diametros BD, AC.
Dico AC, BD diametros in K & L proportionaliter esse divisas.



Demonstratio.

Iungantur GB, BH, FA, AI, HI. Quoniam AB, GF lineae aequidistant, erunt GB, [§.52. above] FA segmenta aequalia; sed GB segmento est aequale segmentum HB. (nam segmentum lineae GKB aequatur segmento HKB, & triangulum GBK triangulo HBK) & FA segmento ob eandem causam aequatur segmento AI, igitur & HB segmentum est aequale segmento AI, & HI aequidistat linea AB [§.45. above], hoc est FG; quare totum segmentum GBH, aequale est toti segmento FAI: adeoque per praecedentem AC, BD diametri in K & L similiter sunt divisas. Quod erat demonstrandum.



PROPOSITION LV.

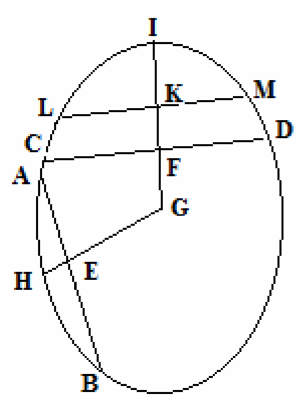
In ellipsi data sit quaecunque diametrorum coniugatio AB, CB , & iungatur AC, cui parallela sit quaevis ED, ex punctis autem D, & E ducantur ordinatim ad diametros DF, EH, DG, El, erunt igitur figurae DFBG, EIBH parallelogramma. Dico parallelogramma illa aequalia esse.

Demonstratio.

Quia per praecedentem BA, BC proportionaliter sunt divisae, erit AB ad BF, ut CB ad BH: & permutando ut AB ad CB, ut BF ad BH; similiter AB per praecedentem est ad BI ut CB ad BG, & permutando ut AB ad CB, sic BI ad BG; ergo BF est ad BH, ut BI ad BG. Ergo [§.14. ch.6], parallelogramma AG, HI aequalia sunt. Quod erat demonstrandum.

PROPOSITION LVI.

Sint AB, BC diametri coniugate; iunctis punctis AC, ducatur ED parallela rectae AC, tum ex D & E rectae ponantur EI, DF, DG, EH ordinatim ad diametros AB, CB, iunganturque EB, DB. Dico EBD sectorem, aequari figurae EIFDKE.



Demonstratio.

Per praecedentem parallelogramma FG, HI: & illorum dimidia, triangula DFB, EIB inter se sunt aequalia; ablato igitur communi LIB, erit ELB triangulum aequale trapezio DFIL; quare addica figurae communi ELDKE, sector EBD aequalis est figurae EIFDKE. Quod erat demonstrandum.

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PROPOSITION LVII.

Secant ABC ellipsim duae quaevis lineae AB, CD, oportet CD lineae parallelam ducere, quae segmentum auferat aequale segmemto AHB.

Constructio & demonstratio.

Divisis AB, CD bifariam in E & F, agantur ex G centro per E & F diametri GH, GI; tum GI dividatur in K, sicut HG divisa est in E, ponaturque per K ordinatim LM, patet per 53.huius LIM, AHB segmenta esse aequalia ; est autem LM, parallela datae CD, igitur dato in ellipsi segmento, &c. Quod erat faciendum.

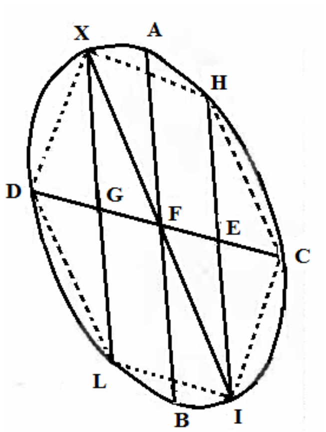
PROPOSITION LVIII.

Secent ABC ellipsim coniugatae duae diametri AB, CD; divisaque CD illarum altera quadrifariam in EG, agantur per E & G rectae HI, KL aequidistantes diametri AB, iunganturque puncta D, K, H, C, I, L, D.

Dico LD, DK, KH, HC, CI, IL lineas segmenta auferre aequalia.

Demonstratio.

Quoniam KL, HI, sunt parallelae rectae AB, quae est diameter coniugata ipsi DC, erunt KL, Hl ordinatim positae ad DC, ergo ut rectangulum DGC, ad rectangulum DEC, ita quadratum KG, ad quadratum HE, adeoque cum rectangula sint aequalia, etiam quadrata erunt aequalia, unde & rectae KG, HE aequales
sunt ; sunt vero & GF, EF aequales & anguli KGF, FEI (quod Hl, KL sint parallelae) aequales sunt; igitur KGF triangulum aequale est triangulo FEI, angulusque IFE aequalis angulo GFK, & quia GFE linea recta est, anguli IFE, KFG ad verticem constituri sunt aequales: unde KFI, puncta sunt in directum, adeoque; sectores KFD, CFI, sunt ad verticem constuti: quia autem ostendi triangulum KFG, aequari triangulo IFE, & simili discursu ostendi possit triangulum quoque ; DKG aequari triangulo ICE, erit triangulum totum DKF, aequale toti triangulo ICF. Atqui & sectores DFK, IFC ad verticem positi sunt aequales. Igitur reliqua etiam segmenta DK, IC inter se aequalia erunt ; eodem modo ostenduntur DL, HC segmenta aequalia. Iterum cum duo latera KG, GF sint duobus lateribus DG, GL aequalia, & anguli lateribus aequalibus contenti ad verticae aequalibus contenti ad vertice aequales ; erunt triangula GKF, DGLinter se aequalia, &



angulus GKF aequalis angulo altero GLD: adeoque KFI, DLlineae parallelae; quare DK , LI segmenta sunt inter se aequalia; est vero iam ostensum
segmenta quoque CI, DK aequalia esse, aequantur igitur tria segmenta DK, LI, CI. ulterius quia DC, LI iungunt aequales & parallelas GL, FEI, ipsae etiam erunt parallelae. Unde rursum segmenta CI, DL aequalia sunt aequantur igitur quatuor segmenta DL, CI, LI, DK. Rursum quia KH, LI, iungunt KL, HI aequales & parallelas, sunt ipsae etiam parallelae: erunt ergo aequalia etiam segmenta KH,Ll. aequantur igitur quinqae segmenta KH, LI, KD, DL, CI. Atqui etiam ostensum est aequalia esse segmentas HC,DL; aequantur igitur omnia sex segmenta. Quod fuerat demonstrandum.

Vocetur autem figura DKHCIL, ellipsi inscripta, polygonum regulare.

PROPOSITION LIX.

Eadem manente figura propositum sit ellipsi hexagonum regulare inscribere.

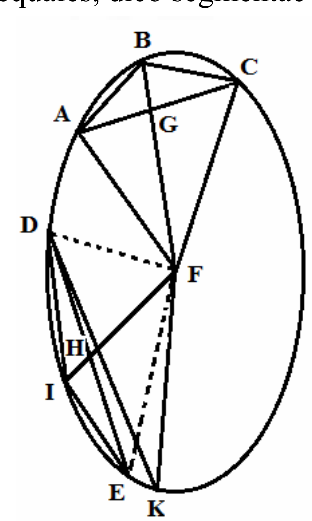
Constructio & demonstratio.

Sumantur duae quaevis diametri coniugatae AB, CD, divisaque CD quadrifatiam in E & G, agantur per E & G lineae HI, KL aequidistances A B: ducanturque DK, KH, HC, CI, IL, LD: patet per praecedentem: rectas illas segmenta auferre aequalia, adeoque figuram hexagonam DK, HC, ILD, esse regularem;
igitur ellipsi hexagonum inscripsimus regulare. Quod erat faciendum.

PROPOSITION LX.

Secent ABC ellipsim duae quaevis lineas AC, DE auferentes segmenta aequalia, ducanturque ad illarum extremitates semidiametri FA, FC, FD, FE.

Dico sectores AFC, DFE, esse aequales, & si sectores fuerint aequales, dico segmentae esse aequalia.



Demonstratio.

Dividis AC, DE bifariam in G & H agantur per G & H, diametri FB, FI, iunganturque AB, BC, DI, EI; quoniam segmenta AC, DE ex hypothesi sunt aequalia, ergo rectae, quae puncta C & E, A & D iungerent , forent parallelae; ergo triangula segmentis AC, DE inscriptorum maxima, sunt aequalia, sed ABC, DIE, sunt inscriptorum maxia, erunt igitur triangula ABC, DIE, aequalia : quia autem FB, FI diametri in G & H, sunt proportionalite divisae : igitur ut ABC triangulum est ad triangulum AFC, sic DIE triangulum est ad triangulum DFE, & permutando ut ABC triangulum est ad triangulum DIE, sic AFC triangulum est ad triangulum DFE. Cum igitur triangula ABC, DIE aequalis sint, etiam triangula AFC, DFE aequalia erunt. Quare additis aequalibus segmentis AC, DE , erunt sectores FAC, FDE aequales.

Sint iam sectores AFC, DFE aequales, iunganturque AC, DE. Dico ABC, DIE segmenta quoque esse aequalia; sin vero : sit alterutrum (puta) DIE minus altero, ducaturque ex D linea DK, segmentum auferens aequale est segmento ABC, sector DFK, aequalis est sectori AFC, id est sectori DFE quod fieri non potest; segmenta ergo AC, DE non inaequalia sunt, sed aequalia. Quod erat demonstrandum.

Corollary.

Ex hac & ex 53 huius sequitur primo, si AFC, DFE sectores quivis fuerint aequales, eorumque subtensae in G & H divisae bifariam, quod diametri per G & H ductae, iiisdem punctis proportionalite dividantur.

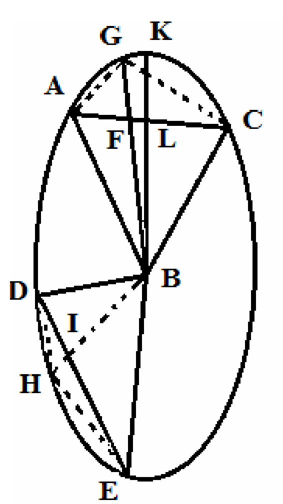
Secundo si sectores duo AFC, DFE fuerint aequales, subtendanturque ipsorum anguli rectis AC, DE: quod triangula AFC , DER sint aequalia.

Tertio hinc tale problema solvitur , dato AFB sectore quocunque, oportet ex F duas educere semidiametros quae sectorem constituant aequalem sectori AFB: pro constructione iuncta AB, ducatur ex F semidiameter quae cunque FD: tum ex D recta ducatur DI segmentum auferens aequale segmento AB iungaturque IF, patet IFD sectorem aequari sectori AFB.

PROPOSITIO LXI.

Sint ABC, DBE sectores aequales : iunctisque AC, DE ducatur ex B quaevis linea BG fecans AC lineam in F: dein sectori ABG fiat aequalis sector DBH secetque; HB linea, rectam ED in I

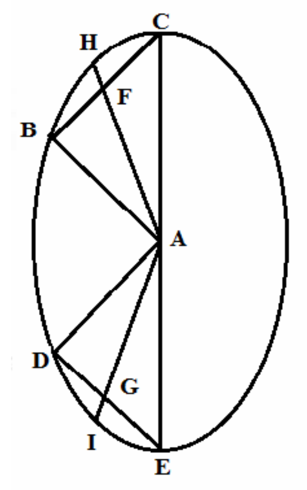
Dico tam AC, DE lineas quam BG, HB in F & I proportionaliter esse divisas.



Demonstratio.

Iungantur AG, GC, DH, HE. Quoniam ex hypothesi sectores tam ABG, DBH, quam ABC, DBE aequales, erunt & reliqui GBC, HBE aequales; quare AGB triangulum aequale triangulo DHB, & triangulum CGB aequale triangulo HEB, quare ut triangulum BAG ad triangulum GCB, sic BDH triangulum ad triangulum BHE, sed (quod facile ext.6. est demonstratu) rationes rectarum AF, FC, & DI, IE, eadem sunt cum rationibus triangulorum BAG, GCB, & BDH, BHE. Ergo etiam, AF est ad FC, ut DI ad IE. Deinde cum trapezia GABC, BDHE aequalia sint (est enim triangulum BAG, triangulo BDH, & triangalum BGC, triangulo BHE, aequale) sit autem & ACB aequale triangulo DEB, erit reliquum triangulum AGC aequale reliquo DHE: igitur ut AGC triangulum ad triangulum ACE, id est ut GF ad FB sic DHE triangulum ad triangulum DEB, id est HI ad IB. Quod erat demonstrandum.

PROPOSITION LXII.

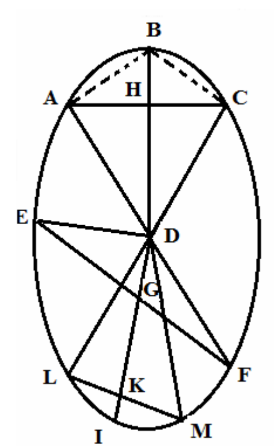


Sint iam AC, DE lineae in F & I proportionaliter divisae: aganturque per F & I semidiametri BG, BH.
Dico ABG, DBH sectores esse aequales.

Demonstratio.

Sin vero: sit alteruter ut ABG minor altero : fiat ABK sector aequalis sectori DBH, secetque BK linea rectam AC in L. quoniam ABK, DBH sectores sunt aequales: erit per primam partem huius AL ad LC, ut DI ad IE. Atqui etiam AF est ad FC, ut DI ad IE, igitur ut AF ad FC, sic AL ad LC, quod fieri non potest : quia punctum L cadit ultra aut citra F; quare sector ABK non est aequalis sectori DBH nec alius quisquam praetcr ABG. Quod erat demonstrandum.

PROPOSITION LXIII.



Sint duae quaevis diametri AB, AC, iunctisque illarum extremitatibus, ducantur duae quaevis aliae diametri AD, AE sic ut iuncta DE, sint ABC, ADE triangula aequalia : diuisis autem BC, DE bifariam in F & G, agantur per F & G, diametri AH, AI: quae si in F & G, proportionaliter sint divisae.
Dico BAC, DAE sectores esse aequales.

Demonstratio.

Quoniam AH, AI diametri in F & G proportionaliter sunt divisae, erunt BHC, DAE segmenta aequalia: sunt autem ex hypothesi triangula ABC, ADE aequala; sectores igitur BAC, DAE sunt inter se aequales.
Quod erat demonstrandum.

PROPOSITION LXIV.

Sint duo sectores DABC, DEIF, ductisque rectis AC, LM, & bisectis in H & G, ducantur diametri DHB, DGI. Sit auten ratio DG ad DI minor ratione DH ad DB.
Dico sectorem DEIF maiorem esse sectore DABC.

Constructio & demonstratio.

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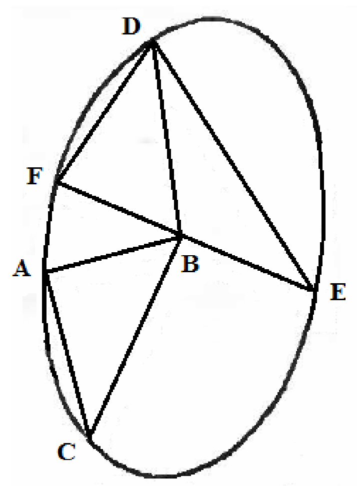
Quoniam DG est ad DI in minori proportione quam DH ad DB. Fiat DK ad DI, ut DH ad DB, maior
igitur erit DK quam DG, & punctum K cadet inter G ac I, per K ducatur ordinatim LKM; iunganturque DL, DM; segmenta igitur LIM, ABC aequalia sunt. Quare sectores etiam DLIM, DABC sunt aequales, et proinde sector DEIF maior est sectore DAVC. Quod erat demonstrandum.

PROPOSITION LXV.

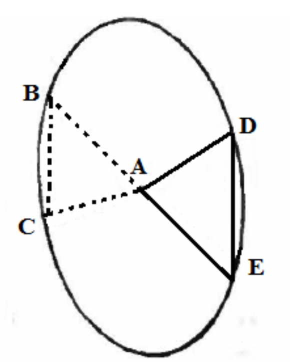
Sint ABC, DBE sectores duo inaequales, sic ut ABC, DBE triangula sint aequalia: Dico ABC, DBE sectores simul sumptos aequari semiellipsi.

Demonstratio.



Cum sectores sint ex hypothesi inaequales, sit maior BDGE, quia ergo triangulum DBE, ex hypothesi aequatur triangulo BAC, erit segmentum DGE maius segmento AC, abscindatur itaque EG segmentum ipsi segmentum AC aequalc, iunganturque BG, GD, & EB producta in F ducatur FD: quia igitur segmenta GE, AC aequalia sunt, etiam sectores aequales sunt , adeoque & triangulum GEB, triangula ACB, hoc est triangulo BDE aequale erit ;quare BE, DG sunt parallelae adeoque segmentum GE, hoc est segmentum AC aequale est segmento DF; sectores itaque BAC, BFD aequantur. Atqui sectores BFD, BDGE, constituunte semiellipsim, ergo & sectores BCA, BDGE, semiellipsim constituunt. Quod erat demonstrandum.

PROPOSITION LXVI.



Sumantur sectores duo BAC, DAE, sic ut triangula ABC, ADE sint aequalia: si sectores illi simul sumpti, maiores fuerint vel minores semiellipsi:
Dico illos inter se aequales esse.

Demonstratio.

Si enim non sint aequales, sit BAC minor sectore DAE: cum igitur triangula ABC, DAE sunt aequalia, erunt BAC, DAE sectores simul sumpti aequales semiellipsi : Quod est contra hypothesim.
Igitur sectores BAC, DAE non sunt inaequales sed aequales. Quod erat demonstrandum.

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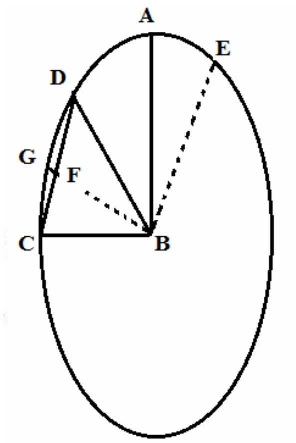
PROPOSITION LXVII.

Sint AB, BC diametri coniugatae, ductaque ex C recta quavis CD , ducatur ex B linea BE parallela rectae CD; iungaturque DB.

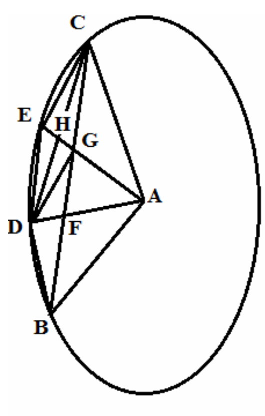


Dico DBC sectorem duplum esse sectoris ABE.

Demonstratio.

Divisa CD bifariam in F, ducatur diameter BG, & quoniam BE aequidistat ordinatim positae CD ad diametrum BG, ipsa etiam est posita ordinatim & quidem per centrum, sunt igitur coniugatae diametri BG, BE. sunt autem ex hypothesi CB, BA etiam coniugatae. Ergo sector CBA par est sectori GBE, demptoque communi GBA, sectores CBG, ABE aequales erunt. Atqui sector CBD duplus est sectoris CBG. Ergo CBD duplus quoque est sectoris ABE. Quod erat demonstrandum.

PROPOSITION XVIII



Si sectores ABD, ADE, AEC sint aequales, ducanturque rectae DE & BC, secans rectas AD, AE in F & G.

Dico BF, DE, EC esse aequales.

Demonstratio.

Ducantur rectae BD , EC, DC, DG. Quia sectores ADE , AEC sunt aequales, etiam triangula CHA, DHA aequalia sunt; ergo aequales sunt rectae DH, HC. Deinde quia sectores ADB, ACE aequales sunt,etiam segmenta BD, EC sunt aequalia. Ergo CB, ED parallelae. Anguli igitur GCH, EDH, aequantur : sunt vero anguli quoque GHC, DHE aequales, & iam ostendi aequales etiam esse rectas DH, HC; igitur CG, ED. aequales sint. Similiter ostendemus BF, DE aequales esse. Constat ergo veritas propositonis.

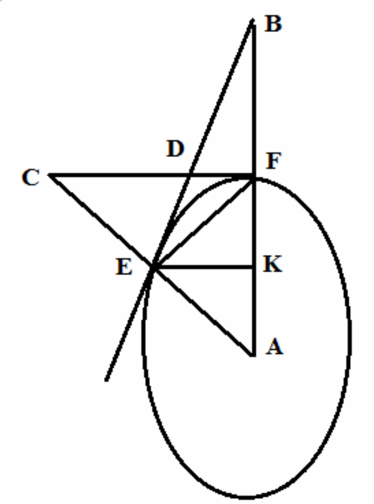
PROPOSITION LXIX.

Iisdem positis producatur DG, donec AG lineae occurat in I.

Dico AGI, AGC, AEC triangula in continua esse analogia.

Demonstratio.

Ex superiori demonstratione, in triangulis CHG, EHD, patet omnia esse aequalia, adeoque latera etiam EH, HG tunc aequalia considerentur modo triangula DHG, BHC, in quibus cum duo latera DH, HE, duobus lateribus CH, HE aequalia sint, angulusque DHG aequalis angulo CHE, ad basim etiam aequales
erunt anguli HGD, CEH, ac proinde DI, CE sunt parallelae adeoque ut AI ad AC, sic AG ad AE; sed est ut AI ad AC, sic AGI triangulum ad triangulum AGC: & ut AG ad AE sic AGC triangulum ad triangulum AEC: igitur ut AGI triangulum ad triangulum AGC: sic AGC triangulum ad triangulum AEC.
Quod erat demonstrandum.



PROPOSITION LXX.

Ellipsim secent binae diametri AE, AF in punctis E & F, ex quibus ducantur FC, EB tangentes ellipsim, occurrentes diametris in C & B, iunganturque; EF.
Dico triangulum ACF triangulo ABE aequale essse.

Demonstratio.

Ex puncto E ad AF duc ordinatim EK, erit haec tangenti F parallela, ideoque triangula AEK, ACF similia sunt, ac proinde duplicatam habent rationem rationis AK ad AF :hoc est quia AK, AF, AB sunt tres continuae proportionales, rationem habent quam AK ad AB, sed etiam est ut AK ad AB , sic triangulum AEK ad triangulum AEB: ergo triangulum AEK ad triangula ACF, AEB eandem habet rationem; aequantur igitur triangula ACF, AEB: Quod erat demonstrandum.

ELLIPSIS

PARS TERTIA

Considerat axium ac diametrorum coniugatarum tam aequalium quam inaequalium proprietates.

PROPOSITIO LXXI.

In ellipsi diametrorum maxima & minima sunt axes.

Demonstratio.

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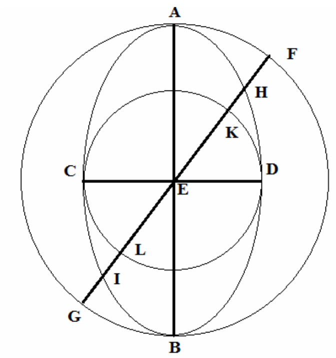
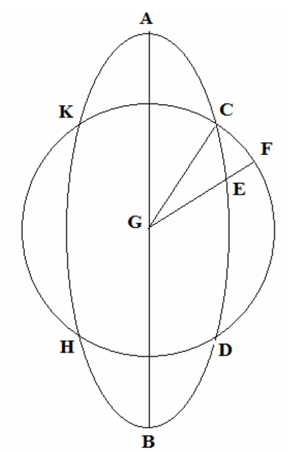


Sint ABC ellipseos axes AB, CD, & AB quidem maior, D vero minor. Dico AB, diametrorum esse maximam, CD vero minimam, centro ellipsis E: intervallo EA circulus describatur AFG: transilit per B, et reliquo sui totus extra ellipsim cadet: ducatur dein per E diameter quaecunque FG occurrens ellipsin H & I circulo autem in F & G. Quoniam circulus AFG totus cadit extra ellipsim, erit FG linea maior recta HI : igitur & AB maior est quam HI. Idem ostenditur de quavis alia diametro; igitur AB axis maximus est diametrorum ellipsis ABC. Quod erat primum.

Rursum centro E intervallo ED circulus describatur DKL occurrens FG lineae in K & L: transilit is per C & reliqua sui parte totus intra sectionem cadet, igitur HI linea maior est quam KL hoc est CD: Quare cum idem de omni alia linea quae per C & D non transit ostendatur, erit CD diameter omnium minima quae in ellipse ADB duci possunt. Quod erat demonstrandum.

Corollorium primum.

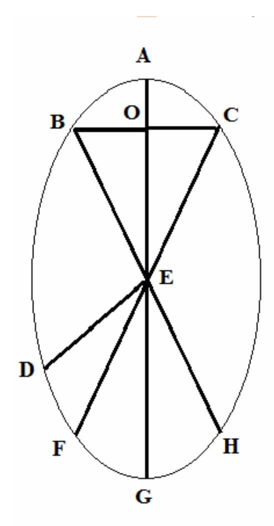


Diameter quae maiori axi est propior, illa maior est; & quae remotior, minor sit ABC ellipseos axis AB, centrum G: ponaturque GC diameter propior axi quam GE. Dico GC maiorem esse ipsa GE centro enim G intervallo GC circulus describatur occurret is ellipsi in quatuor tantum punctis CKHD, quare GE ad peripheriam non pertingit. Unde minor est quam GC. Quod erat demonstrandum.

Corollarium secundum.

Porro diameter axi vicinior est, quae cum axe minorem vel angulum facit vel sectorem, primum patet : alterum e primo sic ostendo. Ellipseos axis maior sit AG faciatque diameter BE cum axe sectorem BEA. minorem sectore DEG, quem cum axe facit diameter DE.

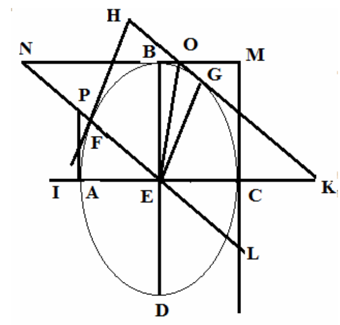
Quoniam igitur sector DEG maior est sectore BEA, fiat sector FEG aequalis sectori BEA, & FE occurrat ellipsi in C, ducaturque BOC, sector BEA aequatur sectori FEG ex constructione hoc est sector ad verticem AEC. Ergo BC bisecta est in O ab axe AG. Anguli ergoad O recti sunt. Patet ergo angulum BEA aequari angulo AEC, hoc est angulo FEG, hoc est minorem esse angulo DEG; liquet igitur ex primo BE quae sectorem facit cum axe minorem, axi propiorem esse quam DE, quae maiorem.



PROPOSITIO LXXII.

Rectangulum sub dimidiis axibus aequale est parallelogammo sub semidiametris coniugatis.

Demonstratio.



Sint ABC ellipsis axes AC, BD: centrum E, & quaevis semidiametri coniugatae EF, EG. Actisque per F & G tangentibus quae conveniant in H, & axi AC occurrant in I & K: ducantur etiam per C & B lineae quae ellipsim contingant in C & B: conveniant autem in M: & HK, EF secent in O, L, N: tum iunctis punctis EO agatur per A tangens, secans EN lineam in P. Quoniam tam NO, KE linea quam OK, NE sibi mutuo aequidistant; erit NO, KE parallelogrammum, diametro OE divisum bifariam: sunt autem triangula a EOB, EOC aequalia, igitur & reliqua triangula EBN, EGK interse aequantur. Rursum cum AP, CL lineae aequidistent, & AE, CE lineae sint aequales, erit ECL triangulo aequale triangulum EAP hoc est EIF. Quoniam igitur triangulum EGK aequatur triangulo EBN, & triangulum IFE triangulo ECL, proportionalia erunt quatuor illa triangula; sunt vero etiam similia inter se, nimirum EGK ipsi IFE, & EBN ipsi ECL, ergo rectae KE, EI, NE, EL, proportionales sunt, quare cum super proportionalibus in directum positis constitura sunt triangula IFE, EGK, IHK inter se similia & triangula CLF, EBN, LMN inter se quoque similia, ut sunt duo triangula IFE, EGK ad duo triangula IFE,EGK aequantur, ut ostendi supra, duobus LCE, EBN. Ergo etiam triangulum IHK triangulo LMN aequale est, ac proinde demptis aequalibus parallelogrammum GEPH sub semidiametris coniugatis, aequatur rectangulo BECH sub dimidiis axibus contento. Quod erat demonstrandum.

Corollorium primum.

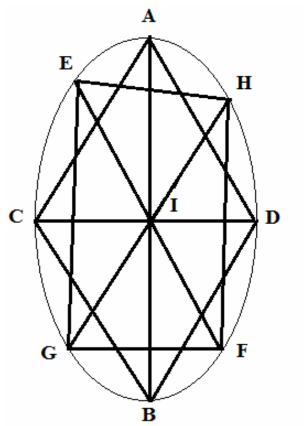
Hinc sequitur si in ellipsi duae quaevis sint diametrorum coniugationes AE, EB, EF, EG triangula super EA, EB, EF, EG in angulis AEB, FEG, esse inter se aequalia: sunt enim dimidia parallelogrammorum aequalium.

Corollorium secundum.

Sequitur secundo parallelogramma sub totis diametris coniugatis, inter se esse aequalia : cum sint quadrupla eorumque hac propositione ostensa sunt aequalia.

PROPOSITIO LXXIII.

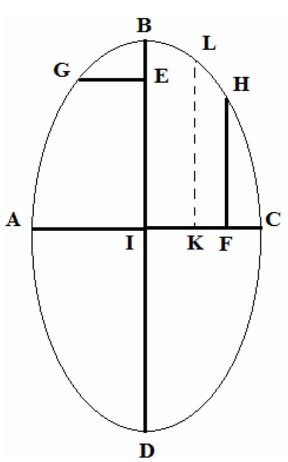
IN ellipsi parallelogrammum quod fit a lineis extrema axium coniungentibus aequale est parallelogrammo contento lineis extrema quarumvis diametrorum coniugatarum coniungentibus.



Demonstratio.

Sint ABC ellipseos axes ABCD, & alia quaevis diametrorum coniugatio EFGH. Iunganturque tam axium, quam diametrorum extremae: dico CADB parallelogrammum aequari parallelogrammo EHFG. Rectangulum ex AB & DI duplum est trianguli ADB: & rectangulum ex AB & CI, duplum est trianguli ACB. Ergo rectangulum ex AB, CD, duplum est parallelogrammi ACBD, sive ACBD parallelogramum, dimidium est rectanguli super AB, CD. Similiter ostendam parallelogrammum EGFH dimidium esse parallelogrammi super EF, GH in angulo EIH: sed parallelogrammum super EF, GH aequale est parallelogrammo super ABCD; igitur & ACBD parallelogrammum aequale est parallogramo EGFH. Quod fuit demonstrandum.

PROPOSITIO LXXIV.



Sint ABC ellipseos axes vel diametri coniugatae, AC, BD. Divisaque BD utcunqae in E, dividatur AC in F proportionaliter, & per E & F ordinatim ducantur lineae EG, FH:

Dico rectangulum AFC aequari quadrato GE, & BED rectangulum quadrato HF, & si quadratum GE sit aequale rectangulo AFC. dico BD, AC proportionaliter esse divisas in E & F.

Demonstratio.

Cum ex hypothesi DE sit ad EB ut AF ad FC, erit permutando DE ad AF, ut BE ad FC. Quare & tota DB, ad totam AC, ut DE ad AF & EB ad FC. igitur rationes BE ad FC, & DE ad AF, simul sumptae duplicatae sunt rationis DB ad AC. Atqui ratio rectanguli BED ad rectangulum AFC, componitur ex rationibus BE ad FC, & DE ad AF. Ergo ratio rectanguli BED ad rectangulum AFC duplicata est rationis BD ad AC, hoc est rationis BI ad AI. Ergo rectangulum BED est ad rectangulum AFC, ut quadratum BI ad quadratum AI, sed idem quoque rectangulum BED est ad quadratum GE, ut rectangulum BID. Hoc est, quadratum BI, ad quadratum AI aequantur igitur quadratum GE & rectangulum AFC. Similiter ostendemus rectangulum BED & quadratum HF aequalia esse.

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Sint iam aequalia quadratum GE & rectangulum AFC: dico BD, AC proportionaliter esse sectas: Nam si non est ut BE ad ED, sic AF ad FC, sit ut BE, ad ED, sic AK ad KC. Erit ergo quadratum GE aequale rectangulo AKC per primam partem huius: Quod fieri non potest, cum quadratum GE ex hypothesi sit aequale rectangulo AFC. Non igitur AC in K aut alibi quam in F erit secta proportionaliter ad BD. Quod erat demonstrandum.

PROPOSITIO LXXV.

Si axes aut diametri coniugatae sint proportionaliter sectae in E & F, & ordinatim ducantur EG, FH.

Dico quadratum FH esse ad quadratum EG, ut quadratum BD, ad quadratum AC.

Demonstratio.

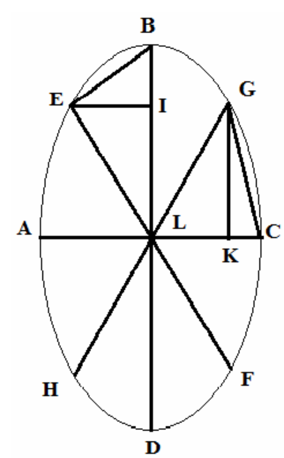
Quadratum FH aequatur rectangulo BED, sed rectangulum BED est ad quadratum EG ut rectangulum BID, hoc est quadratum BI, ad quadratum IA. Ergo etiam quadratum FH est ad quadratum EG, ut quadratum BI ad quadratum IA, hoc est, ut quadratum BD ad quadratum AC. Quod erat demonstrandum.

Quod si ad diametros coniugatas ordinatim positae sint EG, FH, & sit ut quadratum BD ad quadratum AC, ita quadratum FH ad quadratum EG : Dico BD, AC proportionaliter esse sectas in E & F. Si enim negas esse AF ad FC, ut DE ad EB, fiat AK ad KC, ut DE ad EB, & sit ordinarim KL. Ergo ut quadratum BD ad quadratum AC, sit quadratum KL ad quadratum EG: quod fieri non potest, cum ex hypothesi quadratum FH sit ad quadratum EG, ut quadratum BD ad quadratum AC. Non igitur est ut quadratum BD ad quadratum AC, ita quadram KL, aut quodvis aliud praeter quadratum FH, ad quadratum EG. Quod erat demonstrandum.

PROPOSITIO LXXVI.

Sint in ABC ellipsi duae quaevis diametrorum coniugationes AC, BD, EF, GH; ducanturque ex E & G lineae EI, GK ordinatim ad diametros BD, AC.

Dico EI quadratun, aequari rectangulo AKC, & BID, rectangulum aequale esse quadrato GK.



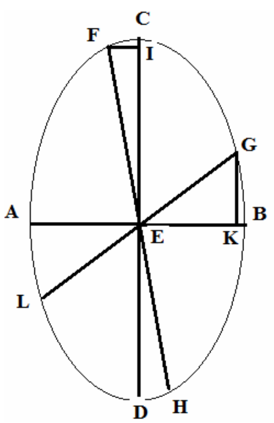
Demonstratio.

Ponantur EB, GC. Quoniam igitur tam AC, BD, quam EF, GH diametri sunt coniugatae, erit sector BLC aequalis sectori ELG, & dempto communi BLG, sector ELB aequalis sectori CLG. Adeoque & LEB triangulum aequale triangulo LGC: & quia BD est coniugata ipsi AC, erit BD parallela ad KG quae est ordinatim posita ad AC. Ergo angulus GKL aequalis angulo BLA. Similiter quia AC est coniugata ipsi BD, erit AC parallela ipsi EI ordinatim positae ad BD; angulus ergo EIB aequalis est angulo BLA, hoc est angulo GKL; igitur cum triangula sint aequalia, erit (ut infra ostendam) ut basis LB ad basim LC, ita KG ad EI, adeoque ut quadratum BL ad quadratum LC, hoc est ut quadratum BD, ad quadratum AC, sic quadratum KG ad quadratum EI. Unde BD, AC lineae in I & K proportionaliter sunt divisae. Quare EI quadratum aequale rectangulo AKC, item BID rectangulum aequale quadrato GK. Quod fuit demonstrandum.

PROPOSITIO LXXVII.

Axium quadrata simul sumpta aequalia sunt quadratis cuiuscunque coniugationis simul sumptis,

Demonstratio.

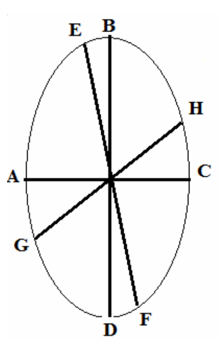


Sint ABC ellipseos axes AB; CD, & alia quaevis diametrorum coniugatio FH, GL. Dico AB, CD quadrata simul sumptis aequari quadratis FH, GL simul sumptis. Ducantur ordinatim lineae, FI, GK quae, quia ad axes ducantur, perpendiculares erunt; centrum autem sectionis ponatur E. Quadratum EC aequale est quadrato EI una cum CID rectangulo id est quadrato GK: quadratum autem EB aequale est quadrato EK una cum rectangulo AKB id est quadrato FI : unde quadrata duo EB, EC simul sumpta aequalia sunt quadratis FI, IE, EK, GK simul sumptis. Sed iisdem quadratis aequalia sunt quadrata SE, EG, quadratis igitur EF, EG aequalia sunt quadrata EB, EC. Quare cum AB, CD quadrata simul sumpta quadrupla sint quadratorum EB, EC & FH, GL quadrata Quadrupla quadratorum EF, EG; patet AB, CD quadrata simul sumpta aequari quadratis FH, GL simul sumptis. Quod erat demonstrandum.

PROPOSITIO LXXVIII.

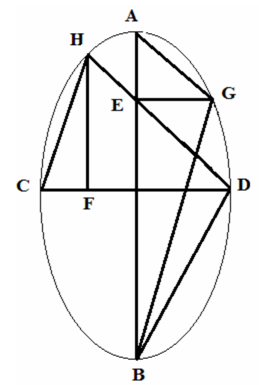
Axes ellipseos simul sumpti minimae sint omnium diametrorum coniugatarum simul sumptarum.

Demonstratio.



Sint axes AC, BD & quaevis diametrorum coniugatio, EF, GH: dico axes simul sumptos minores esse diametris coniugatis simul sumptis. Quoniam AC, BD quadrata simul sumpta, aequalia sunt quadratis EF, GH simul sumptis: sit autem & BD maxima diametorum, AC vero minima, erunt AC, BD simul sumptae minores rectis EF, GH igitur, &c. Quod fuit demonstrandum.

PROPOSITIO LXXIX.



Sint ABC ellipsis axes AB, CD, devisi proportionaliter in E & F: ductisque ordinatim (quae hic sunt perpendiculares) lineis EG, FH: iungantur AG, GB, CH, HD: Dico quatuor quadrata AG, GB, CH, HD, simul sumpta, aequari duobus axium quadratis.

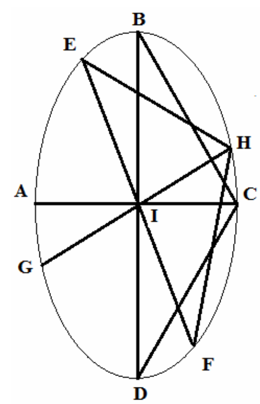
Demonstratio.

Quadratum AB aequale est quadratis AE, EB una cum rectangulo AEB id est quadrato HF bis sumpto quadratum vero CD aequalc est quadratis CF, FD, & CFD rectangulo id est quadrato EG bis sumto, sed iisdem quadratis aequalia sunt quadrata AG, GB, CH, HD, igitur axium quadrata simul sumpta aequalia sunt quadratis AG, GB, CH, HD. Quod erat demonstrandum.

PROPOSITIO LXXX.

Quadrata linearum extrema axium coniungentium aequalia sunt quadratis linearum quae extrema cuiusuis coniugationis coniungut.

Demonstratio.



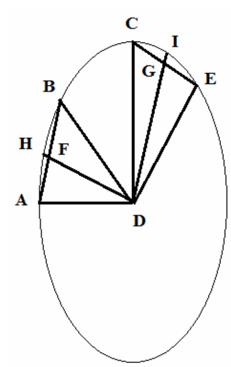
Sint ABC ellipseos axes AC, BD & alia quaevis diametrorum coniugatio EF, GH. iunganturque BC,CD, EH, FH. Dico quadrata BC,CD simul sumpta aequari quadratis EH, FH simul sumptis quadrata BC, CD simul sumpta aequalia sunt quadratis BI, IC bis sumptis quadrata autem EH, HF aequantur quadratis EI, IH bis sumptis: sed quadrata EI, IH simul sumpta sunt aequalia quadratis BI, IC simul sumptis; igitur quadrata BC, CD simul sumpta, aequalia sunt quadratis EH, HF simul sumptis. Quod erat demonstrandum.

PROPOSITIO LXXXI.

Secent ABC ellipsim, cuius centrum D duae diametrorum coniugationes AD, DC, BD, DE, iunctisque punctis AB, CD dividantur AB, CD linae bifariam in F & G, ducanturque DF, DG quae productae occurrant ellipsi in H & I.

Dico HD, ID diametros esse coniugatas.

Demonstratio.



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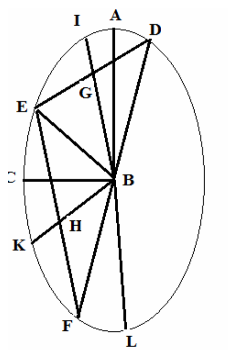
Quoniam AD, DC, BD, DE, diametri sunt coniugatae, erunt ADC, BDE sectores aequales: ablato igitur communi BDC, erunt ADB, CDE reliqui aequales : rursum cum AB lineam secet in F, bifariam diameter DH, erunt tam ADH, BDH sectores, quam AFD, BFD triangula aequalia. Eadem modo ostenditur EDI sector aequalis sectori CDI; sectores igitur ADH, EDI sunt aequales inter se: Addico igitur communi HDI, erit sector ADI aequali sectori HDE, coniugatae ergo sunt DH, DI.

PROPOSITIO LXXXII.

Sint ABC ellipseos axes AB, CD, sit autem & alia quaevis diametrorum coniugatio, DF, EB; quas iungant DE, FE; DE quidem secans axem maiorem, EF vero minorem : ipsas deinde DE, FE bifariarm secent diametri BGI, BHK.
Dico IB diametrum maiorem esse diametro KB.

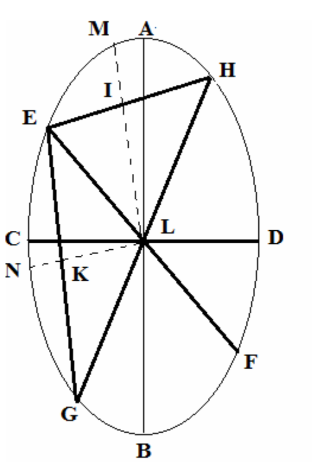


Demonstratio.

Quoniam diametri BI, BK bisecant rectas ED, EF sectores ambo DBE, EBF bisecantur. Quare cum ipsi aequales sint, etiam ipsorum sectores dimidii IBD, KBF aequales erunt. Sector igitur IBD minor est sectore KBL; ergo sector IBA multo minor sectore KBL. Ergo IB propior axi est quam BK, ac proinde maior quam BK. Quod erat demonstrandum.

PROPOSITIO LXXXIII.

Sint ABC ellipseos axes AB, CD, & alia quaevis diametrorum coniugatio EF, GH, iunganturque EH, EG.
Dico lineam EH quae axem maiorem secat, minorem esse lineam EG, quae minorem secat.
Demonstratio.



EH, EG dividantur bifariam per diametros LM, LN, in I & K. Quoniam GH, EF sunt contiugatae; sectores GLE, ELH aequales erunt, ac proinde segmenta GNE, EMH aequalia sunt. Quare LM, LN bisecantes subtensas EH, EG, proportionaliter sunt divisae Ergo MI ad IL ut NK ad KL; & componendo ac permutando ut LM ad LN, sic LI ad LK. Sed LM maior est quam LN; ergo LI etiam maior quam LK. Iam vero cum LN, LM etiam sint coniugatae, & EG sit ordinatim ex constr. posita ad LN, erit LM parallela ad EK. ob similem causam LN, EI parallelae erunt ; parallelogrammum igitur est EI, LK, adeoque LI, KE, LK, EI aequantur. Cum ergo LI ostensa sit maior esse quam LK, erit & KE maior quam LK, hoc
quam EI. Quare dupla eius EG, maior dupla EH. Quod erat

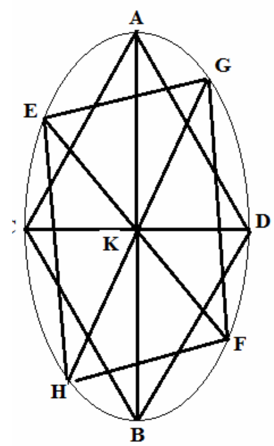
demonstrandum.

PROPOSITIO LXXXIV.

Axium extrema coniungentes simul sumptae maximae sunt omnium quae quarumvis diametrorum coniugatarum conuertunt extrema.

Demonstratio.

Sint axes AB, CD & alia quaevis diametrorum coniugatio EF, GH : iunganturque extrema tam axium, quam aliarum diametrorum. Dico lineas CA, AD, DB, si C simul sumptas maiores esse lineis FG, GF, FH, HE, BC simul sumptas. Quoniam CK ipsi HK aequalis est, EK vero communis, & EH recta maior quam EG, erit angulus EKH, maior angulo EKG: igitur & angulus EKH maior est recto AKC. Sunt autem AKC, EKH triangula aequalia, quare & EH est maior quam AC: eadem modo ostenditur AC linea maior est recta EG. Ergo EG minima est, & EH, maxima linearum EH, AC, AD, EG. igitur cum EH, EG quadrata simul sumptis sint aequalia quadratis AC, AD simul sumptis, erunt EH, EG lineae fimul sumptae minores lineis AC, CD simul sumptis. Eodem modo ostenduntur lineae GF, FH minores lineis CB, BD, ergo, & c. Quod erat demonstrandum,

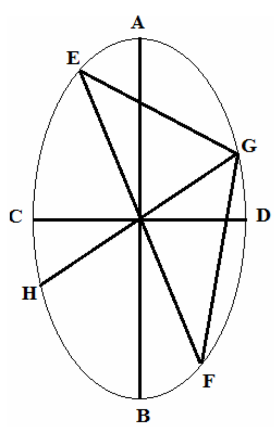


PROPOSITIO LXXXV.

ln nulla ellipsi est invenire diametros coniugatas quae sese ad rectos secent, praeter axes.

Demonstratio.

Sint axes, AB maior, CD minor & alia quaevis diametrorum coniugatio EF, GH: centrum autem ellipsis sit K. Dico neutrum angulorum EKG, GKF rectum esse: iungantur enim puncta EG, GF: Quonian EK, KG duabus rectis FK, KG aequales sunt, & EG minor quam FG. erit angulus, EKG minor angulo GKF; Quare cum erum summa sit duobus rectis aequalis, neuter illorum rectus est: idem de aliis omnibus ostenditur: igitur in nulla ellipsi est invenire, &c. Quod erat demonstrandum.



PROPOSITIO LXXXVI.

Sint ABC ellipsis axes AC, BD & alia quaevis diametrorum coniugatio EF, GH : iunctisque punctis AB, BC, rectae ducantur EH, EG.

Dico angulum ABC qui circa minorem axem existit, maiorem esse angulo GEH, ac proinde maximum esse omnium angulorum qui continentur a lineis extrema diametrorum coniugatarum coniungentibus.

Demonstratio.

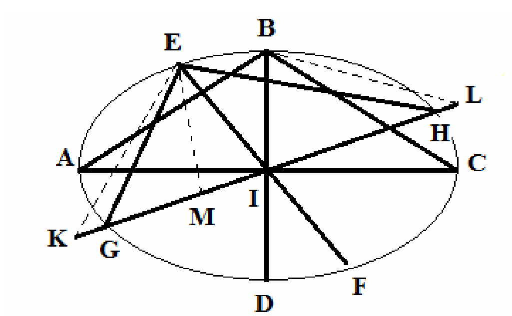
Cum GH linea minor sit axe AC, producatur utrimque aequaliter in K & L ut LK, sit aequalis axi AC, iunganturque puncta EK, EL, & ex E demittatur EM normalis ad LK: erit igitur KEL triangulurom maius triangulo GEH id est triangulo ABC ; quare cum inaequalium triangulorum bases AC, LK sint aequales, erit EM altitudo trianguli KEL, maior IB altitudine trianguli ABC: adeoque angulus KEL maior angulo ABC: angulus igitur GEH multo minor est angula ABC. Quod erat demonstrandum.

PROPOSITIO LXXXVII.

Sint ABC ellipseos axes AC, BD : iunganturque illorum extrema AB, BC, CD, DA : sit autem & alia quaecunque diametrorum coniugatio EF, GH, quarum extrema quoque coniungantur.

Dico angulos ABC, EGF, HFG, BCD arithmetice esse proportionales.

Demonstratio.



Quoniam tam AC quam EF parallelogrammum est, erunt tam ABC est, erunt tam ABC, BCD anguli, quam EGF, GFH duobus rectis aequales: quare & anguli ABC, BCD simul sumpti aequantur angulis EGF, GFH simul sumptis: est autem angulus ABC, ostensus igitur maior angulo EGF, igitur & GFH maior est angulo BCD: & quia, ut iam ostendi, anguli B, & C simul sumpti aequantur angulis G, & F, simul sumptis quo excessu ABC angulus superat angulum EGF, eodem necesse est ut angulus GFH superet angulum BCD. Quod erat demonstrandum.

Corollarium.

Hinc patet angulum BCD qui circa axem maiorem existit, minimum esse omnium angulorum qui sunt a lineis diametrorum coniugatarum extrema coniungentibus.

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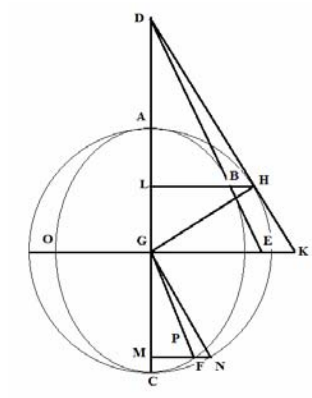
PROPOSITIO LXXXVIII.

Ellipsum ABC cuius axes AC, EG contingat in B recta quaedam DE conveniens cum utroque axe in D & E: ex centro vero G recta demittatur GF parallela lineae DE.

Dico DB, GF, BE lineas esse in continua proportione.



Demonstratio.

Centro G,intervallo AG circulus describatur AHC: & ex D demittatur DK contingens circulum in A occurrens EG axi ellipseos in K, ductaque ex H linea HL normali ad axem quae per Coroll.33. huius transit etiam per B, agatur per F normalis alia FMN, iunganturque puncta NG, HG. Quoniam LH, MN lineae aequidistant, erunt anguli LDB, MGF aequales: sunt autem & anguli BLD, FMG recti per constructionem; igitur & reliquus LBD, reliquo MFG aequalis est. Quare anguli DBH, GFN inter se aequantur. Quia autem triangula DLB, GMF sunt similia, erit ut LB ad M F, sic DB ad G F. sed ex demonstratis in scholio quartae huius, ut LB ad MF, sic BH ad FN. Ergo DB ad GF, ut BH ad FN. Quare cum anguli DBH, GFN iam ostensi sint aequales, similia erunt triangula DBH, GFN. Ergo HD est ad BD, hoc est HK est ad BE, ut GN ad GF, & permutando ut HK ad GN, sic BE ad GF. Deinde cum in triangulo DGK angulus ad G rectus sit & GH ex centro ad contactum ducta, normalis ad DK, erit HK ad GH, ut GH ad HD. Sed GN, GH aequantur, ergo, ut KH ad GN, hoc est sicut ante ostendi, ut BE ad G F, sic GN ad DH. Sed ob similitudinem triangulorum ut GN ad DH, sic GF ad DB. ergo, ut BE ad GF, sic GF ad DB. Quod erat demonstrandum.

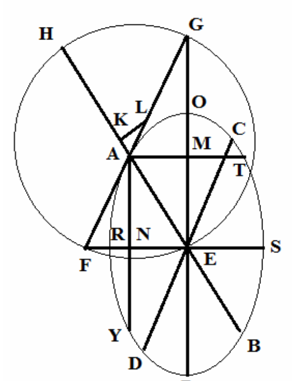
PROPOSITIO LXXXIX.

Iisdem positis si GF sit proportionalis media inter DB, BE.

Dico punctum F esse ad ellipsim.

Demonstratio.

Si punctum F non est ad ellipsim, occurrat ergo ellipsis rectae GF in P, supra vel infra F. Ergo per praecedentem DB, GP, BE sunt continue proportionales; quod fieri non potest, cum DB, GF, BE ponantur continue. Non igitur aliud punctum rectae GF ad ellipsim est, quam F. Quod erat demonstrandum.



PROPOSITIO XC.

Data quavis diametrorum coniugatione, ellipseos axes reperire.

Constructio & demonstratio.

Sint datae diametri coniugatae AB, CD secantes se bifariam in E; actaque per A linea FG quae aequidistat CD, fiat ut AE ad

ED, sic ED ad AH; tum EH divisa bifariam in K, erigatur ex K normalis KL occurrens FG in L, dein centro L intervallo LE circulus describatur EFG, transibit hic per H secabitque FG lineam in punctis quibusdam F & G: iungantur demum puncta FF, EG. Dico lineas EF, EG satisfacere petitioni. Quoniam enim rectae AE, ED, AH sunt ex constructione in continua analogia, erit quadrato ED aequale rectangulum EAH, hoc est FAG rectangulum. Itaque GA, DE, FA rectae in continua etiam sunt analogia, ac proinde, erit punctum D ad ellipsim, cuius axes sunt in lineis EF, EG. Sed & A in eadem est ellipsi, igitur & B, C puncta, in eadem erunt ellipsi.

Porro termini axium ita invenientur: ductis AM, AN normalibus ad EG lineam, inveniatur inter EM, EG media EO, & inter EN, EF media ER : describaturque,per D, A, C puncta ellipsis; quoniam CD ipsi AB est coniugata, adeoque ad ipsam ordinatim posita & FG linea aequidistat ipsi CD, erit FG tangens ellipsim ABC; sunt autem AM, AN normales ad lineas in quibus axes sectionis existunt; & tam EM, EO, EG, quam EN, ER, EF continuae, igitur ellipsis ABC transit per puncto RO: quare R & O termini sunt axium, quos oportuit exhibere.

Scholion.

Propositum est hoc problema a Pappo, lib.8.Mathem.Collect.prop.14.ac veram quidem eius constructionem eam nempe quam ex illo nos iam attulimus, sed non demonstrat.Fredericus Commandinus demonstrationem supplere conatus est, ita scribens:

Producatur AM usque ad T ita ut TM ipsi MA, sit aequalis: producatur etiam AN usque ad Y ut YN sit aequalis NA : erunt puncta TY in ellipsi, ex iis quae demonstrata sunt ab Apollonio in propop:47 a lib. Conic. Sed RS parallela est ipsi AT, est enim angulus in semicirculo rectus, quare & OP ipsi AY parallela erit. Quoniam igitur CD ad AB ordinatim est applicata quae per A ipsi DC parallela ducitur, videlicet FG sectionem in puncto A continget, & cum FG sectionem contingens diametro occurrat in G & AM ordinatim applicetur, erit per 37. prim.Coni.Apollon, rectangulum GEM aquale quadrato ex EO vel EP. Eadem quoque ratione cum AN ordinatim applicetur rectangulum FEN quadrato ex ER vel ES aequale est ergo OP, RS ellipsis coniugati axes erunt.

Haec Commandinus quibus recta ostendit OP & RS conugatos axes esse ellipsis, quae per puncta ATY incedit, & a linea FG in puncto tangitur: verum hoc propositum non fuit. Nam ad inveniendum eius modi coniugatos axes non opus erat ad describendum circulum FEG, facere rectangulum EAH quadrato ED aequale, & secare EH bifariam in K indeque normalem excitare, quae congrediens cum FG in L, centrum praeberet L circuli FEG, sumpto si quidem in linea FG centro quocumque, si per E circulus circumducatur, qui secet lineam FG: non iam quidem in punctis F & G sed in aliis quae ex E recta emittentur angulum rectum continebunt, non secus ac EF & EG, quare si ab A ad has ipsas postremo ducts lineas, normales ducanturque, quales erant AM & AN, duplicentur , ut AT & AY, erunt puncta quae vices punctorum T & Y subibunt in ellipsi, quae per A incedit, tangiturque lineae FG infinitae destinantur. At perspicuum est hanc ellipsin (quod fuerat demonstrandum) per puncta C & D minima transire, propterea quod circuli centrum aliud ab L assumptum sit, nec sit quadratum ED rectangulo sub EA, & alia linea quam AH contentum aequale. Itaque ut ostendatur OP & RS coniugatos axes esse ellipsis, quae per

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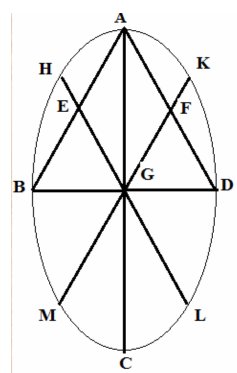
terminos diametrorum coniugatarum AB, & CD incedit, alia ratio est ineunda, quam in demonstratrione nostra iam proposuimus.

Haec hactenus super Commandini demonstratio.

Caeterum ipse Pappi textu temporum iniuriam sescio quid infortunii passus videtur, ita enim habet : Facile autem est inventis quibuscumque coniugationibus diametrorum ellipsis, axis eius organice invenire. Quod quidem hac ratione fiet. Quae verba legitimum sensum non habent, cum ea, quam adfert constructio non organica. Sed omnino Geometrica sit, uti eam legenti satis patet: quare puto omissum verbum Geometrice, sicque legendum: Facile autem est inventis quibuscumque coniugationibus diametrorum ellipsis, axes eius organice invenire, quod quidem Geometrice hac ratione fiet. Deinde addita sunt in ipsa constructione illa verba: cum sit DE maior quam EA: cum enim constructio universalis sit, sive DE minor sive maior, sive ipsi EA aequalis ponatur, ut ex nostra demonstratione colligi potest, quodque Pappum etiam latere nullo modo potuisse certum est , frustra assumitur DE maior ipsa EA. Mirum proinde est hunc errorem Fredericum Commandium non advertisse, praesertim cum illo assumpto in demonstratione sua, quam superius dedimus, usus non fuerit: sed universalem attulerit demonstrationem: unde cum & ipsa desit demonstratio, quam quom Pappus addiderit , dubium non est. Satis manifestum est eorum errore id contigisse, in quorum manus venit haec propositio (quae pene tota, ut existimo, interciderat:) qui eam plane iam mutilam & imperfectam, frustra restituere conati sunt.

PROPOSITIO XCI.



Datis axibus in ellipsi, aequales diametros coniugatas exhibere.

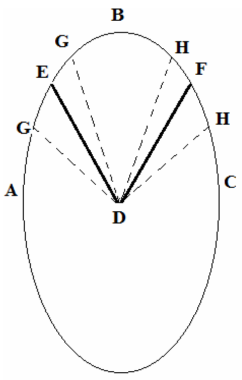
Constructio et demonstratio.

Sint ABC ellipsis axes AC, BD : oporteat autem exhibere diametros coniugatas aequales iunctis AB, AD; dividantur rectae AB, AC bifariam in E & F; & per E & F ex G centro rectae ducantur GH, GK, occurrentes ellipsi in H, K, L, M punctis. Dico illas satisfacere propositioni. Quoniam enim rectae duae EB, BG, aequales sunt duabus lineis FD, DG (sunt autem & anguli, aequalibus lateribus contenti inter se aequales) erunt etiam anguli ad basin, EGB, FGD adeoque & reliqui AGE, AGF aequales. Rursum cum angulus AGB sit rectus & basis AB in E divisa bifariam, si centro E intervallo EA describatur circulus, transibit etiam per B, adeoque EA, EG lineae erunt aequales. Quare & angulus EAG, aequalis angulo EGA, hoc erit AGF. Ergo AB, KM lineae parallelae: eodem modo ostenduntur rectae AD, HL parallelae: unde cum diametri HL, KM mutuas parallelas bisecent, erunt coniugatae. Quia vero angulus HGA est angulo AGK ostensus aequalis, erit quoque HG linea aequalis GK, ut patet ex 18.huius ergo HL, KM diametri sunt coniugatae & aequales exhibuimus ergo, &c. Quod erat demonstrandum.

PROPOSITIO XCII.

In una ellipsi duas tantum est reperire diametros coniugatas aequales.

Demonstratio.

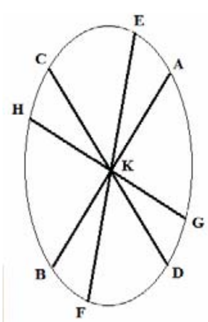


Sit ABC ellipsis centrum D & in ea aequales diametri coniugatae. ED, FD : dico alias diametros coniugatas & aequales in ea exhiberi non posse: sint enim, si potest fieri, praeter ED, FD diametros : aliae aequales & coniugatae GD, HD: erit igitur EDF sectori aequalis sector GDH. Quod fieri non potest , nam GD, HD diametri cum sint aequales necesse est maiores vel minores illas esse diametris ED; FD, adeoque, ambas simul cadere supra vel infra diametros ED, FD: Igitur praeter ED, FD diametros coniugatas aequales, nullas alias aequales in ellipst est exhibere. Quod erat demonstrandum.

PROPOSITIO XCIII.

In ellipsi aequales diametri coniugatae simul sumptae, maximae sunt omnium diametrorum coniugatorum simul sumptarum.

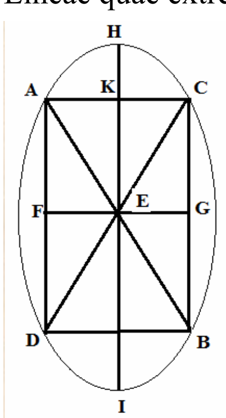
Demonstratio.



Sint AB, CD diametri coniugatae aequales, sit autem & alia quaevis diametrorum coniugatio EF, GH: dico diametros AB, CD simul sumptas maiores esse diametris EF, GH simul sumtis: cum enim sectores AKG, GKE sint inter se aequales ; necesse est unam coniugatarum inaequalium (sit EF) axi viciniorem esse utravis aequalium AB, CD: alteram vero HG; remotiorem unde ex quatuor diametris EF maxima, & GH minima est. Sunt autem EF, GH quadrata simul sumpta aequalia quadratis AB, CD simul sumptis ; igitur AB, CD lineae simul sumptae maiores quoque sunt lineis EF, GH simul sumptis: Quod erat demonstrandum.

PROPOSITIO XCIV.

Lineae quae extrema diametrorum coniugatatum aequalium coniungunt, ab axibus bifariam secantur.



Demonstratio.

Sint AB, CD diametri coniugatea aequales, iunganturque illarum extrema AD, AC, CB, DB. Dico illas ab axibus bifariam secari, divisa enim AD bifariam in F, agatur per E centrum FEG occurrens CB rectae in G. Quandoquidem ergo AB, CD ponantur aequales, harum dimidiae AE, DE, etiam sunt aequales. Aequantur autem ex const: similiter AF, DF. Itaque in trianguli AEF, DEF cum FE sit commune, omnia latera sibi invicem aequantur. Ergo anguli ad F aequales, adeoque rectis & anguli quoque FEA, FED, aequales. Quare anguli etiam GEC, GEB prioribus ad verticem oppositi aequantur. Sunt vero latera rursum CE, EB aequaliter & EG commune utrique triangulo GEC, GEB. Igitur CG, BG aequales, & anguli ad G aequales adeoque recti. Cum ergo FG rectas AD, CE, (quae per 19.huius sunt parallelae) bifariam & ad angulos rectos secet, axis est. Secantur igitur ab axe bifariam recte AD, CE extrema coniugatarum aequaluum connectentes.Eodem modo ostendemus reliquas duas AC, BD ab axe HI bisecari. Constat ergo veritas propositionis.

PROPOSITIO XCV.

Si lineae quae extrema coniugatarum connectunt, ab axibus secentur bifariam: Dico diametros illas esse inter se aequales.

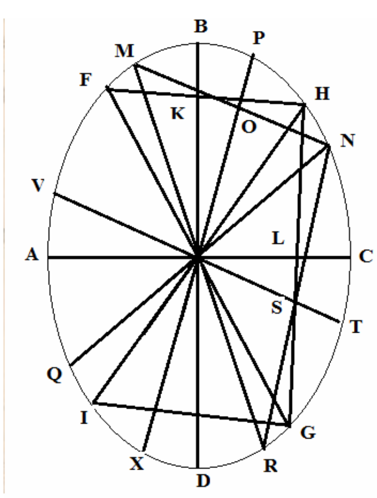
Demonstratio.

Ponatur eadem figura quae prius, sintque AD, CE, AC lineae, extrema coniugatarun connectentes in F, G, & K bifariam & ad rectos divisae axibus Hl, FG. Dico AB, CD, diametros coniugatas esse inter se aequales: cum enim AD, CE per huius sint parallelae & ex hypothesi ab axe in F & G bisecentur, anguli ad F, recti sunt, & latera duo AF, FE aequalia sunt lateribus DF, FE; reliqua igitur latera AE, ED quoque inter se aequalia. Similiter ostendam CE, EB aequales. Unde & totem diametri AB, CD aequales. Quod fuit demonstrandum.

Corollarium.

Hinc patet lineas, quae inaequalium coniugatarum extrema coniungunt, nunquam ab axibus aut alia quavis diametro bifariam & ad rectos secari.

PROPOSITIO XCVI.



Linearum quae extrema coniugatarum quarumvis coniungunt, illa maxima est quae coniugatas aequales connectens, axem minor secat ; minima, quae maiorem.

Demonstratio.

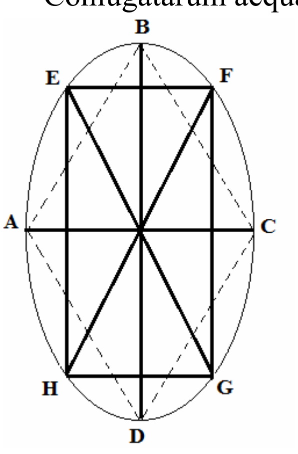
Sint AC, BD axes ellipseos ABC, coniugatae vero aequales FG, HI iunctaque FH maiori axi occurrat in K:

& HG minori in L. Dico HG lineam maximam esse illarum quae cuiuscunque coniugationis extrema coniungunt, & FH minimam. Fiat enim quaevis alia diametrorum coniugatio MR, NQ. Quarum extrema iungant MN, NR, quibus in O & S bisectus ducantur per centrum XOP, VST. Quoniam ergo FG, HI sunt coniugatae, sectores EFBH, EHCG aequantur sunt. Ergo segmenta FBH, HCG aequalia sunt. Quare, cum axes BD, AC etiam bisecent rectas FH, HG, quae aequales coniugatas iungunt, erunt axes ipsi in K & L, proportionaliter secti. Ergo rectangulum BKD aequale est quadrato LH. Simili plane discursu ostendemus rectangulum POX aequari quadrato NS. Deinde quia sectores MEN, FEH aequantur, adeoque segmenta MPN, FBH, suntque ambae FH, MN bisectae in K & O, erunt DB, XP proportionaliter sectae in K & O; sed DB axis maior est quam XP. Ergo rectangulum BKD, hoc est, ut ante ostendi, quadratum HL,maius est rectangulo POX, hoc est quadrato NS. Ergo recta HL maior recta NS. sed HG dupla est ipsius HL, ut ante ostendi, & NR ex const. ipsius NS dupla est. Ergo HG maior quam NR: quia autem NR maior est quam MN; erit HG etiam maior quam MN. Eodem modo ostendemus HG maiorem esse quarumvis aliarum coniugatarum extrema connectentibus. Ergo HG est omnium maxima. Quod erat primum.

Quod autem FH sit omnium minima, discursu plane disimili demonstrabimus. Sumatur enim quaevis alia diametrorum coniugatio MR, NQ, & eadem, quae supra est adhibitis, repetatur constructio. Eodem modo ostendemus quadratum FK minus esse quadrato MO, & rectam FK minorem recta MO, ac proinde FH, minorem quam MN. Est autem RN maior quam MN. Ergo FH etiam minor est quam NR. Atque ita demonstrabimus FH minorem esse quarumvis coniugatarum extrema connectentibus. Omnium igitur minima est. Quod erat secundo loco ostendendum.

PROPOSITIO XCVII.

Coniugatarum aequalium extrema coniungentes simul sumptae minimae sunt omnium quae quascunque diametros coniugatas coniungunt.



Demonstratio.

Sint in ABC ellipti coniugatae aequales EF, GH. Ponatur autem & alia quaevis diametrorum coniugato, EF, GH. Dico lineas quae extrema coniugatarum aequalium coniungunt, simul sumptas minores esse lineis quae extrema alterius coniugationis connectunt. Sunt enim EG, GF quadrata aequalia quadratis AB, BC, insuper & EG linea connectentium minima, & FG maxima per praecedentem; igitur EG, GF lineae minores sunt lineis AB, BC; eodem modo ostenduntur EH, HF lineae minorem lineis AD, DC : igitur lineae, &c. Quod fuit demonstrandum.

PROPOSITIO XCVlll.

Sint ABC ellipseos axes AC, BD & EF una exdiametris coniugatis aequalibus.

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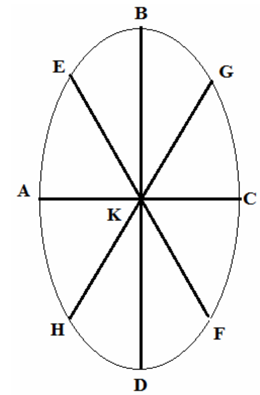
Dico quadrata AK, BK simul sumpta esse dupla quadrati EK.



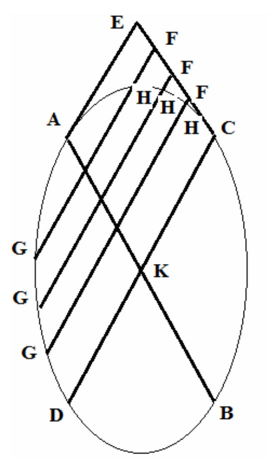
Demonstratio.

Ducatur GH altera diametrorum coniugatarum aequalium. Quoniam AC, BD quadrata simul sumptae qualia sunt quadratis EF, GH simul sumptis, erunt & quadrata AK, BK sub dimidiis axibus, aequalia quadratis EK, GK sub dimidiis diametris aequalibus; sunt autem EK, GK quadrata inter se aequalia, igitur quadrata AK, BK simul sumpta dupla sunt quadrati EK.
Quod erat demonstrandum.

PROPOSITIO XCIX.

Apollonius l. 3· Conic. prop.16.huiusmodi habet theorema: si ellipsim tangant AE, CE, conuenientes in E, & sumpto in sectione puncto G ducatur GHF tangentium uni parallela GHF, erit rectanguluum GFH ad quadratum BC, ut quadratum BC, ut quadratum AE ad quadratum CE.
Verum non similitudo tantum rationum sed spatiorum etiam aequalitas reperietur si tangentes a diametrorum coniugatarum aequalium ductae fuerint.

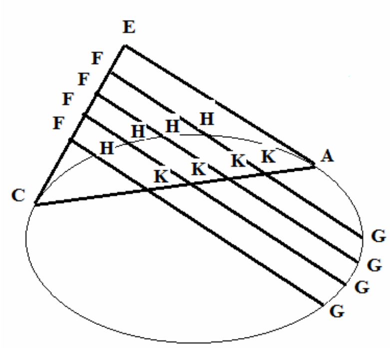
Ellipseos diametri coniugatae aequales sint AB, CD, in quarum terminis A, C, ellipsim tangant duae rectae conuenientes in E, si alterutri ducantur quotius parallelae GH erunt rectangula GFH quadratis FC aequalia.



Demonstratio.

Quoniam CD est diameter coniugata diametri AB, erit ad ipsam ordinatim posita: ergo tangenti AE parallela est. Eodem modo AK tangenti EC parallels est. Figura igitur KAEC est parallelogrammum.Quare cum AK, KC ex hypothesi sint aequales, etiam AE, CE aequales sunt: aequantur igitur quadrata AE, EC. Atque est ut quadratum AE ad quadratum EC, ita rectangulum GFH ad quadratum FC, ergo rectangulum GFH quadrate FC aequale est.
Quod erat demonstrandum.

Et quoniam Theorema illud Apollonii iam habemus in manibus, triam hoc addo quod similiter Apollonius non videtur observasse: nimirum si ductis tangentibus AE, CE, iungantur puncta contactuum A, C, rectangula GFH, quadratis KF aequalia esse.

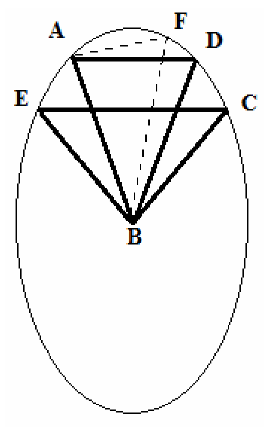


Quoniam FK, AE sunt parallelae, triangula AEC, KFC similia sunt. Ergo AE ad EC, sic KF ad FC. Ergo ut quadratum AE ad quadratum EC, sic quadratum KF ad quadratum FC. Sed etiam ut quadratum AC ad quadratum EC sic rectangula GFH ad quadratum KF. Ergo quadratum KF & rectangulum GFH ad quadratum EC, eandem habent rationem; aequantur igitur. Quod erat demonstrandum.

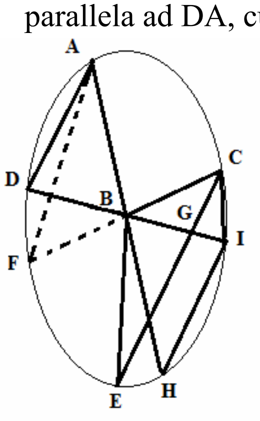
PROPOSITIO C.

Sint AB, BC diametri coniugatae inaequales, & ex A recta quaevis ducatur AD secans ellipsim in D; cui ex C parallela ducatur C, iunganturque EB, DB.
Dico EB, DB diametros esse coniugatas & contra.

Demonstratio.

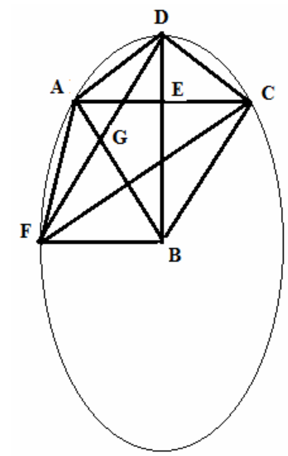


Primo cadant parallelae ad eandem partem ellipseos. Ducantur rectae lineae EA, DC. Quoiam AD, EC lineae sibi mutuo aequidistant, erunt segmenta EA, DE inter se aequalia adeoque ABE, DBC sectores aequales addito igitur communi ABD, erunt EBD, ABC sectores inter se aequales. Quare cum unius sectoris latera BA, BC sint diametri coniugatae, etiam alterius latera EB, BD sunt coniugatae.
Secundo cadant AD, CE parallelae ad partes ellipsis oppositas: producantur semidiametri AB, DB in H & I, iunganturque puncta HI. Quoniam AB, BC sunt conjugatae erit sector ABC quarta pars ellipseos, sed AH diametro bisecat ellipsim, adeoque portio ACH, dimidium est ellipseos. Ergo ABC sector dimidius est semi-ellipseos ACH. Ac proinde aequalis sectori CBH. Quia autem IH per 19.huius est parallela ad DA, cui ex hypothese etiam CE est parallela , erunt IH, CE inter se



parallelae. Ergo segmenta CI, EH adeoque & sectores CBI, HBE aequantur; addito igitur commumi IBH, ante ostendi, sectori ABC. Quare cum sector ABC sit quarta pars ellipseos, sive dimidium semiellipseos, etiam sector IBE erit dimidium semiellipseos, hoc est portionis IED, quam est semiellipsim patet ex Coroll. 45. huius . Ergo sector IBE hoc est sector ABC aequalis est sectori EBD. Quare cum AB, BC sint coniugatae, etiam DB, EB erunt coniugatae.
Sint iam AB, CB,item, EB, DB diametri coniugatae iunganturque AD, EC. Dico AD, EC lineas esse parallelas. Sin vero; ducatur ex A ipsi EC parallela AF iunganturque FB: erit igitur FB diameter coniugata ipse EB per secundam partem huius: sed DB per constructionem coniugata est diametro EB, ergo eidem EB plures diametri sunt coniugatae. Quod lieu non potest. Igitur AF non aequidist ipsi EC. Idem ostenditur de quavis alia. Ergo AD sola parallela est rectae BC. Quod erat demonstrandum.

PROPOSITIO CI.



Sit in ADC ellipsi cuius centrum B quaevis diametrorum coniugatio AB, BC: iunctisque punctis AC, secet AC lineam in E diametrum utcunque BD, cui coniugata ducatur BF, ductaque linea, FD secet AB diametrum utcunque in G.

Dico AC, FD lineas, uti & BD, AB in E & G proportionaliter esse divisas.

Demonstratio.

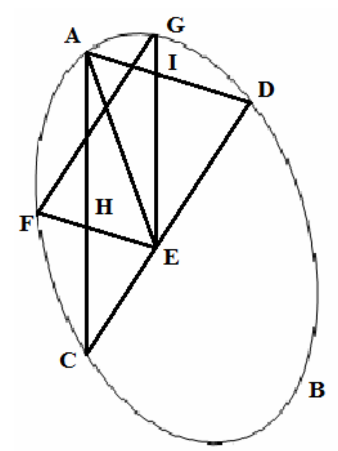
Quoniam tam AB, BC diametri quam DB, FB coniugatae sunt, sectores ABC, FBD aequales erunt : ablato igitur communi ABD, aequales manent DBC, ABF sectores. Unde BD, AB lineae, item AC, FD in E & G proponionaliter sunt divisae.

PROPOSITIO CII.
Iidem positis:
Dico iunctas AD, FC aequidistare.

Demonstratio.

Per praecedentem sectores DBC, ABF ostensi sunt aequales; segmenta igitur DE, AF quoque inter se aequantur: Ergo AD, FC lineae aequidistant. Quod erat demonstrandum.

PROPOSITIO CIII.



Secet ABC ellipsim quaevis diametrorum coniugatio AE, CD: sit autem & alia diametrorum coniugatio, FE, GE quae iunctas AD, AC secet in H & I:

Dico esse ut AH ad HC, sic DI ad IA.

Demonstratio.

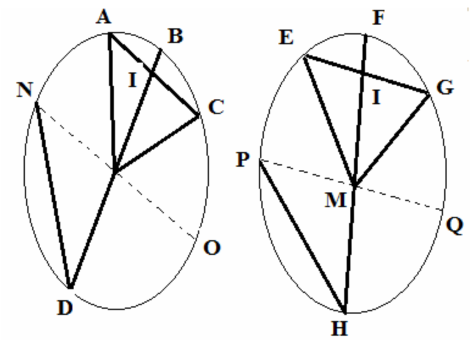
Ducatur FG quae AE secet in K ut AH ad HC, sic FK ad KG: sed ut FK ad KG, sic DI est ad AI, igitur ut AH ad HC sic DI ad AI. Quod erat demonstrandum.

PROPOSITIO CIV.

Secet ABC est ellipsim diameter quaecunque BD : sit autem & EFG, ellipsis similis & aequalis est ipsi ABC: quam secet quaevis alia diameter FH : dein BD, FH diametris proportonaliter divisis in I & K, agantur per I & K, ordinatim lineae AC, EG: aequarum extremitatibus ducantur semidiametri AL, CL, EM, GM.

Dico ALC, EMG triangula esse aequalia.

Demonstratio.

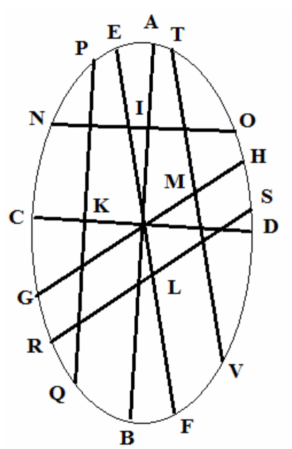


Diametro BD coniugata ducatur NO, & FH diametro PQ: iunganturque ND, PH. Quoniam diametri BD, FH tam in IK quam LM proportionaliter sunt divisae, erit ut rectangulum BID ad rectangulum BLD sic FKH, rectangulum ad rectangulum FMH: quare ut quadratum AI ad quadratum NL sic quadratum EK ad quadrum PM : & ut AI linea ad lineam NL sic EK ad PM: sed etiam est per constructionem ut IL ad BL, id est LD, sic KM ad FM, id est MH; igitur ut triangulum NLD ad triangulum AIL, sic PMH triangulum ad triangulum EKM (quia ex iisdem illorum ratio componitur:) & permutando ut NLD triangulum ad triangulum PMH, sic AIL triangulum ad truangulum EKM. Sed NLD, PMH triangula sunt aequalia, igitur AIL, EKM triangula, adeoque tota ACL, EGM aequantur, Quod erat demonstrandum.

PROPOSITIO CV.

Secent ABC ellipsim duae diametrorum coniugationes AB, CD, EF, GH, & omnes quatuor diametri proportionaliter sint divisae in I, K, M, L punctis, per quae ordinatim ducantur lineae NO, PQ, RS, TV.

Dico quadrata NO, PQ simul sumpta aequari quadratis RS,TV simul sumptis.



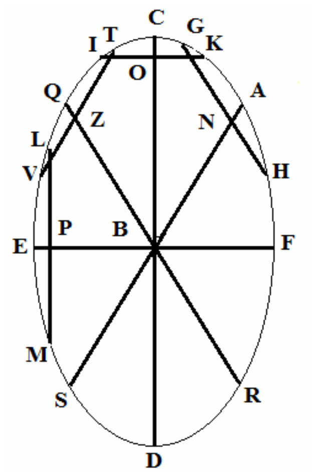
Demonstratio.

Quoniam tam AB, EF, quam CD, HG proportionaliter sunt divisae, erit ut AB quadratum ad rectangulum AIB, sic EF quadratum ad rectangulum ELF, & quadratum CD ad rectangulum CKD , & HG quadratum ad rectangulum HMG. Ex iisdem enim rationibus singulae quadratorum ad rectangula proportiones componuntur. Igitur ut AB, CD quadrata simul sumpta rectangula AIB, CKD simul sumpta, sit quadrata EF, GH simul sumpta, sunt ad rectangula ELF, HMG ; hoc est quadrata PK, NI ad quadrata SL, TM: sed AB, CD quadratis simul sumptis aequalia sunt quadrata EF, GH simul sumpta; igitur & quadratis NI, PK aequalia sunt quadrata SL, TM: ergo NO, PQ quadrata simul sumpta aequalia sunt quadratis SR, TV. Quod erat demonstrandum.

PROPOSITIO CVI.

Secet ABC ellipsim quaevis diametrorum coniugatio CD, EF, quaevis proportionaliter divisis in O & P : ducatur una ex diametris coniugatis aequalibus AS queae dividatur in N, ut CD est divisa in O. Per NOP rectae ducantur ordinatim GH, IK, LM.
Dico IK, LM quadrata simul sumpta esse dupla quadrati GH.

Demonstratio.

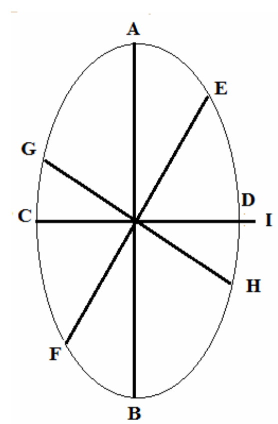


Ducatur altera coniugatarum aequalium QR, quam similiter divisi in Z, ut SA est in N, & CD, EF, in O & P, per punctum Z ponatur ordinatim VT quia igitur QR, AS sunt aequales & similiter sectae, rectangulum QZR aequatur rectangulo ANS. sed rectangula QZR, ANS aequantur quadratis GN, TZ. Ergo quadrata GN, TZ adeoque & quadrata GH, TV aequalia sunt. Sed quadrata ML, IK aequantur quadratis VT, GH. Ergo quadrata M L, IK dupla sunt quadratri. Quod erat demonstrandum.

PROPOSITIO CVII.

Secent ABC ellipsim duae diametrorum coniugationes AB, CD, EF, GH : sitque AB maxima & EF magnitudine secunda.
Dico rationem AB ad EF minorem esse ratione GH ad CD.

Demonstratio.



Quoniam AB, CD quadrata simul sumpta aequalia sunt quadratis EF, GH simul sumptis: non est ut AB ad EF, sic GH ad CD, nam tunc quadrata AB, CD maximae & minimae maiora essent quadratis EF, GH. Fiat igitur ut AB ad EF, sic GH ad CI: eruntque AB, CI quadrata maiora quadratis EF, G H; hoc est quadratis AB, CD. Quare CI linea est maior recta CD, & ratio GH ad CD; id est ex constr. ratio AB ad EF minor est ratione GH ad CD. Quod erat demonstrandum.

PROPOSITIO CVIII.

In data ellipse diametris exhibere coniugatas in data ratione.

Constructio & demonstratio.

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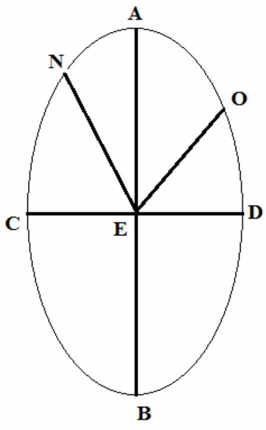
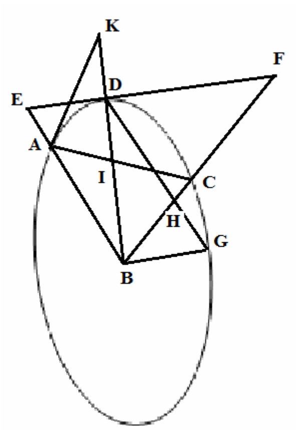


Sint ABC ellipsis axes AB, CD: centrum E : ratio vero data maioris ad minus F ad G, quam non maiorem esse oportet ratione axium AE, EC. Si ratio data aequalis est axium rationi ; res patet: si minor, sumantur HI, IK lineae, rectis AE; EC aequales, ponanturque ad angulum rectum HIK: ductaque HK, & divisa bifariam in L, centro L intervallo LH circulus describatur HIK, qui transibit per puncta I & K, in segmento dein HIK inclinentur duae lineae HIK habentes datam rationem F ad G. Quoniam igitur HM est ad MK ut F ad G, estque ratio F ad G, minor ratione axium, hoc est: ex const. ratione HI ad IK, patet punctum M cadere inter I & H, adeoque rectam HM minorem quam HI, hoc est quam EA, rectam vero MK maiorem quam IK hoc est quam EC. Centro igitur E intervallo HM, descriptus circulus secat ellipsim in puncto aliquo N inter A & C, ducatur igitur diameter NE, eique; coniugata EO. dico NE, OE diametros satisfacere petitioni. Quoniam quadrata NE, OE sunt aequalia quadratis & AE, EC hoc est HI, IK hoc est quadratis HM, MK: sit autem & NE linea aequalis rectae HM. Erit & OE linea aequalis lineae MK. igitur ut HM ad MK, id est ut F ad G, sic NE ad OE. Exhibuimus ergo, &c. Quod erat faciendum.



PROPOSITIO CIX.

Secet ACD ellipsum quaevis diametrorum coniugatio AB, BC assumptoque in peripheria puncto quovis D inter A & C, agatur per D contingens EF, quae cum AB, BC lineis concurrat in E& F; cui per centrum parallela ducatur BG.
Dico DF, BG, DE lineas esse in continua analogia.

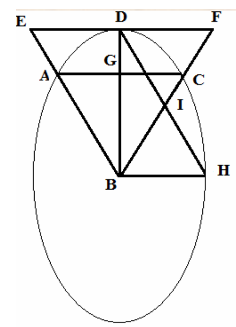
Demonstratio.

Quoniam ex hypothest DF, BG sunt parallelae, triangula DFH, BHG, ut ex elementis patet, erunt similia, ac proinde DF est ad BG, ut DH ad HG; hoc est ut triangulum DHB ad triangulum BHG. Deinde, quoniam BG per centrum est tangenti DF parallela, liquet eam coniugatam esse diametrum ipse DB: sunt vero ex hypothesi etiam AB, BF coniugatae. Ergo sectores BDCG, BADC, adeoque: & triangula BDG, BAC aequantur: Quorum bases DG, AC, cum proportionaliter sint divisae H & I, ut DH sit ad HG, sicut CI est ad IA , constat ex elementis triangula BHD, BIC & BHG, BIA esse aequalia. Ergo cum prius ostenderim DF esse ad BC ut triangulum BHD ad triangulum BHG, erit quoque DF ad BG, ut triangulum BHD ad triangulum BHG. Erit quoque DF ad BG, ut triangulum BIC ad triangulum BIA. Ulterius cum triangulum BHG sit ad triangulum BHD, ut GH ad HD, hoc est, ut AI ad IC, hoc est ut KI ad IB (cum enim CB sit coniugata ipsi AB , & AK, ex constructione tangens, patet AK, CB esse parallelas) hoc est ut triangulum AIK ad triangulum AIB : erit componendo triangulum BDG ad triangulum BHD, ut triangulum AKB ad triangulum AIB; et permutando triangulum BDG ad triangulum AKB, ut triangulum BHD, est sicut ante ostendi, ut triangulum BIC ad triangulum AIB. Sed triangulum AKB est triangulum EDB, hoc est triangulum BDG est ad triangulum EDB, hoc est,quoniam ED, BG sunt parallelae, BG est ad ED, ut triangulum BIC ad triangulum AIB, hoc est sicut ostendi supra, ut DF ad BG. Sunt igitur in ratio continua DF, BG, ED. Quod erat demonstrandum.

PROPOSITIO CX.

Sint AB, BC diametri quaecunque coniugatae, assumptoque; in peripheria, inter A & C puncto quouis D, agatur per D contingens, occurrens AB, BC diametris in E & F, iunctaque AC occurrat diametro DB in G.
Dico rectam DF ad DE, rationem habere duplicatam, eius quam habet CG ad GA.

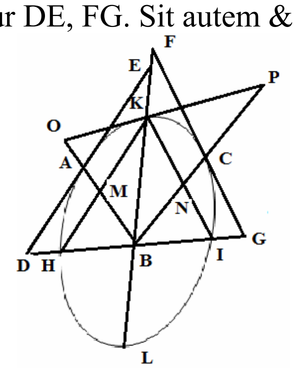
Demonstratio.



Ducatur ex B linea BH parallela ipsi EF, & ex DE recta DH occurrens FB in I. Erunt igitur per praecedentem, continuae FD, BH, ED. Adeoque ratio FD ad ED, duplicata rationis FD ad BH, id est DI ad IH, quia DF, BH per constructionnem aequidistant: rursum cum HB recta aequidistet tangenti DF, erunt DB, BH diametri coniugatae; sunt autem ex constructione etiam AB, BC coniugatae; igitur ut DI ad IH, sic CG ad GA: quare & ratio FD ad DE, duplicata est rationis CG ad GA. Quod erat demonstrandum.

PROPOSITIO CXI.

Sint AB, BC diametri coniugatae & per A & C tangentes ducantur DE, FG. Sit autem & alia quaevis diametrorum coniugatio HI, KL, quae producta occurrat tangenibus DE, FG in E, F, D, & G.
Dico lineas DE, FG in A & C proportionaliter esse divisas, nimirum esse EA ad AD, ut GC ad CF.



Demonstratio.

Ducantur lineae HK, KI quae rectas AB, BC secent in M & N. Ratio EA ad AD duplicata est rationis KM ad MH, & GC ad CF, duplicata est rationis IN ad NK. Atqui ratio KM ad MH aequalis est rationi IN ad NK. Ergo rationes EA ad AD, & GC ad CF aequalium rationum duplicatae, sunt aequales, proportionaliter ergo sectae sunt DE, GF in punctis C, A. Quod erat demonstrandum.

Corollarium.

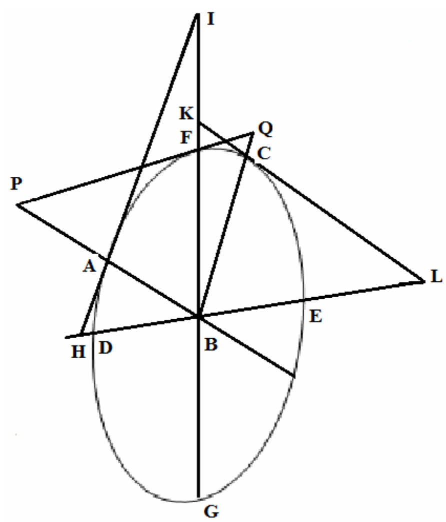
Quod si per K ducatur tertia tangens, conveniens cum BA, BC coniugatis in O & P, dico fore OK ad KP, ut EA ad AD. Quod ducta recta AC eodem modo quo usi sumus demonstrabitur.

Itaque tres tangentes DE, OP, FG similiter sunt divisae sic ut EA sit ad AD, sicut OK ad KP, & GC ad CF.

PROPOSITIO CXII.

Sint duae diametrorum coniugationes A B, BC, FG, DE, aganturque per A & C tangentes HI, KL, quae FG, DE diametris occurrant in H, I, K, L punctis.

Dico esse ut BC ad BA sic HI ad KL.



Demonstratio.

Quoniam recta BC aequidistat rectae HI & KL linea ipsi AB , erunt tam HA, BC, AI quam KC, AB, CL lineae in continua analogia. Cum ergo sic ut HA ad AI , prima ad tertiam, sic KC ad CL, prima ad tertiam: erit etiam HA ad BC, prima ad secundam ut KA est ad BC, prima ad secundam : quare permutando est HA ad KC, ut BC ad AB, cum igitur, ante ostenderim HA esse ad AI, ut KC ad CL, adeoque invertendo componendo, ac permutando sit ut HA ad KC, sic HI ad KL, erit ut BC ad BA, ita HI ad KL.

Corollarium.

Eodem modo ostenditur, si per F agatur tangens quae cum AB, BC convenient in P & Q esse ut BC ad BD, sic HI ad PQ.

PROPOSITIO CXIII.

Sint duae diametrorum coniugationes AC: BD, EF, EG: actisque per F & G tangentibus quae diametris AC, BD occurrant in H, I, L, M ducatur recta FG secans BE diametrum in K.

Dico LK, KE, KI lineas esse continuas.

Demonstratio.

Quoniam FE, utpote coniugata ipsi EG, aequidistat tangenti LG , erit LK ad KE ; ut GK ad KF. Similiter quoniam EG, utpote coniugata ipsi EF, parallela sit tangenti FI, est ut GK ad KF, sic KE ad KL; igitur ut LK ad KE, sic KE est ad KI. Quod erat demonstrandum.

PROPOSITIO CXIV.

Iisdem positis:

Dico IHE, LME triangula esse aequalia.

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Demonstratio.

Ducatur recta A B quae FE lineam secet in N : & ex E rectae ducantur EO, EP, normales ad lineas HI, LM. Quoniam LM linea aequidistat ipsi FE (est enim LM tangens, & FE coniugata ipsi EG,) erit angulo FEG aequalis angulus EGP. Eodem modo erit angulus OFE aequalis angulo FEG. quare anguli OFE, EGP sunt inter se aequales: sunt autem EPG, EOF anguli recti; igitur triangula EGP, EFO similia. Quare ut EG ad EF, sic EP ad EO. Sed est ut EG ad EF, sic HI ad LM. Igitur ut EP ad OE, sic reciprocem HI ad LM. Ergo IHE, LHM triangula sunt aequalia. Quod erat demonstrandum.

PROPOSITIO CXV.

Iisdem positis triangulum EGM, triangulo EFI, & EGL triangulum, triangulo EFH aequale est.

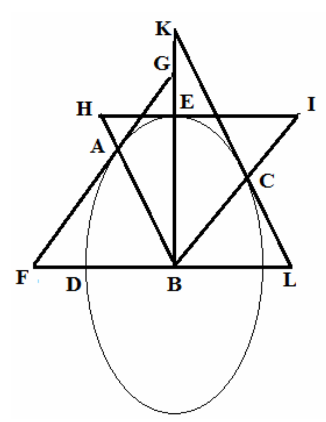
Demonstratio.

Est enim ut HF ad FI, sic LG ad GM, & componendo ut Hl ad FI, sic LM ad GM: sed est ut HI ad FI, sic HIE triangulum ad triangulum FIE, & ut LM ad GM, sic ELM triangulum ad triangulum EGM, igitur ut HIE triangulum ad triangulum FIE, sic ELM triangulum est ad triangulum EGM. Quare FIE, EGM triangula sunt aequalia. Eodem modo ostenduntur reliqua EGL, HFE aequalia.

PROPOSITIO CXVI.

Si ellipsim ADC duae secent diametrorum coniugationes AB, BC, EB, BD: aganturque per A, E, C. Contingentes FG, HI, KL quae diametris quidem EB, BD occurrat in G, K, F, L. Diametris vera AB, BC, in H & I.

Dico triangula FGB, HBI, KLB esse inter se aequalia.



Demonstratio.

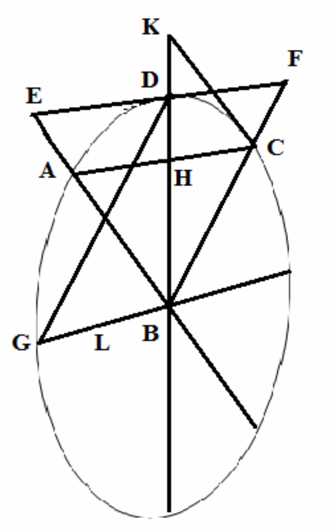
Nam triangulum BIE aequatur triangulo BKC, hoc est per 115, huius triangulo BFA : & triangulum BHE, aequatur triangulo GAB, ergo triangulum totum BIH aequatur toti BFG. Quare cum etiam FGB, KBL aequalia sint; liquet tria triangula esse aequalia.

PROPOSITIO CXVIl.

Sint ellipseos duae diametri coniugatae AB, BC, &. ex puncto aliquo inter A & C assumpto, scilicet D ducatur tangens DE coniugatis AB, BC productis occurrens in E & F: deinde ex centro B ducatur BG, ipsi EF aequidistans, sitque GB media inter ED, DF.

Dico punctum G esse in peripheria ellipseos, cuius diametri coniugatae AB, BC, & tangens ED.

Demonstratio.



Ex centro ad contactum ducatur BD, & quia GB hypothesi est parallela tangenti ED, erit GB ordinatim posita ad BD, & quidem ad centrum. Unde BD, GB sunt diametri coniugatae. Iungantur deinde AHC & D G, ex C ducatur CK parallela ipsi AB,
occurrens BD protractae in K : quae sectionem C contingit, eritque ut AH ad HC, ita BH ad HK: sed ut AH ad HC, ita est ABH triangulum ad triangulum HBC, & ut BH ad HK, ita est triangulum HBC ad triangulum HCK: ergo ut triangulum ABH ad ipsum HBC ita est triangulum HBC ad triangulum HCK: ergo componendo, ac permutando triangulum gulum ABC, ad triangulum BCK est ut triangulum BHC ad triangulum HCK, id est ut iam ostensum, ut triangulum ABH ad triangulum HBC, id est ut linea AH ad lineam HC: rectae: sed aequalia sunt triangula BCK, BDF,ergo etiam erit triangulum ABC ad triangulum BDF, ut AH ad HC. Ulterius quoniam DF, GB sunt parallelae, erit triangulum GDB, ad DBF triangulum, ut G B ad DF: sed, quoniam ex hypothesi ED, GB, DF sunt continuae, ratio GB ad DF, est dimidiata rationis ED ad DF. Ergo ratio trianguli GDB ad triangulum DBF dimidiata est rationis ED ad DF. Atque ratio AH ad HC, hoc est ut ostensum supra, ratio trianguli ABC ad triangulum DBF, dimidiata quoque est rationis ED ad DP. Ergo triangulum GDB est ad triangulum DBF, ut triangulum ABC ad idem triangulum DBF. Aequantur igitur triangula GDB, ABC. Ergo & parallelogrammum contentum semidiametris, ut supra ostendi, coniugatis, GB, BD in angulo GBD aequatur parallelogrammo contento sub semidiametris coniugatis AB, BC. Ergo punctum G est ad ellipsim. Quod erat demonstraandum.

PROPOSITIO CXVIII.

Sint rursum binae diametri coniugatae BA, BC. Et sumpto in perimetro ellipsis puncto D inter A, ac C, tangat ellipsim EF in D, occurrens diametris in E & F. Deinde ex centro B ducatur ad perimetrum BG parallela tangenti.

Dico ED, GB, DF esse in continua analogia.

Demonstratio.

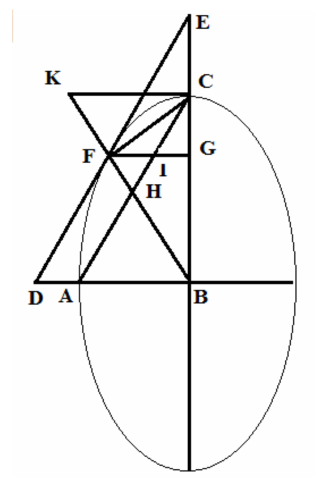
Si non sit aliqua LB minor vel maior quam CB media inter ED, DF. Ergo per praecedentem punctum L est ad ellipsim, quod fieri non potest, cum ex hypothesi punctum G ad ellipsim existat. Nulla igitur praeter GB, media est inter ED, DF. Ergo ED, GB, DF sunt continuae proportionales. Quod erat demonstrandum.

PROPOSITIO CXIX.

Sint in ellipsi diametri coniugatae AB, BC iungaturque; AC cui parallela fiat linea DE tangens est ipsum F, & occurrens diametris coniugatis protractis in D & E.
Dico triangulum EDB trianguli CAB duplum esse.

Demonstratio.

Ex centro B per tactum F ducatur diameter BHF, cui occurrat in K recta CK ellipsim tangens in C, deinde ex tactu F ponatur ordinatim FIG. Quoniam AC parallela est ex hypothesi tangenti DE, erit ad diametrum BF ordinatim posita. Ergo KB, FB, HB sunt continuae proportionales. Ergo est ut KB ad FB, sic KF ad FH. Sed KB est ad FB, & triangulum ECB ad triangulum FCB, hoc est, quia triangula KCB, EFB aequantur, ut triangulum EFB ad idem triangulum CFB, hoc est EB ad CB. Igitur ut EB ad CB, sic KF ad FH. Deinde quia ex constr. KC tangit, & FG est posita ordinatim ad BC, rectae EB, CB, GB sunt continuae. Ergo ut EB ad CB, sic EC ad CG. Sed etiam est ut iam ostendi, sic ut EB ad CB, ita KFad FH. Ergo ut KF ad FH, sic EC ad CG. Ergo ut triangulum KCF ad triangulum FCH: sic triangulum EFC ad triangulum CFG.



Atqui cum tota KCB, EFB aequalia sint, ablato communi FBC reliqua KCF, EFC aequalia sunt. Ergo & FCH, CFG aequalia sunt; ablato igitur communi FIC, aequalia remanent FIH, CIG, quibus si commune addis BHIG, FGB aequabitur CHB. Iam vero quia AC ordinatim posita est, ut supra ostendi, ad BF, bisecta est AC in H, adeoque & triangulum CAB duplum est: trianguli CHB, & DE parallela ad AC etiam bisecatur in F: est vero FG, utpote ducta ordinarim ad CB, parallela ad CB, diametrum coniugata ipsi C B. Ergo & BE bisecatur in G. Proindeque EFB duplum est GFB. Atqui CHB, FGB ostensa sunt aequalia. Ergo & eorum dupla CAB, EFB aequalia sunt. Sed triangulum DEB duplum est trianguli EFB, est enim, DE bisecta in F. Ergo triangulum DEB duplum quoque est trianguli CAB. Quod erat demonstradum.

ELLIPSEOS PARS QUARTA

Sectionis polos :& lineam a puncto in axe dato ad peripheriam, brevissimam designat .

Partem hanc, quae de poles est, aggressuri, paucis praemittemus est quae ad inventionem polorum ab Apollonio libro tertia propositione 41& 45 demonstrata sunt ; & quidem hoc necessarium esse duxi; tum quod ad illorum intestigentiam quae Apollonius in rem hanc contulit, nec omnium captui ita patent, plurimum conducant; tum quod ad rem nostram plane iudicem necessaria. Apollonius igitur ut in axe ellipseos polos exhibeat,haec utitur constructione propositione 45 & 6. Quartae, inquit, parti figurae aequale rectangulum comparetur ex utraque parte; id est, sectonis axis AC ita secetur in duobus punctis G & H, ut tam AGC quam CHA rectangulum aequale sit quartae partifigurae: quo posito ulterius ostendit G & H polos esse sectionis: quos punctae vocat ex comparatione facta; videlicet ex comparatione rectangulorum sub segmentis axeos, cum quarta parte figurae. Figuram porro hic vocat Apollonius rectangulum quod sit sub latere recto axeos maioris & ipso axe: atque illud cum quarta sui parte ad usus seruiret eximios; videlicet inventionem polorum &c. singulari prae reliquis rectangulis appellatione figuram appellavit antiquitas: huius autem quartae parti aequale est quadratum semiaxeos minoris: quod Pergaeus lib.tertio, propos.42. praeclare demonstravit, & nos verbo uno sic ostendimus.

Lemma.

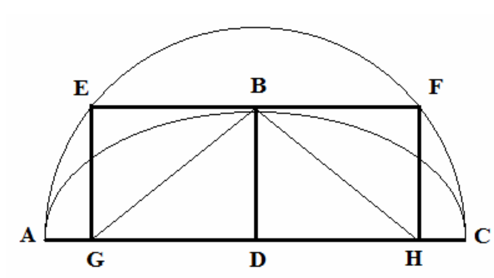
Per undecimam huius, figura sive rectangulo sub axe maiore & latere illius recto aequale est quadratum axeos minoris; sed quadrati minoris axis quarta pars est quadratum dimidii axis minoris; igitur quadratum dimidii axeos minoris aequale est quartae parti figurae. Unde cum voce illa in hac parte utar, quarta pars figurae, intelligi volo quadratum semiaxeos minoris.

Occurrent in hac parte propositiones aliquot eadem cum illis quas Apollonius de focis, demonstravit: quod eo consilio feci, ne quid in hac materia studiosus lector desideraret.

PROPOSITIO CXX.

Sint ABC ellipsis axes AC, BD, actaque per B tangente EF: centro BD intervallo DA circulus describatur AEFC, qui tangenti occurrat in E & F. dein ex E & F, normales demittantur EG, FH ad axem AC.

Dico tam AGC quam AHC rectangulum aequale esse quartae parti figurae



Demonstratio.

Quoniam tam EB linea aequidistat rectae AD quam EG ipsi BD, erunt EG, BD lineae aequales : est autem AGC rectangulum aequale quadrato EG, quod recta EG ducta sit ad diametrum circuli normalis; igitur & quadrato BD aequale est rectangulum AGC. eodem modo est FH quadrato, hoc est quadrato BD aequale rectangulum AHC, sed BD quadratum est aequale quartae parti figurae, igitur tam AGC quam AHC rectangulum est aequale quartae parti figurae. Quod erat demonstrandum.

Corollarium.

Hinc patet AG, HC lineas aequales & GH bifariam esse divisam in D.

PROPOSITIO CXXI.

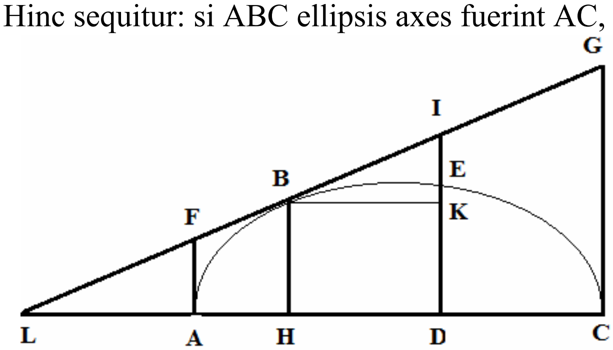
Iisdem positis ducantur rectae BG, BH.
Dico BG, BH lineas simul sumptas axi AC esse aequales.

Demonstratio.

Quoniam aequales sunt (§120.Cor.) AG, HC,quadratum AG aequatur rectangulo ex AG & HC. lterum quia aequales sunt GD, DH, aequabitur rectangulum AGD bis sumptum rectangulo AGH. Quare cum quadratum AD aequale sit quadratis DG, AG & rectangulo AGD bis, idem quadratum AD aequabitur quadrato DG & rectangulo ex AG , HC una cum rectangulo AGH. Atqui rectangula AG, HC, & AGH, aequantur rectangulo AGC. Ergo quadratum AD aequale est quadrato DG una cum rectangulo AGC; hoc est quadratis DG, GE, hoc est quadratis DG, DB. sed iisdem aequatur quadratum GB, aequantur igitur quadrata AD, GB, ac proinde rectae AD, G B aequales sunt. Eodem modo demonstrabitur rectas CD, HB aequales esse. Ambae igitur GB, BH simul sumptae axi sunt aequales. Quod erat demonstrandum.

Corollarium.

Hinc sequitur: si ABC ellipsis axes fuerint AC, BD, & ex B vertice minoris axis rectae dimittantur BG, BH, aequales lineis AD, DC, secantes axem AC in G & H. Quod tam AGC quam AHC rectangulum, aequale sit quartae parti figurae, adeoque G & H sectionis poli sunt.



PROPOSITIO CXXII.

Ellipsim ABC, cuius diametri coniugatae AC, DE contingant in A & C, & alio puncto quovis B tres lineae AF, CG, FG: & FG quidem occurrat AF, CG rectis in F & G. productaque ED occurrat FG lineae in I, & ex B, recta demittatur BH ordinatim ad AC diametrum:

Dico rectangulum super AF, CG lineis aequari rectangulo super BHID.

Demonstratio.

Producatur FG linea donec cum axe conveniat in L. Quoniam DH, DA, DL lineae sunt in continua ratione, erit ut LD ad AD, hoc est ad DC sic LA ad AH, & invertendo componendo ut CL ad DL, sic HL ad AL, sed est ut CL ad DL, sic CG ad DI, & ut HL ad AL, sic HB ad AF, igitur ut CG ad DI, sic HB ad FA: adeoque rectangulum super AF, CG lineis aequale rectangulo BH, ID. Quod erat demonstrandum.

Corollarium.

Hinc sequitur rectangulum AF, CG vel HB, ID, aequale esse quartae parti figurae: ducatur enim BK parallela axi AC: erit rectangulum DK, DI aequale quadrato ED; igitur & AF, CG rectangulum est aequale quadrato ED hoc est quartae parti figurae.

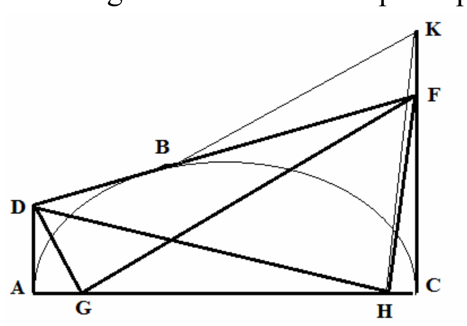
PROPOSITIO CXXIII.

Ellipsim ABC cuius axis AC, contingant in A & C, & alio quovis puncta B, lineae AD, CF, DF: & D F quidem conveniat cum AD, GF lineis in D & F dividatur autem linea AC in G & H , ut AGC, AHC rectangula sint aequalia quartae parti figurae ducanturque; lineae DG, GF,DH, HF.

Dico angulos DGF, DHF esse rectos; & si sint recti: dico DF, lineam tangere ellipsim.

Demonstratio.

Rectangulum DACF est aequale quartae parti figurae, hoc est rectangulo AGC. Ergo ut



AG ad AD sic FC ad CG , sunt autem anguli DAG, FCG recti; igitur DAG, FCG triangula similia : & angulus ADG aequalis angulo CGF, est autem angulus ADG, una cum angulo AGD aequalis uni recto, cum DAG angulus in triangulo ADG sit rectus: igitur & angulus CGF, una cum angulo AGD uni recto sunt aequales: ergo reliquus DGF est rectus: eodemmodo ostenditur angulus DHF rectus. Quod erat demonstrandum.

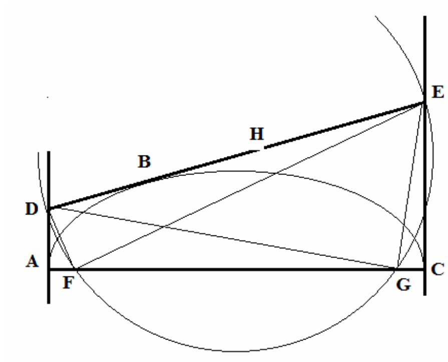
PROPOSITIO CXXIV.

Ellipsim ABC cuius axis AC contingant in A, C, B, punctis AD, CE, DE: & DE quidem occurrat rectis AD:CE in D & E. Fiant autem quartae parti figurae, aequalia rectangula AFC, AGC, seceturque ED bifariam in H:

Dico circulum centro H intervallo D& E, descriptum transire per F & G.

Demonstratio.

Iungantur puncta DF, FE, DG, GE. Quoniam tam angulus DFE, quam GE est rectus, & DE linea utcumque subtendens divisa bifariam in H, patet circulumm centro H intervallo HD descriptum transire per F& G. Quod erat demonstrandum.



Corollarium.

Hinc sequitur angulos EDG, FDA esse inter se aequales , est enim angulus ADF in demonstrationc praecedentis aequalis ostensus angulo GFE: sed angulo GFE aequatur angulus EDG cum eidem arcui EG insistat, ergo anguli EDG, FDA sunt inter se aequales.

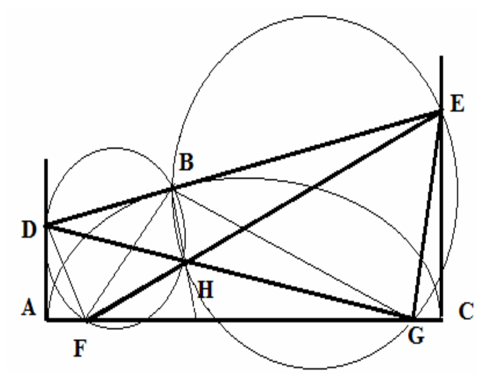
PROPOSITIO CXXV.

Ellipsim ABC cuius axis AC contingant in A, C, B, lineae AD, CE, DE: & DE quidem conveniat cum AD, CE lineis in D & E. Fiant autem AFG, AGC rectangula aequalia quartae parti figurae: ductisque lineis FE, GD quae se intersecent in H, ex puncto H ad contactum B, ducatur recta HB.

Dico HB normalem esse ad tangentem DE.

Demonstratio.

Ducatur recta ED, CE. Anguli DFE, ECD recti erunt. Iam super H D, HE lineis ut diametris circuli describantur DBH, EBH, Quoniam DH, HE lineae non sunt in directum, patet DBH, EBH circulos se invice secare in puncto aliquo B. Iunctis igitur punctis HB; ducantur rectae DB, EB: erunt anguli DBH, EBH recti, adeoque DB,EB linea in directum, & HB lineae normalis rectae DE. sunt autem ut ante ostendi DFE, EGD anguli recti ; igitur DE linea est tangens. Quare recta BH est normalis ad ED tangentem. Quod erat demonstrandum.

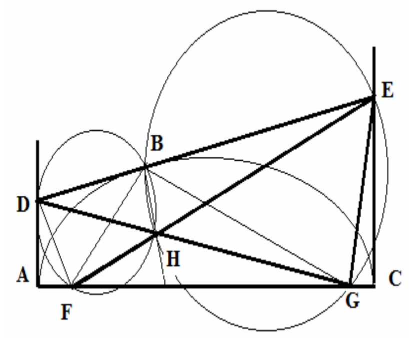


PROPOSITIO CXXVI.

Eadem manente figura: ducantur FB, BG.

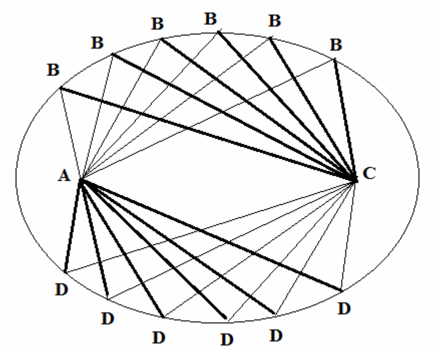
Dico angulos DBF, EBG ad contingentem esse aequales.

Demonstratio.



Quoniam anguli DFH, EGA recti sunt, transibunt per F & G. circuli DFH, EGH transit autem uterque etiam per B: quia anguli EBH, DBH sunt recti: igitur tam anguli DBF, DHF quam EBG, EHG anguli sunt inter se aequales: sed angulus DHF aequalis est angulo EHG: ergo & DBF aequatur angulo EBG. Quod erat demonstrandum.

Scholion

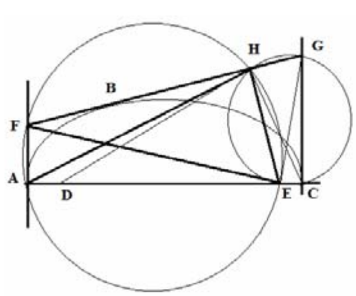


Cum punctum B in peripheria assumptum, sit quodcunque sequitur lineas omnes ex F in peripheriam ellipsis ductas, reflectendas in G. Quare & puncta FG poli seu foci a nonnullis vocantur: quae ab Apollonio puncta ex comparatione facta dicuntur, porro haec in ellipticis mirabiles habent proprietates : inter reliquas placuit sequentiem hic adiungere. Sint A, C , foci ellipsios, quorum distantia par sit intervallo oculorum, ponaturque in A oculus sinister, & dexter in C. Dico illum per totum speculum videri ab oculo sinistro in A ; demonstratio patet: species enim obiecti A, per totum speculum diffusa, reflectuntur in C, & species obiecti C per totum diffusae reflectuntur in A. Quare obiectum A per totum apparebit speculum, oculo C, uti & obiectum C, oculo A. Hunc sequitur quod minimum & visibile positum in C, maximum apparebit oculo in A posito: quia apparebit per totam speculis superficiem diffusum.

PROPOSITIO CXXVII.

Ellipsim ABC, cuius axis AC & poli DE contingant in punctis A, C, B rectae AF, CG, FG; & FG quidem conveniat cum AF, CG lineis in F& G. erigatur ex E, linea EH normalis ad tangentem FG, iunganturque puncta AH, CH.
Dico angulum AHC rectum esse.

Demonstratio.



Ductis lineis FE, EG describantur super FE , EG diametris circuli FHE, HGC: ac
circulus quidem FHE,cum anguli EHF, EAF sint recti, transibit per H, F, A, puncta ; circulus vero HGC: cum

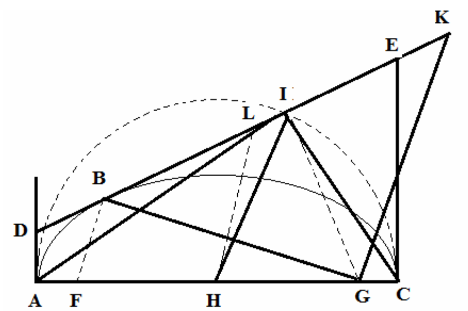
DHE, ECG anguli quoque recti sint transibit per H, C. Erunt igitur tam anguli AHF, AEF quam EHC, EGC, anguli aequales : sed angulus GCE per demonstrata in 121. huius aequalis est angulo FEA, igitur & angelo FHE recto aequalis angulus AHC, quare & ipse rectus. Quod erat demonstrandum.

PROPOSITIO CXXVIII.

Ellipsim ABC, cuius axis AC contingant in A, C, B, lineae AD, CE, DE, ac DE quidem occurrat AD, CE lineis in D & E : sint autem poli F, G, centrum H ductaque ex F recta FB ad punctum contactus ducatur ex H linea HI parallela rectae FB occurrens ED lineae in I.

Dico HI lineam aequalem lineae HC, & si HI occurrens ED rectae, sit aequalis HC. Dico HI lineam aequidistare FB.

Demonstratio.



Fiat BI aequalis IK: iunganturque BG, GK, & rectae ducantur AI,IC: Quoniam IB, IK sunt aequales, erit BI ad IK, ut FH ad HG: adeoque BF, KG lineae parallelae, & angulus BKG aequalis angulo DBF, hoc est IBG: quare BG,GK lineae aequales: sunt autem & duo reliqua latera BI,IG aequalia duobus lateribus KI, IG. Angulus ergo BIG, aequalis angulo KIG: adeoque GI linea normalis tangenti DE, & angulus AIC rectus. Quare circulus centro H intervallo HC descriptus transibit per I, eritque HI linea aequalis lineae HC. Quod erat primum.

Reliquis manentibus, sit iam HI linea quae occurrat tangenti ED in I, aequalis lineae HC. Dico HI rectam aequidistare lineae BF: sin vero; ducatur ex H linea HL, parallela rectae FB occurrens ED tangenti in L; erit igitur HL linea aequalis lineae HC, hoc est HI. Quare circulus centro H intervallo HC descriptus transilit per I & L puncta. Quod impossibile igitur HL non est parallela ipsi FB: nec quaevis aliae praeter HI lineam. Quod erat demonstrandum.

PROPOSITIO CXXIX.

Sit ABC ellipseos axis AC, poli autem D, E ex D & E rectae inflectantur DB, EB convenientes in puncto quodam peripheriae B.

Dico DB, EB lineas simul sumptas aequari axi AC.

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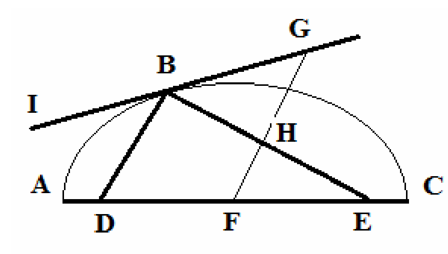
Demonstratio.

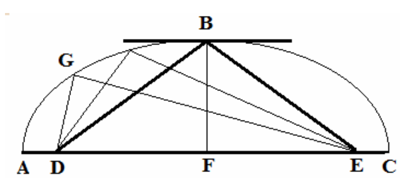
Sit F centrum ellipseos; actaque per B tangente BG: ducatur recta FG parallela lineae DB secans EB lineam in H. Quoniam BD, FG lineae parallelae, erit angulus FGB aequalis angulo DBI hoc est EBG, adeoque HB, HG lineae aequales: rursum cum sit ut DE ad FE, sic BE ad HE, sitque DE dupla FE, erit & EB, dupla rectae BH id est HG: sed etiam BD dupla est FH, cum sic ut DE ad FH, sic DB ad FH; igitur EB, BD lineae simul sumptae duplae sunt rectae FG hoc est FC: quare & aequales axi AC. Quod erat demonstrandum.



PROPOSITIO CXXX.

Triangulorum isoperimetrorum maximum est isoscelium.

Demonstratio.



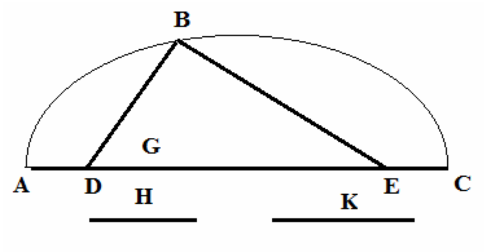
Describatur ellipsis quaecunque ABC cuius axes AC, FB poli D, E, iunganturque puncta DB, BE tum super ED bali triangula constituantur quaecunque DGE, quorum vertices G sint in peripheria. Quoniam tam DB, BE lineae quam DG, GE simul sumptae sunt aequales axi AC: patet DBE, DGE triangula esse isoperimetra: dico autem illorum esse maximum triangulum DBE agatur enim per B tangens: quae cum in uno tantum puncto B ellipsi occurrat & reliqua sui parte tota cadat extra, patet DGE triangula quae terminantur in ellipsi minorem habere altitudinem triangulo DBE, adeoque illo esse minora : est autem DBE triangulum isosceles, quia DF, FB latera aequalia sunt lateribus EF, FB & anguli illis contenti recti; igitur triangulorum isoperimetrorum maximum est isosceles. Quod erat demonstrandum.

PROPOSITIO CXXXI.

Oporteat e focis ellipseos DE duas inclinare ad idem punctum perimetri quae datam contineant rationem H ad K.

Debet autem data ratio maior esse ratione AD ad DC, minor vero ratione AE ad EC.

Constructio & demonstratio.

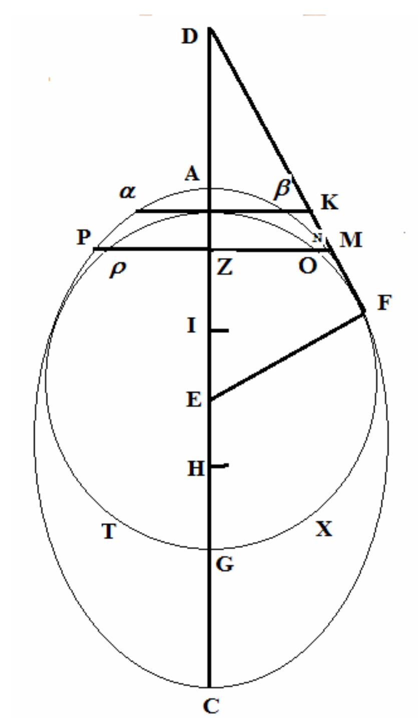


Secetur axis AC in G, secundum datam rationem H ad K, quae cum ponatur maior ratione AD ad DC, & minor ratione AE ad EC, manifestum est AG lineam maiorem esse recta AD: minorem vero AE, ac proinde punctum G cadere inter polos D, E erigatur igitur ex D ad peripheriam linea DB aequalis rectae AG. Iunganturque puncta BE. Dico factum esse quod petitur. Cum enim rectae duae DB, BE simul sumptae sint aequales axi AC, sit autem per constructionem DB linea aequalis lineae AG, erit BE reliqua aequa aequalis reliquae GC: igitur DB est ad BE, ut AG ad GC, id est ut H ad K. Inclavimus igitur, &c. Quod erat faciendum.

PROPOSITIO CXXXII.

Ellipsim ABC contingat in B linea BD conveniens cum axe maiore CA, in D; ex B autem contactu, normalis ad contingentem ponatur BE, occurrens axi in E.

Dico EB lineam brevissimam esse illarum quae ex E puncto ad peripheriam ellipseos duci possunt.



Demonstratio.

Centro E intervallo EB circulus describatur FBG occurrens axi in F & G centrum ellipseos sit H. Quoniam DB ellipsim contingens cui axe maiori convenit in D, & angulus DBE rectus est BE linea non transit Per H centrum ellipseos : si enim E centrum est, recta EB, normaliter ad contingentem posita axis erit coniugatus axi AC, (cum aequidistances omnes contingenti DB bifariam & ad rectos divideret) adeoque DB aequidistaret axi AC: non igitur E centrum est ellipseos, nec EB diameter : quia vero DB cum axe convenit ad partes A, EB linea minor est semidiametro, sibi parallela : adeoque & minor est semiaxe HC,& multo minor recta EC, quare circulus radio EB descriptus, occurrit axi in G intra ellipsim, punctum igitur G, supra C est.

Rursum cum HC id est AH, maior sit ostensa quam EB id est EG, ablato communi EH, manet AE maior quam HG, posita autem EI aequali EH, recta FI aequatur HG, igitur AE quoque maior est FI: ablato ergo ex FE & AE, communi IE, manet IA maior quam IF: unde & F punctum cadit intra ellipsim infra A. Ulterius ponatur per F, contingens FK, cui aequidistet PQNM, erit igitur ut MB quadratum ad quadratum BK sic QMO rectangulum

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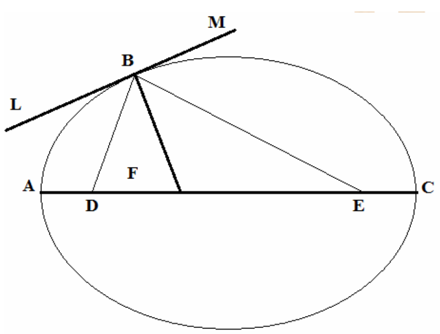
ad quadratum FK; sed ut MB quadratum ad quadratum BK sic PMN rectangulum ad rectangulum αKβ ; igitur ut QMO rectangulum ad quadratum FK sic PMN rectangulum est ad rectangulum αKβ: & permutando, invertendo ut FK quadratum ad rectangulum αKβ, sic QMO, rectangulum est ad rectangulum PMN: est autem FK quadratum maius rectangulo αKβ, igitur & QMO rectangulum maius est rectangulo PMN : iterum QMO rectangulum una cum quadrato ZO aequale est quadrato ZM, & PMN rectangulam una cum quadrato ZN, eidem quadrato Z M aequale est; aequale igitur est rectangulum QMO & una cum quadrato ZO, rectangulo PMN, una cum quadrato ZN; a quibus si inaequalia auferantur rectangula QMO, PMN , inaequalia remanent quadrata ZO, ZN: & quia QMO rectangulum maius est rectangulo PMN, quadratum ZO minus est quadrato ZN: & ZQ minus quadrato PZ; puncta igitur O & Q intra ellipsim sunt: similiter ostendentur puncta XT, & quaevis alia perimetri circuli FHG esse intra ellipsin ; circulus igitur FBG totus intra ellipsim cadit: unde cum rectae omnes ex E centro circuli ad ellipsis peripheriam duci; prius circulo occurrant quam ellipsi: adeoque semidiametris eiusdem maiores sint igitur EB, quae in B puncto communi ellipsi & circulo terminatur omnium illarum brevisssima est quae ex E puncto ad ellipsis peripheriam duci possunt. Quod erat demonstrandum.

PROPOSITIO CXXXIII.

A puncto (H) in axe ellipseos assignato lineam ad perimetrum brevissimam ducere.



Constructio & demonstratio.

Sint D, & E foci ellipseos. Axem AC seca in F, ita ut AF sit ad FC, sicut DH est ad HE. Tum ex polo ad perimetrum aptetur DB aequalis ipsi AF, iunganturque HB. Dico HB esse brevissimam.

Ducatur enim ex polo E ad B recta EB, & LM tangens ellipsim in B, DB, BE aequantur axi. sed DB aequalis est AF. Ergo BE aequalis est FC. Ergo DB est ad BE, ut AF ad FC, hoc est ex const. ut DH ad HE. Ergo anguli DBH, EBH aequantur, aequantur autem & anguli ad contingentem DBL, EBM, toti igitur anguli HBL, HBM aequales sunt;normalis igitur est HB ad tangentem, ergo per praecedemem brevissima omnium quae ex puncto H ad perimetrum duci possunt. Factum igitur est quod petebatur.

Si punctum F incidat in polum D, aut inter A, & D; tunc brevissima ex dato puncto H ad perimetrum erit pars axis , ut patet constructionem ac demonstrationem priorem consideranti.

PROPOSITIO CXXXIV.

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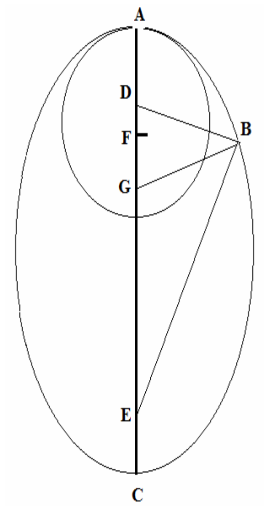
In data ellipsi circulum describere maximum eorum qui ellipsim in termino axis contingunt & ab ellipsi comprehenduntur.

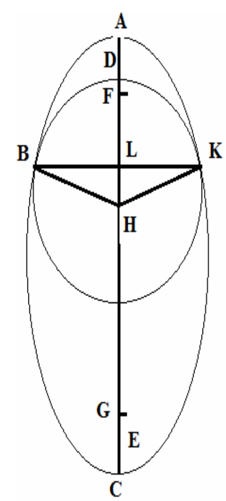


Constructio & demonstratio.

Poli ellipseos sint D & E. Fiat ut CD ad DA; sic EF ad FD. Dico circulum centro F intervallo A descriptum eum esse qui petitur. Cum enim ex const. sit CD, ad DA, ut EF ad FD. Patet ex praeccd. FA esse brevissimam omnium, quae a puncto F ad perimetrum duci possunt; circulus igitur centro F per A descriptus tangit ellipsim, quod erat primum: quod autem tangentium intra ellipsim maximus sit,sic ostendo. Sume ulterius punctum aliquod G pro centro maioris circuli, quoniam igitur EG est ad GD, in minori ratione quam EF ad FD, hoc est quam CD ad DA, sit exemp.grat. ut EG ad GD, sic CF ad FA: eritque FA necessario maior quam DA: adeoque punctum F cadet ultra polum D versus E: si igitur ex polo D ad perimetrum aptetur DB aequalis GA, iungaturque GB, patet ex praeced. GB fore minimam omnium quae ex G ad perimetrum ducuntur. Quare GA maior est quam GB, circulus ergo centro G per A descriptus extra ellipsim cadit. Similiter ostendemus quemlibet circulum alium maiorem circulo qui intervallo FA ante descriptus est, cadere extra ellipsim: ergo ille omnium intra ellipsim tangentium, maximus est. In data igitur ellipsi, &c. Quod erat faciendum.

Corollarium.

Ex huius propositionis discursu clare constat circulos omni intervallo descriptos quod minus est intervallo FA ellipsim intra contingere in puncto A. Si centro inter F & A constituto pertingant usque ad A, verticem axeos. Illi etenim circuli contingent eum qui radio FA descriptus est, eoque minores erunt; quare etiam ellipsim intra contingent cuius axis AC.



PROPOSITIO CXXXV.

Sit ABC ellipseos axis maior AC & in illo poli D, E, fiatque ut CD ad DA sic EF ad FD, & DG ad GE.

Dico ex quovis puncto rectae FG circulos posse describi qui ellipsim intus in duobus punctis contingant: centra vero illorum consistere inter F & G exclusis terminis.

Demonstratio.

Sumatur enim quodvis in FS recta punctum H, & ex linea ducatur HB, brevissima illarum a quae ex H ad peripheriam duci poterunt ; dein ex B ordinatim ad axem ponatur B L K, & iunge HK, HB, patet per elementa HK iuncta aequari HB, adeoque circulum centro H intervallo HB descriptum transire per K & B: & cum HK, HB lineae sint brevissimae per constructionem, patet circulum BDK , totum cadere intra ellipsim ac proinde eam in B & K, punctis contingere. Quod autem centra circulorum ellipsim in duobus punctis contingentium consistant inter F & G exclusis terminis, ex eo patet quod FA, GC lineae brevissimae sint illarum quae ex F & G, ad peripheriam duci poterunt, adeoque circuli centro F vel G, & intervallo quovis maiore quam sit FA vel GC descripti ellipsim secent: radiis vero FA vel GC descripti maximi sint, illorum qui ellipsim intus in uno tantum puncto contingunt.

PROPOSITIO CXXXVI.

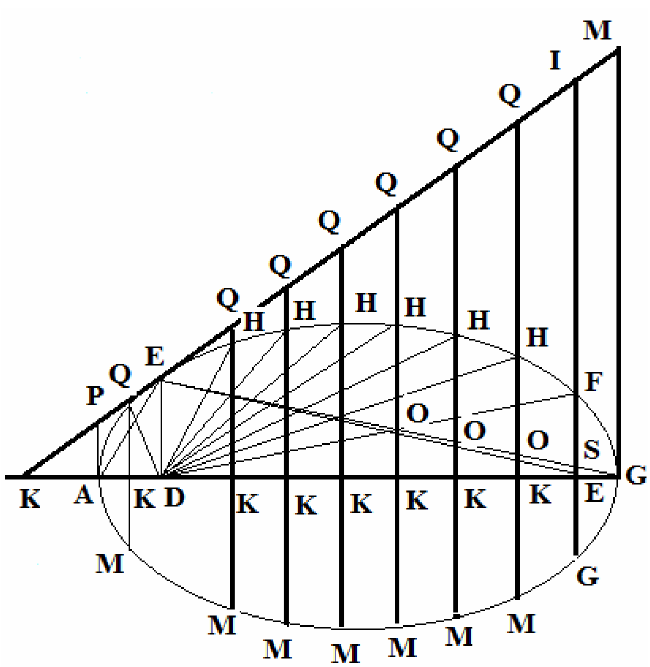
Eadem manente figura: propositum sit in axe punctum designare quo centro circulus describatur, qui ellipsim in dato puncto intus contingat.

Constructio & demonstratio.

Sit datum in peripheria punctum B, per quod si acta intelligatur contingens, demittatur ex B linea B H, normalis ad tangentem, occurrens axi in H. Manifestum est H punctum satisfacere petitioni; nam cum HB linea brevissima sit earum quae ex H ad peripheriam duci possunt, continget circulus centro H intervallo HB descriptus ellipsimin puncto B; igitur, &c. Quod erat faciendum.

PROPOSITIO CXXXVII.

Ellipseos axis sit A, C, poli D, E, ex quibus ductae sint DB, EFG normales axi; ellipsim autem tangat KBI in B, occurrens axi in K, rectae vero GF in I, iungaturque DF.
Dico rectangulum FIG quadrato DE aequale esse.



Demonstratio.

Quoniam DB , EB ductae sunt e polis ad contactum, anguli KBD, IBE aequales sunt , sed, quia DB, EI parallelae , angulus KBD angulo EIB aqualis est, aequantur igitur anguli IBE, EIB, ergo BE, IE aequales sunt. Deinde quia rectangula CEA, CDA aequalia sunt, etiam quadrata EF. DB sunt aequalia, adeoque & rectae EF,DB aequales. Quare cum

DB, BE aequentur EF,FE etiam BE, DF aequales erunt. Atqui BE ostensa est aequalis EI. Ergo & DF est aequalis EI, & quadrata proinde DF, EI aequalia sunt. Sed quadrata DE, EF aequantur quadrato DF; &, quia GF bisecta est in E, eique adiecta FI, rectangulum GIF cum quadrato EF aequatur quadrato EI. Quadrata ergo DE, EF aequantur rectangulo GIF cum quadrato EF. Dempto igitur communi quadrato EF, remanent aequalia rectangulum GIF & quadratum DE. Quod erat demonstrandum.

PROPOSITIO CXXXVIII.

Iisdem positis ducantur quotcunque aliae Q, H, K, M, normales axi.
Dico rectangula H, Q, M, quadratis DK esse aequalia.

Demonstratio.

Ellipsim tangat CN in C, occurrens tangenti BN in N, ducaturque BC secans QM in O, & IG in S quoniam NC, IG, QM sunt normales axi, aequidistant ; ergo per ea quae propos 99.huius demonstravimus, rectangulum FIG, aequatur quadrato SI, & rectangulum HQN quadrato QO aequale est quare ut quadratum IS ad quadratum QO, hoc est ut quadratum SB ad quadratum OB, hoc est ut quadratum ED ad quadratum KD, ita rectangulum FIG ad rectangulum GQM; & permutando ut rectangulum FIG ad quadratum ED, ita rectangulum HQM ad quadratum KD, sed rectangulum FIG per praeced. aequatur quadrato ED. Ergo rectangulum quoque HQM aequatur quadrato KD.
Similiter demonstrabimus ad alteram partem poli D, rectangula HQM aequatur quadratis KD esse aequalia tangat enim ellipsim AP, in A occurrens tangenti in P, & tactus iungat AB secans QM in R, in triangulo BCN. Ducatur aliqua QO, parallela NC normali ad axem, ita se habens ad QB, ut QR est ad QB, erit igitur permutando ut QB ad BQ hoc est ut KD ad DK, ita QO ad QR. Ergo ut quadratum KD ad quadratum DK, ita quadratum QO ad quadratum QR, hoc est rectangulum HQM ad rectangulum HQM; permutando igitur ut rectangulum HQM ad quadratum KD, ita rectangulum HQM ad quadratum DK.
Atqui supra demonstrarum est rectangulum HQM (illud nempc quod est versus C) aequari quadrato KD, ergo rectangulum quoque HKM quod est versus A, aequatur quadrato DK. Omnia igitur rectangula HKM, &c. Quod erat demonstrandum.

Corollarium.

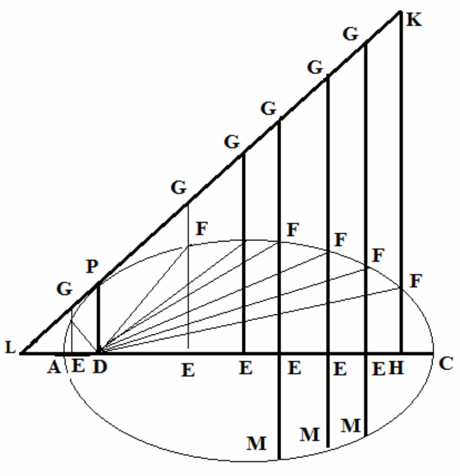
Ex discursu demonstrationis iam allatae licet colligere quadrata tangentium CN, AP, quadratis CD, DA esse aequalia.

PROPOSITIO CXXXIX.

Data sit ellipsis cuius axis AC, poli D, H, ex polo D ducta sit ad perimetrum DP normalis axi, & in P ellipsim tangat linea GPG. Ducantur autem quotcunque normales axi GFE, iunganturq; DF, DF.

Dico lineas omnes DF, lineis omnibus GE aequales esse.

Demonstratio.



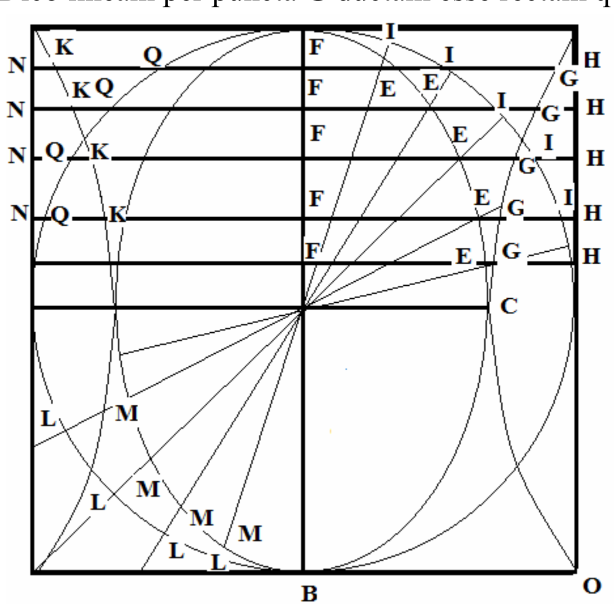
Producatur una rectarum GE in M, per praeced. rectangulum FGM aequatur quadrato DE; addito igitur communi quadrato EF, aequantur quadrata DE, EF, hoc est quadratum DE, rectangulo FGM cum quadrato EF, hoc est, quadrato GE. Quia igitur quadratum DF aequatur quadrato GE, etiam recta DF rectae GE aequalis est. Eodem discursu reliquae omnes DF, reliquis omnibus GE aequales sunt. Quod erat demonstrandum.

In libro de hyperbola, tria sequentia theoremata licet sint demonstranda quod ab hyperbolae proprietatibus dependeant, ob miram tamen cum ellipticis affectionibus connectionem visum est non alienum hoc loco proponere.

PROPOSITIO CXL.

Eadem manente figura, si rectis DF e polo ductis aequentur lineae EFG normales ad axem AC.

Dico lineam per puncta G ductam esse rectam quae ellipsim contingat in P.



Demonstratio manifesta est ex propositione praecedenti, contingit in C.

Demonstrationem vide in lib. de hyperbola.

PROPOSITIO CXLI.

Translated from Latin by Ian Bruce; 24/7/2021.Free download at 17centurymaths.com.

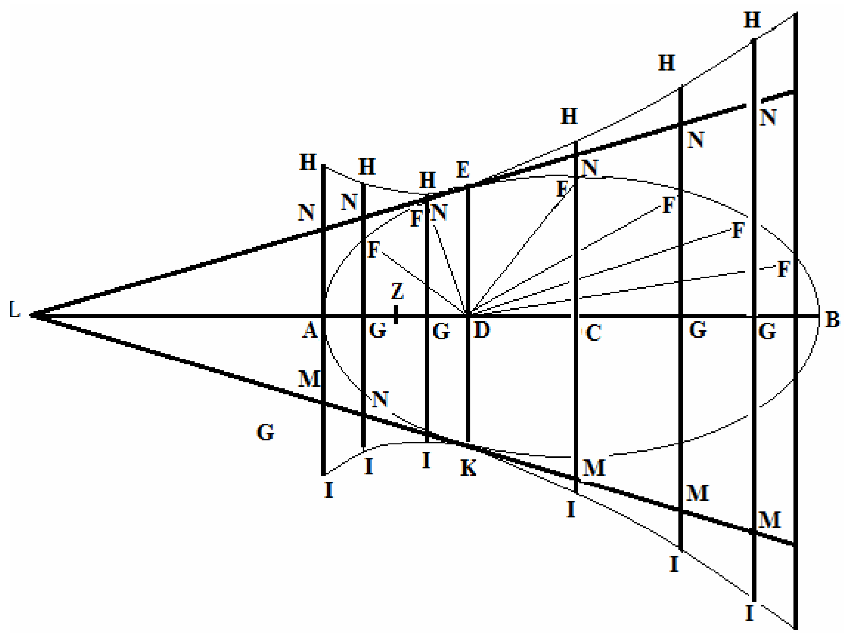
Data sit ellipsis axem
habens AB, centrum D sumatur in axe punctum quod prima sit centrum ellipseos, ex eo ducatur ad perimetrum normalis axi DC ac deinde quotcunque aliae DE, DE : quibus aequales
fiant lineae FEG axi normales.

Dico lineam per puncta
G descriptam esse hyperbolam quae idem habeat cum ellipsi centrum D, eamque contingit in C.

Demonstrationem vide in lib. de hyperbola.

PROPOSITIO CXLII.

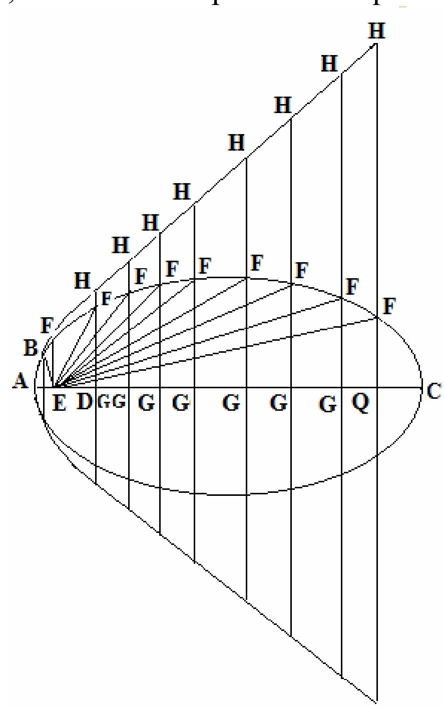
Translated from Latin by Ian Bruce; 24/7/2021.Free download at 17centurymaths.com.



Data sit ellipsis axem habens AB, centrum C, polos X, Z in axe sume punctum aliquod D inter centrum C & polum D, ex quo ducatur ad perimetrum DE normalis axi, ac deinde quaevis aliae DF, DF; quibus aequales fiant GFH axi normales.

Dico lineam per puncta H, H descriptam esse hyperbolam, quae ellipsim tangat in F.

Demonstrabitur in libro de hyperbola.



PROPOSITIO CXLIII.

Data rursum sit ellipsis axem habens AC, polos D, Q, in axe sumatur punctum E inter polum D & verticem A, ex quo ducatur ad perimetrum normalis axi EB : & quotvis aliae EF, quibus aequales fiant GFH axi normales.

Dico lineam quae per puncta H describitur hyperbolam esse quae ellipsim ambiat & tangat in puncto B.

Demonstrationem dabimus in libro de hyperbola.

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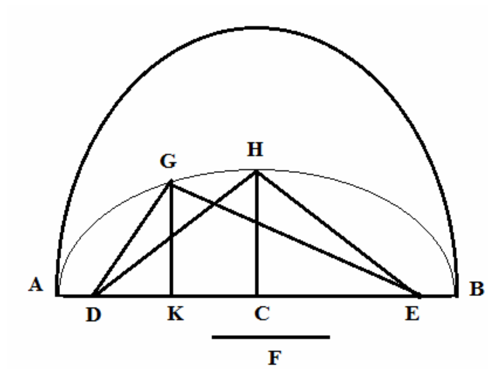
PROPOSITIO CXLIV.

Data basi aggregato laterum & altitudine triangulum exhibere.

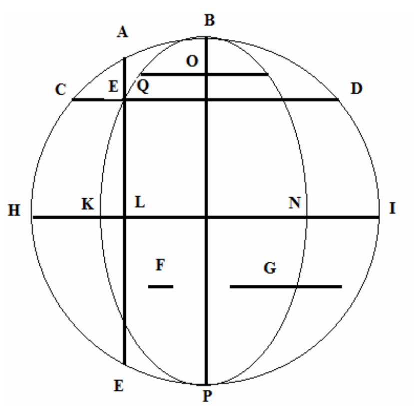
Constructio & demonstratio.



Dato aggregato laterum ponatur AB, aequalis, qua bifariam divisa in C fiat DE aequalis basi trianguli bifarium divisa in C sic ut utrimque relinquantur aequales AD, BE, altitudini autem sit aequalis F, ex lateribus AC, CB, DE fiat triangulum DHE (nam AC, CB, simul sumptae maiores sunt DE,) erit DHE isosceles. Deinde fiat ut quadratum HC ad F quadratum, ita rectangulum ACB ad AKB, & erigatur KG aequalis F parallela HC, & iungantur DG,GE. Dico DGE, esse triangulum quaesitum, quoniam ACB, rectangulum est ad AB rectangulum, ut quadratum HC ad quadratum F, hoc est, quadratum, erunt puncta A,G,H,B ad eandem ellipsim cuius AB, sit axis: & quia AD, ipsi EB, itemque DH, HE, aequales sunt ipsi AB, erunt DE, puncta ex comparison facta sive foci ellipseos, quare DGE, latera aequalia sunt axi AB, hoc est aggregatio laterum estque basis data DE, & altitudo F hoc est GK. Igitur exhibuimus triangulum quod quaerebatur.

PROPOSITIO CXLV.

Rectam AB, subtensam cuiusvis arcus circuli ABC, altera secare CD, eidem ad angulos rectos ut CE ad ED, datam obtineat rationem F ad G.



Constructio & demonstratio.

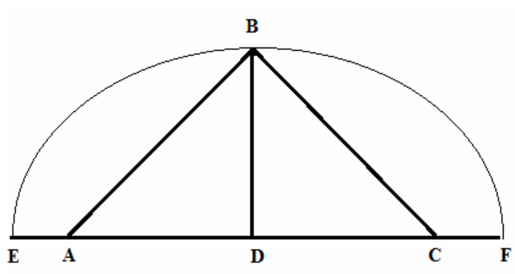
Libro de circulorum proprietatibus proposuimus hoc problema : sed quoniam eius demonstratio ab elliptica proprietate dependet, id circo in hunc locum eam distulimus , constructio vero est. Ducatur diameter HI normalis ad AB secans AB in L. Fiatque ut F ad G, sic HK ad KI: sumpa deinde IM aequali ipsi HK dividatur diameter HP, in O puncto ut dividitur KM in L: deinde rectangulo NOP, fiat aequale quadratum LE; tandem ducatur per E, recta normalis CED. Dico CED, divisam in E, secundum rationem F ad G. Quoniam sunt lineae NP, KM ad angulos rectos bifariam divisae; igitur descripta ponatur circa illas tamquam axes ellipsis NKP, erunt itaque OQ, LE, ordinatim positae ad singulos axes:& quia axes similiter divisae sunt in O & L, estque rectangulum NOP aequale quadrato LE, patet a punctum E esse ad ellipsim

per puncta N, K, P, M descriptam. Ergo est ut HK ad KI, hoc est ut F ad G, sit CE ad ED, patet rectam
D applicatam esse in circulo normaliter ad CD, ut CE ad ED datam rationem obtineat F ad G. Quod fuit demonstrandum.

PROPOSITIO CXLVI.

Data recta AC & altitudine BD, ellipsim describere cuius poli sint A, & C.

Constructio & demonstratio.



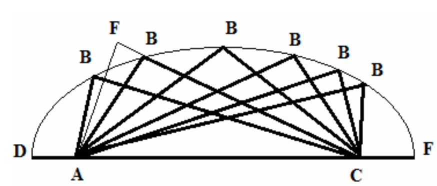
Fiat super AC linea in altitudine BD triangulum isosceles ABC dein AC linea utrimque aequaliter producatur in E & F: ut tota EF sit aequalis duabus AB, BC, cum per E, B, F, puncta ellipsis describatur. Dico illam esse quae petitur. Quoniam EF linea divisa est bifariam in D & non bifariam in A: erit EAF rectangulum una cum quadrato AD, aequale quadrato ED hoc est per constructionem quadrato AB; sed etiam quadrato AB aequalia sunt quadrata AD, BD; dempto igitur communi quadrate AD, manet EAF rectangulum aequale quadrato BD id est quartae parti figurae. Eodem modo ostenditur quadrato BD aequari rectangulum FCE : quare A & C, foci sunt descriptae ellipseos EBF. data igitur linea & altitudine, &c. Quod erat faciendum.
Corollarium.

Hinc sequitur dato quovis triangulo isosceli ABC continente ad verticem, angulum quemcunque, describi posse ellipsim cuius foci sint extrema basis trianguli dati ABC. Demonstratio patet ex propositione.

PROPOSITIO CXLVII.

Super AC linea descripti sint quotcunque triangula isoperimetra ABC, AGC.
Dico puncta G, B, B esse ad eandem ellipsim cuius poli sint A & C.

Demonstratio.



Producatur AC utrimque aequaliter in D & E, ut tota DE sit aequalis duabus AB, BC, tum per puncta D, E, G ellipsis describatur. Dico illam transire per reliqua puncta B, B sin vero; transeat supra vel infra B, ac primum supra per punctum F, producta CB donec peripheriae occurrat in F, iungantur AF. Quoniam igitur GF puncta ad ellipsim sunt, cuius poli A & C, erunt AGC, AFC triangula isoperimetrae est autem AGC triangulum per constructionem isoperimetrum triangulo ABC; igitur AFC, ABC triangula sunt isoperimetra, quod fieri non potest. quare DGE, ellipsis non transit supra B, sed nec infra B, cadere eodem modo demonstrabitur. Ergo per B, B puncta; igitur GBB sunt ad ellipsim cuius poli sunt A, C. Quod erat demonstrandum.

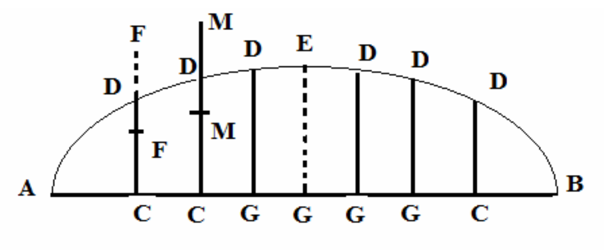
ELLIPSEOS PARS QUINTA

Varias exhibet ellipsis geneses.

PROPOSITIO CXLVIII.

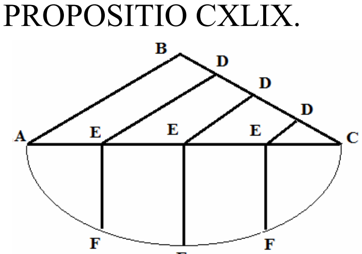
Sit AB linea utcunque divisa in C, & ex C quotcunque erigantur parallelae CD, fiatque ut ACB rectangulum ad rectangulum ACB sic DC quadratum ad quadratum DC.
Dico ADB puncta esse ad eandem ellipsim vel circulum.

Demonstratio.



AB bisecta in G, erigatur GE parallela ipsis CD, fiatque ut rectangulum ACB ad rectangulum ACB, sic quadratum CD ad quadratum GE, intelligatur deinde descripta esse ellipsis cuius diametri coniugatae sint AG, GE si ergo ellipsis non transit per punctum D, occurrat rectae CD supra vel infra D in F quia AG, EG sunt diametri coniugatae; erunt DC ipsi EG parallelae ordinatim positae ad diametrum AB. Quare cum ellipsis dicatur transire per F, erit quadratum FC ad quadratum EG ut rectangulum ACB ad rectangulum AGB, hoc est ex constr. ut quadratum DC ad quadratum EG: Quod est absurdum. Non igitur punctum ullum F supra aut infra D, est ad ellipsim, sed ipsum D punctum ad ellipsim est. Sumatur iam aliud quodlibet punctum D, exempli gratia punctum propius sequens; si rursum ellipsis non transit per D, occurrat rectae CD supra vel infra D in M. Quoniam est ex hypothesi ut quadratum DE ad quadratum DE, sic rectangulum ACB ad rectangulum ACB, & ex constr. ut quadratum DE ad quadratum EG, ita rectangulum ACB ad rectangulum ACB, erit ex aequali ut quadratum DE ad quadratum EG, ita ACB rectangulum ad rectangulum AGB: sed etiam est quadratum MC ad quadratum EG, ut rectangulum ACB ad rectangulum AGB cum punctum M ponatur esse ad ellipsim. Ergo quadrata DC, MC ad quadratum EG, eandem habent rationem. Quod est absurdum: Non igitur punctum M aut aliud ullum praeter D ad ellipsim est; simili discursu reliqua puncta D ad ellipsim esse demonstrabimus. Ex quibus constat veritas Theorematis.

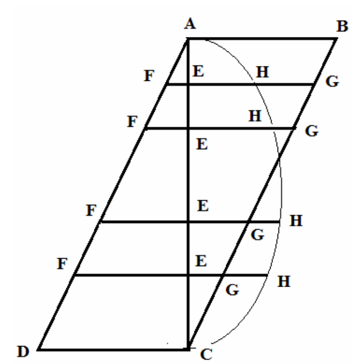
PROPOSITIO CXLIX.



Sit ABC triangulum quodcunque, divisoque latere BC utcunque in punctis DD: ducantur ex D rectae DE parallelae AB, & ex A demittantur lineae EF sit ut quadrata EF,sint BDE rectangulis aequalia.
Dico puncta A, F, C esse ad eandem ellipsim.

Demonstratio.
Ut BDE rectangulum ad rectangulum BDE, sic BDC, rectangulum est ad rectangulum BDC, hoc est rectangulum AEC ad rectangulum AEC: sed ut BDE rectangulum est ad rectangulum BDE, sic EF quadratum est ad quadratum EF, igitur ut rectangulum AEC est ad rectangulum AEC, sic EF quaratum est ad quadratum EF. Ergo AFC, puncta sunt ad ellipsim.

PROPOSITIO CL



Sit AB, CD parallelogrammi diameter AC, ducanturque; AB lateri quotcumque parallelae FG, secantes AC lineam in E : dein fiant inter FE, EG mediae E H.
Dico puncta A, C, & omnia puncta H esse ad eandem ellipsim, nisi sint ad circulum.

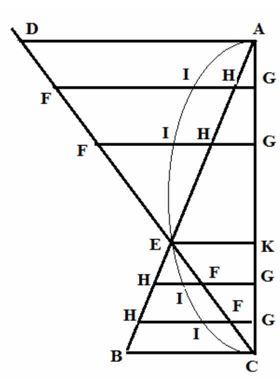
Demonstratio.

Ratio FEG rectanguli ad rectangulum FEG est composita ex ratione FE ad FE, id est AE ad AE; & ex EG ad EG, id est EC ad EC: sed ex iisdem componitur ratio rectanguli AEC ad AEC rectangulum, igitur ut FEG rectangulum ad rectangulum FEG; hoc est quadratum EH ad quadratum EH, sic AEC rectangulum est ad rectangulum AEC. Quare AH, HC puncta sunt ad ellipsim. Quod erat demonstrandum.

Corollarium.

Si AB recta sit normalis ad AC; & illi fuerit aequalis, erunt AHH puncta ad eundem circulum; erit enim AEC rectangulum aequale rectangulo FEG, hoc est quadrato EH. adeoque puncta HH ad circulum.

PROPOSITIO CLI.



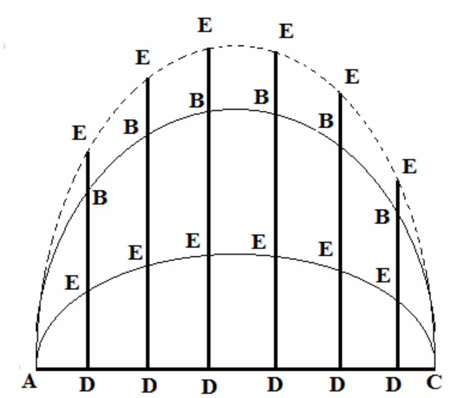
Secent se in E duae quaevis lineae AB, CD quas coniungant duae parallelae AD, BC; iungantur item puncta AC: tum rectae ducantur FG parallelae lineis AD, BC, occurrentes AB lineae in HH, fiantque; HGF, rectangulis equalia quadrata GI.
Dico puncta I, I, I esse ad ellipsim.

Demonstratio.

Ratio HGF rectanguli ad rectangulum HGF est composita ex ratione HG ad HG, id est AG ad AG, &ex ratione FG ad FG, id est GC ad GC : sed ex iisdem est composita ratio rectanguli AGC ad AGC, igitur ut HGF rectangulum est ad rectangulum HGF, hoc est quadratum IG ad quadratum IG, sic AGC rectangulum ad rectangulum AGC: quare I, I puncta sunt ad ellipsim. Quod erat demonstrandum.

Corollarium.

Ducatur ex E puncto intersectionis recta EK parallela lineae AD: si AK, EK, KC fuerint continuae, & EK linea normalis ad rectam AC, dico I, I, puncta esse ad circulam cum enim sit ut AKC rectangulum ad AGC rectangulum, sic EK quadratum ad quadratum IG: (id enim eodem discursu probabimus, quo rectangula AGK ostendimus esse ad rectangula AGK ut quadrata GI ad quadrata GI) erit permutando ut AKC rectangulum ad quadratum EK sic AGC rectangulum ad quadratum GI. Adeoque quadratum IG aequale rectangulo AGC. Igitur I,I sunt ad circulum.



PROPOSITIO CLII.

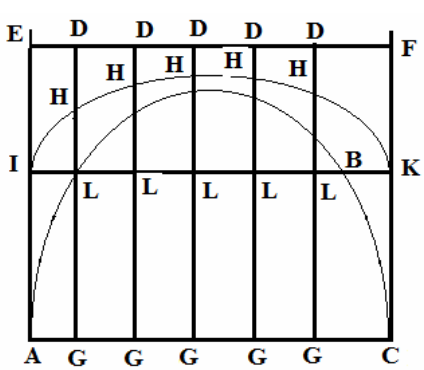
Sit ABC semicirculi diameter AC, divisa utcunque in DD, & ex D normales erigantur DE, fiantque ut BD ad BD, sit ED ad ED.

Dico puncta E, E esse ad eandem ellipsim.

Demonstratio.

Ut quadratum DB ad quadratum DB, sic ED quadratum est ad quadratum ED: sed ut quadratum BD ad quadratum BD sic ADC rectangulum est ad rectangulum ADC: igitur ut quadratum ED ad quadratum ED sic ADC rectangulum est ad rectangulum ADC. Quare E,E puncta sunt ad ellipsim. Quod erat demonstrandum.

PROPOSITIO CLIII.



Super ABC semicirculi diametro AC rectangulum describatur AF: ductisque lineis DG parallelis lateri AE quae circulo occurrant in BB, ducatur qaaevis IK parallela rectae ED occurrens DG lineis in LL; fiatque ut AI ad IE sic BH ad HD.

Dico puncta HH esse ad eandem ellipsim.

Demonstratio.

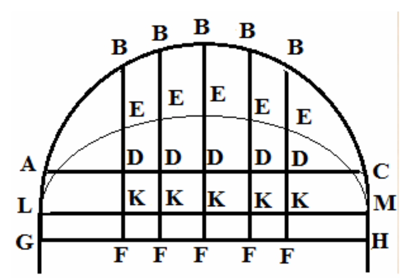
Ut AE ad AI, hoc est GD ad LD, sic BD est ad DH, igitur permutado dividendo iterumque permutando ut GB ad LH, sic BD ad DH, atqui BD est ad HD, ut HD ad HD, sunt enim ambae rationes BD ad HD, eadem rationi AE ad IE; quare ut GB ad LH, sic GB ad LH: & permutando ut GB ad GB, sic LH ad LH: & ut quadratum GB ad quadratum GB , sic LH quadratum ad quadratum LH: est autem ut quadratum GB ad quadratum GB sic AGC rectangulum ad rectangulum AGC, id est IIK, rectangulum ad rectangulum IIK, igitur ut LH, quadratum est ad quadratum LH, sic IIK rectangulum est ad rectangulum IIK : quare H, H puncta sunt ad ellipsim. Quod erat demonstrandum.

PROPOSITIO CLIV.

Sit ABC segmentum circuli quodcunque, cuius AC subtensa, divisa utcunque in DD, erigantur ex D normales DB, fiatque ut BD ad BD sic ED ad ED.

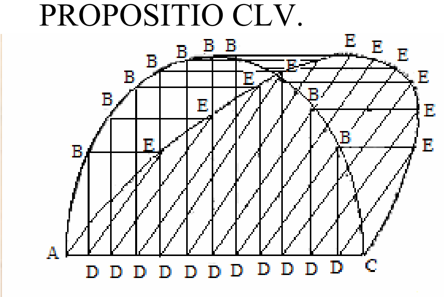
Dico puncta E,E esse ad ellipsim.

Demonstratio.



Perfecto semicirculo ABC: ducatur GH diameter circuli GBH parallela lineae AC, quae BD, lineas productas secet in FF: fiatque ut BD ad DF, sic ED ad DK: tum ex G & H rectae erigantur GL, HM parallelae lineis BF secantes KK lineam in L & M, Quoniam est ut BD ad DF, sic DE ad DK, erit permutando, ut BD ad ED, sic DF ad DK. Atqui BD est ad DE, ut BD ad DE. igitur ut DF ad DK, sic DF ad DK : quare puncta K, K ad eandem lineam, & quidem parallelam lineae GH. Rursum cum sit ut BD ad DK, igitur ut ED ad DK, erit componendo & permutando BF ad EK, ut DF ad DK. igitur ut BF ad ad EK, & rursum permutando, ut BF ad BF, sic EK ad EK, & ut quadratum BF ad quadratum BF ad quadratum BF, sit EK, quadratum ad quadratum EK : sed ut BF quadratum ad quadratum BF, sit HFG rectangulum est ad rectangulum HFG, id est MKL rectangulum ad rectangulum MKL, igitur ut MKL rectangulum ad rectangulum MKL sit quadratum EK ad quadratum EK. Quare E, E puncta sunt ad ellipsim. Quod erat demonstrandum.

PROPOSITIO CLV.



Esto ABC semicirculi diameter AD, quam in D secent quotcunque normales BC dein rectis BD fiant aequales BE parallelae diametro AC.

Dico puncta E, E esse ad ellipsim cuius diameter est AC.

Demonstratio.

Ducantur rectae DE: quoniam anguli BDC recti sunt, & BE parallelae, anguli quoque DBE erunt rectiquadrata igitur DE, aequantur quadratis BD, BE, hoc est, quia BD, BE sunt aequales, dupla sunt quadratarum BD. Ergo ut quadratum BD ad quadratum BD, hoc est: ut rectangulam ADC ad rectangulum ADC, ita quadratum DE ad quadratum DE. Sunt vero & DE rectae inter se parallelae: cum enim anguli DBE recti sint; & latera BD, BE aequalia ; erunt BDE semirecti. cum ergo etiam BDC rectus sit, reliqui EDC sunt semirecti, adeoque aequales: unde DE parallelae. Puncta igitur E, E sunt ad ellipsim. Quod autem AC sit diameter, facile apparebit si perfecto circulo ellipsis eadem constructione ad partem alteram tam producatur, tunc enim parallelae omnes DE a recta AC bifariam dividentur.

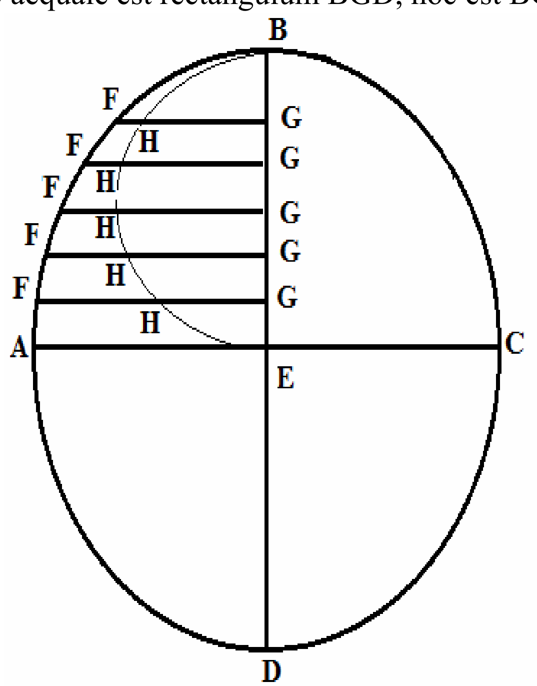
PROPOSITIO CLVI.

Circulum ABC secent ad angulos rectos diametri AC, BD ductisque rectis FG quae AC, diametro aequidistent, demittantur ex B lineae BH aequales rectis FG secantes FG lineas in HH.

Dico puncta B, H, E esse ad eandem ellipsim.

Demonstratio.

Quoniam FG quadrato aequale est rectangulum BGD, hoc est BGE, a rectangulum bis



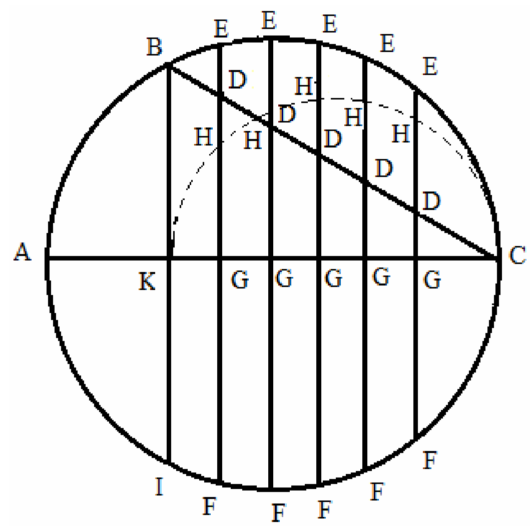
sumptum una cum quadrato BG, erit & quadratum HB aequale rectangulo BGE bis sumpto una cum quadrato BC: sed HB quadratum est aequale quadratis HG, BG: ablato igitur communi quodrato BG manet HG, quadratum aequale rectangulo

BGE bis sumpto; similiter reliqua quadrata HG dupla sunt rectangulorum BGE; igitur ut quadratum HG ad quadratum HG: sit BGE rectangulum est ad rectangulum BGE quare puncta B,E, & omnia puncta H, ad eandem sunt ellipsim. Quod erat demonstrandum.

PROPOSITIO CLVII.

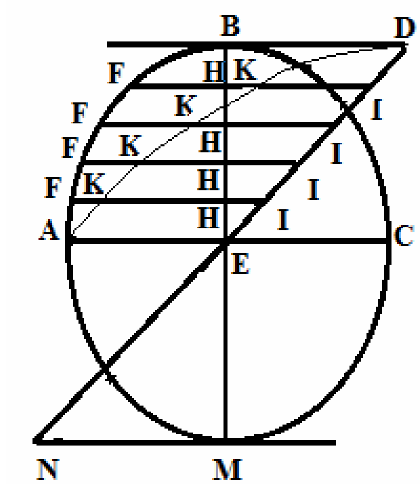
Esto ABC circuli diameter AC & ex C recta quaevis ducta CB occurrant circuli perimetro in B, dein ex B demissa recta BI quae AC, diametrum ad rectos angulos secet in K. Ducantur quotcunque lineae EF parallelae rectae BI occurrentes AC diametro in G, & lineae BC in D: fiantque EDF rectangula aequalia quadratis GH.

Dico KHC puncta esse ad eandem ellipsim.



Demonstratio.

Ut EDF rectangulum ad rectangulum EDF, sic BDC rectangulum est ad rectangulum BDE, id est rectangulum KGC ad rectangulum KGC: sed (quemadmodum alternando patet ex hypothesi) ut EDF rectangulum ad rectangulum EDF, sit HG quadratum est ad quadratum HG: igitur ut KGC rectangulum est ad rectangulum KGC, sit H G quadratum est ad quadratum HG. quare HGC puncta sunt ad ellipsim. Quod erat demonstrandum.



PROPOSITIO CLVIII.

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Secent ABC circulum ortogonaliter diametri AC, BE actaq; per B tangente BD ducatur per E centrum recta quovis ED, occurrens tangenti BD in puncto quovis D. Dein rectae ducantur FHI, parallelae tangenti BD, occurrentes EB diametro in HH, & ED lineae in I, I :fiatque ut FH ad FH, sit IK ad IK.

Dico AKD puncta esse ad eandem ellipsim.

Demonstratio.

Producta BE diametro in M; producatur & DE linea donec actae per M tangenti occurrat in N. Quoniam BD, NM, HI lineae aequidistant, erit ut rectangulum BHM ad rectangulum BHM, sic DIN rectangulum ad rectangulum DIN ;sed est ut BHM rectangulum ad rectangulum BHM , sit FH quadratum ad quadratum FH, id est quadratum IK ad quadratum IK: igitur ut DIN rectangulum ad rectangulum DIN, sic est quadratum IK ad quadratum I K. Quare AKD puncta sunt ad ellipsin. Quod erat demonstrandum.

PROPOSITIO CLIX.

Circulum ABC cuius diameter AC contingant duae lineae AD, BD secantes sese ortogonaliter in D: iunctisque punctis AB, agatur per C tangens sese ortogonaliter in D: iunctisque punctis AB, agatur per C tangens CL, occurrens AB lineae in L; dein rectae ducantur quotqunque FE parallelae lineae DB occurrentes AB lineae in HH, & circulo in EE; tum per E rectae ducantur GK parallelae lineae AD occurrentes AC diametro in K & AL lineae in II. Fiantque FE lineis aequales EG.

Dico puncta ALL esse ad ellipsim.

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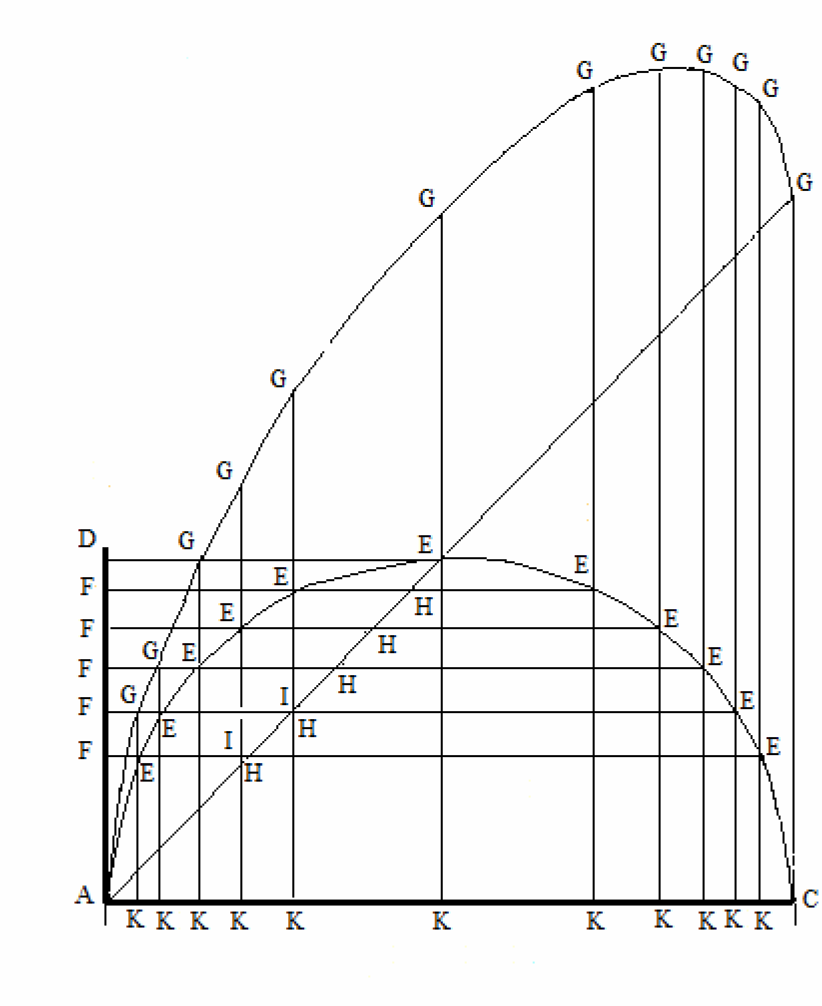

Demonstratio.

Ut AD est ad DB, sic AF est ad FH: sed AD, DB lineae sunt aequales, igitur & AF, FH lineae aequantur; quare & EK, FH lineae aequantur ; Rursum cum sit ut AF ad FH, sic EI ad EI ad EH, erunt EI, EH lineae inter se aequales: est autem ex constructione FE lineae aequalis lineae EG: igitur tota IG, est aequalis tota FH, hoc est FA id est EK; quare ut quadratum EK ad quadratum EK, sic IG est ad quadratum IG: sed ut EK quadratum est ad quadratum EK, sic ACK rectangulum est ad rectangulum ACK, id est AIL rectangulum ad rectangulum AIL, igitur ut quadratum IG est ad quadratum IG, sic AIL rectangulum est ad rectangulum AIL. Quare ALL puncta sunt ad ellipsim. Quod erat demonstrandum.

Quod si eadem constructio ad alteram partem continuetur, perficietur ellipsis altera fui parte, quae intra circulum cadet. Ubi hoc notatu dignum occurrit, quod licet circulus & ellipsis sese invicem secent, eandem tamen rectam DA patet ex hypothesi: quod eandem contingat etiam ellipsis, inde fit manifestum quod omnia perimetri elliptici puncta sint in lineis GK quae inter puncta C & A, ipsi DA ducuntur parallelae.

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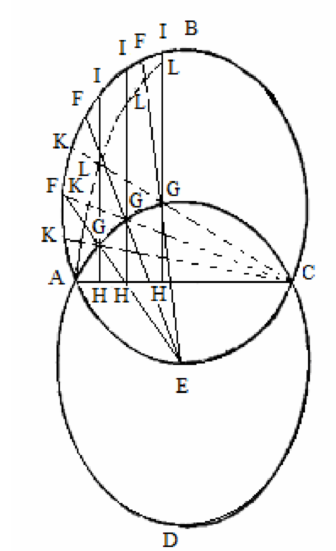
PROPOSITIO CLX.



Secent se duo circuli ABC, ADC ut illorum alter ABC transeat per E centrum circuli AB, iuncitisque ; punctis AC ducantur ex E lineae quaecunque EF occurrentes circulo ABC in punctis F & ADC circulo ABC in punctis F & ADC circulo in punctis G: tum per G rectae agantur HI normales ad lineam AC, occurrentes AC lineae in HH, & circulo ABC in II: fiantque rectis GF aequales lineae GL.
Dico puncta ALL esse ad ellipsim.

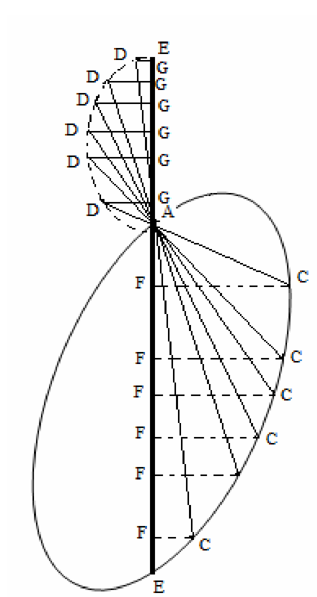
Demonstratio.

Ducantur ex C per G lineae CGK, ut KGC rectangulum est ad rectangulum KGC, sic FGE rectangulum est ad rectangulum FGE sed est & ut KGC rectangulum ad rectangulum KGC, sit GH linea ad lineam GH, & ut FGE rectangulum ad rectangulum FGE, sic FG linea ad lineam FG, igitur ut GH ad GH, sic FG ad FG, id est LG ad LG, & componendo permutando LH ad LH, ut GH ad GH. Quare puncta ALL sunt ad ellipsim.

PROPOSITIO CLXI.

Sit ABC ellipsis diameter quaecunque AB, actisque per A lineis CD, quae ellipsi occurrant in CC, fiat ut AC ad AC, sic AD ad AD, & ut AC ad AD, sic AB ad AE.
Dico puncta A, D, E ad eandem ellipsim esse.

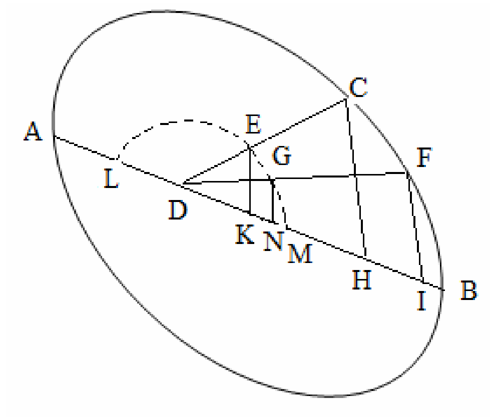
Demonstratio.



Quoniam DG, FG sunt parallelae, triangulaque proinde FCA, DGA similia, erunt ut lineae CA ad lineas AD singulae ad singulas, ita singulae FC ad singulas GD. Atqui singulae CA sunt ad singulas AD ut BA ad AE. Ergo singulae FC sunt ad singulas DG, ut BA ad AE. Quare quam rationem habet una FC ad unam DG, eandem habent singulae reliquae FG ad singulas reliquas DG. Igitur permutando ut sunt FC ad FC, ita GD sunt ad GD, adeoque ut sunt quadrata FC ad quadrata FC, ita quadrata GD sunt ad quadrata GD; similiter demonstrabimus, ut AF funt ad AF, sit esse AG ad AG. Unde ut reliquae FB sunt ad reliquas FB, ita reliquae GE sunt ad reliquas GE. Quare cum rectangula AFB rationem habeant ad sese invicem compositam ex rationibus AF ad AF, & FB ad FB, quae ostensae sunt eadem esse rationibus AG ad AG,

& GE ad GE , ex quibus componitur ratio rectangulorum AGE; erunt ut rectangula AFB ad rectangula AFB, sit rectangula AGE ad rectangula AGE. Atqui rectangula AFB sunt ad rectangula AFB, ut quadrata FC ad quadrata FC; hoc est per superius demonstrata, ut quadrata GD ad quadrata GD; ergo rectangula AGE sunt ad rectangula AGE, ut quadratae GD ad quadrata GD. Puncta igitur DA, ADB sunt ad ellipsim. Quod erat demonstrandum.

PROPOSITIO CLXII.



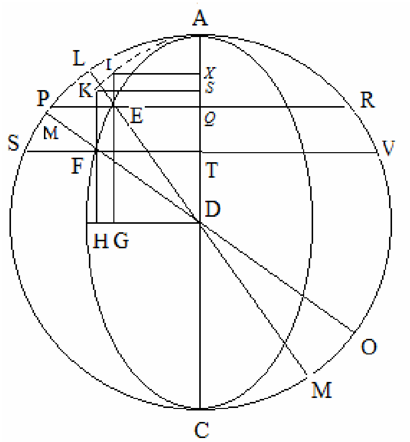
Esto ABC ellipsis diameter quaevis AB, divisa utcunque in D & ex D ad peripheriam rectae ducantur DC, DF quae proportion aliter dividantur in E & G: dein AD dividatur in L , & DB in M, ut DC, DF divisae sunt in E & G.
Dico L EGM puncta esse ad ellipsim.

Demonstratio.
Ducantur ex C & F ordinatim lineae CH, FI ad AB, diametrum quibus ex E & G parallelae ducantur EK, GN ut DE ad DE, sic DF ad GD, igitur ut CH ad EK, sic FI ad GN, & EK ad GN, ut CH ad FI: igitur & quadratum EK ad quadratum GN ,ut CH quadratum ad quadratum FI. Deinde, quoniam DI est ad DN, ut DF ad DG, & DA ad DL,ut DF ad DG, erit ut DI ad DN, sit DA ad DL, ergo ut una anticedens
DI ad unum consequens DN, (hoc est: ut DF ad DG, hoc est ut DC ad DE, hoc est ut DH ad DK) ita ambae antecedentes, hoc est: tota AI ad ambas consequentes, hoc est totam LN, similiter inferemus AH esse ad IK, ut DH ad DK. Unde AI est ad LN, ut AH ad DK, & permutando AI est ad AH, ut LN ad IK. Praeterea, quoniam est ut DF ad DG (hoc est: ut tota DB ad totam DM) sit ablata DH ad ablatam DN, erit & reliqua IB ad reliquam NM, ut tota DB ad totam DM. Similiter inferemus HB esse ad KM, ut DB ad DM. Ergo IB ad NM, ut HB ad KM, permutando igitur ac inuertendo HB ad IB, ut KM ad NM. Cum igitur ostenderimus rationes AH ad AI, & AK ad AN, item rationes HB ad IB, & KM ad NM easdem esse, rationes quoque rectanguli AHB ad rectangulum AIB, & rectanguli AKB ad rectangulum ANB, ex rationibus illis aequalibus compositae, eaedem erunt; sed rectangulum AHB est ad rectangulo AIB, ut quadratum CH ad quadratum FI; hoc est per superius demonstrata ut quadratum EK ad quadratum GN. Ergo rectangulum IKM est ad rectangulum LNM, ut quadratum EK ad quadratum GN. Ergo puncta L, E, G. M sunt ad ellipsim. Quod erat demonstrandum.

PROPOSITIO CLXIII.

Sint ABC ellipsis axes AC, BD: ductisque ex D semidiametris quibusvis DE, DF, agantur per E & F lineae IG, KH aequales ipsis ED, FD, parallelae vero axi AC, occurrentes axi BD in G & H.
Dico puncta AIK esse ad ellipsim cuius axis.

Demonstratio.



Super AC ut diametro describatur circulus ALC, & DE, DF lineae utrimque producantur donec circuli perimetro occurrant in L, M, N, O: actisque per E & F, lineis PER, SFV quae circulo occurrant in R, V & AC diametro in Q & T, & aequidistent axi BD, ducantur ordinatim ad axem AC lineae IX,KZ. Quoniam PER, SFV lineae in E & F, proportionaliter sunt divisae, ratio rectanguli PER ad rectangulum SFV duplicata est rationis PE ad SF, adeoque erit PER rectangulum ad rectangulum SFV ut quadratum PE ad quadratum SF, id est ut quadratum EQ ad quadratum FT, id est ut quadratum IX ad quadratum KZ. Quare cum rectangula LEN, PFO aequalia sint rectangulis PER, SFV, etiam LEN rectangulum est ad rectangulum MFO, ut quadratum IX ad quadratum KZ: deinde cum IG hoc est XD aequalis ED. & DC aequalis DN, erit XC aequalis EN; est vero & tota AC aequalis toti LN. Ergo reliqua AX reliquae LE aequalis est : adeoque AXC rectangulum aequale rectangulo LEN: eodem modo ostenditur rectangulum AZCa aequari rectangulo MFO, erit igitur ut AXC rectangulum ad rectangulum AZC, sic quadratum IX ad quadratum KZ. Quare AIKC, puncta ad ellipsim. Quod erat demonstrandum.

ELLIPSIS

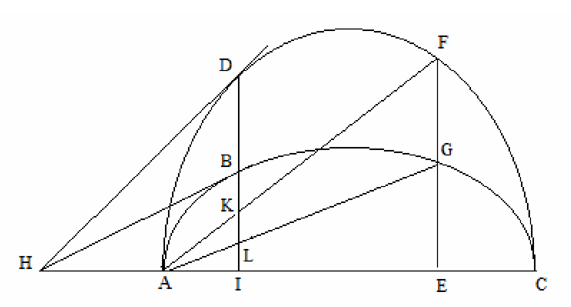
PARS SEXTA

Circulum cum ellipsi comparat.

PROPOSITIO CLXVIII.

Habeant ABC ellipsis, & circulus ADC communem axem AC, ductaquae ordinatim EF, occurrat circulo in F & ellipsi in G: iunganturque AF, AG dein ducta DH parallela AF quae circulum contingat in D; occurratque axi in H; demittatur ex D ordinadim lineae DI ad diametrum AG secans ellipsim in B & AF, AG in K & L: iunganturque HB.
Dico AG lineam aequidistare rectae HB.

Demonstratio.



Ut EG ad EF, sic IL est ad IK, sed ut EG ad EF, sic IB est ad ID, igitur ut IL ad IK, sic

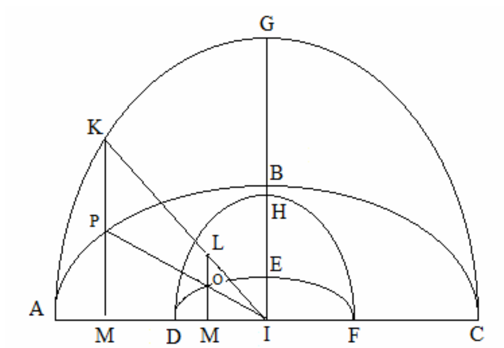
IB ad ID, & permutatando ut IK ad ID, sic IL ad IB,
est autem ut IK ad ID, sic IA ad IH (quia AF, HD ex hypothesi aequidistant) igitur ut IL ad IB, sic IA ad IH. Quare AG, HB lineae sunt parallelae. Quod erat demonstrandum.

PROPOSITIO CLXIX.

Sint ABC, DEF ellipses similes, similiterque ad idem centrum I constitutae : & super AC, DF diametris , circuli describantur AGC, DHF: ponatur autem ex centro quaedam IK occurrens circulis in L & K, punctis , ex quibus normales demissae LM, KN, secent ellipses in O & P: ducanturque lineae IO, OP.

Dico esse ut IL ad LK, sic IO ad OP.

Demonstratio.



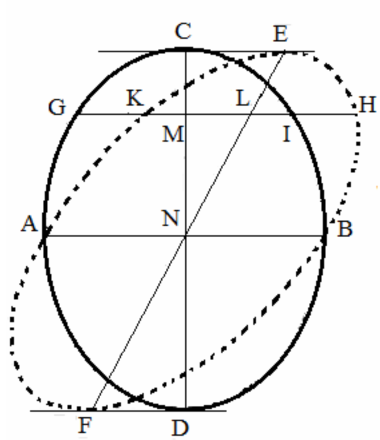
Erigatur ex I centro normalis IG occurrens ellipsibus in E, B, circulis vero in H & G. Quoniam tam circuli AGC, DBF quam ellipsos ABC, DEF similes sunt similiterque ad idem centri constitutae, ut IG ad IB, sic IH est ad IE, sed ut GI ad BI, sit KN ad PN; & ut HI ad EI, sic LM ad OM: igitur ut KN ad PN, sic LM est ad OM, & permutando ut KN ad LM, hoc est IN ad IM, sic PN ad OM, in directum sunt I, O, P. Quare cum KN, LM ad AC, sint perpendiculares, ac proinde inter se parallelae, erit ut IL ad LK, sic IO ad OP. Quod erat demonstrandum.

PROPOSITIO CLXX.

Circulum ABC secent diametri duae AB,CE ad rectos sese angulos decussantes, actaeque per C & D lineae quae circulum contingant in C & D contingant etiam ellipsim AEB cuius aliqua sit diameter AB; dein quaevis ducatur GH parallela CE, occurrens circulo in G & I, ellipsi vero in K & H, & CD lineae in M.

Dico lineas GI, HK esse aequales.

Demonstratio.



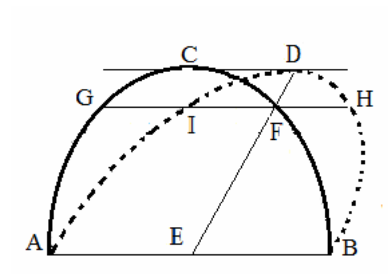
Ex contactu ponatur ad centrum EN , quae producta incidet in punctum contactus F ad hanc diametrum ut patet ex alibi hoc in libro demonstratis, erunt ordinatim positae KH, AB, unde rectangulum ELF est ad rectangulum ENF, ut quadratum LH ad quadratum

NB, sed rectangulum ELF est ad rectangulum ENF, ut rectangulum CMD ad rectangulum CND, (cum enim CE, DF & KH ex hypothesi sint parallelae, rectangulorum illorum rationes ex iisdem rationibus componuntur,) ac rectangulum CMD, est ad rectangulum CND, ut quadratum MI, ad quadratum NB, quadratum igitur MI est ad quadratum NB, ut quadratum IH ad quadratum NB; aequantur ergo quadrata MI, LH, adeoque & rectae MI, LH earumque duplae GI, KH aequales sunt. Quod erat demonstrandum.

PROPOSITIO CLXXI.

Semicirculum ABC cuius diameter AB & centrum E, contingat recta CD, aequidistans AB, & per A & B, puncta ellipsis describatur quae diametrum habeat AB, & rectam CD contingat in puncto quovis D; ex D vero ponatur DE occurrens circuli peripheriae in F, agaturque per F parallela GH, secans ellipsim in H & I, circulum vero in G & F.

Dico lineam GH in I & F, trifariam esse divisam.



Demonstratio.

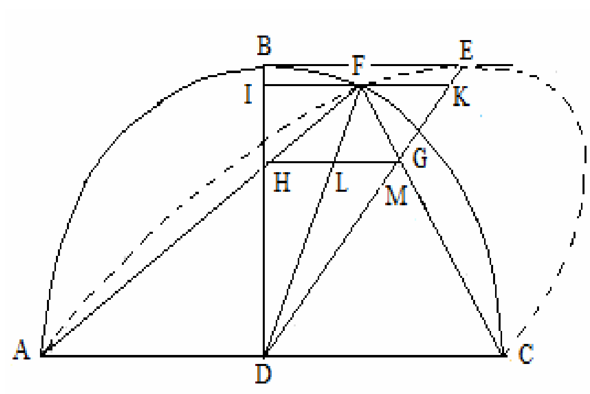
Quoniam HI per praecedentem est aequalis GF, ablata communi IF, manet FH, aequalis GI; sed ipsi FH aequatur IF (quia HI ordinatim posita est ad diametrum DE) aequantur igitur GI; IF, FH lineae. Q.e.d.

PROPOSITIO CLXXII.

Circulum ABC cuius diametri AC, BD se ad rectos decussent in D, contingat in B linea BE, ducta deinde per A & C ellipsis AEC, contingens BE in E, occurrat circulo in F, & posita ex contactu ad centrum recta ED ducantur AF, CF : & AF quidem secans BD diametrum circuli in H, CF vero ellipseos diametrum ED in G.

Dico iunctam GH aequidistare BE.

Demonstratio.



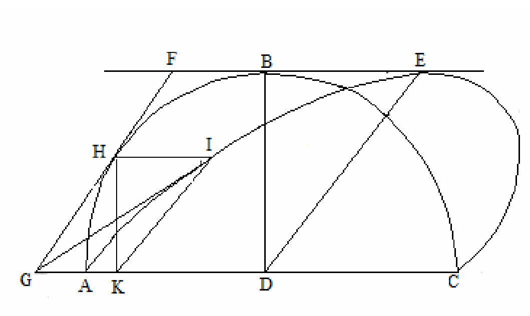
Ponatur per F, linea IK aequidistans EB, iunctisque punctis FD, ex H recta ducatur HG, parallela IK, occurrens FD lineae in L & ED in G. Quoniam IK aequidistat tangenti E B, IF, FK lineae inter se aequales sunt; quare & HG, linea aequidistans

IK in L, divisa quoque est bifariam; si iam punctum G non sit commune lineis FC, ED, HG, occurrat HG ipsi in M: Quoniam ergo HM aequidistat EB adeoque AC, ut AD ad DE, sic HL ad LM, quare MH in L divisa est bifariam: sed & HG, in L bifariam est divisa; puncta igitur G & M, unum idemque sunt: unde G commune lineis FC, ED aequidistant; igitur HG, BE. Quod erat demonstrandum.

PROPOSITIO CLXXIII.

Circulum ABC cuius diametri AC, BD sese ad rectos decussant, contingat in B recta BE, ductaque ex E recta linea ED, describatur ellipsis AEC, contingens FE, in E ponatur quoque FG contingens circulum in H, occurrens diametro AC, in G: dein ex H ponatur HI, parallela BE occurrens ellipsi in I, iunganturque IG.
Dico IG lineam contingere ellipsim in I.

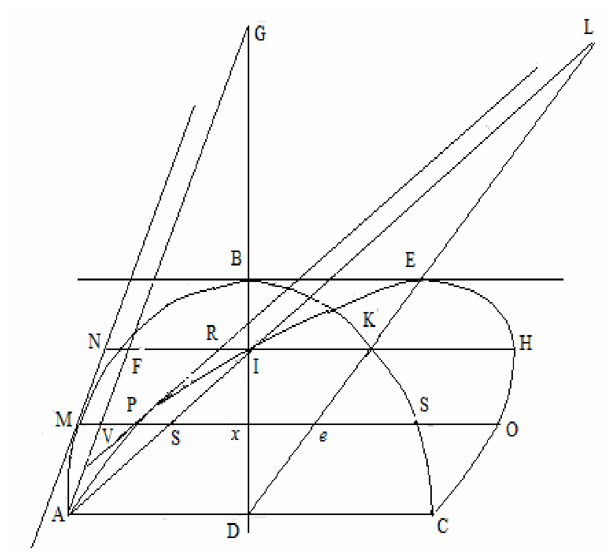
Demonstratio.



Ponatur IK aequidistans ED, erit illa ordinatim posita ad diametrum AC, cum AC, DE diameri sint coniugatae: ex K vero erigatur KH, parallela BD, occurrens HI lineae in H. Quoniam tam HK, BD, quam IK, ED aequidistant, erit ut quadratum ED ad quadratum IK, sit BD quadratum ad quadratum HK: sed ut quadratum ED ad quadratum IK, sit ADC rectangulum ad rectangulum AKC, ut igitur rectagulum ADC ad rectangulum AKC, sic BD quadratum ad quadratum HK. Quare punctum H in peripheri circuli est : igitur cum HK sit normalis & HG contingens, ut GC ad CK, sit CK ad AK: est autem IK ad diametrum AC, ordinatim posita; ergo GI linea est tangens. Quod erat demonstrandum.

PROPOSITION CLXXIV.

The diameters of the circle ABC cross each other at right angles at D, contingat in B linea BE: dein per A & C puncta ellipsis describatur contagens BE lineam in E, cuius una ex diametris sit AC iunctisque ED, ducatur ex A secans AF, & per F agatur FH, parallela BE, occurrens ellipsi in I. Tum per A & I ducatur recta AI, contingat autem circulum recta MN in M, parallela



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secand AF, & ex M ducatur MO, parallela tangenti BE, occurrens ellipsi in P; ponaturque per P, rectae BE aequidistans PR, secans FH lineam in R.

Dico PR lineam, contingere ellipsim in P.

Demonstratio.

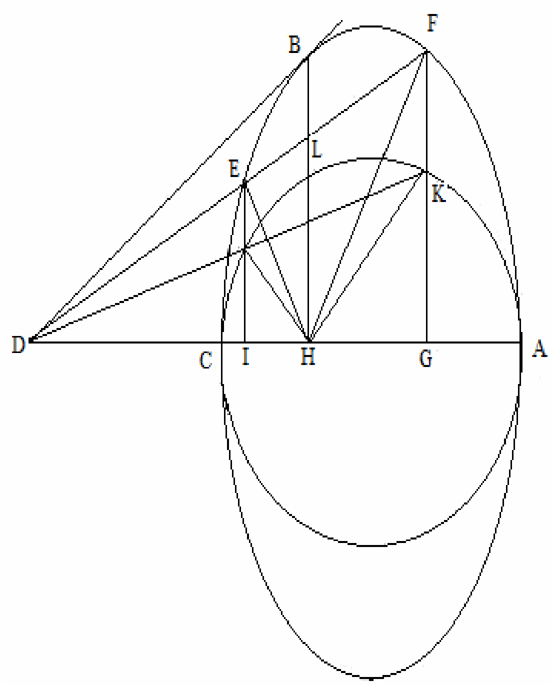
Secantes AF, AI ipsis DB, DE occurrant in G & L. Deinde FH occurrat ipsi BG in T, & tangenti in N, & rectae DE in K. Similiter MO occurrat ipsi AF in V, & AI in S, & DE in Q, GD in X. Quoniam AGD, ALD triangula, eandem habent basim AD, suntque FT , KI aequales, erunt triangula AGD, ALD inter easdem parallelas. Rursum cum MO linea aequidistet FK, erunt VX, SQ lineae aequales; est vero, & MX iterum ipsi PQ aequalis; ergo & reliqua MV, reliquae PS aequalis est: sed rectae MV aequatur linea NF, & PS, est aequalis RI, igitur & NF, RI lineae sunt inter se aequales : est vero & FT ipsi KI aequalis, igitur NT, KR aequales sunt. Sed, quia MN tangens cadit tota extra circulum, NT maior est quam FT, hoc est quam KI; ergo & KR maior est quam KI. ergo punctum R, cadit extra ellipsim; eodem modo si tam supra quam infra MO, parallelae ducantur quotcunque, ostendentur omnia puncta rectae PR, cadere extra ellipsim praeter punctum P; recta igitur PR, tanget ellipsim. Quod erat demonstandum.

PROPOSITIO CLXXV.

Esto ABC ellipsis axis AC utcunque productus in D, ductaque ; ex D linea DB, quae ellipsim contingat in B ponatur secans altera DF, occurrens ellipsi in E & F:demissis deinde EF ex E, B, F normalibus FG,BH, El ad axem AC, iungantur FH, EH.
Dico FGH, EIH triangula esse similia.

Demonstratio.

Super AC diametro describatur circulus AKC, occurrens rectis FG, BH, EI in KLM, ducantur autem rectae MH, KH, EH, FH: cum igitur sit ut FG ad EI, (hoc est GD ad ID), sic KG ad MI, patet MK productam convenire in D : est autem DL contingens, igitur HKG, HMI triangula similia sunt. Quare ut KG ad MI sic HG ad HI: sed est ut KG ad MI, sic FG ad EI, igitur ut FG ad EI, sic HG est ad HI:sunt autem anguli lateribus proportionalibus contenti recti; triangula igitur FGH, EIH sunt similia. Quod erat demonstrandum.



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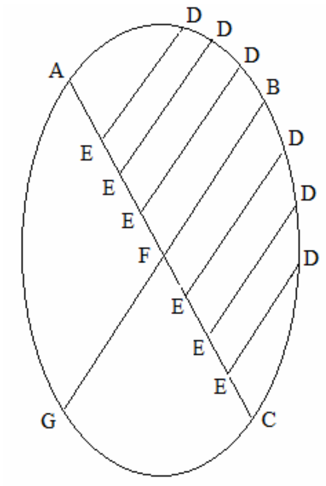
PROPOSITIO CLXXVI.



Sit in ABC ellipsi diameter AC una coniugatarum aequalium : ad quam ordinatim ponantur quotcunque DE.
Dico AEC rectangula aequari quadratis DE.

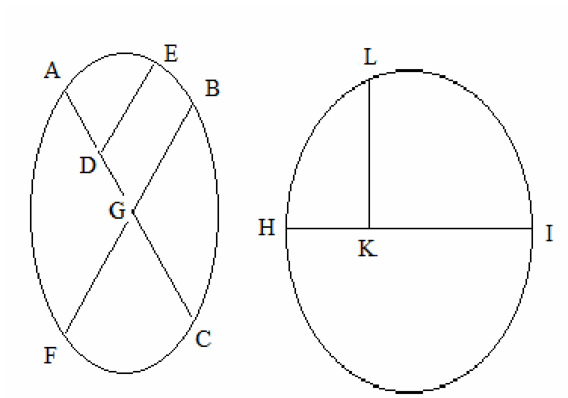
Demonstratio.

Ponatur BC altera diametrorum coniugatarum aequalium : centrum autem sectionis sit F: erit igitur AEC rectangulum ad quadratum ED, ut AFC rectangulum ad quadratum FB: sed AFC rectangulum id est quadratum AF aequatur quadrato FB, cum diametri sint aequales, rectangulum igitur AEC aequale est quadrato DE: quod erat demonstrandum.

PROPOSITIO CLXXVII.

Secet ABC ellipsim una ex diametris coniugatis aequalibus AC, quam in D secet ordinatim linea ED, sumptaque HI linea quae sit aequalis AC, descriptoque super AI circul HLI, dividatur HI in K, ut AC est divisa in D, & ex K normalis erigtur KL.
Dico ED, KL quadrata esse inter se aequalia.

Demonstratio.



Ducatur coniugatarum aequalium altera FB: sectionis autem centrum sit G, rectangulum ADC, est ad quadratum ED. ut AGC rectangulum hoc est quadratum AG est ad quadratum GB : sed AG, GB quadrata sunt aequalia. Igitur & ADC rectangulum est aequale quadrato ED: rursum cum HI, AC lineae ponantur aequales & proportionaliter in D & K divisae, erit ADC rectangulum hoc est quadratum ED, aequale rectangulo HKI, id est quadrato LK. Quod erat demonstrandum.

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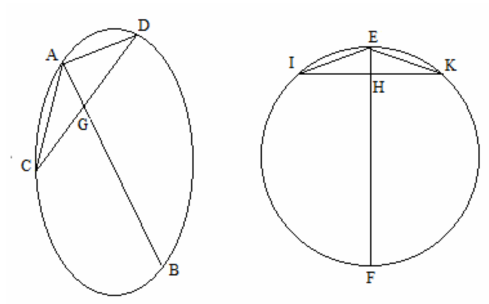
PROPOSITIO CLXXVIII.

Secet ellipsim una ex diametris coniugatis aequalibus AB, ad quam ponatur recta CD ordinatim, iunganturque AC, AD; cum super EF aequali rectae AB, & in H proportionaliter divisa ipsi AB describatur circulus EFK: actaque per H normali IK, iungantur EI, EK.

Dico quadrata AC, AD simul sumpta, aequari quadratis IE, EK simul sumptis.



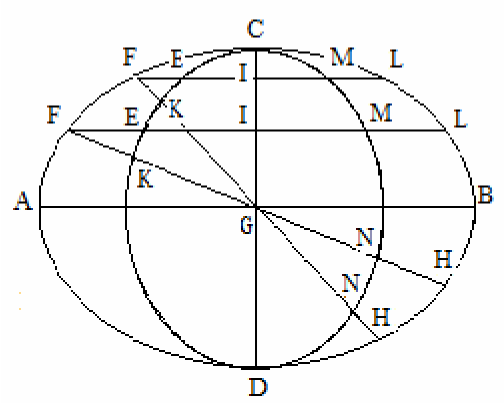
Demonstratio.

Quadrata AC, AD simul sumpta aequalia sunt quadratis CG, AG bis sumptis, & quadrata IE, EK aequalia sunt quadratis IH, HE bis sumptis; sed per praecedentem CG, IH quadrata sunt aequalia, suntque item aequalia inter se quadrata et AG, EH, quod AG, EH rectae aequales sint ex constructioneae quadrata igitur CG, AB bis sumpta aequantur quadratis GH, EH bis sumptis; igitur & quadrata duo CA, AD simul sumpta aequalia sunt quadratis IE, EK simul sumptis.Quod erat demonstrandum.

PROPOSITIO CLXXIX.

Sint ABC ellipsis axes AB, CD &. super axe minore CD, circulus describatur CED ; positis FL ordinatim ad axem CD, quae circulo occurrant in E & M, axi autem in I, & ellipsi in L; ducamur ex F per G, centrum, FH occurrentes circulo in K & N, ellipsi autem in H.

Dico esse ut quadratum FI ad quadratum FI, sic FKH rectangulum ad rectangulum FKH.



Demonstratio.

Ex scholio quartae huius libri patet has duas proportiones FE ad FE, & EL ad EL eadem esse cum ratione EI ad EI quare cum ratio rectanguli FEL ad rectangulum FEL, componatur ex rationibus FE ad FE, & EL ad EL, erit ratio rectanguli FEL ad rectangulum FEL, duplicata rationis EI ad EI, ac proinde eadem quae quadrati EI ad quadratum El: sed rectangula FEL sunt rectangula MFE, hoc sit rectangula NFK, id sit FKH. Ergo rectangula FKH sunt ad se invicem ut quadrata EI id est quadrata FI.Quod erat demonstrandum.

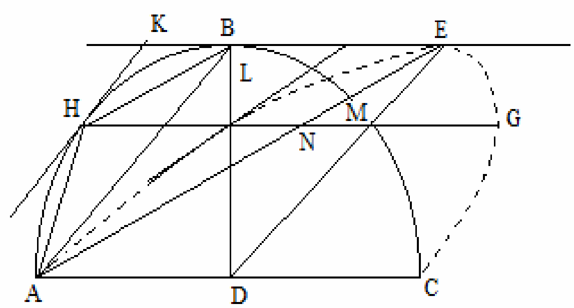
PROPOSITIO CLXXX.

Circulum ABC cuius diametri AC, BD se decussant ad rectos, contingat in B linea BE descripta dein ellipsi per A, C puncta quae contingat BE lineam in E, iungantur puncta AE, AB.

Dico AFE, AHB segmenta esse aequalia.

Demonstratio.

Ducta HK parallela ipsi AB quae circulum contingat in H ducatur ex linea HG, parallela tangenti BE occurrens ellipsi in F & rectis AB, AE, ED, BD in M,N, I,O, iunganturque ; punctae BH, AH, EF, FA, dein per F ducatur linea FL aequidistans ipsi AE: Quoniam tam HK tangens aequidistat rectae AB, quam EL ipsi AE, sit autem & FL tangens, erunt AHB, AFE triangulorum maxima quae segmentis AHB, AFE inscribi possunt; ac proinde plus quam dimidia suorum segmentorum. Rursum cum triangula ABD, AED sint super eadem basi & intra easdem parallelas constituta, & HM linea aequidistet basi AD, erunt OI, NM lineae aequales; sed & totae HI, FM sunt aequales; igitur & reliquae HO, FN inter se aequantur: Quare tam triangula HOB ,NEF, quam triangula HAO, NFA, adeoque tota triangula BHA, EFA sunt aequalia. Eodem modo si residuis segmentis triangula inscribantur, ostendemus triangula residuo circuli inscripta aequari triangulis residuo ellipseos inscripta, & utraque maiora dimidiis esse suorum segmentorum. Quare cum dicta triangulorum inscriptio, utrimque semper aequalium & maiorum dimidiis segmentorum sine termino continuari possit, segmenta AHB, AFE aequalia sunt. Quod erat demonstrandum.



Corollarium.

Iisdem positis sequitur semicirculum ABD aequalem esse semiellipsi AEC, est enim segmentum AHB ostensum aequale segmento AFE, sunt autem triangula ABD, AED super eadem basi & inter easdem parallelaes constituta inter se aequalia, igitur quadrans circuli ABC aequalis est quadranti ellipsis AED. Ergo semicirculus ABC aequalis est semiellipsi AEC. Quod erat ostendendum.

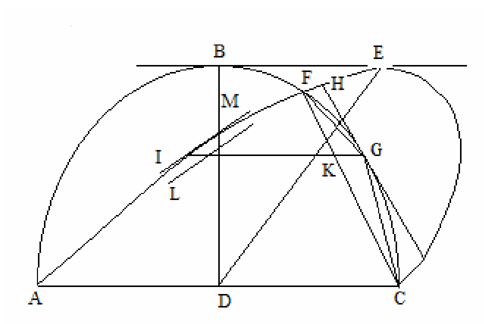
PROPOSITIO CLXXXI

Habeant ABC semicirculus & AEC semiellipsis communem diametrum AC & tangentem BE, parallelam diametro AC: secet autem AEC ellipsis circulum ABC in F, ducanturque lineae AF, CF.

Dico segmenta AIF, CGF esse aequalia.

Demonstratio.

Ducta GH linea parallela rectae CF, quae circulum in G contingat, ducatur ex G lineae et parallela rectae AC, occurrens lineis CF, AF in K & L; ellipsi vero in I: atque per I linea IM, quae AF lineae aequidistat: iungantur puncta AI, FI, CG, FG: Quoniam IM linea aequidistat secanti AF, erit IM recta tangens,ideoque AIF triangulum eorum maximum AIF segmento possunt inscribi: quod autem CGF triangulum eorum sit maximum quae CGF segmento circuli inscribuntur manifestum est: utrumque ergo triangulum plus est quam dimidium sui segmenti. Deinde, quia IL, KG sunt aequales, ut facile ex 174 huius deducetur, suntque IG, AC parallelae, triangula IAL, GCK sunt aequalia: sunt vero ob eandem causam aequalia triangula IFL,GFK. Tota igitur AIF,CGF aequalia sunt. Similiter demonstrabimus segmentis reliquis ellipticis ac circularibus inscribi posse, sive termino triangulo maiora dimidiis segmentorum & aequalia inter se; aequalia sunt segmenta AIF, CGF. Quod erat demonstrandum.



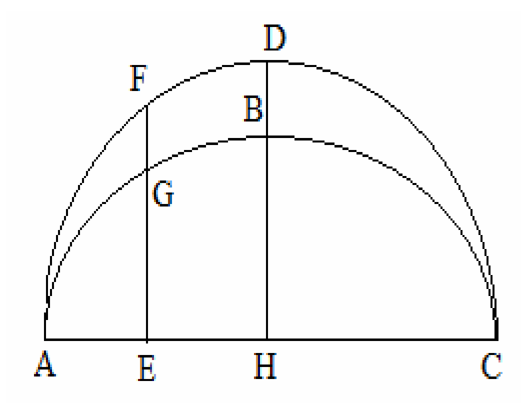
PROPOSITIO CLXXXII.

Habeant ABC ellipsis & circulus ADC eundem axem AC, ducaturque recta quaevis EF normalis ad axem AC, occurrens ellipsi in G.

Dico esse ut EG ad EF, sit ABC ellipsim ad circulum ADE.

Demonstratio.

Ex centro H normalis erigatur HBD occurrens ellipsi in B & circulo in D, ut HB ad HD, sic EG est ad EF: sed ut HB ad HD, sic ABC ellipsis est ad circulum ADE: igitur ut EG ad EF, sic ellipsis ABC est ad circulum ADE. Quod erat demonstrandum.

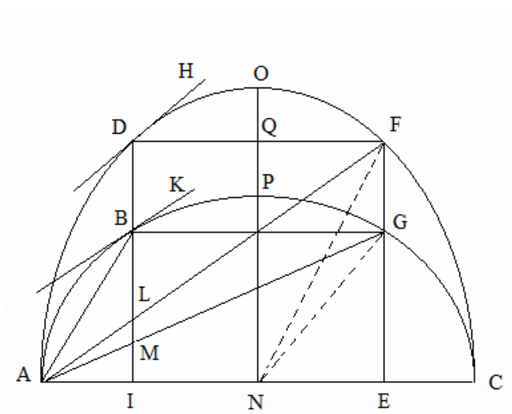


PROPOSITIO CLXXXIII.

Habeant ABC ellipsis & circulus ADC eundem axem AC ad quem ordinatim posita sit recta EF occurrens ellipsi in G & circulo in F:

Dico segmentum ADFE esse ad segmentum ABGE ut ADC circulus est ad ellipim ABC.

Demonstratio



Iungantur AF, AG, ducaturque recta DH aequidistans ipsi AF, contingens circulum in puncto D: ex quo ordinatim demittatur ad axem recta DI, occurrens ellipsi in B & AF,AG lineis in L & M, agaturque per B recta BK parallela ad AG, quae per 154, ellipsim tanget. Dein, lineae ducantur AD, DF, AB, BG, ut EF ad EG, sic IL est ad IM, sed est ut EF ad EG, sic ID ad IB. Igitur ut ID ad IB, sic IL ad IM; ergo & reliqua DL ad reliquam BM, ut tota ID ad totam IB; est autem ut DL ad BM, sic triangulum DAL, ad triangulum BAM, & (quia parallelae sunt DI, FE,) triangulum DFL ad triangulum BGM, igitur triangulum DAL est ad triangulum BAM, ut ID ad IB, & triangulum DFL est ad triangulum BGL ut ID ad IB. Ergo totum triangulum ADF est ad totum triangulum ABG, ut ID ad IB. Quia autem DH, BK sunt tangentes, triangula ADF , ABG maxima sunt eorum quae inscribi segmentis possunt, ac proinde maiora se sigmentorum dimidiis. Similiter demonstrabimus segmentis residuis tam circuli quam ellipseos inscribi posse triangula, quae sint maiora residuorum dimidiis, & rationem habeant, quam ID ad IB. Quare cum hoc fieri possit sine termino, segmentum ADF est ad segmentum ABG, ut ID ad IB: est vero & triangulum AFE ad triangulum AGE, ut EF ad EG, hoc est ut ID ad IB. Ergo totum segmentum ADFE est ad totum segmentum ABGE, ut ID ad IB, hoc est ut circulus ad ellipsim. Quod erat demonstrandum.

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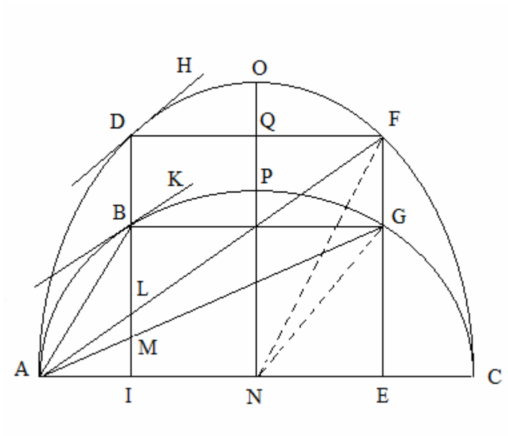
PROPOSITIO CLXXXIV.

Eadem manente figura ducatur ex centro N quaevis diameter NF, demissaque ex F normali FE quae ellipsim secet in G, iungantur NG:
Dico ANF sectorem esse ad sectorem ANG, ut circulus ADE ad ellipsim ABC.

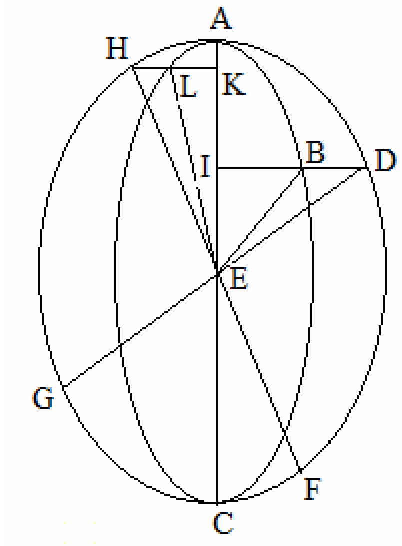


Demonstratio.

Segmentum ADFE est ad segmentum ABGE, ut FE ad GE : & triangulum FNE est ad triangulum GNE, ut FE ad GE. Ergo & reliquum nempe sector ANF est ad reliquum nempe sectorem ANG, ut FE ad GE, hoc est ut circulus ad ellipsim. Quod erat demonstrandum.

PROPOSITIO CLXXXV.

Sit ellipsis ABC, & circulus ADC, habentes communem axem AC: & per centrum commune E ducantur duae diametri circuli sese ad angulos rectos intersecantes, occurrentesque et circulo in punctis D, F, G, H: deinde ex D & H, ordinatim ad AC applicentur lineae DI, HK occurrentes ellipsi in B & L:
iunganturque: EB, EL.
Dico LEB quadrantem ellipseos esse, sicut HED est quadrans circuli.



Demonstratio.

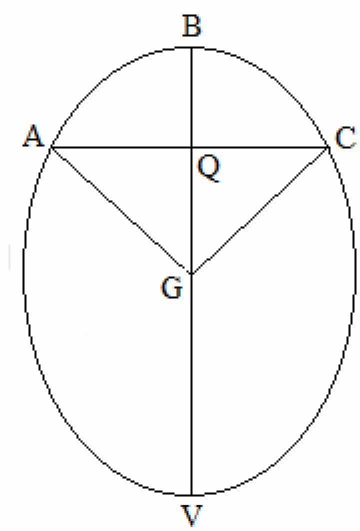
Segmentum ABI ad segmentum ADI est ut IB ad ID, utque eadem IB ad ID, ita est triangulum IEB ad triangulum IED. Ergo ut IB ad ID, ita est sector AEB ad sectorem AED. Simila modo ostendemus sectorem LEA ad sectorem HEA, esse ut KL ad KH, id est, ut IB ad ID: ergo ut IB ad ID, ita est totus sector BEL ad totum sectorem DEH, sed ut IB ad ID, hoc est minor axis ellipseos ad diametrum circuli, per 5.Archimedis de Sphaer. ita est ellipsis ABC ad circulum ADC; ergo sector BEL ad sectorem DEH , ut ellipis ad circulum , & invertendo ac permutando ut sector DEH ad circulum ADC, ita est sector BEL ad ellipsin ABC; sed sector DEH est quadrans circuli, ergo & sector BEL quadrans ellipseos erit, adeoque erunt BE, EL diametri coniugatae, ergo, &c. Quod erat demonstrandum.

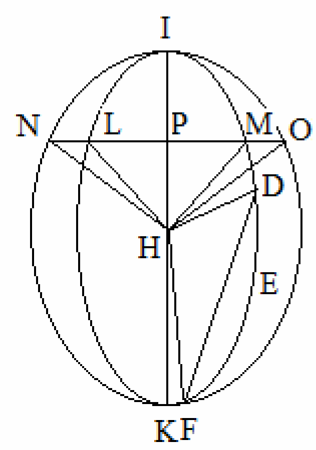
Nota idem demonstrari si ADC sit ellipsis cuius axis AC, sintque HF, DC ipsius coniugatae diametri; si vero DEH non sit quadrans ellipseos ut sector DEH ad circulum, ita erit sector BEL ad ellipsin, ut ex demonstratione constat.

PROPOSITIO CLXXXVI.

Esto circuli ABC quicunque sector AGC, & ellipsis DEF sector DHF: sit autem sector ad sectorem ut circulus ad ellipsim : ducanturque rectae AC, DF.
Dico segmentum ABC esse ad segmentum DEF, ut circulus ABC est ad ellipsim DEF, & contra.

Demonstratio.





Invento ellipseos maiore axe IK, describatur super IK diametro circulus, ductaque; ad axem ordinatim LM quae auferat segmentum LIM aequale segmento DEF, occurrat circulo in N & O, ducanturque rectae LH, MH, NH, OH. Quoniam segmentum MIL aequale est segmento DEF, erit sector LHM aequalis sectori DHF. Rursum cum sit ut ellipsis ad circulum NOK, sic LHI sector ad sectorem NHI, adeoque & LHM sector ad sectorem NHO, sitque LHM triangulum ad triangulum NHO, ut LM ad NO, hoc est ut ellipsis ad circulum NOK, erit ut ellipsis ad circulum NOK, sic LIM segmentum ad segmentum NIO, & permutando circulus NOK ad segmentum NIO, ut ellipsis ad segmentum LIM. Iam vero circulus ACV est ad sectorem AGC, ex hypothesi ut ellipsis ad sectorem HDF, hoc est(ut ostenda supra) ut ellipsis ad sectorem LHM, hoc est ut circulus NOK ad sectorem NHO. Ergo etiam circulus ACV ad segmentum ABC, ut circulus NOK ad segmentum NIO, hoc est, ut anti ostendi, ut ellipsis ad segmentum LIM, hoc est (quoniam segmenta LIM, DEF sunt ex constructione aequalia) ut ellipsis ad segmentum DEF. Igitur permutando ut circulus ACV ad ellipsim, sic segmentum ABC ad scgmentum DEF. Quod erat demonstrandum.
Iam vero si fuerit segmentum ABC ad segmentum DEF, ut est circulus ABC ad ellipsim lDF: dico & sectorem AGC esse ad sectorem DHF, ut est circulum ABC ad ellipsim IDF: invento enim ut ante ellipseos DEF maiore axe IK, super IK ut diametro describatur circulus NOK : ductaque ordinatim LM quae segmentum LIM auferat aequale segmento DEF, fiant reliqua ut prius. Segmentum igitur NIP est ad segmentam LIP, ut circulus NOK ad elipsim IDF IDF. Itaque segmentum NIO ad segmentum LIM, ut circulus NOK ad elipsim IDF: & permutando, circulus NOK ad segmentum NIO, ut ellipsis ad

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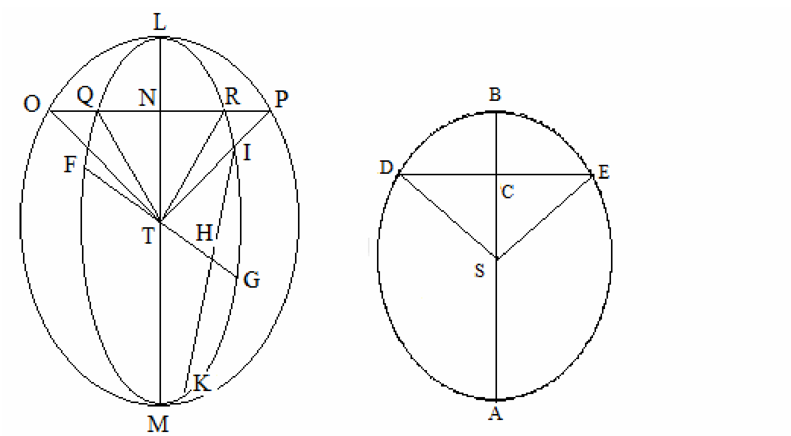
segmentum LIM, hoc est: ex hypothesi ut circulus ACV ad segmentum ABC. Cum ergo sit circulus NOK ad segmentum NIO, ita circulus ACV ad segmentum ABC, erit iam ut circulus NIK ad sectorem NHO, ita circulus ACV ad sectorem AGC. Atqui ut circulus NOK ad sectorem NHO, sic ellipsis ad sectorem LHM, hoc est: quoniam segmenta LIM, DEF sunt aequalia ad sectorem DHF. Ergo ut circulus ACV ad sectorem ABC, ita ellipsis ad sectorem DHF. Quod erat demonstrandum.

PROPOSITIO CLXXXVII.

Esto ADB circuli diameter AB divisa utcunque in C, & per C normalis posita DE, sit autem & FG, diameter quaecunque ellipseos divisa in H , ut AB est in C, & per H ordinatim ducta IK.

Dico segmentum DBE esse ad segmentum IGK ut est circulus ADB ad ellipsim FLG, & contra.

Demonstratio.



Invento ellipseos axe maiore LM, describatur super LM circulus LOM, divisaque LM in N, ut AB est divisa in C, agatur per N normalis OP, secans ellipsim in Q & R: ducanturque semidiametri OT, QT, RT, PT. Rectangulum BCA ad quadratum DE, ut rectangulum LNM ad quadratum ON, & permutando rectangulum BCA est ad rectangulum LNM, ut quadratum DC ad quadratum ON, sed ratio rectanguli BCA ad rectangulum LNM, componitur ex rationibus BC ad LN, & CA ad NM. Ergo ratione BC ad LN, & CA ad NM simul sumptae aequantur rationi quadratorum DC, ON, hoc est rationi ad DE,ON bis sumptae. Atqui rationes BC ad LN, & CA ad NM, sunt eaedem sive aequales, cum sint BA, LM ex hypothesi proportionaliter divisae; ergo earum una BC ad LN, eadem est rationi DE ad ON : sed, cum sit ut BC ad LN, sic CA ad NM, erit quoque ut BC ad LN, sic BA ad LM. Quare BA ad LM, id est SD ad OT, ut DC est ad ON: sunt autem anguli DCS, ONT recti; igitur triangula DCS, ONT sunt similia, angulique DSC, OTN aequaeles: quare & anguli DSE, OTP illorum duplis sunt aequales & DSE, OTP sectores sunt similes, adeoque & segmenta DBE,OLP sunt similia ; igitur ut segmentum OLP ad circulum LOM, sit segmentum DBE ad circulum DBA: sed etiam ut segmentum OLP ad circulum LOM, sic segmentum OLR ad ellipsim. Quare ut segmentum DBE ad circulum ADB, sic QLR segmentum sit ad ellipsim FLG, hoc est: ex constructione segmentum IGK ad ellipsim FLG, & permutando est segmentum DBE ad segmentum IGK, ut circulus ADB ad ellipsim FLG. Quod erat primum.

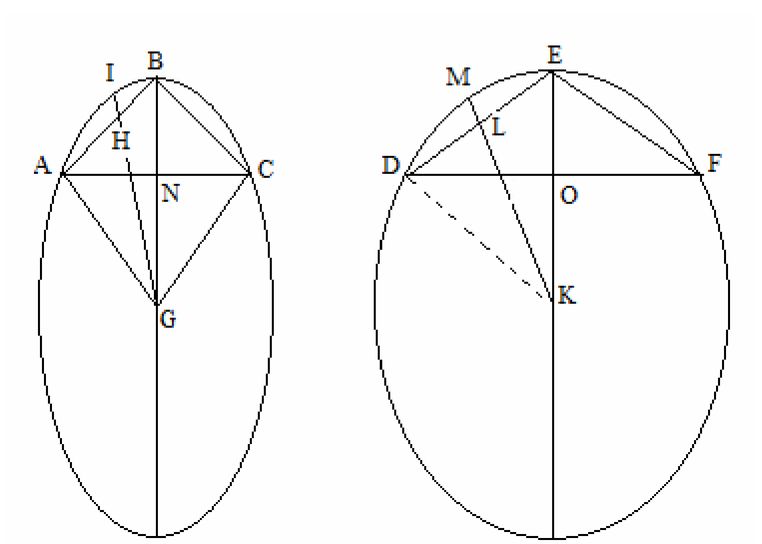
Sit iam segmentum DBE ad segmentum IGK, ut circulus ADB ad ellipsim FLR: dico BS,GT lineas in C & H, proportionalitcr esse divisaes, ponantur eadem omnia quae prius: Quoniam est segmentum DBE ad segmentum IGK, id est QLR ut circulus ADB ad ellipsim FLR, & permutando DBE segmentum ad circulom ADB, ut segmentum QLR ad ellipsim FLR, sit autem & OLP segmentum ad circulum OLM, ut QLR segmentum ad ellipsim FLR, erit ut segmentum OLP ad circulum LOM, sic DBE segmentum ad circulum ADB. Quare & sectores DSE, OTP sunt similes & anguli DSE, OTP adeoque & illorum dimidii DSC, OTN aequales: sunt autem & anguli DCS, ONT recti; igitur triangula DCS, ONT similia & ut DS ad OT, id est BS ad LT, sic SC ad NT. Quare BS, LT in C & N proportionaliter sint divisae; sed tum per constructionem segmenta QLR, IGK sint aequalia, erit ut LT in N, sic GT divisa in H. igitur ut BC ad CS, sic GH ad HT. Quod erat demonstrandum.

PROPOSITIO CLXXXVIII.

Esto ABC ellipseos segmentum quodcunque ABC; sumatur autem in circulo DEF segmentum DEF, quod ita se habeat ad suum circulum, ut ABC segmentum ad ellipsim suam : dein ABC, DEF segmentis triangula inscribantur maxima ABC, DEF, divisisque AB, DE lineis bifariam in H & L, agantur per H & L, diametri GI, KM.

Dico illas in H & L proportionaliter esse sectas.

Demonstratio.



Ex B & E ducantur diametri BG, EK, iunganturque DK, KF, GA, GC, & quoniam e verticibus maximorum triangulorum ductae sunt diametri, in circulo quidem patet DF bisecati in C, in ellipsi autem bisecari quoque; AC colliges ex 42 huius. Quare tam in ellipsi quam in circulo sectores AGC, DKF bisecantur. Ergo sector AGB est ad sectorem DKE, ut sector AGC ad sectorem DKF. Iam vero tum permutando hypothesim, segmentum ABC sit ad segmentum DEF, ut ellipsis ad circulum, etiam sector AGC erit ad sectorem DKF, hoc est (ut iam ostendi) sector AGB, ad sectorem DKE, ut ellipsis ad circulum. Et quoniam est sector AGB ad sectorem DKE, ut ellipsis ad circulum, erit quoque segmentum AIB ad segmentum AME, ut ellipsis ad circulum. Ergo diametri IG, MK proportionaliter in H & L sunt divisae. Quod erat demonstrandum.

PROPOSITIO CLXXXIX.

Eadem manente figurae: si fuerit segmentum ABC ad segmentum DEF, ut ABC ellipsis ad circulum DEF.

Dico esse & triangulum maximum ABC ad triangulum maximum DEF ut ABC ellipsis est ad circulum DEF.

Demonstratio.

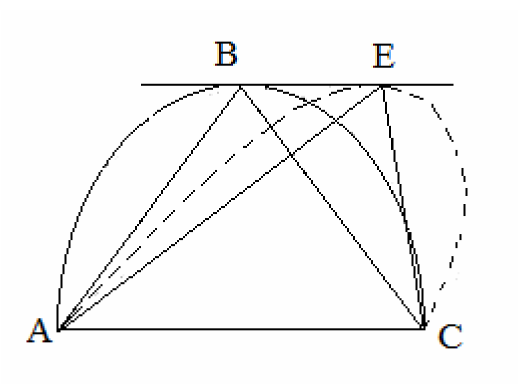
Vt ABC ellipsis est ad circulum DEF, sic ostendimus in priori segmenta AIB, BC esse ad segmenta DME, EF. Quare cum etiam ex hypothesi sit, ut ellipsis ad circulum sic totum segmentum ABC ad totum segmentum DEF, igitur & reliquum triangulum ABN est ad reliquum triangulum DEO ut ABC, ellipsis ad circulum DEO. Quod erat demonstrandum.

PROPOSITIO CXC.

Esto ABC semicirculo inscriptum triangulum maximum ABC; sit autem & AEC semiellipsi quae communem AC habeat diametrum triangulum inscriptum maximum AEC: si fuerint ABC, AEC triangulia:

Dico & semicirculum ABC aequalem semiellipsi AEC.

Demonstratio.

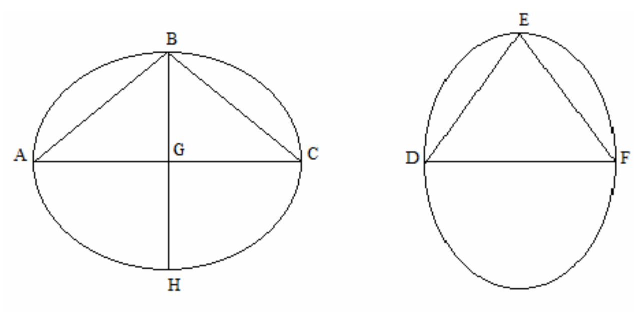


Iungantur puncta B & E. Quoniam triangula ABC, AEC super eadem basi descripta per hypothesim sunt aequalia, erit iuncta BE parallela rectae AC; adeoque cum tam ABC, quam AEC, sit triangulum maximum, continget recta BE & circulum & ellipsim : igitur semicirculus ABC aequatur semiellipsi AEC. Quod erat demonstrandum.

PROPOSITIO CXCI.

Ellipsis est ad circulum vel ellipsim ut triangulum maximum inscriptum semiellipsi ad triangulum maximum inscriptum semicirculo aut semiellipsi.

Demonstratio.



Cum enim sit ut semiellipsis ad semicirculum aut semiellipsim, ita tota ellipsis ad totum circulum aut ellipsim; erit ut triangulum maximum AEC semiellipsi inscriptum ad triangulum maximum DEF semicirculo aut semiellipsi inscriptum, ita ellipsis ad circulum ad ellipsim. Quod erat demonstrandum.

PROPOSITIO CXCII.

Dato circulo vel ellipsi DEF ellipsim aequalem exhibere. Et datae ellipsi circulum aequalem.

Constructio & demonstratio.

Semicirculo vel semiellipsi DEF inscribatur triangulum maximum DEF, cui fiat aequale aliud quodcumque triangulum ABC, divisaque AC bifariam in G, ducatur BG, & protrahatur BG in H, ut BG, GH sint aequales, & si AC, BH sese ad rectos intersecent, describatur ellipsis ABCH cuius axes sint ACBH, si autem non ad rectos sese intersecent, datis ACBH coniugatis diametris axes inveniantur circa quos describatur ellipsis ABCH Dico ellipsin ABCH circulo vel ellipsi DEF aequari: est enim triangulum ABC maximum eorum quae semiellipsi interscribi possunt quia ACBH ponuntur diametri coniugatae. Quare cum triangulum maximum semiellipsis inscriptum aequale sit triangulo maximo circulo vel ellipsi inscripto, erit ellipsis ABC circulo vel ellipsi DEF aequalis, ergo quod petebatur.

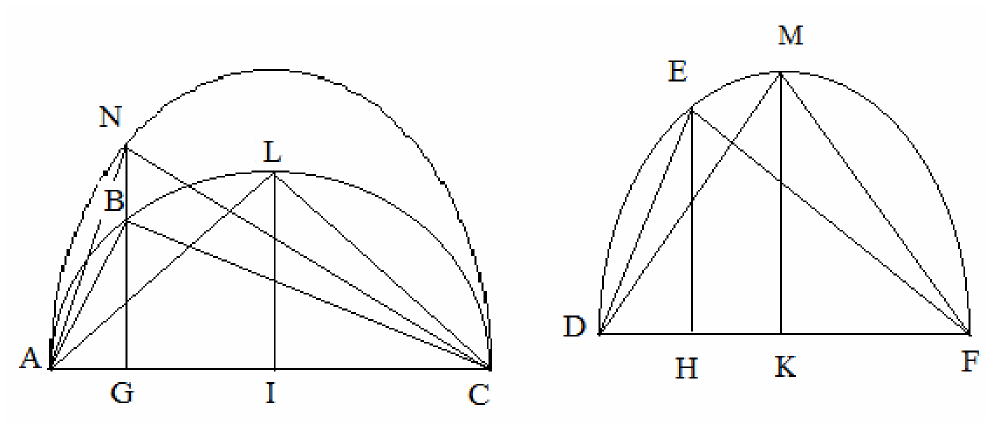
Ex his secundae partis constructio & demonstratio est manifesta.

Corollarium.

Hinc patet infinitas dari ellipses circulo vel ellipsi ABC aequales, quia triangulo ABC dantur infinita triangula aequalia.

PROPOSITIO CXCIII.

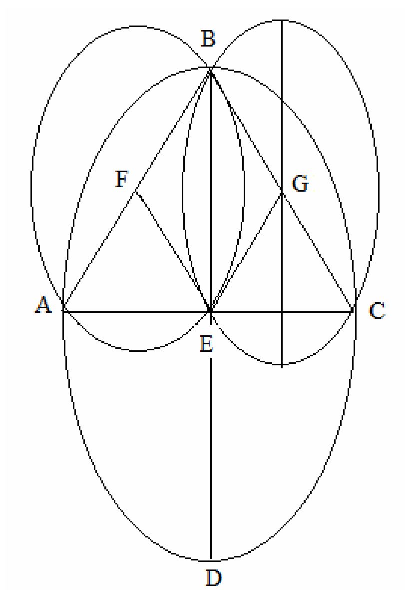
Sit semiellipsis ABC cuius axis sit AC, semicirculus autem DEF, semiellipsi ABC aequalis ponatur; inscribantur deinde semiellipsi & semicirculo triangula ABC, DEF inter se aequaliae & ex B & E, normales ad basim demittantur BG: EH.
Dico lineas AC, ADF in G & H, similiter divisas esse.



Demonstratio.

Ex centris I & K, ad diametros AC, DF normales erigantur IL, KM iunganturque; AL, LC, & DM, MF:quoniam semiellipsis ABC semicirculo DMF, aqualis est, erunt & maxima triangula illis inscripta nempe ALC, DMF inter se aequalia ; erit igitur ut triangulum ALC, ad triangulum ABC, id est ut LI ad BG, ita triangulum DMF ad triangulum DEF, id est ita linea MK ad EH, adeoquae ut quadratum LI ad BG quadratum, ita erit quadratum MK ad ipsum EH: sed ut quadratum LI ad quadratum BG, ita est rectangulum AIC ad rectangulum AGC: & ut quadratum MK ad EH, quadratum, ita est DKF rectangulum ad rectangulum DHF, ergo ut AI quadratum ad rectangulum AGC, ita est: quadratum DK ad rectangulum DHF: & permutando ut quadratum AI ad ipsum DK quadratum, sive ut quadratum AC ad DF ita est rectangulum AGC ad rectangulum DHF, constat igitur ex Sereni I.1. prop. 12, lineas AC, DF in G & H, proportionaliter esse divisas. Quod erat demonstrandum.

PROPOSITIO CXCIV.



Secent ABC ellipsim diametri quaevis coniugatae AC, BD iunctisque; punctis AB, CB, dividantur AB, CB lineae bifariam in F & G: ducanturque; ex E centro lineae EF, EG: dein tam puncta AEB, quam CEB ellipses describantur, quarum coniugatae sint diametri AB, EF,CB,EG :

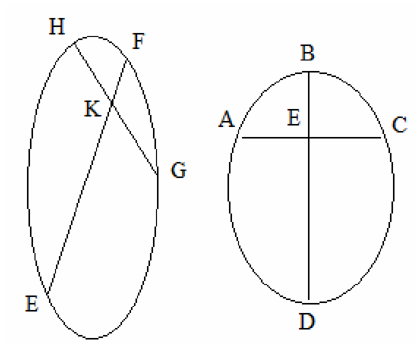
Dico ABC ellipsim aequalem esse duabus ellipsibus AEB, CEB.

Demonstratio.

Quoniam tam ABEF, CBEG, ACBE diametri sunt coniugatae, erunt ABC, AEB, CEB triangula maxima quae suis semiellipsibus inscribi possunt : est autem ABC triangulum duplum trianguli CEB, igitur & ellipsis ABC dupla est: ellipsis CEB. similiter ostendam ellipsim ABC duplam esse ellipsis BEA aequantur ; igitur ellipses BEA, CEB ac proinde ellipsis ABC singularum dupla aequatur utrique simul sumptae. Quod erat demonstrandum.

PROPOSITIO CXCV.

Circulum ABC secet recta quaevis AC auferens segmentum ABC:oportet in data ellipsi EFG, ad datam diametrum EF ordinatim ducere HG, quae segmentum auferat HFG, quod ad ellipsim eam habeat rationem quam ABC segmentum ad circulum ABC.



Constructio & demonstratio.

Divisa in circulo ABC recta AC bifariam in E, agatur per I normaliter diameter
BD: dein in ellipsi EFG,dividatur EF diameter in K, ut divisa est BD in I, agaturque
per K ordinatim linea HG ad FE. diametrum, patet HFG segmentum esse ad ellipsim EFG, ut ABC segmentum est ad circulum ABC. Dato igitur in circulo segmento, &c. Quod erat faciendum.

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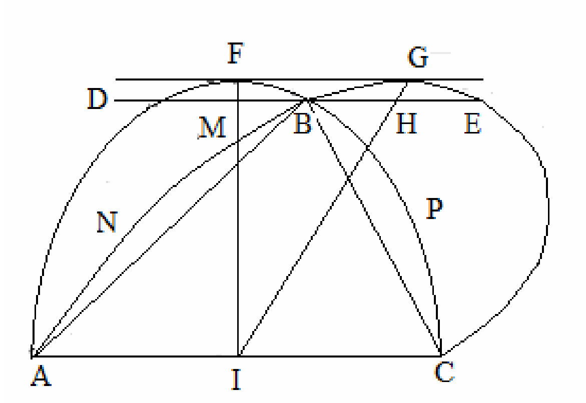
PROPOSITIO CXCVI.

Esto ABC semicirculo inscriptum triangulum quodcunque ABC, oporteat super BC linea segmentum describere ellipticum, aequalae segmento circulari ADB.

Constructio & demonstratio.

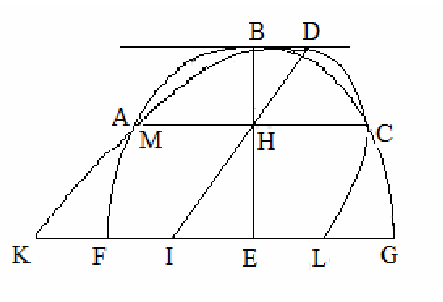


Ducatur recta FG parallela diametro AC contingens circulum in F, dein per B recta agatur DE parallela diametro AC, fiatque BE aequalis ipsi DB, qua divisa in H bifariam, ducatur per H ex I, centro circuli recta IG occurrens FG, lineae in G: tum per A, G, C puncta ellipsis describatur: cuius diametri coniugatae sint AC, IG, & quoniam FG est ipsi AC per extremitatem diametri IG parallela continget ellipsim in G. Quare ellipsis circulo aequalis est. Deinde quoniam FG, MH sunt parallelae, facile ostendemus ex elementis rectangulum sub GH, & reliqua parte diametri, esse ad rectangulum sub GI, & reliqua parte diametri ut rectangulum sub FM, & reliqua parte diametri ut rectangulum sub FI & reliqua parte diametri, & reliqua parte diametri est ad rectangulum sub FI & reliqua parte diametri ut quadratum DM ad quadratum AI, hoc est: ut quadratum BH ad quadratum AI, ergo hoc est ut rectangulum sub GH & reliqua parte diametri est ad rectangulum sub GI, & reliqua parte diametri, ut quadratum BH ad quadratum AI. Ellipsis ergo transit per punctum B. Iam quia ellipsis circulo aequalis est: ut ostendi supra, ablato communi segmento curvilineo ANBPC, aequalia remanent segmenta BPCE, ANBD, quibus si addas segmenta BPC, ANB aequae aequalia sunt, segmentum ellipticum BCE basim habenti rectam BC aequabitur segmento circuli ADB, basim habenti rectam AB. Factum igitur est quod petebatur.

PROPOSITIO CXCVII.

Esto ABC segmentum quodcunque circulare, oportet super AC subtensa segmentum constituere ellipticum, dato ABC segmento aequale, cuius una e diametris coniugatis sit data quae sit maior diametro FG circuli ABC.

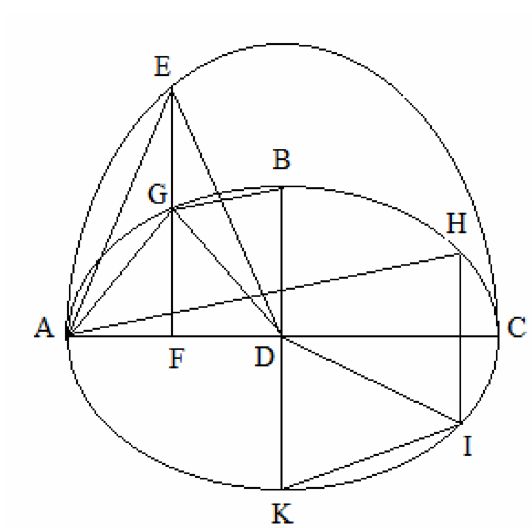
Constructio & demonstratio.



Invento E centro circuli ABC ducatur per E diameter FG aequidistans rectae AC erectaque ex E normali EB , quae AC lineam bifariam dividit in H , agatur per B tangens BD, dein per H recta ducatur ID, aequalis datae occurrens tangenti in D & FG diametro in I: fiatqae IK aequalis rectae FE, & IL aequalis EG, tum per KDL puncta ellipsis describatur, occurrens AC lineae: utcunque in M. Dico factum esse quod petitur. Quoniam enim LK, DI diametri sint coniugatae & BD parallela ipsi LK, patet ED contingere ellipsim KDL, adeoque rectam MH aequalem ipsi AH: ellipsis igitur transibit per punctum A. Eodem modo ostenditur transire per C: igitur segmentum ellipticum ADC aequale est segmento circulari ABC; super data igitur linea AC, &c;. Quod erat faciendum.

PROPOSITIO CXCVIII.

Ellipsim a dato in peripheria puncto, in datos numero sectores aequales dividere.



Constructio & demonstratio.

Data sit ellipsis ABC axis maior AC, datum in peripheria punctum B, oporteat ab hoc ellipsim secare in sectores tot aequales quot volueris, v.g. in sex.
Diametro AC describe se semicirculum, inscribe AE, latus poligoni tot habentis latera quot aequales petebantur sectores. Quoniam autem petebantur sectores aequales sex, erit AE latus hexagoni. Ponatur ergo EFG normalis ad AC, iunganturque
ED, GD, BD, GB: & GB aequidistet AH: dico factum esse quod petitur: ut circulus

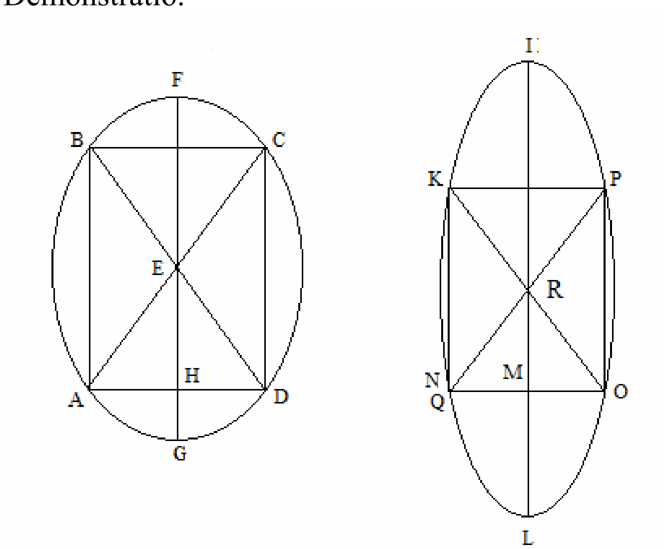
ad ellipsin ita est: segmentum EFA ad segmentum GFA ; id est ut FG ad FE estque item ut FE ad FG, ita triangulum EDF ad triangulum GDF:ergo ut circulus ad ellipsin, ita est sector EDA, ad sectorem GDA: sed sector EDA est pars tertia semicirculi, cum linea AE sit latus hexagoni; ergo & sector GDA tertia pars semiellipseos erit. Cum autem GB ipsi AH aequidistet, erunt segmenta AG, BH: adeoque & sectores GDA, BDH aquales inter se : hoc est sexta pars ellipseos totius; fiant modo segmento BH aequalia segmenta HI, IK iunganturque DH, DI, DK sectores igitur DBH, DHI, DIK aequantur adeoque singuli sunt sexta pars ellipseos. & simul sumpti semiellipsim constituunt BCK, reliqum igitur semiellipsim BAK seca ut divisa est BCK, eritque tota ellipsis in sex aequales sectores divisa. Quod facere oportebat.

PROPOSITIO CXCIX.

Esto circulo ABC cuius centrum E inscriptum polygonum quodvis regulare ABCD ductaque diameter FG secet latus quodvis AD bifariam in H: sit autem & IKL ellipseos diameter quaecunque IL divisa in M sicut FG est divisa in H, agaturque per M ordinatim linea NO ad diametrum IL.

Dico rectam NO, esse unum ex lateribus polygoni regularis inscribendi ellipsi tot laterum, quot est polygonum circulo ABC inscriptum. Polygonum autem ellipticum regulare voco, cuius singula latera abscindunt elliptica segmenta aequalia.

Demonstratio.



Ducatur ex O linea OP auferens segmentum aequale segmento NLI, & ex P rectae PK quae segmentum auferat aequale segmento NLO, dein ex K linea KQ auferens segmentum aequale segmento NLO, erit Q punctum idem cum puncto N. Iunctis enim in circulo ABC punctis EA, EB, FC, ED ducantur in ellipsi semidiametri, RK, RN, RO, RP. Quoniam diametri FG, IL sunt in H & M, proportionaliter divisae, & AD, NO lineae per H & M, actae ordinatim ad diametros FG, IL; erit ut sector AED ad circulum ABC, sic NRO sector ad ellipsim IKL: sunt autem tam AFD, DEC, CEB, BEA sectores, quam NRO, ORP, PRK, KRQ sectores inter se aequales, (quia NO, OP, PK, KQ segmenta sunt aequalia) igitur ut sectores quatuor circulares ad suum circulum sic elliptici sectores quatuor ad suam ellipsim: sed toti circulo aequantur sectores circulares, igitur & ellipsi sunt aequales sectores elliptici.Quare punctum Q idem est cum puncto N & KNOP polygonum est regulare tot laterum quot est polygonum circulo inscriptum. Quod erat demonstrandum.

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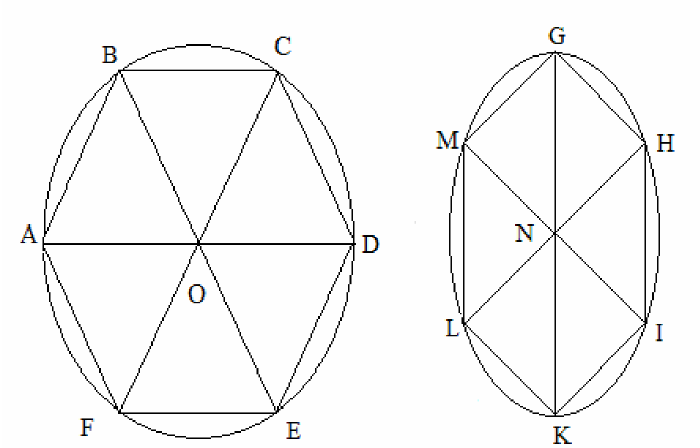
PROPOSITIO CC.

Esto circulo ABC inscriptum polygonum quodcunque regulare A, B, C, D, E, F: sit autem & ellipsi GHI inscriptum polygonum regulare G, H, I, K, L, M, totidem laterum, quot est polygonum circulo inscriptum.

Dico segmentum circulare ab aliquo laterum polygoni ablatum, esse ad segmentum ellipticum, ab aliquo laterum polygoni elliptici ablatum, ut est circulus ABC ad ellipsim GHI.

Demonstratio.



Sit O centrum circuli & N centrum ellipsis: ducanturque ex O & N, semidiametri ad angulos sui polygoni: Quoniam GH, HT, IK, &c. segmenta in ellipsi per constructionem sunt aequalia, erunt & sectores GNH, HNI, INK, &c. aequales: sunt autem & sectores circuli AOB, BOC, COD, &c. aequales & pares numero ellipticis, igitur est sector AOB ad sectorem GNH, ut omnes sectores circulares, id est circulus ABC ad omnes sectores ellipticos, id est ellipsim GHI, quare & segmentum AB est ad segmentum GH, ut circulus ad ellipsim. Quod erat demonstrandum.

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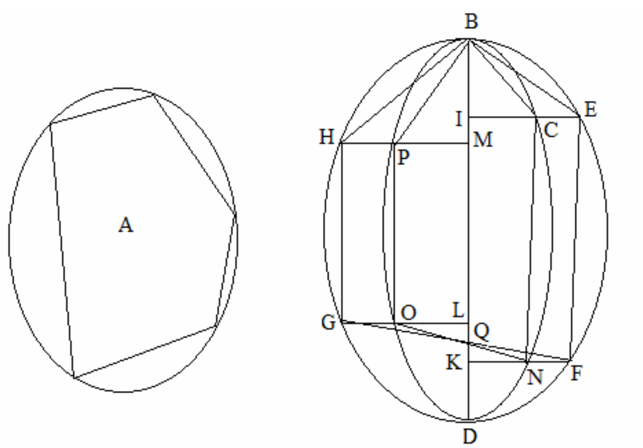
PROPOSITIO CCI.

Circulo A inscriptum sit quodcunque polygonum, oportet datae ellipsi BCD polygonum pari numero laterum proportionate inscribere, hoc est quod & eandem ad ellipsim proportionem habeat quam circulare ad circulum; & cuius singula latera, segmenta auferant quae ad ellipsim suam talem habeant rationem quam habent segmenta circulariam, singulis lateribus polygoni circularis ablata ad suum circulum.



Constructio & demonstratio.

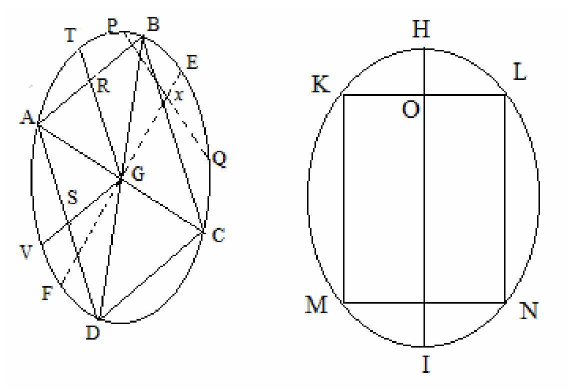
Invento ellipsis maiore axi BD, describatur diametro BD circulus BEF, cui polygonum inscribatur BEFGH, simile illi, quod A circulo inscriptum est polygono: secetque FG latus, axem BD in Q; dein ductis ex E, F,G, H ad axem BD normalibus EI, FK, GL, HM, quae ellipsim secent in C, N, O, P. Ducatur rectae BC, CN, NQ, OQ, OP, PB. Dico factum esse quod petitur. Quoniam GL, FK sunt parallelae, erunt triangula GLQ, KFQ similia; adeoque ut GL ad KF, hoc est ut OL ad KN, sic LQ ad QK sunt vero & anguli OLQ, FKQ, recti; ergo triangula OLQ, KFQ sunt similia. Ergo OQ, QF sunt in directum; figura igitur BC, NO, PB est polygonum ellipsi inscriptum. Ut IE linea est ad lineam IC, sic IBE triangulum est ad triangulum IBC, sed est ut IE ad IC, sic IEFQ trapezium est ad trapezium ICNQ; igitur tota figura BEFQ est ad figura BCNQ, ut IE linea ad lineam IC, id est segmentum IBE ad segmentum IBC, id est ut circulus BEP ad ellipsim BCD. Eodem modo ostenditur figura BHGQ esse ad figuram BPOQ, ut circulus BEC ad ellipsimm BCD. Quare erit totum polygonum BEFGH ad polygonum HCNOP, ut circulus BEF ad elliptim BCD, & permutando, ut BEFGH polygonum ad circulum BEF, hoc est ex const. ut polygonum circulo A inscriptum ad suum circulum, sit BCNOP polygonum ad ellipsim BCD: Quod erat primum.

Rursum cum tam segmentum IBE sit ad segmentum IBC, quam IBE triangulum ad triangulum IBC, ut est circulus BED ad ellipsum BCD, erit & segmentum BE ad segmentum BC vt BED circulus ad ellipsim BCD : & permutando ut segmentum BE ad circulum BED, sic BC segmentum est ad ellipsim BCD: idem similiter de reliquis segmentis ostenditur : igitur datae ellipsi BCD polygonum inscribsimus, &c. Quod erat faciendum.

PROPOSITIO CCII.

Esto ABC ellipsi inscriptum quadrilaterum regulare ABCD, ductaeque per G centrum uni e diametris coniugatis aequalibus EF; aequalis sumatur recta HI : qua diametro circulus describatur HIK, cui inscribatur quadratum KLMN.

Dico quadrata KL, LN, NM, MK simul sumpta aequari quadratis AB, BC, CD, DA simul sumptis.



Demonstratio.

Divisa KL bifariam in O ducatur per O diameter HI : dein applicetur ad FE diametrum ordinatim linea PQ, segmentum auferens aequale segmento AB: divisisque AB, AD bifariam in R & S, ducantur semidiametri GRT, GSV iungaturque puncta AG, BG, CG, DG. Quoniam segmenta AB, BC, CD, DA per constructionem sunt aequalia, erunt & sectors AGB, BGC, CGD, AGD aequales, adeoque AG, BG diametri coniugatae. Praeter cum ex const. GT, GV bisecent e centro rectas AB, AD sectores AGT, AGV, dimidia pars sunt sectorum AGB, AGD, hoc est semiellipsis; sector igitur TGV quarta pars est ellipseos, ergo GT, GV sunt coniugatae & AB, AD lineae ordinatim ad illas positae: igitur tum per constructione segmentum PEQ aequale sit segmento ATB, sive AFD erit PQ quadratum bis sumptum, aequale quadratis AB, AD simul sumptis, adeoque PQ quadratum quarto sumptum aequale quadratis AB, BC, CD, DA. Rursum cum segmentum PQ sit ad segmentum KL, ut ABC ellipsis ad circulum KLM, EF, HI diametri sunt in X & O, proportionalter divisae, adeoque PQ linea aequalis lineae KL. Igitur & quadratum KL quater sumptum aequale est quadratis AB, BC,CD, AD. Quod erat demonstrandum.

PROPOSITIO CCIII.

Esto ABC ellipsi inscriptum quodcunque polygonum regulare ABCD, EFGH : ductaque IK una ex diametris coniugatis aequalibus; describatur circulus LMN habens diametrum aequalem diametro IK: dein circulo inscribatur polygonum regulare tot laterum quot est polygonum ellipsi inscriptum.

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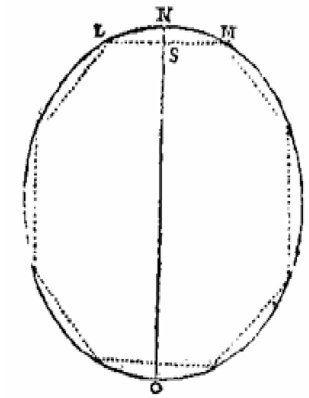
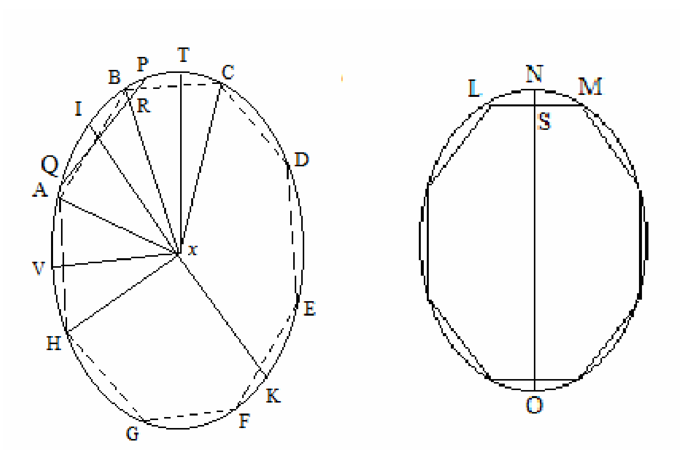
Dico omnia quadrata laterum polygoni ellipsi inscripti, simul sumpta aequari quadratis laterum polygoni circularis simul sumptis.



Demonstratio.



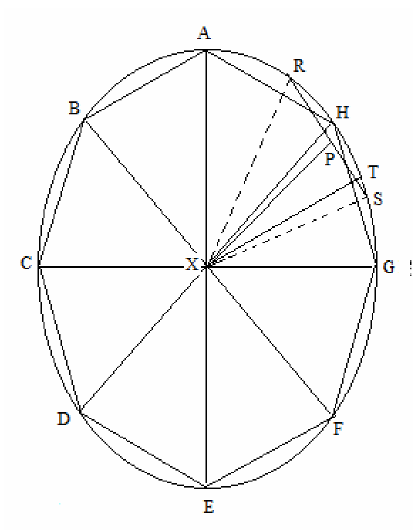
Statuamus E, G, ellipsi & circulo inscripta esse octogona regularia, eadem quippe demonstratio polygonis omnibus conveniet, in circulo LMN ducatur diameter NO secans LM, lineam bifariam in S; diametrum vero IK secet ordinatim linea PQ segmentum auferens aequale segmento AB, tum ducantur semidiametri HX, AX, BX, CX, item TX, VX quae lineas AH, CB dividant bifariam AXB, BXC, &c. aequales: sunt autem illi sumpta aequales toti ellipsi, igitur sector. Quoniam segmenta AB, BC, CD, &c. sunt ex constructione aequalia, erunt & sectore A es duo AXB, B XC hoc est quarta pars sectorum, erunt quadrans ellipis ABC. Iam quia XT, XV ex centro ductae bisecant BC, AH, erunt sectores CXT, XAV dimidii sectorum aequalium BXC, AXH , ac proinde inter se aequales; addito igitur communi sectore AXT, totus sector VXT, sectori toti AXC aequatur, quare cum AXC sit quadrans ellipseos, erit & VXT. Ergo VX, TX diametri sunt coniugatae, ad quas AHCB sunt ordinatim positae auferentes segmenta aequalia; igitur quadratum PQ bis sumptum, est aequale quadratis AH, CB simul sumptis : eodem modo ostenditur idem quadratum PQ bis sumptum aequari quadratis AB, GH simul sumptis: adeoque PQ quadratum quarto sumptum aequatis CB, BA, AH, HG, id est quadratis GF, FE, ED, DC. quare & quadratum PQ sumptum octies quot laterum est polygonum; aequale est quadratis laterum totius polygoni elliipsi inscripti. Iterum cum sit ut ellipsis ABC ad circulum LMN, ita segmentum AB id est PQ ad segmentum LM, erunt IK, NO diametri in R & S, proportionaliter divisae. & quadratum PQ aequale quadrato LM. Ergo quadratum PQ octies sumptum aequatur quadatis laterum polygoni circularis. Sed quadratum PQ octies sumptum equatur etiam, ut supra ostendi, quadratis laterum polygoni elliptici; ergo quadrata laterum polygoni elliptici simul sumpta aequantur quadratis laterum polygoni circulis simul sumptis. Quod erat demonstrandum.

Corollarium.

Hinc patet: si eidem ellipsi duo inscribantur polygona parium numero laterum, quadrata laterum unius polygoni simul sumpta, aequalia esse quadratis laterum alterius polygoni simul sumptis.

PROPOSITIO CCIV.

Ellipsi inscriptum sit polygonum regulare, una autem coniugatarum aequalium sit QP, ad quam sit ordinatim RS.
Dico quadrata laterum polygoni simul sumpta esse ad totum polygonum ut linea RS ad dimidium rectae TQ.
Demonstratio.



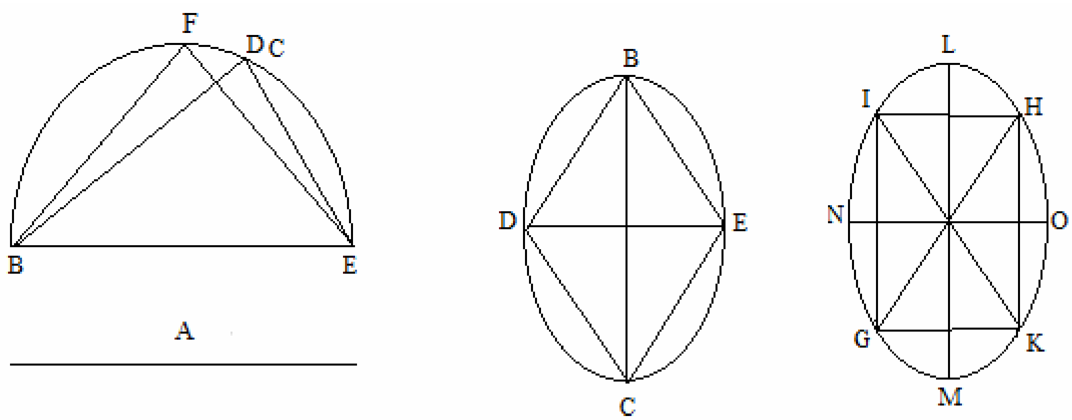
Ducantur ex A, B, C, D, E, F, G, H, R, S punctis semidiametri. Quoniam RS segmentum per constructionem est aequale segmento AB, erit & triangulum RQS aequale triangulo AQB: similiter ostendam triangula singula BQC, CQD, &c. aequari triangulo RQS: adeoque RQS triangulum octies sumptum aequale toti polygono: est autem RS quadratum octies sumptum aequale quadratis omnium laterum polygoni, igitur ut RS quadratum octies sumptum est ad triangulum RQS octies sumptum hoc est ut RS quadratum semel sumptum ad R, Q, S, triangulum semel sumptum, ita omnia quadrata laterum polygoni ad totum polygonum. sed cum RS quadratum sit ad rectangulum super RS, TQ, ut RS linae ad lineam TQ, erit RS quadratum ad triangulum RQS. dimidium rectanguli RS, TQ, ut RS linea ad dimidium rectae TQ, igitur, & omnia quadrata laterum polygoni sunt ad totum polygonum ut RS linea ad dimidium lineae TQ. Quod erat demonstrandum.

PROPOSITIO CCV.

Datis axibus & diametro ellipseos invenire illius coniugatam & positione cum datis axibus in eadem constituere ellipsi.

Constructio & demonstratio.

Sit A diameter data, & axes dati BC, DE. oportet invenire diametrum coniugatam ipsi A, quam cum datis axibus oportet in eadem collocare ellipsi; axes ED, HC ad angulum ponantur rectum ECB, iunctaque BE , super ea semicirculus describatur ECB, in quo datae, A aequalis aptetur EF, ducaturque FB: quoniam igitur EC, CB axium quadrata aequalia sunt quadratis cuiusvis coniugationis in ellipsi,eademque axium quadrata aequentur quadratis EF,FB, & EF aequalis A una sit ex diametris, recta FB diametro est coniugata FE.: cxhibuimus igitur diametro A, coniugatam, quod primo faciendum fuit.



Iungantur deinde axium extrema DBEC quae parallelogrammum exhibeant DBEC: cui aequale fiat parallelogrammum a IHKG, quod IK, HG diametros habeat rectis EF, FB aequales. Data igitur diametrorum IK HG coniugatione, exhibeantur positione axes LM, NO adeoque & ellipsis LMN. Erit illa aequalis ellipsi BEC cuius axes dati sunt BC, ED ; cum enim per extrema coniugationis positione datae, unica tantum ellipsis transeat, & IK, GH coniugatio positione sit in ellipsi LMN, eademque; pertineat ad ellipsin BEC, ellipses LMN, BEC adeoque & axes aequales sunt: ellipsis igitur LMN, illa est: in qua coniugatas EF, FB;id est IK,GH positione cum datis axibus BC, DB id est: LM, NO collocare oportebat.

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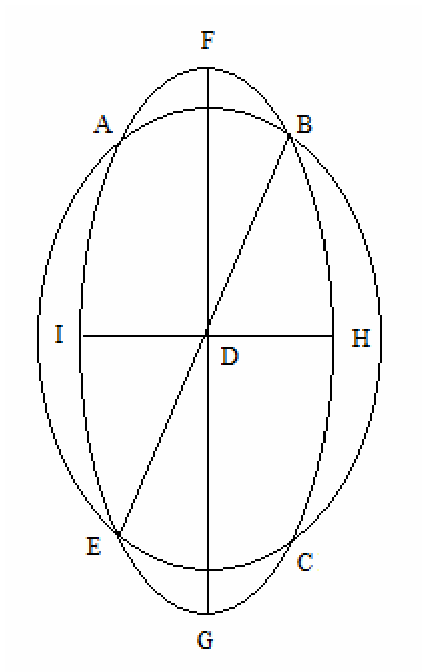
PROPOSITIO CCVI.

Data ellipsi & circulo illam intersecante puncta intersectionis geometrice exhibere. Oportet autem circulum & ellipsim idem habere centrum.

Constructio & demonstratio.



Ellipsim ABC cuius centrum D, intersecet circulus AB, CE oportet intersectionis puncta exhibere; quoniam igitur circulus & ellipsis commune habent centrum D, si ex illo ad intersectionis aliquod punctum recta intelligatur duci, erit illa semidiameter circuli & ellipsis, diametrus igitur aliqua, circulo & ellipsi communis , sit illa AC; deinde cum ellipus data sit, dati quoque sunt axes FG, HI datis igitur axibus & diametro AC, inveniatur illius coniugata & per praecedentem coniugatio illa cum datis axibus in eadem. positionc collocetur ellipsi, illarum una, puncta assignabit intersectionum. Quod erat praestandum.

Finis libri quarti